

ON A CERTAIN POISSON FORMULA¹

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Introduction. Let G denote a locally compact commutative group with a lattice Γ , G^* its dual, and $\langle g, g^* \rangle = g^*(g)$ for every (g, g^*) in $G \times G^*$; let Γ_* denote the annihilator of Γ in G^* and dg the Haar measure on G such that G/Γ is of measure 1. Finally, let F denote an L^1 -function on G and F^* its Fourier transform defined by

$$F^*(g^*) = \int_G F(g) \langle g, g^* \rangle dg .$$

Then, under suitable conditions on F , we have

$$\sum_{\gamma \in \Gamma} F(\gamma) = \sum_{\gamma^* \in \Gamma_*} F^*(\gamma^*) ,$$

in which both sides are absolutely convergent. This is a classical Poisson formula.

Let X denote a locally compact commutative group, dx a Haar measure on X , and f a continuous mapping of X to the above group G ; for every Φ in the Schwartz-Bruhat space $S(X)$ of X , define a function $F^* = F_\Phi^*$ on G^* as

$$F_\Phi^*(g^*) = \int_X \Phi(x) \langle f(x), g^* \rangle dx .$$

Then, under suitable conditions on f , the Haar measure dx decomposes into a family of tempered measures $d\mu_g$, where $\text{Supp}(d\mu_g)$ is contained in $f^{-1}(g)$ for every g in G , such that the above Poisson formula holds for $F = F_\Phi$ defined by

$$F_\Phi(g) = \int_X \Phi(x) d\mu_g(x) .$$

This variant is due to Weil [9]; it is an "abstract form" of the Siegel formula.

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Let k denote a global field, i.e., a number field or a function field of one variable with a finite constant field; let the subscripts A and k denote the adelization relative to k and the taking of k -rational points, respectively; let ψ denote a non-trivial character of k_A/k and identify k_A with its dual by $(i, i^*) \rightarrow \psi(ii^*)$. We shall fix a universal domain K containing k and identify K with an affine line over the prime field. We shall change our notation slightly: let X denote an affine space and f a morphism of X to K defined over k ; let $|dx|_A$ denote the Haar measure on X_A such that X_A/X_k is of measure 1. Then we can take k_A as $G = G^*$, k as $\Gamma = \Gamma_*$, X_A as X , $|dx|_A$ as dx , and f_A as f . During the past several years, we became interested in proving a Poisson formula (of Weil's type) in the above setup. In this paper, we shall consider the special case where f is homogeneous of degree at least 2 and "strongly non-degenerate" in the sense that it is submersive everywhere except at the origin 0 of X . We shall show, in that case, that the Poisson formula holds if $\text{char}(k)$ does not divide $\deg(f)$ and

$$\dim(X) > 2 \deg(f) .$$

In this formula, everything is explicitly defined; for a complete statement, we refer to § 5, Th. 5. It appears that the simplicity of the above condition is quite remarkable. We have included an additional section on some numerical coefficients of certain asymptotic expansions.

1. A review of some results. We shall keep the notation in the introduction. Let v denote a valuation on the global field k and k_v the corresponding local field; let ψ_v denote the product of the canonical injection $k_v \rightarrow k_A$ and the non-trivial character ψ of k_A/k . We recall that X is an affine space defined over k ; we introduce coordinates in X with respect to a k -base of X_k . Let $(x_1, \dots, x_n), (y_1, \dots, y_n)$ denote coordinates of x, y in X ; then $[x, y] = x_1 y_1 + \dots + x_n y_n$ defines a non-degenerate symmetric bilinear form on $X \times X$. Let X_v denote the vector space over k_v of k_v -rational points of X ; let $|dx|_v$ denote the autodual (or "self-dual") measure on X_v relative to the bicharacter $(x, y) \rightarrow \psi_v([x, y])$ of $X_v \times X_v$. Then the restricted product measure of all $|dx|_v$ becomes the autodual measure on X_A relative to the bicharacter $(x, y) \rightarrow \psi([x, y])$ of $X_A \times X_A$; this measure coincides with the Haar measure $|dx|_A$ on X_A such that X_A/X_k is of measure 1.

If we take 1 as a k -base of k , what we have said can be applied to the universal domain K instead of X : we shall denote by $|di|_v$ the autodual measure on k_v relative to the bicharacter $(i, i^*) \rightarrow \psi_v(ii^*)$ of $k_v \times k_v$ and by $|di|_A$ the restricted product measure of all $|di|_v$, etc. We shall denote by $|\cdot|_v$ the absolute value on k_v defined by $|d(i_0 i)|_v = |i_0|_v |di|_v$ for every $i_0 \neq 0$ in k_v .

We recall that $f: X \rightarrow K$ is a morphism defined over k ; it gives rise to a continuous mapping, in fact a k_v -analytic mapping, $f_v: X_v \rightarrow k_v$ for every v . If there is no ambiguity, we shall denote f_v also by f . Let X' denote the set of points of X where f is submersive, i.e., where the cotangent vector df does not vanish; then X' is a Zariski open subset of X defined over k . We observe that f is strongly non-degenerate if and only if $X - X' \subset \{0\}$; for a moment, we shall only assume that $X - X' \subset f^{-1}(0)$. We put $U(i) = f^{-1}(i) \cap X'$ for every i in K ; we have $U(i) = f^{-1}(i)$ if $i \neq 0$. For every i in k_v , let $U(i)_v$ denote the set of k_v -rational points of $U(i)$; then $U(i)_v$ becomes a k_v -analytic manifold, and the union of all $U(i)_v$ coincides with the similarly defined open subset X'_v of X_v . Moreover, there exists a Borel measure $|\theta_i|_v$ on each $U(i)_v$ such that

$$\int_{X_v} \phi(x) |dx|_v = \int_{k_v} \left(\int_{U(i)_v} \phi(x) |\theta_i(x)|_v \right) |di|_v$$

for every continuous function ϕ on X_v with compact support contained in X'_v ; the measure $|\theta_i|_v$ admits an explicit analytic expression; cf. [9], pp. 12–13.

We define a function F_Φ on $k_v^\times = k_v - \{0\}$ for every Φ in the Schwartz-Bruhat space $\mathcal{S}(X_v)$ of X_v as

$$F_\Phi(i) = \int_{U(i)_v} \Phi(x) |\theta_i(x)|_v .$$

We also define a function F_Φ^* on k_v as

$$F_\Phi^*(i^*) = \int_{X_v} \Phi(x) \psi_v(f_v(x) i^*) |dx|_v .$$

Finally, for every quasicharacter ω of k_v^\times which is bounded around 0, we put

$$Z(\omega, \Phi) = \int_{X_v} \omega(f_v(x)) \Phi(x) |dx|_v .$$

In AE we developed a coherent theory of the above three types of functions; in the following, we shall recall some of our results:

Suppose first that k_v is an \mathbf{R} -field, i.e., v is archimedean; then, for every quasicharacter ω of k_v^\times and t in k_v^\times , we have

$$\omega(t) = |t|_v^s (|t|^{-1}t)^p,$$

in which s is in \mathbf{C} and p in \mathbf{Z} ; we have $p = 0, 1$ if $k_v = \mathbf{R}$. Conversely, for every s in \mathbf{C} and p in \mathbf{Z} , the above prescription defines a quasicharacter of k_v^\times . The complex power (or the "local zeta function") $Z(\omega, \Phi)$ has a meromorphic continuation to the whole complex Lie group of quasicharacters of k_v^\times with poles only on the negative real axis of the s -plane. If $-\lambda$ is a pole of $Z(\omega, \Phi)$ and

$$\sum_{i=1}^{m_\lambda} b_{\lambda,i}(p)(s + \lambda)^{-i}$$

the principal part of its Laurent expansion around $-\lambda$, we have the following asymptotic expansion:

$$F_\Phi^*(\gamma^{-1}t) \approx \sum_\lambda \sum_{i=1}^{m_\lambda} a_{\lambda,i}^*(|t|^{-1}t) |t|_v^{-\lambda} (\log |t|_v)^{i-1}$$

as $|t|_v \rightarrow \infty$. The constant γ on the left hand side is an element of k_v^\times defined by $\psi_v(t) = e(\gamma t)$, $e(2 \operatorname{Re}(\gamma t))$ for every t in $k_v = \mathbf{R}, \mathbf{C}$, respectively. The coefficients $a_{\lambda,i}^*(u)$ on the right hand side are determined by $b_{\lambda,i}(p)$ as follows: put $m_v = 2, 2\pi, d = \frac{1}{2}, 1$ for $k_v = \mathbf{R}, \mathbf{C}$, respectively, and

$$b_p(s) = i^{|p|} (2d\pi)^{d(1-2s)} \Gamma(ds + \frac{1}{2}|p|) / \Gamma(d(1-s) + \frac{1}{2}|p|);$$

then

$$a_{\lambda,i}^*(u) = (1/m_v) \sum_p \left(\sum_{j=1}^{m_\lambda} ((-1)^{i+j} / (i-1)! (j-i)!) b_{\lambda,j}(p) \cdot (d^{j-i} b_p(s) / ds^{j-i})_{s=\lambda} \right) u^p.$$

We refer to AE-II, Th. 2 for the proof (in the case where $\gamma = 1$).

Suppose next that $K = k_v$ is a p -field, i.e., v is non-archimedean; let R denote the maximal compact subring of K , P its maximal ideal, and $R/P = \mathbf{F}_q$; let π denote an element of $P - P^2$ and write an arbitrary element t of K^\times as $\pi^e u$ with e in \mathbf{Z} and u in $K_1^\times = R - P$; then, for every quasicharacter ω of K^\times , we have

$$\omega(t) = z^e \chi(u) ,$$

in which z is in C^\times and χ is a character of K_1^\times . Conversely, for every z in C^\times and χ in the dual of K_1^\times , the above prescription defines a quasi-character of K^\times ; and the complex power $Z(\omega, \Phi)$ becomes a rational function of z provided that $\text{char}(k) = 0$. If we write

$$Z(\omega, \Phi) \equiv \sum_{\alpha} \sum_{i=1}^{m_{\alpha}} b_{\alpha, i}(\chi) (1 - \alpha^{-1}z)^{-i} \quad \text{mod } C[z, z^{-1}] ,$$

we have $|\alpha| > 1$ and

$$\begin{aligned} F_{\Phi}^*(\pi^e u) = & \left[(1 - q^{-1})^{-1} \sum_{\alpha, i} \sum_{j=1}^{m_{\alpha}} \binom{i - d - e - e_x - 1}{i - 1} b_{\alpha, i}(\chi) g_{x^{-1}} \alpha^{e_x} \chi(u) \right. \\ & \left. + \sum_{\alpha} \sum_{i=1}^{m_{\alpha}} \left(\sum_{j=0}^{\infty} \binom{i - d - e + j - 1}{i - 1} \alpha^{-j} \right) b_{\alpha, i}(1) \right] \alpha^{d+e} \end{aligned}$$

for all small e , in which d is the largest integer such that $\psi_v = 1$ on P^{-d} and e_x is the smallest positive integer such that $\chi = 1$ on $1 + P^{e_x}$; g_x for $\chi \neq 1$ is a complex number of absolute value $q^{-\frac{1}{2}e_x}$ and $g_1 = -q^{-1}$. Again we refer to AE-II, Th. 2 for the proof (of an equivalent statement).

About the function F_{Φ} , we have only to know the following: if k_v is an R -field, F_{Φ} is an infinitely differentiable function on k_v^\times such that $F_{\Phi}(i)$ tends to 0 as $|i|_v \rightarrow \infty$ more rapidly than any negative power of $|i|_v$; if k_v is a p -field, F_{Φ} is a locally constant function on k_v^\times with bounded support in k_v . Moreover, the limit $F_{\Phi}(0)$ of $F_{\Phi}(i)$ as $|i|_v \rightarrow 0$ exists for every Φ in $\mathcal{S}(X_v)$ if and only if F_{Φ}^* is an L^1 -function on k_v (for every Φ); cf. AE-II, Th. 2.

Finally, we recall that the information about $Z(\omega, \Phi)$ comes from the existence of a "Hironaka resolution" of the singularities of f . If X^* is the projective space obtained from X by "adding" a hyperplane at infinity, say E , and \mathcal{I} the sheaf of ideals associated with the divisor of zeros of the extension f^* of f to a function on X^* , the Hironaka resolution of (E, \mathcal{I}) is the one defined by his "Main Theorem II (N)" in [3], p. 176. In the case where f is strongly non-degenerate, without any assumption on the characteristic of k , the Hironaka resolution exists and is unique; as a morphism, it is simply the monoidal transformation of X^* with the origin 0 of X as its center; and it is "tame" if $\text{char}(k)$ does not divide $\deg(f)$. In particular, the above "provision" can be replaced by this condition. In the following sections, we shall tacitly

assume that f is strongly non-degenerate; we put $\dim(X) = n$ and $\deg(f) = m \geq 2$.

2. Asymptotic formulas. We shall first consider the case where k_v is an \mathbf{R} -field and (leaving the ambiguity of a numerical constant) determine the "first term" of the asymptotic expansion of $F_\Phi^*(i^*)$ as $|i^*|_v \rightarrow \infty$; we recall that Φ is an arbitrary Schwartz function on X_v .

THEOREM 1. *We have*

$$F_\Phi^*(i^*) \approx c_v \Phi(0) |i^*|_v^{-n/m} + \dots$$

as $|i^*|_v \rightarrow \infty$, in which $c_v = c_0 + c_1 \operatorname{sgn}(i^*)$ if $k_v = \mathbf{R}$; c_0, c_1 , and c_v for $k_v = \mathbf{C}$ are independent of Φ and i^* .

Proof. Consider the complex power $Z(\omega, \Phi)$; then it becomes a finite sum of the following four types of integrals:

$$\begin{aligned} \int_{k_v} \omega(t^m) |t|_v^{n-1} \phi_1(t) |dt|_v, & \quad \int_{k_v^*} \omega(y_1^m y_2) |y_1|_v^{n-1} \phi_2(y) |dy|_v, \\ \int_{k_v} \omega(t) \phi_3(t) |dt|_v, & \quad \int_{X_v} \omega(f_v(x)) \phi_4(x) |dx|_v, \end{aligned}$$

in which all ϕ 's are Schwartz functions; of these ϕ_1, ϕ_2, ϕ_3 have compact supports; $\operatorname{Supp}(\phi_i)$ does not contain 0; and $\phi_1(0) = \text{const.}$ $\Phi(0), \phi_2(0, t) = \Phi(0)\theta(t)$ with the "const." and the θ both independent of Φ . Therefore, if $\omega(t) = |t|_v^s (|t|^{-1}t)^p$ for every t in k_v^\times , then the poles of the meromorphic continuation of $Z(\omega, \Phi)$ are among the following sequences in the s -plane:

$$\begin{aligned} -n/m - (1/2dm)\text{-times } 0, 1, 2, \dots, \\ -1 - (1/2d)\text{-times } 0, 1, 2, \dots, \end{aligned}$$

in which $d = \frac{1}{2}$ or 1 according as $k_v = \mathbf{R}$ or \mathbf{C} . Moreover, the order of a pole $-\lambda$ is at most 2. For our purpose, we have only to examine the principal parts at those poles which are not smaller than $-n/m$.

In the case where $k_v = \mathbf{R}$, the principal parts in question are as follows:

- (1) $A_p \Phi(0)(s + n/m)^{-1}$;
- (2) $A'_{\lambda, p}(s + \lambda)^{-1}$, where λ is a positive integer at most equal to n/m and $A'_{\lambda, p} \neq 0$ only if $\lambda \not\equiv p \pmod{2}$;
- (3) $A''_p \Phi(0)(s + n/m)^{-2}$ if n/m is an integer, where $A''_p \neq 0$ implies $n/m \not\equiv p \pmod{2}$.

Moreover A_p in (1) and A_p'' in (3) are independent of Φ . Each one of these principal parts contributes to the asymptotic expansion of $F_\Phi^*(\gamma^{-1}t)$ as $|t|_v \rightarrow \infty$, in which $\psi_v(t) = e(\gamma t)$. The contribution can be determined by the formula in the previous section: from (1) we get

$$\left(\frac{1}{2}\right) \left(\sum_p A_p b_p(n/m) (\operatorname{sgn} t)^p \right) \Phi(0) |t|_v^{-n/m};$$

from (2) we get no contribution because $A'_{\lambda,p} \neq 0$ implies $\Gamma(\frac{1}{2}(1 - \lambda + p)) = \infty$, hence

$$b_p(\lambda) = i^p \pi^{(1/2)(1-2\lambda)} \Gamma(\frac{1}{2}(\lambda + p)) / \Gamma(\frac{1}{2}(1 - \lambda + p)) = 0;$$

and, for a similar reason, from (3) we only get

$$-\left(\frac{1}{2}\right) \left(\sum_p A_p'' (db_p(s)/ds)_{s=n/m} (\operatorname{sgn} t)^p \right) \Phi(0) |t|_v^{-n/m}.$$

This implies our theorem for $k_v = R$.

In the case where $k_v = C$, the principal parts in question are as follows:

- (1) $A_p \Phi(0)(s + n/m)^{-1}$, where $A_p \neq 0$ implies $p = 0$;
- (2) $A'_{\lambda,p}(s + \lambda)^{-1}$, where $\lambda = 1 + \frac{1}{2}|p| + i$ for some non-negative integer i such that λ is at most equal to n/m ;
- (3) $A_p'' \Phi(0)(s + n/m)^{-2}$ if n/m is an integer, and $A_p'' \neq 0$ implies $p = 0$.

Moreover A_p in (1) and A_p'' in (3) are independent of Φ . Each one of these principal parts contributes to the asymptotic expansion of $F_\Phi^*(\gamma^{-1}t)$ as $|t|_v \rightarrow \infty$, in which $\psi_v(t) = e(2 \operatorname{Re}(\gamma t))$: from (1) we get

$$(1/2\pi) A_0 b_0(n/m) \Phi(0) |t|_v^{-n/m};$$

from (2) we get no contribution (as in the previous case); and from (3) we only get

$$-(1/2\pi) A_0'' (db_0(s)/ds)_{s=n/m} \Phi(0) |t|_v^{-n/m}.$$

This implies our theorem for $k_v = C$. q.e.d.

Remark. The constants c_0, c_1 for $k_v = R$ and c_v for $k_v = C$ have the following properties: c_0, c_v are real and c_1 is pure imaginary; and $c_1 = 0$ if m is odd. These properties can be proved in two ways. One way is to make the above proof more precise: we observe that the “const.”

in $\phi_1(0) = \text{const. } \Phi(0)$ and $\theta(t)$ in $\phi_2(0, t) = \Phi(0)\theta(t)$ are real. This implies that A_p and A_p'' are real. Moreover, in the case where $k_v = \mathbf{R}$, we get $mp \equiv 0 \pmod{2}$ from $A_p \neq 0$ and also from $A_p'' \neq 0$. The properties of c_0, c_1 , and c_v follow immediately from these. Another way is simply to manipulate the asymptotic formula in the theorem: we observe that the complex conjugation applied to $F_\Phi^*(i^*)$ has the effect of replacing (Φ, i^*) by $(\bar{\Phi}, -i^*)$. If $k_v = \mathbf{C}$, the asymptotic formula as $|i^*|_v \rightarrow \infty$ of the complex conjugate of $F_\Phi^*(i^*)$ can be obtained in two different ways; and we get $\bar{c}_v = c_v$. If $k_v = \mathbf{R}$, we similarly get $\bar{c}_0 + \bar{c}_1 = c_0 - c_1$; this implies that c_0 is real and c_1 pure imaginary. In the integral defining $F_\Phi^*(i^*)$, we replace x by $-x$; then $F_\Phi^*(i^*)$ and $\Phi(0)$ remain unchanged. If m is odd, however, this has the effect of changing the sign of i^* . Therefore, by passing to the asymptotic formula, we get $c_1 = 0$.

We shall consider the case where $K = k_v$ is a p -field and determine $F_\Phi^*(i^*)$ for all large $|i^*|_v$; we recall that Φ is an arbitrary locally constant function with compact support on X_v .

THEOREM 2. *We have*

$$F_\Phi^*(\pi^e u) = \left[\sum_{\alpha^m = q^n} \left(\sum_{\chi^m = 1} c_{\alpha, \chi} \chi(u) \right) \alpha^e \right] \Phi(0)$$

for all small e in \mathbf{Z} and for every u in K_1^\times , in which $c_{\alpha, \chi}$ are certain constants independent of Φ and $i^* = \pi^e u$.

Proof. We proceed as in the proof of Th. 1: in the non-archimedean case, we only have the first three types of integrals where ϕ 's are locally constant functions with compact support; and $\phi_1(0) = \text{const. } \Phi(0)$, $\phi_2(0, t) = \Phi(0)\theta(t)$. Therefore, if $\omega(\pi^e u) = z^e \chi(u)$ for every e in \mathbf{Z} and u in K_1^\times , we get

$$\begin{aligned} Z(\omega, \Phi) \equiv & \sum_{\alpha^m = q^n} A_{\alpha, \chi} \Phi(0) (1 - \alpha^{-1} z)^{-1} + A' (1 - q^{-1} z)^{-1} \\ & + A'' \delta_{mn} \Phi(0) (1 - q^{-1} z)^{-2} \quad \text{mod } \mathbf{C}[z, z^{-1}], \end{aligned}$$

in which $A_{\alpha, \chi}, A', A''$ are constants and δ_{mn} is the Kronecker delta; $A_{\alpha, \chi}, A''$ are independent of Φ ; $A_{\alpha, \chi} = 0$ unless $\chi^m = 1$ and $A' = A'' = 0$ unless $\chi = 1$. By applying the formula we recalled in the previous section, we get

$$F_{\Phi}^*(\pi^e u) = \left[\sum_{\alpha^m = q^n} ((1 - q^{-1})^{-1} \sum_{\chi^m = 1} A_{\alpha, \chi} g_{\chi^{-1} \alpha^e \chi}(u) \right. \\ \left. + (1 - \alpha^{-1})^{-1} A_{\alpha, 1} \alpha^{d+e} + (1 - q^{-1})^{-2} A'' \delta_{mn} q^{d+e} \right] \Phi(0)$$

for all small e . This can be rewritten as stated in the theorem. q.e.d.

Th. 1 and Th. 2 have the following implication:

COROLLARY. *We have*

$$|F_{\Phi}^*(i^*)| \leq \text{const.} \max(1, |i^*|_v)^{-n/m}$$

for every i^* in k_v , in which the “const.” depends on Φ but not on i^* .

We might mention that such an estimate was difficult to obtain in the archimedean case. In fact, even for a special f , known estimates are considerably less precise than ours.

3. Asymptotic formulas (“good case”). We have introduced coordinates in X with respect to a k -base of X_k . In terms of the coordinates (x_1, \dots, x_n) of x , $f(x)$ becomes a homogeneous polynomial of degree m with coefficients in k . Since f is strongly non-degenerate, there exist a positive integer ρ and n^2 polynomials $A_{ij}(x)$ in $k[x_1, \dots, x_n]$ satisfying

$$x_i^{\rho} = \sum_{j=1}^n A_{ij}(x) f_j(x)$$

for $1 \leq i \leq n$, in which $f_j(x) = \partial f / \partial x_j$. For a non-archimedean valuation v on k , we have started using the notation $K = k_v, R, P$, etc.; we shall use the notation $R^{(n)}$, instead of R^n , to denote $R \times \dots \times R$; we similarly define $P^{(n)}$ etc. Also we shall denote by X_v° the $R^{(n)}$ considered as a (compact open) subset of X_v . We choose v so that the coefficients of $f(x)$ and $A_{ij}(x)$ for all i, j are in R and $d = 0$, i.e., $\psi_v = 1$ on R but not on P^{-1} . We have excluded only a finite number of valuations on k and achieved the following: (1) $f(x)$ is in $R[x_1, \dots, x_n]$; (2) if $\bar{f}(x)$ is the element of $F_q[x_1, \dots, x_n]$ obtained from $f(x)$ by reducing its coefficients modulo P and $\bar{a}_1, \dots, \bar{a}_n$ elements of any extension field of F_q not all 0, at least one $\bar{f}_i(x) = \partial \bar{f}(x) / \partial x_i$ does not vanish at $\bar{a} = (\bar{a}_1, \dots, \bar{a}_n)$; (3) X_v° is of measure 1.

We shall assume that v is such a “good valuation” on k ; and, for the sake of completeness, we shall prove the following two elementary lemmas:

LEMMA 1. Let t denote an element of R of order r and $N_e(t)$ the number of $a \bmod P^e$, where a is in $R^{(n)}$, satisfying $f(a) \equiv t \bmod P^e$; then, if $e > r$, we have

$$q^{-(n-1)e}N_e(t) = \left(\sum_{0 \leq mi < r} q^{-(n-m)i} \right) q^{-(n-1)}(N_1(0) - 1) \\ + \delta'_{r0} q^{-(n/m-1)r-(n-1)} N_1(\pi^{-r}t),$$

in which $\delta'_{r0} = 1$ or 0 according as $r \equiv 0$ or $r \not\equiv 0 \bmod m$; if $r = \infty$, i.e., if $t = 0$, we have

$$q^{-(n-1)e}N_e(0) = q^{e+n[-e/m]} + \left(\sum_{0 \leq mi < e} q^{-(n-m)i} \right) q^{-(n-1)}(N_1(0) - 1),$$

in which $[\]$ is the Gauss symbol.

Proof. Suppose first that $t \neq 0$. If $f(a) \equiv t \bmod P^e$, $a \equiv 0 \bmod P^i$, and $e > r$, then we get $mi \leq r$; hence

$$N_e(t) = \sum_{0 \leq mi \leq r} q^{(m-1)ni} \cdot \text{card},$$

where the “card” is the number of $a \bmod P^{e-mi}$ satisfying

$$f(a) \equiv \pi^{-mi}t \bmod P^{e-mi}, \quad a \not\equiv 0 \bmod P.$$

By the usual lifting process of a solution $\bmod P$ to solutions modulo higher powers of P , we get

$$\text{“card”} = \begin{cases} (N_1(0) - 1)q^{(n-1)(e-mi-1)} \\ N_1(\pi^{-r}t)q^{(n-1)(e-mi-1)} \end{cases}$$

according as $mi < r$ or $mi = r$; the rest is clear.

Suppose next that $t = 0$. If $a \equiv 0 \bmod P^i$ and $mi \geq e$, we obviously have $f(a) \equiv 0 \bmod P^e$; hence

$$N_e(0) = 1 + \sum_{e \leq mi < me} (1 - q^{-n})q^{n(e-i)} + \sum_{0 \leq mi < e} q^{(m-1)ni} \cdot \text{card},$$

where the “card” is the number of $a \bmod P^{e-mi}$ satisfying $f(a) \equiv 0 \bmod P^{e-mi}$, $a \not\equiv 0 \bmod P$. Hence, by the usual lifting process, we get

$$\text{“card”} = (N_1(0) - 1)q^{(n-1)(e-mi-1)};$$

the rest is straightforward. q.e.d.

LEMMA 2. Let Φ denote the characteristic function of X_o° ; then for every i^* in K^\times of order $-e \leq 0$, we have

$$F_{\phi}^*(i^*) = \begin{cases} q^{n[-e/m]} \\ q^{-n((e-1)/m+1)} \sum_{t \bmod P} \psi_v(\pi^{e-1} i^* t) N_1(t) \end{cases}$$

according as $e \not\equiv 1$ or $e \equiv 1 \pmod{m}$.

Proof. By Lemma 1 and by the orthogonality of characters (of a finite commutative group), we get

$$\begin{aligned} F_{\phi}^*(i^*) &= \int_{X_0} \psi_v(f(x)i^*) |dx|_v \\ &= \sum_{t \bmod P^e} \psi_v(ti^*) q^{-ne} N_e(t) \\ &= q^{-e} (q^{e+n[-e/m]} \\ &\quad + \left(\sum_{e-1 \leq m i < e} q^{-(n-m)i} \right) q^{-(n-1)} (N_1(0) - 1) \\ &\quad + \delta'_{e-1,0} q^{-(n/m-1)(e-1)-n+1} \sum' \psi_v(ti^*) N_1(\pi^{-(e-1)} t)), \end{aligned}$$

where the summation \sum' is relative to $t \bmod P^e$ satisfying $\text{ord}(t) = e - 1$. The rest is straightforward. q.e.d.

In the notation of Lemma 2, we have

$$(*) \quad |F_{\phi}^*(i^*)| \leq |i^*|_v^{-n/m}$$

for every i^* in $K - P$ provided that $e = -\text{ord}(i^*) \not\equiv 1 \pmod{m}$; we shall examine the case where $e \equiv 1 \pmod{m}$: we change our notation slightly and denote by t an element of F_q , by ψ any non-trivial character of F_q , and by $N(t)$ the number of solutions of $\bar{f}(x) - t = 0$ in F_q^n . Then by Lemma 2 we can rewrite (*) as

$$(**) \quad \left| q^{-n} \sum_t \psi(t) N(t) \right| \leq q^{-n/m},$$

in which the summation is taken over F_q ; it is equal to the sum of $\psi(\bar{f}(a))$ for a running over F_q^n . In the special case where $m = 2$, it is well known (and easy to show) that (**) holds with the equality sign. On the other hand, if $n \geq 3$, then $\bar{f}(x) - tx_0^m$ is absolutely irreducible for every t in F_q and $\bar{f}(x) - tx_0^m = 0$ defines a non-singular projective hypersurface over F_q . Therefore we can apply the "Riemann-Weil hypothesis" proved recently (after the works of Grothendieck and others) by Deligne [1]; see also Dwork [2]:

In the above notation, "there exist complex numbers $\alpha_1, \alpha_2, \dots$ of absolute value $q^{(1/2)n-1}$ such that

$$N(0) = q^{n-1} + (-1)^n(q-1) \sum_i \alpha_i ;$$

also there exist complex numbers $\alpha_1(t), \alpha_2(t), \dots$ of absolute value $q^{(1/2)(n-1)}$ depending on $t \neq 0$ such that

$$N(t) = q^{n-1} - (-1)^n \left(\sum_i \alpha_i + \sum_j \alpha_j(t) \right);$$

the number of α_i 's and the number of $\alpha_j(t)$'s depend only on m and n .

If we use this result, we get

$$q^{-n} \sum_t \psi(t) N(t) = (-1)^n q^{-n} \left(q \sum_i \alpha_i - \sum_{t \neq 0} \psi(t) \left(\sum_j \alpha_j(t) \right) \right),$$

hence

$$\left| q^{-n} \sum_t \psi(t) N(t) \right| \leq c \cdot q^{-(1/2)(n-1)}$$

for some constant c depending only on m and n . Since we have $\frac{1}{2}(n-1) - n/m \geq 1/6$ for $n > m \geq 3$, the inequality:

$$c \cdot q^{-(1/2)(n-1)} > q^{-n/m}$$

implies $c^6 > q$; the number of such v 's is finite. We have thus obtained the following theorem:

THEOREM 3. *Suppose that $n > m$, i.e., $\dim(X) > \deg(f)$; then there exists a finite set S of valuations on k containing the set S_∞ of archimedean valuations such that if v is not in S and Φ is the characteristic function of X_v° , we have*

$$|F_\Phi^*(i^*)| \leq \max(1, |i^*|_v)^{-n/m}$$

for every i^* in k_v .

We recall that we have the standing hypothesis that f is strongly non-degenerate and $m \geq 2$.

4. A tempered measure on X_A . We shall start from a local consideration; the following lemma follows from what we have reviewed in §1, from the corollary in §2 (of Th. 1 and Th. 2), and from AE-II, Th. 3:

LEMMA 3. *Suppose that $n > m$; then the image measure of $|\theta_0|_v$ under the inclusion $U(0)_v \rightarrow X_v$ is tempered and*

$$F_{\Phi_v}(i) = \int_{X_v} \Phi_v(x) |\theta_i(x)|_v$$

defines a continuous L^1 -function F_{Φ_v} on k_v for every Φ_v in $\mathcal{S}(X_v)$. Moreover its Fourier transform $F_{\Phi_v}^*$ is also a continuous L^1 -function on k_v ; and we have

$$F_{\Phi_v}(i) = \int_{k_v} F_{\Phi_v}^*(i^*) \psi_v(-ii^*) |di^*|_v$$

for every i in k_v .

For every i in k_v , we have recalled the definition of $U(i)_v$ in §1. Suppose that v is a good valuation. If i is v -integral and different from 0, we define $U(i)_v^\circ$ as the (compact open) subset of $U(i)_v$ consisting of v -integral points; if $i = 0$, we define $U(0)_v^\circ$ as the subset of $U(0)_v$ consisting of v -primitive points, i.e., v -integral points with v -units among their coordinates. Then, in the notation of the previous section, we have

$$\begin{cases} \int_{U(i)_v^\circ} |\theta_i|_v = q^{-(n-1)} N(\bar{i}) \\ \int_{U(0)_v^\circ} |\theta_0|_v = q^{-(n-1)} (N(0) - 1) \end{cases}$$

provided that \bar{i} , the residue class of $i \bmod P$, is different from 0. We can apply Deligne's result to the right hand sides; and we get

$$\left| \int_{U(i)_v^\circ} |\theta_i|_v - 1 \right| \leq c \cdot q^{-(1/2)(n-1)}, c \cdot q^{-(1/2)n+1}$$

according to the cases, in which c is a constant independent of v ; the upper bound $c \cdot q^{-(1/2)n+1}$ can obviously be used in both cases.

We shall change our notation slightly and denote by i an element of k ; then the above estimates hold for almost all v . Consider the following infinite product:

$$\prod_v' \int_{U(i)_v^\circ} |\theta_i|_v$$

extended over the set of good valuations or over its subset defined by $|i|_v = 1$ according as $i = 0$ or $i \neq 0$. Then, if $\frac{1}{2}n - 1 > 1$, i.e., if $n > 4$, it is convergent (in the usual sense if we exclude a finite number of factors which may be 0). Therefore the restricted product measure

$|\theta_i|_A$ of all $|\theta_i|_v$ is defined on the adelization $U(i)_A$ of $U(i)$ provided that $n > 4$; this provision can be replaced by $n > 3$ for $i \neq 0$. Moreover the image measure of $|\theta_i|_A$ under $U(i)_A \rightarrow X_A$ is tempered for every $i \neq 0$; by Lemma 3 it is also tempered for $i = 0$ if $n > m$. Therefore, if $n > \max(m, 4)$, then

$$F_\phi(i) = \int_{U(i)_A} \Phi(x) |\theta_i(x)|_A$$

is defined for every Φ in $\mathcal{S}(X_A)$ and i in k .

THEOREM 4. *Suppose that $n > \max(m + 1, 4)$ and let C denote a compact subset of $\mathcal{S}(X_A)$; then the series*

$$\sum_{i \in k} F_\phi(i)$$

for Φ varying in C has a “dominant series.”

Proof. There exists a finite set S of valuations on k containing S_∞ and an element $\phi_v \geq 0$ of $\mathcal{S}(X_v)$ for each v , which is equal to the characteristic function of X_v° for every v not in S , such that

$$|\Phi(x)| \leq \phi(x) = \left(\prod_v \phi_v \right)(x)$$

for every Φ in C and x in X_A . For the proof, we refer to [8], Lemma 5; see also [5], Lemma 7. We may assume that every v not in S is a good valuation. We have only to show that the series of $F_\phi(i)$ for i running over k^\times is convergent.

First of all, we have

$$F_\phi(i) = \prod_v F_{\phi_v}(i)$$

for every i . We observe that if v is not in S_∞ , the image of $\text{Supp}(\phi_v)$ under f_v is compact. Therefore the set of all i for which $F_{\phi_v}(i) \neq 0$ is bounded in k_v . In particular, if v is not in S , then $F_{\phi_v}(i) \neq 0$ implies $|i|_v \leq 1$. Since $F_{\phi_v}(i) = 0$ implies $F_\phi(i) = 0$, we may restrict i in k^\times by the condition that $|i|_v \leq 1$ for every v not in S and $|i|_v \leq \text{const.}$ for every v in $S - S_\infty$. Then, in the case where k is a function field, we just get a finite set; hence there is no problem. Therefore we may assume that k is a number field. In that case, i is contained in a fractional ideal of k ; and its image in

$$k_\infty = k \bigotimes_Q R = \prod_{v \in S_\infty} k_v$$

is a lattice in this vector space.

If v is not in S and $0 < |i|_v \leq 1$, by Lemma 1 we have

$$F_{\phi_v}(i) = q^{-(n-1)e} N_e(i)$$

for any $e > \text{ord}_v(i)$. Therefore $F_{\phi_v}(i)$ is equal to $q^{-(n-1)} N_1(i)$ if $\text{ord}_v(i) = 0$; it has

$$(1 - q^{-(n-m)})^{-1} q^{-(n-1)} N_1(0) + q^{-(n-m)-(n-1)} \max_{|t|_v=1} N_1(t)$$

as an upper bound if $\text{ord}_v(i) > 0$ provided that $n > m$. We evaluate these further by Deligne's result and we get

$$F_{\phi_v}(i) \leq (1 - 2q^{-2})^{-1} (1 + cq^{-(1/2)n+1})$$

for some constant c independent of v and i provided that $n > m + 1$. Therefore, if $n > \max(m + 1, 4)$, we get

$$\prod_{v \in S} F_{\phi_v}(i) \leq c_1$$

for some constant c_1 independent of i .

On the other hand, if $n > m$, then F_{ϕ_v} is a continuous function on k_v for every v ; it has a compact support for v not in S_∞ ; and $F_{\phi_v}(i_v)$ tends to 0 as $|i_v|_v \rightarrow \infty$ more rapidly than any negative power of $|i_v|_v$ for v in S_∞ . This follows from Lemma 3 and from what we have reviewed in §1. Therefore we get

$$\prod_{v \in S} F_{\phi_v}(i) \leq c_2 \prod_{v \in S_\infty} \max(1, |i|_v)^{-2}$$

for some constant c_2 independent of i . If we denote the summation over the above-mentioned lattice in k_∞ by \sum' , by putting these together, we get

$$\sum_{i \in k^\times} F_\phi(i) \leq c_1 c_2 \sum'_i \prod_{v \in S_\infty} \max(1, |i|_v)^{-2};$$

and the right hand side is convergent, say, by [5], Lemma 12. q.e.d.

We recall that each $F_\phi(i)$ is a tempered measure on X_A . Therefore Th. 4 shows that the series of $F_\phi(i)$ for i running over k also defines a tempered measure on X_A . We shall add the following remark:

Remark. Suppose that k is a number field and denote by k° its principal order, by Q an integral ideal of k , and by $|Q|$ its absolute norm; assume that $f(x)$ is in $k^\circ[x_1, \dots, x_n]$ and, for every $i \neq 0$ in k° , let $N_Q(i)$ denote the number of $a \bmod Q$, where a is in $(k^\circ)^{(n)}$, satisfying $f(a) \equiv i \bmod Q$. Then the limit (if it exists) of $|Q|^{-(n-1)}N_Q(i)$ as Q becomes divisible by any given integral ideal of k is called the "singular series" associated with $f(x)$ and i ; cf. [7]. This is related to $F_\phi(i)$ as follows:

We decompose X_A into $X_0 \times X_\infty$, where X_0 is the restricted product of X_v for all v not in S_∞ and X_∞ the product of X_v for v in S_∞ ; similarly, we decompose $U(i)_A$ into $U(i)_0 \times U(i)_\infty$ and $|\theta_i|_A$ into $|\theta_i|_0 \otimes |\theta_i|_\infty$. Let Φ_0 denote the characteristic function of the product of X_v° for all v not in S_∞ , Φ_∞ an arbitrary Schwartz function on X_∞ , and $\Phi = \Phi_0 \otimes \Phi_\infty$. Then we have

$$F_\phi(i) = \left(\lim_Q |Q|^{-(n-1)} N_Q(i) \right) \cdot \int_{X_\infty} \Phi_\infty(x) |\theta_i(x)|_\infty$$

provided that $n \geq 4$.

5. Poisson formula. We shall turn our attention to another type of functions; we shall first prove the following general lemma:

LEMMA 4. *Let r denote a non-negative integer, ε a positive real number, and σ_v for each valuation v on k a real number; suppose that $\sigma_v > r$ for all v and $\sigma_v \geq r + 1 + \varepsilon$ for almost all v . Then the series*

$$\sum_i \prod_v \max(1, |i_v|)^{-\sigma_v},$$

in which $i = (i_1, \dots, i_r)$ runs over k^r , is convergent.

Proof. In the case where k is a number field, this lemma was proved in [5] as Prop. 1. Suppose, therefore, that k is a function field of genus g ; let F_{q_0} denote the algebraic closure in k of the prime field. Since k contains a prime divisor of arbitrarily large degree, we can choose one, say p_∞ , satisfying $\deg(p_\infty) \geq 2g$; let $|\cdot|_\infty$ denote the usual absolute value at p_∞ . For any non-negative integer e , let $L(ep_\infty)$ denote the vector space over F_{q_0} of elements of k with poles only at p_∞ and with orders at most e ; then $L(p_\infty)$ contains an element t not in F_{q_0} .

Let k° denote the integral closure of $F_{q_0}[t]$ in k and λ, α real numbers satisfying $\lambda \geq 1, \alpha > 1$; then we have

$$\sum_{i \in k^\circ} \max(\lambda, |i|_\infty)^{-\alpha} \leq c \cdot \lambda^{1-\alpha}$$

for some constant c independent of λ . This is a counterpart of [5], Lemma 12; and it can be proved as follows:

We observe that k° is the union of $L(ep_\infty)$ for $e = 0, 1, 2, \dots$. Moreover, if q denotes the power of q_0 with $\deg(p_\infty)$ as its exponent, we have $|t|_\infty = q$. Therefore the left hand side is equal to

$$q_0 \lambda^{-\alpha} + \sum_{e=1}^{\infty} \text{card}(L(ep_\infty) - L((e-1)p_\infty)) \max(\lambda, q^e)^{-\alpha},$$

in which $L(0) = F_{q_0}$. And by the Riemann-Roch theorem we get

$$\text{card}(L(ep_\infty)) = (q_0)^{1-e} q^e$$

for $e = 1, 2, \dots$. If we split the above summation into two parts by $q^e \leq \lambda$ and $q^e > \lambda$, we can convince ourselves that both parts are of order $\lambda^{1-\alpha}$.

Once we have that, the rest can be proved in the same way as "Prop. 1." In fact, since the group of units of k° is $(F_{q_0})^\times$, the proof is simpler. q.e.d.

Let Φ denote an arbitrary element of $\mathcal{S}(X_A)$ and C a compact subset of k_A ; then, if $n > m$, there exists a finite set S of valuations on k with the following properties: Th. 3 holds for S and every v not in S is a good valuation; Φ decomposes into the product of the characteristic function of X_v° for all v not in S and a Schwartz-Bruhat function Φ_S of the product X_S of X_v for v in S ; for every v not in S , the image of C under the canonical projection $k_A \rightarrow k_v$ is contained in the unit disc R .

Let $|dx|_S$ denote the product measure of $|dx|_v$ for all v in S ; then, by using the corollary in § 1 (of Th. 1 and Th. 2) and Th. 3, we get

$$\begin{aligned} |F_\Phi^*(i^* + i)| &\leq \prod_{v \notin S} \max(1, |i|_v)^{-n/m} \\ &\quad \cdot \left| \int_{X_S} \Phi_S(x) \prod_{v \in S} \psi_v(f(x)(i_v^* + i)) |dx|_S \right| \\ &\leq c \cdot \prod_v \max(1, |i|_v)^{-n/m} \end{aligned}$$

for every $i^* = (i_v^*)_v$ in C and i in k , in which the constant c is independent of i^* and i . Actually we can show that such a constant exists even if we let Φ vary in a compact subset of $\mathcal{S}(X_A)$. At any rate, by taking $r = 1$ and $\sigma_v = n/m$ in Lemma 4, we see that the following series:

$$\Theta(i^*) = \sum_{i \in k} F_\Phi^*(i^* + i)$$

for i^* varying in C has a dominant series if $n > 2m$. Since k_A/k is compact, we may assume that $k_A = k + C$. In this way, we see that the above series defines a continuous function on k_A/k and that F_Φ^* is a continuous L^1 -function on k_A . Let

$$\Theta(i^*) = \sum_{i \in k} a(i) \psi(ii^*)$$

denote the Fourier expansion of $\Theta(i^*)$; then we get

$$a(i) = \int_{k_A} F_\Phi^*(i^*) \psi(-ii^*) |di^*|_A.$$

By applying Lemma 3, we see that the right hand side coincides with $F_\Phi(i)$ for every i ; hence

$$\sum_{i \in k} F_\Phi(i) \psi(ii^*) = \sum_{i \in k} F_\Phi^*(i^* + i)$$

for every i^* in k_A provided that the left hand side is absolutely convergent. Since $n > 2m$ implies $n > \max(m + 1, 4)$, the absolute convergence is guaranteed by Th. 4. And, by putting $i^* = 0$, we get the following theorem:

THEOREM 5. *Suppose that $n > 2m$, i.e., $\dim(X) > 2 \deg(f)$; then for every Φ in $\mathcal{S}(X_A)$ we have*

$$\sum_{i \in k} \int_{U(i)_A} \Phi(x) |\theta_i(x)|_A = \sum_{i^* \in k} \int_{X_A} \Phi(x) \psi(f(x)i^*) |dx|_A,$$

in which both sides are absolutely convergent.

We observe that the integrand $\Phi_{i^*}(x) = \Phi(x) \psi(f(x)i^*)$ on the right hand side is in $\mathcal{S}(X_A)$ and that the above tempered measure on X_A takes the same value at $\Phi(x)$ and $\Phi_{i^*}(x)$. The correspondence $\Phi \rightarrow \Phi_{i^*}$ uniquely extends to a unitary operator on $L^2(X_A)$. This invariance property is preserved even if we introduce a tempered measure $E(\Phi)$ as

$$E(\Phi) = \sum_{i^* \in k} \int_{X_A} \Phi(x) \psi(f(x)i^*) |dx|_A + \Phi(0) .$$

In the special case where $m = 2$, if we put $f(x, y) = f(x + y) - f(x) - f(y)$, the new measure $E(\Phi)$ is also invariant under the following twisted Fourier transformation:

$$\hat{\Phi}(x) = \int_{X_A} \Phi(y) \psi(f(x, y)) |dy|_A ;$$

cf. [9], p. 64. In this way, we see that $E(\Phi)$ is invariant under a group of unitary operators which is isomorphic to $SL_2(k)$. It is an interesting problem to examine whether this classical result has a generalization of some kind to the case where $m \geq 3$.

6. The constants $i_v(\psi_v \circ f)$. We shall add some remarks on the numerical constants which appeared in the asymptotic formulas for $F_\Phi^*(i^*)$ as $|i^*|_v \rightarrow \infty$: consider $\psi_v(f(x))$ as a tempered distribution $T(x) = T_v(x)$ on X_v as

$$T[\Phi] = \int_{X_v} \Phi(x) \psi_v(f(x)) |dx|_v$$

for every Φ in $\mathcal{S}(X_v)$; let T^* denote its Fourier transform defined by $T^*[\Phi] = T[\Phi^*]$, where

$$\Phi^*(x) = \int_{X_v} \Phi(y) \psi_v([x, y]) |dy|_v .$$

Then T^* is an analytic function on X_v for $k_v = \mathbf{R}, \mathbf{C}$. We have learned this fact from L. Ehrenpreis; it can be proved as follows:

Suppose first that $k_v = \mathbf{R}$; for our purpose, we may assume that $\psi_v(t) = e(t)$ for every t in \mathbf{R} . If $P(\xi)$ is a polynomial in n letters ξ_1, \dots, ξ_n with complex coefficients, we shall denote by $P(\partial/\partial x)$ the differential operator obtained from $P(\xi)$ by replacing each ξ_i by $\partial/\partial x_i$; as before, we put $f_i(x) = \partial f/\partial x_i$ for $1 \leq i \leq n$. We observe that the distribution T^* on $X_v = \mathbf{R}^n$ satisfies

$$f_i(\partial/\partial x) T^* + (2\pi(-1)^{1/2})^{m-1} x_i T^* = 0$$

for $1 \leq i \leq n$, in which $m = \deg(f) \geq 2$. Therefore the analyticity of T^* follows from the following theorem:

“Let $P_1(\xi), P_2(\xi), \dots$ denote a finite number of homogeneous polynomials of the same positive degree in ξ_1, \dots, ξ_n with real coefficients

such that they do not vanish simultaneously at any point of $\mathbf{R}^n - \{0\}$; let D_1, D_2, \dots denote linear partial differential operators of the form

$$D_i = P_i(\partial/\partial x) + \text{lower terms},$$

in which the coefficients of the lower terms are analytic on some open subset U of \mathbf{R}^n . Then any distribution S on U satisfying $D_i S = 0$ for every i is analytic on U ." (This theorem itself follows from the standard theorem in, e.g., [4], p. 178 by observing that

$$D = \sum_i P_i(\partial/\partial x) D_i$$

is "elliptic" and $DS = 0$.)

Suppose next that $k_v = \mathbf{C}$; let (z_1, \dots, z_n) , instead of (x_1, \dots, x_n) , denote coordinates on X_v ; put $x_i = \operatorname{Re}(z_i)$, $y_i = \operatorname{Im}(z_i)$, and $F(x, y) = 2 \operatorname{Re}(f_v(z))$; then $F(x, y)$ has the same property as the $f_v(x)$ for $k_v = \mathbf{R}$: it is a homogeneous polynomial of degree m in $x_1, \dots, x_n, y_1, \dots, y_n$ with real coefficients such that its $2n$ partial derivatives do not vanish simultaneously at any point of $\mathbf{R}^{2n} - \{0\}$. Therefore T^* is analytic on (the underlying real vector space of) $X_v = \mathbf{C}^n$.

THEOREM 6. *In the notation of Th. 1, we have $T_v^*(0) = c_0 + c_1$ or c_v according as $k_v = \mathbf{R}$ or \mathbf{C} .*

Proof. We have

$$c_v \Phi(0) = \lim_{t \rightarrow \infty} |t|_v^n F_\Phi^*(t^m)$$

for every Φ in $\mathcal{S}(X_v)$. We choose an element of $\mathcal{S}(X_v)$ with compact support; call it ϕ and take its Fourier transform ϕ^* as Φ ; then we get

$$\begin{aligned} c_v \phi^*(0) &= \lim_{t \rightarrow \infty} \int_{X_v} \phi(x) T^*(t^{-1}x) |dx|_v \\ &= T^*(0) \int_{X_v} \phi(x) |dx|_v. \end{aligned}$$

We have used the fact that $T^*(t^{-1}x)$ tends uniformly to $T^*(0)$ on the compact subset $\operatorname{Supp}(\phi)$ of X_v . Since ϕ is an arbitrary element of $\mathcal{S}(X_v)$ with compact support, we get $c_v = T^*(0)$. q.e.d.

We shall consider the case where $K = k_v$ is a p -field and prove the following counterpart of Th. 6:

THEOREM 7. *The constant $T_v^*(0)$ is meaningful; and, in the notation of Th. 2, we have*

$$T_v^*(0) = \sum_{\alpha^m = q^n} \left(\sum_{\chi^m = 1} c_{\alpha, \chi} \chi(1) \right).$$

Proof. As a tempered distribution on X_v , T^* is a finitely additive function on the family of all compact open subsets of X_v . Let e denote a non-negative integer and ϕ_e the characteristic function of $(P^e)^{(n)}$; then the integral of ϕ_e over X_v is $q^{-(1/2)d+e)n}$ and, if $\Phi = \phi_0$, we have

$$\lim_{e \rightarrow \infty} q^{(1/2)d+e)n} T^*[\phi_e] = \lim_{e \rightarrow \infty} q^{(d+e)n} F_\Phi^*(\pi^{-(d+e)m}).$$

According to Th. 2, the expression on the right hand side (under the limit sign) is equal to

$$\sum_{\alpha^m = q^n} \left(\sum_{\chi^m = 1} c_{\alpha, \chi} \chi(1) \right)$$

for all large e . We observe that this finite sum does not depend on the choice of the coordinates by which the sequence ϕ_0, ϕ_1, \dots is defined. Therefore we may call the above limit the “derivative” of T^* at 0 and denote it by $T^*(0)$. q.e.d.

We shall change our notation and denote $T_v^*(0)$ by $i_v(\psi_v \circ f)$ for every v . Suppose that v is a good valuation and $a_v \neq 0$ an element of $K = k_v$ of order e_0 ; then we get

$$i_v(\psi_v \circ a_v f) = \begin{cases} q^{n[e_0/m]} \\ q^{n((e_0+1)/m-1)} \sum_{t \bmod P} \psi_v(\pi^{-e_0-1} a t) N_1(t) \end{cases}$$

according as $e_0 \not\equiv -1$ or $e_0 \equiv -1 \pmod{m}$. This can be derived from Lemma 2. In particular, by taking $a_v = 1$, we get $i_v(\psi_v \circ f) = 1$. Therefore we can define $i(\psi \circ f)$ as

$$i(\psi \circ f) = \prod_v i_v(\psi_v \circ f);$$

the product on the right hand side is really a finite product. In the case where $m = 2$, $i_v(\psi_v \circ f)$ is equal to Weil's $\gamma_v(\psi_v \circ f)$ except for the square root of the absolute value in k_v of the discriminant of f ; cf. [8], p. 161. We recall that the product of $\gamma_v(\psi_v \circ f)$ is 1; cf. op. cit., p. 179. By an elementary product formula, the product of the absolute

value in k_v of the discriminant of f is 1; hence we get $i(\psi \circ f) = 1$. We shall give an example indicating that a theorem concerning $i(\psi \circ f)$ (even if it exists) can not have such a simple form for $m \geq 3$.

EXAMPLE. We take $k = \mathbf{Q}$, $\psi_\infty(t) = e(-t)$ for every t in $\mathbf{Q}_\infty = \mathbf{R}$, $\psi_p(t) = e(\langle t \rangle)$ for every t in \mathbf{Q}_p , where $\langle t \rangle$ denotes the "fractional part" of t . Then, for any a_∞ in \mathbf{R}^\times , we get

$$i_\infty(\psi_\infty(a_\infty x^m)) = (2/m)\Gamma(1/m)(2\pi|a_\infty|)^{-1/m} - \text{times} \\ e(-\text{sgn}(a_\infty)/4m) \text{ or } \cos(\pi/2m)$$

according as m is even or odd. Let χ denote a character of the group of units of \mathbf{Z}_p and put

$$g_x = p^{-e_x} \left(\sum_{u \bmod p^{e_x}} \chi(u) e(p^{-e_x} u) \right),$$

in which $u \not\equiv 0 \pmod p$ and e_x is (as in §1) the smallest positive integer such that $\chi = 1$ on $1 + p^{e_x} \mathbf{Z}_p$. Then, for any $a_p = p^{e_0} u_0$ in \mathbf{Q}_p^\times of order e_0 satisfying $0 \leq e_0 < m$, we get

$$i_p(\psi_p(a_p x^m)) = 1 + (1/m) \sum_{a^m = p} \left(\sum_{\chi^m = 1} g_x^{-1} \alpha^{e_0 + e_x} \chi(u_0) \right).$$

If we take $m = 2$, as a special case of what we have recalled, we get $i(\psi(ax^2)) = 1$ for every a in \mathbf{Q}^\times . In particular, we have

$$i(\psi(x^2)) = (1 + i) \cdot \frac{1}{2} (1 - i) = 1.$$

However, if we take $m = 4$ and $a = 1$, we get

$$i(\psi(x^4)) = (1 + \zeta) \cdot \frac{1}{2} \Gamma\left(\frac{1}{4}\right) (2\pi)^{-(1/4)} \zeta^{-1},$$

in which $\zeta = e(1/16)$; this is not even a real number. Finally, if we take $m = 3$ and $a = 9$, we get $i_3(\psi_3(9x^3)) = 0$, hence $i(\psi(9x^3)) = 0$. (We can show that the situation does not improve even if we extend \mathbf{Q} by adjoining m -th roots of unity.)

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