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ORDER OF FUNCTIONS BOUNDED ON A SPIRAL

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Section 1. Introduction

The purpose of this paper is to improve a result of Schnitzer and Seidel [6] and to continue an analogy between entire functions and the class of functions which are holomorphic and unbounded in the unit disk, but bounded on a boundary spiral. The existence of such functions was established by Valiron [7].

The author wishes to thank Prof. W. Seidel for his help in this paper.

Throughout this paper the unit disk $|z| < 1$ in the z-plane shall be denoted by D and the upper half-plane $\mathscr{I}(\zeta) > 0$ in the ζ-plane shall be denoted by H where $\mathscr{I}(\zeta)$ and $\mathscr{R}(\zeta)$ stand for the imaginary and real part of ζ respectively. The maximum of the modulus of the function $f(z)$ on the circle $|z| = r$ shall be denoted by $M(r, f)$.

DEFINITION 1. A *spiral* in D is a set $S = \{z(t) | 0 \leq t < 1\}$ where $z(t)$ continuous and one-to-one on $0 \leq t < 1$, $0 < |z(t)| < 1$ on $0 < t < 1$, and $\lim_{t\to 1} |z(t)| = 1$. Moreover, for each branch, $\arg z(t)$, of the argument of $z(t)$, $0 < t < 1$, it is required that either $\lim_{t\to 1} \arg z(t) = +\infty$ or $\lim_{t\to 1} \arg z(t) = -\infty$.

The class of functions which are holomorphic and unbounded in D, but bounded on the spiral S shall be denoted by $\mathscr{V}(S)$.

Some analogies will now be given between the class $\mathscr{V}(S)$ and the class of entire functions.

Every non-constant entire function has infinity as an asymptotic value. Valiron obtained the result that every function in $\mathscr{V}(S)$ has infinity as an asymptotic value [7].

The Picard theorem asserts that a non-constant entire function assumes every complex number with at most one exception in each neighborhood of the boundary point infinity. It is known that each $f \in \mathscr{V}(S)$

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assumes every complex number infinitely often with at most one exception in the disk D [2, Theorem 1]. It is true that f assumes every complex number with at most one exception in each neighborhood of each boundary point.

The analogies obtained in this paper will be between the order of growth of functions in $\mathscr{V}(S)$ and the order of growth of entire functions. In particular, analogies to Wiman's theorem [3, p. 39] and the Ahlfors-Denjoy Theorem are found [1]. Before stating these analogies it is necessary to introduce some preliminary notions.

Let D be the unit disk and S be a spiral in D. Some assumptions can be made about the spiral near $z(0)$ which are convenient. Let $z(0) = 0$

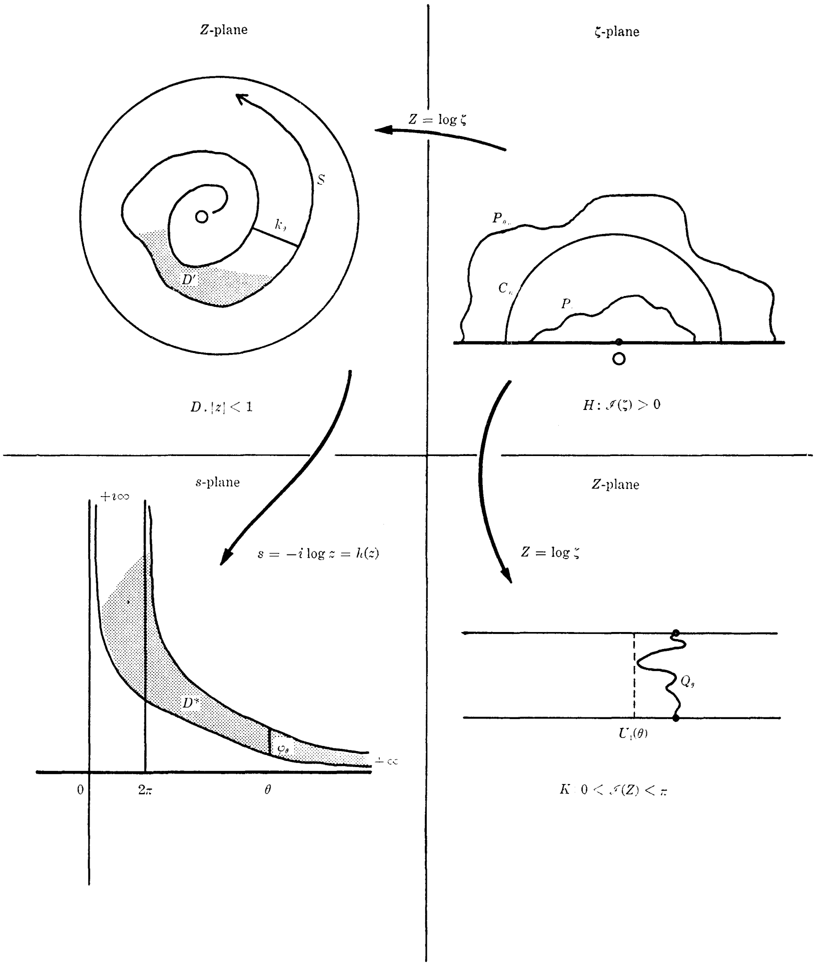

Fig. 1

and let $\lim_{t \to 1} \arg z(t) = +\infty$ for each branch of the argument of $z(t)$. Let $\lim_{t \to 0} \arg z(t)$ exist for each such branch. With these conventions let the symbol $\arg z(t)$ stand for that branch of the argument of $z(t)$ for which $0 \leq \lim_{t \to 0} \arg z(t) < 2\pi$. In addition, suppose that $\arg z(t) > 0$ for $t > 0$. These assumptions are allowable because the shape of the initial part of the spiral is irrelevant to the results in this paper.

Let S be a spiral in D with the above restrictions. The set $D' = D - S$ is a simply connected region. There is a largest interval $(0, a)$, $a > 0$, contained in D'. For any x on this segment let $h(x) = 2\pi + i(-\log x)$ where $-\log x$ is real. From this determination let $s = h(z) = -i \log z$ be continued throughout D' and a one-to-one conformal mapping $s = \theta + i\tau = h(z)$ is obtained. Let $D^* = h(D')$. Denote the boundary point of D^* which corresponds to 0 under $s = h(z)$ by $+i\infty$ and the boundary point of D^* which corresponds to the prime end of D', $|z| = 1$, under $s = h(z)$ by $+\infty$ (see Figure 1).

DEFINITION 2. Let D, S, D' and D^* be defined as above. Let ϕ_θ be the unique vertical straight line crosscut of D^* above $s = \theta$, $\theta > 2\pi$, which separates $+i\infty$ from $+\infty$ in the sense of Ahlfors [1, pp. 5–6]. Let k_θ be the inverse image of ϕ_θ under $s = h(z)$. Let $k(\theta)$ and $\varphi(\theta)$ be the lengths of the straight line segments k_θ and φ_θ, respectively.

Throughout this paper the symbols just introduced shall retain their meaning.

DEFINITION 3. For $f \in \mathscr{V}(S)$ define $K(\theta) = \sup_{z \in k_\theta} |f(z)|$ and, for $\theta_0 > 2\pi$, let

$$\lambda(S) = \underline{\lim}_{\theta \to \infty} \frac{\log \log K(\theta)}{\pi \int_{\theta_0}^{\theta} \frac{dt}{k(t)}},$$

and

$$\rho(S) = \overline{\lim}_{\theta \to \infty} \frac{\log \log K(\theta)}{\pi \int_{\theta_0}^{\theta} \frac{dt}{k(t)}}.$$

It is easy to see that both $\lambda(S)$ and $\rho(S)$ do not depend on the lower limit of the integral. The lemma below follows from straight forward limit arguments and the fact that

$$\lim_{x \to 0} \frac{\log(1 + x)}{x} = 1 .$$

From part (3) it follows that $k(t)$ in both expressions of Definition 3 may be replaced by $\phi(t)$.

LEMMA 1. *For $t > 2\pi$, and for $\varphi(t)$ and $k(t)$ defined as in Definition 2 relative to the spiral S in D, the following relations hold. If $k(t) = x - y$, then $\varphi(t) = \log x - \log y$ and*

$$\text{(a)} \quad \varphi(t) > k(t) , \tag{1}$$

$$\text{(b)} \quad \lim_{t \to \infty} \frac{k(t)}{\varphi(t)} = 1 , \tag{2}$$

$$\text{(c)} \quad \lim_{\theta \to \infty} \frac{\displaystyle\int_{\theta_0}^{\theta} \frac{dt}{\varphi(t)}}{\displaystyle\int_{\theta_0}^{\theta} \frac{dt}{k(t)}} = 1 . \tag{3}$$

If the lower order relative to S of a function $f \in \mathscr{V}(S)$ is defined to be $1/2\ \lambda(S)$ then it shall be proved in Section 2, Theorem 1 that the lower order is not less than $1/2$.

Wiman's theorem states that non-constant entire function which is bounded on a half line has lower order not less than $1/2$ [3].

Thus it is seen that the analogy of class $\mathscr{V}(S)$ to entire functions extends to notions of order.

In Section 3 the analogue to the Ahlfors-Denjoy theorem [1] will be obtained. In Section 4 the analogue to the extension of the Ahlfors-Denjoy theorem proved by MacIntyre [4] will be obtained.

Section 2 contains a proof that the theorem of Schnitzer and Seidel follows from the analogue to Wiman's theorem. (See the corollary to Theorem 1).

Section 2. Wiman's Theorem in $\mathscr{V}(S)$.

All the notations and conventions developed in Section 1 shall be freely used in this section. Two more conformal transformations are needed.

Let $z = g(\zeta)$ be a conformal transformation mapping the upper half plane $H : \mathscr{I}(\zeta) > 0$ onto the region D' which is contained in the unit disk $|z| < 1$. Let $g(0) = 0$ and the boundary point of H at infinity correspond to the prime end, $|z| = 1$, of D'. Although $z = g(\zeta)$ is not fully normalized it is to be considered in any given situation as a fixed mapping.

Map the half-plane H onto the strip $K: 0 < \mathscr{I}(Z) < \pi$ in the Z-plane by the mapping $Z = \log \tau$. In the sequel the following notations shall be adhered to. Let P_θ and Q_θ be the images of k_θ in H and K, respectively, under the appropriate combined mappings (See Figure 1). For $\theta > 2\pi$, let

$$U_1(\theta) = \operatorname*{Inf}_{Z \in Q_\theta} \mathscr{R}(Z) . \tag{4}$$

It is now possible to prove the analogue to Wiman's theorem.

THEOREM 1. *Let $f \in \mathscr{V}(S)$, then $\lambda(S) \geq 1$.*

Proof. By Ahlfors' First Fundamental Theorem [1, p. 10], there exists a $\theta' > 2\pi$ and a $\theta_1' > \theta'$ such that for $\theta > \theta_1'$,

$$U_1(\theta) > \pi \int_{\theta'}^{\theta} \frac{dt}{\varphi(t)} . \tag{5}$$

Assume, contrary to the theorem, that $\lambda(S) < 1$. Suppose that $\theta_0 = \theta'$ and, by Lemma 1, replace $k(t)$ by $\varphi(t)$ in the definition of $\lambda(S)$. Then

$$\varlimsup_{\theta \to \infty} \frac{\log \log K(\theta)}{\pi \int_{\theta'}^{\theta} \frac{dt}{\varphi(t)}} < 1 . \tag{6}$$

There exists a sequence $\theta_1, \theta_2, \cdots$, with $\theta_1' < \theta_1 < \theta_2 < \cdots$, and a γ, $0 < \gamma < 1$, such that

$$\log \log K(\theta_n) < \pi\gamma \int_{\theta'}^{\theta_n} \frac{dt}{\varphi(t)} , \qquad n = 1, 2, \cdots . \tag{7}$$

From (5)

$$\pi \int_{\theta'}^{\theta_n} \frac{dt}{\varphi(t)} < U_1(\theta_n) , \qquad n = 1, 2, \cdots . \tag{8}$$

Let C_n be the semicircle in the half-plane $H: \mathscr{I}(\zeta) > 0$ with radius $\exp\left[\pi \int_{\theta'}^{\theta_n} \frac{dt}{\varphi(t)}\right]$. By the continuity and normalization of the conformal transformations and (8), the set P_{θ_n} is outside of C_n and so P_{θ_n} tends uniformly to ∞ as $n \to \infty$.

Let $F(\zeta) = f(g(\zeta))$, then $F(\zeta)$ is holomorphic in H and is continuous and bounded on the real axis $\mathscr{I}(\zeta) = 0$. It is clear that $K(\theta_n) = \sup_{\zeta \in P_{\theta n}} |F(\zeta)|$.

Let ζ be an arbitrary point of P_{θ_n}. Then, from (7) and (8),

$$\begin{aligned} |F(\zeta)| &\leq \sup_{\zeta' \in P_{\theta n}} |F(\zeta')| = K(\theta_n) \\ &\leq \exp\left\{\exp\left[\pi\lambda\int_{\theta'}^{\theta_n} \frac{dt}{\varphi(t)}\right]\right\} \\ &\leq \exp\{\exp[U_1(\theta_n)]\}^r \\ &\leq \exp[|\zeta|^r]\,. \end{aligned}$$

The function $F(\zeta)$ satisfies all the conditions of the Phragmén-Lindelöf Principle [5] and so is bounded. But because $f \in \mathscr{V}(S)$, $F(\zeta)$ cannot be bounded. This contradiction establishes Theorem 1.

COROLLARY. *Schnitzer and Seidel* [6]. *Let $r = a(t)$ be a continuous strictly increasing function in $0 \leq t < \infty$ such that $0 \leq a(t) < 1$, $a(0) = 0$, and $\lim_{t\to\infty} a(t) = 1$. Let $a = \lambda(r)$ be the inverse function of $r = a(t)$. Setting $S = \{z(t) | z(t) = a(t)e^{it},\ 0 < t < \infty\}$ let $f \in \mathscr{V}(S)$. Then the following relation holds:*

$$\varlimsup_{r \to 1} \frac{\log\log M(r, f)}{\lambda(r)} = \infty\,.$$

Proof. With the notation and hypotheses of the corollary, $k(\theta) = a(\theta) - a(\theta - 2\pi)$ is continuous and positive. Thus L'Hospital's rule applied to

$$R(\theta) = \frac{\pi\displaystyle\int_{\theta_0}^{\theta} \frac{dt}{k(t)}}{\theta}$$

implies

$$\lim_{\theta\to\infty} R(\theta) = \infty\,.$$

Because k_θ is contained in $|z| \leq a(\theta)$, the inequality $M(r, f) \geq K(\lambda(r))$ holds. These two facts and a direct elementary argument using Theorem 1 give the corollary.

Remark. It is seen that Theorem 1 is a generalization of the corollary in two ways. First, it applies to unrestricted spirals. Second, in the case of spirals for which the corollary applies, Theorem 1 implies the corollary.

Remark. The corollary states that the growth of $M(r, f)$ depends on the growth of $\lambda(r)$. If $r = a(\theta)$ grows rapidly to 1, then $\lambda(r)$ grows slowly to ∞ and, consequently, $M(r, f)$ is allowed to grow more slowly. By considering the following example, and by applying Theorem 1, it is seen that not only is the rate that $r = a(\theta)$ tends to 1 involved but also the manner in which $a(\theta)$ tends to 1.

EXAMPLE. Let ε_n be a sequence of positive numbers with $\lim_{n\to\infty} \varepsilon_n = 0$. Let r_n be an increasing sequence of positive numbers with $\lim_{n\to\infty} r_n = 1$. Suppose that $r_n + \varepsilon_n < r_{n+1}$. Let $r(\theta)$ be an arbitrary increasing continuous function, $0 \leq \theta < \infty$, with the exceptions that

$$r(\theta) = \begin{cases} r_n\,, & (4n-2)\pi \leq \theta \leq (4n-2)\pi + \dfrac{\pi}{4} \\ r_n + \varepsilon_n\,, & 4n\pi \leq \theta \leq 4n\pi + \dfrac{\pi}{4}\,. \end{cases}$$

Let $S = \{r(\theta)e^{i\theta} \mid 0 \leq \theta < \infty\}$ be a spiral in D and suppose $f \in \mathscr{V}(S)$. If $k(t)$ is defined in terms of S, then

$$\pi\int_{2\pi}^{4n\pi+\pi/4} \frac{dt}{k(t)} \geq \frac{\pi^2}{4}\sum_{i=1}^{n}\frac{1}{\varepsilon_n} = M_n\,.$$

Given $\varepsilon > 0$ there exists by Theorem 1 an N such that for $n \geq N$,

$$K\left(4n\pi + \frac{\pi}{4}\right) \geq \exp\{\exp(1-\varepsilon)M_n\}\,.$$

By the arbitrariness of $r(\theta)$ in the intervals $4n\pi + \pi/4 \leq \theta \leq (4n+2)\pi$ and the arbitrariness of the sequence ε_n one may construct a spiral which grows extremely rapidly to $|z| = 1$ while M_n, and hence $K(\theta)$, is forced to grow extremely rapidly.

Section 3. Ahlfors-Denjoy Theorem in $\mathscr{V}(S)$.

The methods Ahlfors used to prove the Denjoy conjecture are used here to prove a similar result in $\mathscr{V}(S)$. The result is that an entire function which tends to n different asymptotic values on n asymptotic paths has order exceeding $2n$. See Theorem 2 for the comparable result in $\mathscr{V}(S)$.

Some preliminary facts are needed.

LEMMA 2. [1, pp. 20, 21]. *Let $T_1, T_2, \cdots, T_{n-1}$ be $n-1$ continuous simple paths lying in $H: \mathscr{I}(\zeta) > 0$ each of which connects the point 0 to ∞. Suppose $T_i \cap T_j = \{0\}$ if $i \neq j$. The set $H - \bigcup_{i=1}^{n-1} T_i$ is an open set which consists of n simply connected components $\Omega_1, \Omega_2, \cdots, \Omega_n$. Let $\zeta = \varphi_j(z')$, $j = 1, 2, \cdots, n$, be n conformal transformations each of which maps the half-plane $\mathscr{I}(z') > 0$ onto Ω_j such that $\varphi_j(0) = 0$ and $\varphi_j(\infty) = \infty$. Then, given $\delta > 0$, there exists among the numbers $1, 2, \cdots, n$ an index*

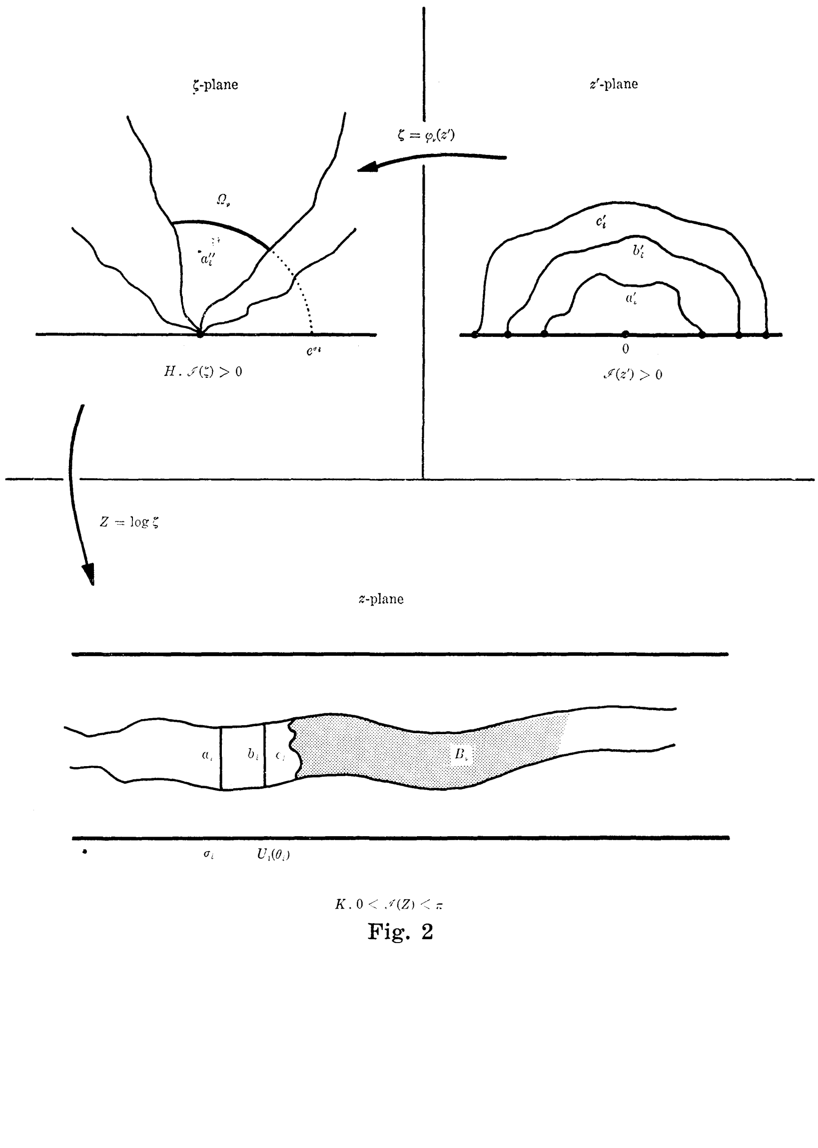

Fig. 2

ν and a sequence $\{a_i'\}$ of simple arcs lying in the upper half-plane $\mathscr{I}(z')$ > 0 which possess the following properties. Each a_i' connects a point of the positive real z'-axis to a point of the negative real z'-axis. As i tends to ∞ the sets a_i' tend uniformly to ∞. For each i and $z' \in a_i'$,

$$|\varphi_\nu(z')| < |z'|^{1/(n+\delta)} . \tag{9}$$

Figure 2 gives a picture of the situation in Lemma 2 and is referred to in the proof of Theorem 2.

Although the proof which is essentially the same as that given by Ahlfors [1] shall be omitted, some of the notation is presented for the computations which follow. For a given $\delta > 0$ there exists the corresponding ν from Lemma 2. The function $Z = \log \zeta$ mapping $H: \mathscr{I}(\zeta) > 0$ onto the strip $K: 0 < \mathscr{I}(Z) < \pi$, also maps the region Ω_ν of the lemma onto a simply connected strip region in K which will be denoted by B_ν. The sequence of sets $\{a_i'\}$ in $\mathscr{I}(z') > 0$ determined by the given δ and defined in the statement of Lemma 2 correspond (in the proof of the lemma) under the mapping $Z = \log \varphi_\nu(z')$ to a sequence of vertical straight line crosscuts of B_ν, each of which separates the point $+\infty$ from the point $-\infty$ in the sense of Ahlfors [1]. The crosscut corresponding to a_i' will be denoted by a_i. The segment a_i lies above a point on the real axis which will be denoted by σ_i (See Figure 2). That is for each i

$$\sigma_i = \mathscr{R}(Z) , \qquad Z \in a_i . \tag{10}$$

The proof of Lemma 2 shows that

$$\lim_{i \to \infty} \sigma_i = \infty . \tag{11}$$

THEOREM 2. *Let $f(z)$ be holomorphic in D, $S_1, S_2, \cdots, S_n$ be spirals in D as defined in Section 1, and let $a_1, a_2, \cdots, a_n$, be distinct finite complex numbers. If $f(z) \to a_i$ as $|z| \to 1$ with $z \in S_i$, $i = 1, 2, \cdots, n$, then $\rho(S_i) \geq n$, $i = 1, 2, \cdots, n$.*

Proof. It may be assumed that $S_i \cap S_j = \{0\}$ if $i \neq j$. Assume the contrary of the theorem for some spiral which is taken to be $S = S_n$.

Define $\varphi(\theta)$, $z = g(\zeta)$, $k(\theta)$, P_θ, Q_θ and $U_1(\theta)$ as in Section 1 with respect to the spiral $S = S_n$.

According to the proof of Theorem 1 there exists a θ_0 such that for all sufficiently large θ,

$$U_1(\theta) > \pi \int_{\theta_0}^{\theta} \frac{dt}{\varphi(t)}. \tag{12}$$

By the mapping $z = g(\zeta)$ the spirals $S_1, S_2, \cdots, S_{n-1}$ are transformed into sets $T_1, T_2, \cdots, T_{n-1}$ in $H: \mathscr{I}(\zeta) > 0$ which satisfy the hypotheses of Lemma 2. Let δ be any positive number.

According to the lemma there is a sequence $\{a_i'\}$ and a $\nu \in \{1, 2, \cdots, n\}$ such that the inequalities (9) hold. There is also the sequence $\{\sigma_i\}$ of real numbers with $\lim_{i\to\infty} \sigma_i = \infty$ satisfying (10).

Consider the function

$$G(\theta) = \pi \int_{\theta_0}^{\theta} \frac{dt}{\varphi(t)}.$$

$G(\theta)$ is a continuous increasing function with $\lim_{\theta\to\infty} G(\theta) = \infty$. By the intermediate value theorem, there exists a sequence $\{\theta_i\}$ with $\lim_{i\to\infty} \theta_i = \infty$ such that for some positive integer N_1 and $i \geq N_1$,

$$G(\theta_i) = \pi \int_{\theta_0}^{\theta_i} \frac{dt}{\varphi(t)} = \sigma_i. \tag{13}$$

With each θ_i, there are the sets k_{θ_i}, P_{θ_i}, and Q_{θ_i} in D', H and $K: 0 < \mathscr{I}(Z) < \pi$, respectively. Now the crosscut a_i of B_ν above σ_i lies in B_ν which lies in K. The reader is referred to Figures 1 and 2. Define $c_i = Q_{\theta_i} \cap B_\nu$ and b_i to be the straight line crosscut of B_ν above the real number $U_1(\theta_i)$ of (4). By definition of $U_1(\theta_i)$, c_i lies entirely to the right of b_i. But by (12) and (13), a_i lies to the left of b_i. Under the mapping $z' = \varphi_\nu^{-1}(e^Z)$, a_i' is the image of a_i as mentioned above. Let b_i' and c_i' be the images of b_i and c_i, respectively, under this same mapping. The set b_i' divides the half-plane $\mathscr{I}(z') > 0$ into regions H_1 and H_2 which are bounded and unbounded, respectively. By the normalization of the conformal transformations, a_i' lies in H_1 and c_i' lies in H_2.

Define $w = F(z') = f(g(\varphi_\nu(z')))$, $\mathscr{I}(z') \geq 0$, where $\varphi_\nu(z')$ is to be taken as the extension of the mapping of $\mathscr{I}(z') > 0$ into H to the boundary. The function $F(z')$ tends to different limits as $z' \to +\infty$ and $z' \to -\infty$ with z' real by the hypotheses of the theorem so that by a well known theorem of Lindelöf it is unbounded in $\mathscr{I}(z') > 0$. This unboundedness is what shall be contradicted.

Since a_i' tends to infinity uniformly by Lemma 2, it is seen that

$\sup_{z' \in a'_i} |F(z')|$ exceeds the bound of $|F(z')|$, for z' on the real axis, provided i exceeds some positive integer N_2. By the maximum principle, for $i \geq \max(N_1, N_2)$,

$$\operatorname*{Max}_{z' \in a'_i} |F(z')| \leq \operatorname*{Max}_{z' \in b'_i} |F(z')| \leq \operatorname*{Max}_{z' \in c'_i} |F(z')| .$$

Since $c_i \subset Q_{\theta_i}$

$$\operatorname*{Max}_{z' \in c'_i} |F(z')| \leq \operatorname*{Max}_{z \in Q_{\theta_i}} |F(z')| = \operatorname*{Max}_{z \in k_{\theta_i}} |f(z)| = K(\theta_i) .$$

Thus

$$\operatorname*{Max}_{z' \in a'_i} |F(z')| \leq K(\theta_i) , \qquad \text{if } i \geq \max(N_1, N_2) . \tag{14}$$

By assumption, $\rho(S) = A < n$, and by (3) of Lemma 1, $k(t)$ may be replaced by $\varphi(t)$. The value of θ_0 obtained in inequality (12) may be used in the definition of $\rho(S)$. Then

$$\varlimsup_{\theta \to \infty} \frac{\log \log K(\theta)}{\pi \int_{\theta_0}^{\theta} \frac{dt}{\varphi(t)}} = \rho(S) = A < n . \tag{15}$$

Therefore, given any $\varepsilon < 0$, there exists an N_3 such that if $i \geq N_3$, then

$$K(\theta_i) \leq \exp \left\{ \exp \left[(A + \varepsilon)\pi \int_{\theta_0}^{\theta_i} \frac{dt}{\varphi(t)} \right] \right\} . \tag{16}$$

Let $z' \in a'_i$ be an arbitrary point and let $N = \operatorname{Max}(N_1, N_2, N_3)$. Applying inequalities (14), (16), and (12) together with (13), for $i \geq N$ one has

$$|F(z')| \leq \exp [\exp \sigma_i]^{A+\varepsilon} . \tag{17}$$

Let a''_i be the image in Ω_ν of a'_i under the mapping $\zeta = \varphi_\nu(z')$ (See Figure 2). Then a''_i is also the image of the straight line segment a_i under the mapping $\zeta = e^Z$. Since $\mathscr{R}(Z) = \sigma_i$ for $Z \in a_i$, one obtains for the arbitrary point $z' \in a'_i$,

$$|\varphi_\nu(z')| = \exp \sigma_i .$$

That is for $i \geq N$, (17) becomes

$$|F(z')| \leq \exp \{|\varphi_\nu(z')|^{A+\varepsilon}\} . \tag{18}$$

But Lemma 2, inequality (9), holds for $z' \in \alpha_i$ and $\varphi_\nu(z')$. Hence, for $i \geq N$,

$$|F(z')| \leq \exp |z'|^{(1/n+\delta)(A+\varepsilon)} . \tag{19}$$

Since $z' \in \alpha_i'$ was an arbitrary choice, the inequality (19) holds for every $z' \in \alpha_i'$, provided only that $i \geq N$.

Since $A < n$, choose ε and δ such that $(A + \varepsilon)(1/n + \delta) = \gamma < 1$. Then one obtains the sequence $\{\alpha_i'\}$ and the N such that

$$|F(z')| \leq \exp |z'|^\gamma , \quad z' \in \alpha_i' , \quad i \geq N .$$

Since $|F(z')|$ is bounded on the real axis, it is a bounded function by the Phragmén-Lindelöf Principle. But $F(z')$ has already been shown to be unbounded. This contradiction establishes Theorem 2.

Section 4. MacIntyre's extension in $\mathscr{V}(S)$.

MacIntyre [4] proves the following theorem: If $f(z)$ is entire and is bounded on n disjoint arcs each connecting zero to infinity and if f is unbounded in all the regions bounded by these arcs then the lower order of f exceeds $n/2$. Theorem 3 is the comparable result in $\mathscr{V}(S)$.

THEOREM 3. *If $f(z)$ is in $\mathscr{V}(S_i)$, $i = 1, 2, \cdots, n$ where $S_1, S_2, \cdots, S_n$ are spirals in D such that $S_i \cap S_j = \{0\}$ if $i \neq j$, and if $f(z)$ is unbounded in each of the regions of $D - \bigcup_{i=1}^n S_i$, then*

$$\lambda(S_i) \geq n \qquad i = 1, 2, \cdots, n .$$

The proof of this theorem, which follows Lemmas 3 and 4, uses results of Ahlfors and MacIntyre.

LEMMA 3. [4]. *Let $f(z)$ be holomorphic and $|f(z)| \leq M$ in a simply connected region G which is contained in the sector $0 \leq \arg z \leq \pi/p$, $|z| \leq R$, where $p > 1/2$. Suppose that part of the boundary of G is on $|z| = R$ and that $|f(z)| \leq m$ in some neighborhood of each point of the part of the boundary of G which is in $|z| < R$. Then for any point $re^{i\theta}$ in G,*

$$\log |f(re^{i\theta})| \leq 2/\pi \left\{ \log m \operatorname{Arctan} \frac{1 - (r/R)^{2p}}{2(r/R)^p} \right.$$
$$\left. + \log M \operatorname{Arctan} \frac{2(r/R)^p}{1 - (r/R)^{2p}} \right\} . \tag{20}$$

MacIntyre proves this lemma for the case $R = 1$ in the reference cited. The lemma is obtained by considering $f(Rw)$ in $|w| \leq 1$.

Suppose $T_1, T_2, \cdots, T_{n+1}$ are in $\bar{H}: \mathscr{I}(\zeta) \geq 0$, where $T_2, T_3, \cdots, T_n$ are simple arcs each of which connects 0 to ∞, T_1 is the positive real axis, T_{n+1} is the negative real axis, and $T_i \cap T_j = \{0\}$, $i \neq j$. Then $H - \bigcup_{\nu=2}^{n} T_\nu$ is open and has n simply connected components $\Omega_1, \Omega_2, \cdots, \Omega_n$. Fix the order by supposing that $T_\nu \cup T_{\nu+1}$ bounds Ω_ν, $\nu = 1, 2, \cdots, n$. The reader is referred to Figure 3. Let ν be any of the numbers $1, 2, \cdots, n$. Given $R > 1$, T_ν has a last intersection with $|\zeta| = 1$ and a first intersection with $|\zeta| = R$ which shall be denoted by A_ν and B_ν, respectively. Let T'_ν be that part of T_ν between A_ν and B_ν, γ_ν the arc of $|\zeta| = 1$ between A_ν and $A_{\nu+1}$ (in H), and Γ_ν the arc of $|\zeta| = R$ between B_ν and $B_{\nu+1}$ (in H). Denote by $D_\nu(R)$ the region bounded by $T'_\nu \cup \Gamma_\nu \cup T'_{\nu+1} \cup \gamma_\nu$.

By means of conformal transformations and the Schwarz-Christoffel

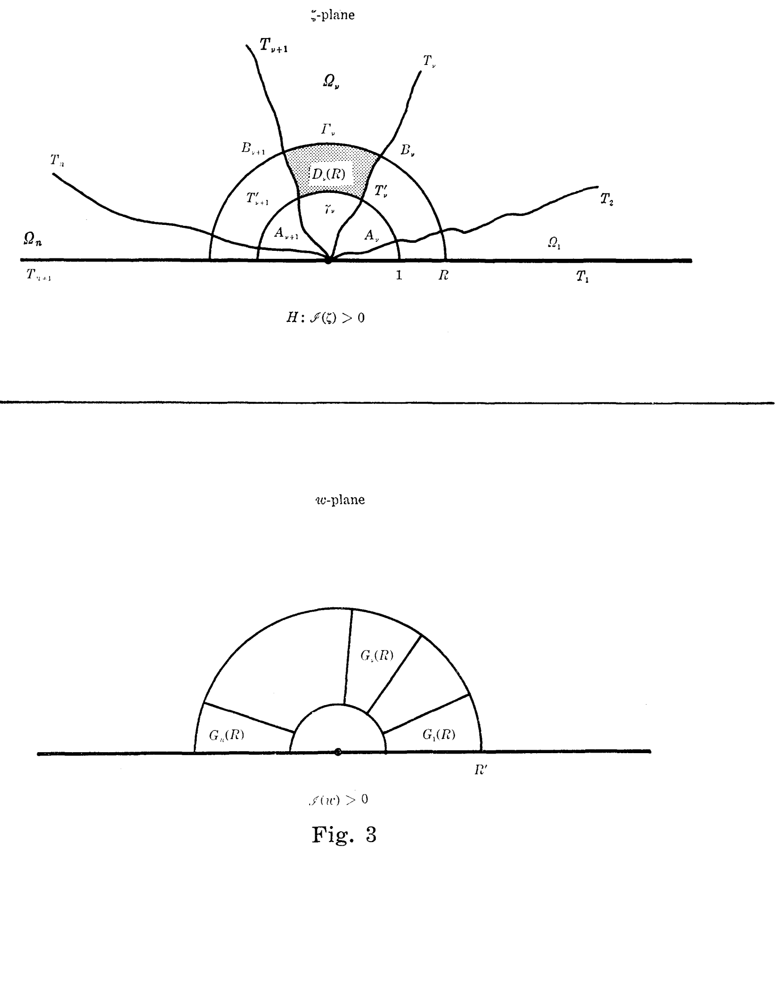

Fig. 3

transformation the region $D_\nu(R)$ can be mapped onto a set $G_\nu(R)$ which is a sector of the annulus $1 \leq |w| \leq R'$ for some R'. The arc Γ_ν can be made to correspond to that part of the boundary of $G_\nu(R)$ which lies on $|w| = R'$. The sets $G_\nu(R)$, $\nu = 1, 2, \cdots, n$, can be arranged so that they do not overlap and the union of their closures is $1 \leq |w| \leq R'$, $\mathscr{I}(w) \geq 0$. See MacIntyre [4] for details. The n conformal transformations so obtained shall be denoted as one by $\zeta = \psi_R(w)$. The number R' will henceforth be referred to as the number associated with R.

LEMMA 4. *Given $R > 1$, the number R' associated with R satisfies*

$$R' \geq R . \tag{21}$$

A proof of this lemma and a discussion of the formation of the sets $G_\nu(R)$ above can be found in MacIntyre [4].

Proof of Theorem 3. It is first proved that for any $\theta_0 > 2\pi$,

$$\varliminf_{\theta \to \infty} \frac{\log K(\theta)}{\exp\left[n\pi\int_{\theta_0}^{\theta} \frac{dt}{\varphi(t)}\right]} > 0 , \tag{22}$$

where $K(\theta)$ and $\varphi(\theta)$ are defined in terms of any one of the spirals $S_1, S_2, \cdots, S_n$. Assume that this inequality is false for some $\theta_0 > 2\pi$ and some S_i which is taken for convenience to be $S = S_1$. Then there is a sequence $\{\varepsilon_i\}$ with $\varepsilon_i > 0$ and $\lim_{i \to \infty} \varepsilon_i = 0$, and a sequence $\{\theta_i\}$ with $\lim_{i \to \infty} \theta_i = \infty$ such that

$$\log K(\theta_i) \leq \varepsilon_i \exp\left[n\pi\int_{\theta_0'}^{\theta_i} \frac{dt}{\varphi(t)}\right] , \qquad i = 1, 2, \cdots , \tag{23}$$

where $K(\theta)$ and $\varphi(\theta)$ are defined as usual relative to S. With $z = g(\zeta)$ defined as in Section 1 relative to S, P_{θ_i} is the image of the k_{θ_i} in H: $\mathscr{I}(\zeta) > 0$ under g. By Ahlfors' Theorem [1] again, there exists $\theta_0' > \theta_0$ such that if i is sufficiently large say $i \geq N$, then P_{θ_i} lies outside the circle

$$C_i : |\zeta| = \exp\left[\pi\int_{\theta_0'}^{\theta_i} \frac{dt}{\varphi(t)}\right] .$$

Setting

$$\varepsilon_i' = \varepsilon_i \exp\left[\pi\int_{\theta_0}^{\theta_0'} \frac{dt}{\varphi(t)}\right], \qquad i = 1, 2, \cdots, \tag{24}$$

and

$$R_i = \exp\left[\pi\int_{\theta_0'}^{\theta_i} \frac{dt}{\varphi(t)}\right], \qquad i \geq N, \tag{25}$$

inequalities (23) becomes

$$\log K(\theta_i) \leq \varepsilon_i' R_i^n, \qquad i \geq N. \tag{26}$$

During the proof of this theorem there will be several occasions when it is convenient to extract subsequences having certain properties from the original sequence $\{\theta_i\}$. Each such subsequence has the limit infinity and this is the only property which is essential to the proof. Thus to keep the notation simple, it shall always be assumed that the subsequence is renumbered to be the original sequence. For example, P_{θ_i} lies outside C_i and inequalities (26) shall hold for every i. Under $z = g(\zeta)$ the spirals $S_1, S_2, S_3, \cdots, S_n$ correspond to $n+1$ paths in $\overline{H}$ which satisfy Lemma 3. With the notation of Lemma 4, for each $R_i > 1$ of (25) there is the associated R_i' which, by (21), satisfies

$$R_i' \geq R_i, \qquad i = 1, 2, \cdots. \tag{27}$$

For i fixed, consider the sets $G_\nu(R)$, $\nu = 1, 2, \cdots, n$, defined in the discussion following Lemma 3. These sets lie in the half annulus $1 \leq |w| \leq R_i'$, $\mathscr{I}(w) \geq 0$. Since each of the n sets $G_\nu(R)$ is a sector of this half annulus, there is a ν_i such that $G_{\nu_i}(R_i)$ lies in a sector of opening smaller than or equal to π/n.

As i varies, a sequence $\{\nu_i\}$ is obtained, each number of which is selected from the finite set $\{1, 2, \cdots, n\}$. Thus there is among these numbers an index ν' and a subsequence $\{i_k\}$ of $\{i\}$, such that each $G_{\nu'}(R_{i_k})$ lies in a sector of opening less than or equal to π/n. As decided above, it is assumed that $G_{\nu'}(R_i)$ has this property for every i. With this ν', consider $\Omega_{\nu'}$ and the function $f(g(\zeta))$ restricted to $\Omega_{\nu'}$. This function is bounded on the boundary of $\Omega_{\nu'}$ and bounded in $|\zeta| \leq 1$. Without loss of generality it can be assumed that the bound in both cases is 1.

With each R_i, there is the "function" $\zeta = \psi_{R_i}(w)$ which maps the region $G_{\nu'}(R_i)$ onto the region $D_{\nu'}(R_i)$. For simplicity let $G_i = G_{\nu'}(R_i)$, $D_i = D_{\nu'}(R_i)$ and $\chi_i = \psi_{R_i}$, since ν' is now fixed.

Define $F_i(w) = f(g(\chi_i(w)))$, $w \in G_i$, $i = 1, 2, \cdots$. It should be recalled that the boundary arc of G_i on $|w| = R'_i$ corresponds under $\zeta = \psi_i(w)$ to the boundary arc $\Gamma_{\nu'}$, of D_i on $|\zeta| = R_i$. Hence, for the remaining part of the boundary of G_i,

$$|F_i(w)| \leq 1 .$$

Moreover,

$$\sup_{|w|=R'_i} |F_i(w)| = \sup_{\substack{|\zeta|=R_i \\ \zeta \in D_i}} |f(g(\zeta))| , \tag{28}$$

and

$$K(\theta_i) = \sup_{\zeta \in P_{\theta_i}} |f(g(\zeta))| . \tag{29}$$

The set P_{θ_i} lies outside of $|\zeta| = R_i$, by definition (25) of R_i and application of Ahlfors' Theorem. Moreover $\sup_{\substack{|w|=R_i \\ w \in \overline{G_i}}} |F_i(w)|$ exceeds 1 for all sufficiently large i (assume these facts for every i). Thus it follows from (28) and (29) that

$$\sup_{\substack{|w|=R_i \\ w \in \overline{G_i}}} |F_i(w)| \leq K(\theta_i) . \tag{30}$$

Let $\zeta_0 \in \Omega_{\nu'}$, with $|\zeta_0| > 1$. With $G = G_i$, $p = n$, $R = R'_i$, $m = 1$, and $M = K(\theta_i)$ apply Lemma 3. Inequality (20) becomes

$$\log |F_i(w_i)| \leq \frac{2}{\pi} \log K(\theta_i) \operatorname{Arctan} \frac{2(|w_i|/R'_i)^n}{1 - (|w_i|/R'_i)^{2n}} ,$$

where w_i is defined to be the image of ζ_0 in G_i, under $\zeta = \psi_i(w)$. That the sequence w_i has a bounded subsequence can be shown using normal families. It is, of course, assumed that

$$|w_i| \leq M_0 < \infty , \qquad i = 1, 2, \cdots .$$

There is an $M_1 < \infty$ such that

$$\log |F_i(w_i)| \leq M_1 \log K(\theta_i) R_i'^{-n} . \tag{31}$$

But $R'_i \geq R_i$ so that (31) becomes

$$\log |F_i(w_i)| \leq M_1 \log K(\theta_i) R_i^{-n} . \tag{32}$$

Using inequality (26) and equality (25) of the definition of R_i, it follows that for every i,

$$\log |F_i(w_i)| \le M_1 \varepsilon_i' . \tag{33}$$

But $F_i(w_i) = f(g(\zeta_0))$ is a constant which by (33) must satisfy $|f(g(\zeta_0))| \le 1$ since $\lim_{i \to \infty} \varepsilon_i' = 0$. This means that $f(g(\zeta))$ is bounded in $\Omega_{\nu'}$. Clearly, from the hypotheses of the theorem, $f(g(\zeta))$ is not bounded in $\Omega_{\nu'}$. This contradiction yields (22).

It has now been established, that for some $a > 0$,

$$\varliminf_{\theta \to \infty} \frac{\log K(\theta)}{\exp\left[n\pi \int_{\theta_0}^{\theta} \frac{dt}{\varphi(t)}\right]} > a > 0 ,$$

with $K(\theta)$ and $\varphi(\theta)$ defined for any of the spirals S_i in the statement of Theorem 3 and $\theta_0 > 2\pi$ arbitrary. Thus, for large enough θ,

$$\log \log K(\theta) > \log a + n\pi \int_{\theta_0}^{\theta} \frac{dt}{\varphi(t)} ,$$

or,

$$\frac{\log \log K(\theta)}{\pi \int_{\theta_0}^{\theta} \frac{dt}{k(t)}} > \frac{\log a}{\pi \int_{\theta_0}^{\theta} \frac{dt}{k(t)}} + n \frac{\pi \int_{\theta_0}^{\theta} \frac{dt}{\varphi(t)}}{\pi \int_{\theta_0}^{\theta} \frac{dt}{k(t)}} .$$

By (3) of Lemma 1, it follows that

$$\lambda(S_i) = \varliminf_{\theta \to \infty} \frac{\log \log K(\theta)}{\pi \int_{\theta_0}^{\theta} \frac{dt}{k(t)}} \ge n ,$$

where $k(t)$ and $K(\theta)$ are defined relative to S_i, $i = 1, 2, \cdots, n$. This completes the proof of Theorem 3.

COROLLARY. *Theorem* 3 *implies both Theorem* 1 ($n = 1$) *and Theorem* 2.

It should be remarked that for $f \in \mathscr{V}(S)$ neither $\rho(S)$ nor $\lambda(S)$ are guaranteed to be finite. It is expected that there are functions for which they are finite for two reasons. The first reason is that the "right" results are obtained and the second reason is that the estimates of the Phragmén-Lindelöf principle and Ahlfors' Theorem may be asymptotically sharp in the case of some spirals. By right results is meant the analogies between class $\mathscr{V}(S)$ and entire functions.

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