T. Higa
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ON THE TOPOLOGICAL STRUCTURE OF AFFINELY CONNECTED MANIFOLDS

TATSUO HIGA

Introduction

The purpose of the present paper is to investigate the relationship between the topological structure and differential geometric objects for affinely connected manifolds.

Let M be a compact, connected and oriented Riemannian manifold, $P^r(M)$ the vector space of all parallel r-forms on M and $b_r(M)$ the r-th Betti number of M. Since every parallel form is harmonic, it follows from the Hodge—de Rham theory that the inequality $\dim P^r(M) \leqq b_r(M)$ holds for all $r = 1, \cdots, \dim M$ (cf. [3], [5]). We shall generalize these inequalities to compact affinely connected manifolds.

Next, let M be a non-compact manifold. A connected submanifold N of M is called a *soul* if $\dim N < \dim M$ and if the inclusion $i: N \to M$ is a homotopy equivalence. J. Cheeger and D. Gromoll proved the following remarkable theorem. If M is a complete Riemannian manifold with non-negative sectional curvature then M has a compact soul (see [1] Theorem 1.11 and 2.1). We shall give another kind of sufficient conditions for M to have a (compact) soul.

Finally, a connected manifold M is said to be *reducible* if M is diffeomorphic to a product manifold $M_1 \times M_2$ with $\dim M_i \geqq 1$, $i = 1, 2$. Otherwise, M is said to be *irreducible*. We shall find a differential geometric condition for M to be reducible. We note that de Rham's Decomposition Theorem ([2]) furnishes a prototype for this condition (for irreducible manifolds, see [8]).

In order to obtain our results in a unified manner, we introduce certain family of functions on a connected manifold M with an affine connection Γ. A function f on M is called an *affine function* if, for every geodesic $c(t)$ with an affine parameter t, there are real constants a and b

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such that $f(c(t)) = at + b$ for all t. We can regard each affine function as a "harmonic mapping". In fact, if Γ is symmetric, then every affine function satisfies formally the defining equation of harmonic mappings (see [4] p. 116).

It is shown that the set $A(M, \Gamma)$ of all affine functions on M is a finite-dimensional real vector space and satisfies $1 \leqq \dim A(M, \Gamma) \leqq \dim M + 1$. By making use of $A(M, \Gamma)$, we shall define another finite-dimensional real vector space $V(M, \Gamma)$. Let $P^1(M, \Gamma)$ be the vector space of all parallel 1-forms on M and $k(M)$ the non-negative integer defined to be the largest k such that $H_k(M, \boldsymbol{Z}_2) \neq 0$, where $H_k(M, \boldsymbol{Z}_2)$ denotes the singular homology group of M with coefficient $\boldsymbol{Z}_2$. For simplicity, we suppose that Γ is complete and symmetric. Then we can state our main results as follows.

A. *If M is compact, then* $\dim P^1(M, \Gamma) \leqq b_1(M)$.

B. *If* $\dim A(M, \Gamma) > 1$, *then M has a soul. Moreover, we have* $k(M) \leqq \dim M - \dim A(M, \Gamma) + 1$. *The equality holds if and only if M has a compact soul N with* $\dim N = \dim M - \dim A(M, \Gamma) + 1$.

C. *If* $m = \dim V(M, \Gamma) > 0$, *then there exists a totally geodesic submanifold M' of M such that M is diffeomorphic to $\boldsymbol{R}^m \times M'$.*

We remark that M is not always affinely isomorphic to the product affinely connected manifold $\boldsymbol{R}^m \times M'$. However, in the Riemannian case, we can prove more (see [7]).

In Section 1 and Section 2, we shall study some basic properties of affine functions. In Section 3, we shall state our main theorems in a rigorous form. The proof of the theorems will be given in Section 4. The crucial point of the proof lies in a careful use of geodesics. In the last section, we shall consider affine symmetric spaces and prove the following result. Let $M = G/H$ be an affine symmetric space with the canonical connection Γ on G/H (see [10] Chap. III). If G is solvable and M is simply connected, then $\dim A(M, \Gamma) > 1$.

Throughout this paper, all manifolds and differential geometric objects on them are assumed to be differentiable of class C^∞. For brevity's sake, we shall often use the adjective "smooth" instead of "differentiable".

§ 1. Affine functions

Let M be a connected smooth manifold with an affine connection Γ. For a smooth curve $c(t)$ in M, we denote by $\dot{c}(t)$ the tangent vector to the curve at $c(t)$ and by $D\dot{c}(t)/dt$ the covariant derivative of $\dot{c}(t)$. A smooth curve $c(t)$ in M defined on an open interval I is called a *geodesic* if $D\dot{c}(t)/dt = 0$ on I. If c is a geodesic (as a point set), any parameter t with respect to which $c = c(t)$ is a geodesic is called an *affine parameter* of c. In this paper, all geodesics under consideration are assumed to be parametrized by affine parameter. The connection Γ is said to be *complete* if every geodesic can be extended to a geodesic $c(t)$ defined for all $t \in \boldsymbol{R}$, where $\boldsymbol{R}$ denotes the field of real numbers.

DEFINITION 1.1. A smooth function f on M is called an *affine function* on M if, for every geodesic $c(t)$, there are real constants a and b such that $f(c(t)) = at + b$ for any t whenever it is defined.

This definition does not depend on the choice of an affine parameter t because any other affine parameter t' is given by an affine transformation $t' = ct + d$, where $c \neq 0$ and d are real constants.

PROPOSITION 1.1. *Let f be an affine function on M. If the differential $(df)_x$ of f at some point $x \in M$ vanishes, then f is a constant function on M.*

Proof. Let N denote the subset of M consisting of all points y such that $(df)_y = 0$. Clearly, N is non-empty and closed in M. Let us take any $y \in N$ and any geodesic $c(t)$ with $c(0) = y$. Then we can put $f(c(t)) = at + b$ $(a, b \in \boldsymbol{R})$. Hence we have

$$a = \frac{d}{dt} f(c(t))|_{t=0} = (df)_y(\dot{c}(0)) = 0 .$$

This means that f is constant on every geodesic starting from the point y. Let U be a convex neighborhood of y (see [9] vol. 1, p. 149). Since every point of U can be joined to y by a geodesic segment, f is constant on U. Thus $U \subset N$ and hence N is open in M. Since M is connected, we can conclude that f is constant on M.

Let $A(M, \Gamma)$ denote the set of all affine functions of M. Then it is clear that $A(M, \Gamma)$ is a linear subspace of the real vector space of all smooth functions on M. Every real number can be identified with a constant function on M, so we get the natural inclusion $i : \boldsymbol{R} \to A(M, \Gamma)$.

PROPOSITION 1.2. *$A(M, \Gamma)$ is finite-dimensional and satisfies* $1 \leqq \dim A(M, \Gamma) \leqq \dim M + 1$. *Moreover, if M is compact, then* $\dim A(M, \Gamma) = 1$.

Proof. Let us fix a point x of M. Let $F: A(M, \Gamma) \to T_x^*(M)$ denote the linear mapping given by $F(f) = (df)_x$ $(f \in A(M, \Gamma))$, where $T_x^*(M)$ is the cotangent space to M at x. Then Proposition 1.1 implies that the sequence $0 \to \boldsymbol{R} \xrightarrow{i} A(M, \Gamma) \xrightarrow{F} T_x^*(M)$ is exact. This proves the first and second assertions. The last assertion follows easily from Proposition 1.1 and the fact that every smooth function on a compact manifold has a critical point.

PROPOSITION 1.3. *If Γ is complete, then every bounded affine function f on M is a constant function on M.*

Proof. Let $x \in M$ and let $c(t)$ $(t \in \boldsymbol{R})$ be any geodesic with $c(0) = x$. Then we can put $f(c(t)) = at + b$ $(a, b \in \boldsymbol{R})$. Now the function $t \mapsto |at|$ on $\boldsymbol{R}$ is bounded, so $a = 0$. Thus we have $(df)_x(\dot{c}(0)) = 0$ and hence $(df)_x = 0$. Therefore the assertion follows immediately from Proposition 1.1.

PROPOSITION 1.4. *Let $1, f_1, \cdots, f_n$ be elements of $A(M, \Gamma)$. Then the following two statements are equivalent*:

1) $1, f_1, \cdots, f_n$ *are linearly independent in* $A(M, \Gamma)$;
2) $df_1, \cdots, df_n$ *are linearly independent at each point of* M.

Proof. Suppose 1). Let x be any point of M and assume that $\sum_{i=1}^n a_i(df_i)_x = 0$ for real constants $a_1, \cdots, a_n$. Then we have $(d(\sum_{i=1}^n a_i f_i))_x = 0$, so by Proposition 1.1 there is a constant b such that $\sum_{i=1}^n a_i f_i + b = 0$. Hence we get $a_i = 0$ for all i, which implies 2). The converse is obvious.

Now we set $a(M, \Gamma) = \dim A(M, \Gamma) - 1$.

PROPOSITION 1.5. *Let $\boldsymbol{R}^n$ be the n-dimensional affine space with the standard flat affine connection Γ_0. Then we have $a(\boldsymbol{R}^n, \Gamma_0) = n$.*

Proof. Let $(x_1, \cdots, x_n)$ be the canonical coordinate system on $\boldsymbol{R}^n$. Then the coordinate functions $x_1, \cdots, x_n$ belong to $A(\boldsymbol{R}^n, \Gamma_0)$. Moreover, it follows from Propositions 1.2 and 1.4 that $1, x_1, \cdots, x_n$ form a basis of $A(\boldsymbol{R}^n, \Gamma_0)$. Hence we have $a(\boldsymbol{R}^n, \Gamma_0) = n$.

Let M' be another connected smooth manifold with an affine connection Γ'.

DEFINITION 1.2 (cf. [11]). A smooth mapping $h: M \to M'$ is said to be *totally geodesic* if, for every geodesic $c(t)$ of M, $h(c(t))$ is a geodesic of M'.

For a smooth mapping $h: M \to M'$ and a smooth function f on M', we denote by $h^*(f)$ the smooth function on M given by $h^*(f) = f \circ h$. Then we have immediately the following proposition.

PROPOSITION 1.6. *If $h: M \to M'$ is a totally geodesic mapping, then $h^*(A(M', \Gamma')) \subset A(M, \Gamma)$ and $h^*: A(M', \Gamma') \to A(M, \Gamma)$ is a linear homomorphism. If moreover h is surjective, then $h^*: A(M', \Gamma) \to A(M, \Gamma)$ is injective.*

PROPOSITION 1.7. *Let M_i be a connected smooth manifold with an affine connection Γ_i $(i = 1, 2)$. Let $\Gamma_1 \times \Gamma_2$ denote the product affine connection on $M_1 \times M_2$. Then we have*

$$a(M_1 \times M_2, \Gamma_1 \times \Gamma_2) = a(M_1, \Gamma_1) + a(M_2, \Gamma_2).$$

Proof. For simplicity, we write $A = A(M_1 \times M_2, \Gamma_1 \times \Gamma_2)$ and $A_i = A(M_i, \Gamma_i)$, $i = 1, 2$. Since the natural projection $p_i: M_1 \times M_2 \to M_i$ is totally geodesic, $p_i^*: A_i \to A$ is an injective homomorphism $(i = 1, 2)$. Let us fix a point (x_0, y_0) of $M_1 \times M_2$ $(x_0 \in M_1, y_0 \in M_2)$. Let $h_i: M_i \to M_1 \times M_2$, $i = 1, 2$, denote the smooth mappings given by $h_1(x) = (x, y_0)$ $(x \in M_1)$ and $h_2(y) = (x_0, y)$ $(y \in M_2)$, respectively. Clearly, we have $h_i^*(A) \subset A_i$, $i = 1, 2$. For any $f \in A$, we set

$$f = f - p_1^*(h_1^*(f)) - p_2^*(h_2^*(f)) + f(x_0, y_0).$$

Then $\tilde{f}$ lies in A and satisfies $\tilde{f}(x_0, y_0) = 0$. It is not hard to verify that $d\tilde{f}$ vanishes at (x_0, y_0). It follows from Proposition 1.1 that $\tilde{f}$ vanishes identically on $M_1 \times M_2$. Hence,

$$f = p_1^*(h_1^*(f)) + p_2^*(h_2^*(f)) - f(x_0, y_0).$$

This formula means that $A = p_1^*(A_1) + p_2^*(A_2)$. Since $p_1^*(A_1) \cap p_2^*(A_2)$ consists of all constant functions on $M_1 \times M_2$, it follows that $\dim A = \dim A_1 + \dim A_2 - 1$. This proves Proposition 1.7.

§ 2. Parallel 1-forms and affine functions

Let M be a connected smooth manifold with an affine connection Γ. Let T be the torsion tensor, R the curvature tensor and ∇ the covariant differentiation of Γ. Γ is said to be *symmetric* if T vanishes identically on M. Let f be any smooth function on M. We set

$$H_f(X, Y) = (\nabla_X df)(Y) + \tfrac{1}{2} df(T(X, Y))$$

for all vector fields X and Y on M. Then it is easy to see that H_f is a symmetric covariant 2-tensor on M.

LEMMA 2.1. *For any smooth function f on M and any smooth curve $c(t)$ in M, we have*

$$\frac{d^2}{dt^2}f(c(t)) = H_f(\dot{c}(t), \dot{c}(t)) + df\left(\frac{D\dot{c}(t)}{dt}\right).$$

Proof. For simplicity, let us denote by $F(t)$ the second derivative of $f(c(t))$ and set $H_f = H_f(\dot{c}(t), \dot{c}(t))$. We can assume that the curve $c(t)$ lies in a coordinate chart $(U, (y_1, \cdots, y_m))$ of M ($m = \dim M$). Let Γ^k_{ij}, $i, j, k = 1, \cdots, m$, denote the components of Γ with respect to the coordinate system and set $c^i(t) = y_i \circ c(t)$, $i = 1, \cdots, m$. Then we have

$$F(t) = \sum_{i,j=1}^{m} \frac{\partial^2 f}{\partial y_i \partial y_j} \cdot \frac{dc^i}{dt} \cdot \frac{dc^j}{dt} + \sum_{k=1}^{m} \frac{\partial f}{\partial y_k} \cdot \frac{d^2 c^k}{dt^2}$$

and

$$H_f = \sum_{i,j=1}^{m} \left(\frac{\partial^2 f}{\partial y_i \partial y_j} - \frac{1}{2} \sum_{k=1}^{m} (\Gamma^k_{ij} + \Gamma^k_{ji}) \frac{\partial f}{\partial y_k} \right) \cdot \frac{dc^i}{dt} \cdot \frac{dc^j}{dt}.$$

Hence $F(t) - H_f$ is given by

$$\sum_{k=1}^{m} \frac{\partial f}{\partial y_k} \left(\frac{d^2 c^k}{dt^2} + \sum_{i,j=1}^{m} \Gamma^k_{ij} \frac{dc^i}{dt} \cdot \frac{dc^j}{dt} \right),$$

which proves the formula.

LEMMA 2.2. *Let f be a smooth function on M. If df is parallel, then $df(T(X, Y)) = 0$ and $df(R(X, Y)Z) = 0$ hold for all vector fields X, Y and Z on M.*

Proof. This can be obtained from the following simple calculations:

1) $$\begin{aligned} df(T(X, Y)) &= df(\nabla_X Y) - df(\nabla_Y X) - df([X, Y]) \\ &= X(df(Y)) - Y(df(X)) - df([X, Y]) = 0; \end{aligned}$$

2) $$\begin{aligned} df(R(X, Y)Z) &= df(\nabla_X \nabla_Y Z) - df(\nabla_Y \nabla_X Z) - df(\nabla_{[X,Y]} Z) \\ &= XY(df(Z)) - YX(df(Z)) - [X, Y](df(Z)) = 0. \end{aligned}$$

PROPOSITION 2.1. *Let f be a smooth function on M. Then:*

(1) *f is an affine function on M if and only if H_f vanishes identically on M;*

(2) *If df is a parallel 1-form, then f is an affine function;*

(3) *If Γ is symmetric, then, for every $f \in A(M, \Gamma)$, df is a parallel 1-form.*

Proof. (1) Suppose that f is an affine function. Let x be any point of M and let $c(t)$ be any geodesic with $c(0) = x$. From Lemma 2.1, we have

$$H_f(\dot{c}(0), \dot{c}(0)) = \frac{d^2}{dt^2} f(c(t))|_{t=0} = 0$$

and hence $H_f(u, u) = 0$ for any $u \in T_x(M)$, where $T_x(M)$ denotes the tangent space to M at x. Since H_f is symmetric, we finally have $H_f(u, v) = 0$ for all $u, v \in T_x(M)$. This implies that H_f vanishes on M. In a similar way, we can prove the converse.

(2) Since df is parallel, it follows from Lemma 2.2 that H_f vanishes on M. Hence f is an affine function.

(3) Since Γ is symmetric, we have $(\nabla_X df)(Y) = H_f(X, Y) = 0$ for all vector fields X and Y on M, so df is parallel.

Remark. Let M be a connected Riemannian manifold and let f be a smooth function on M. Then H_f is the Hessian of f and the Laplace-Beltrami operator Δ is given by $\Delta f =$ Trace of H_f. f is said to be harmonic if $\Delta f = 0$. By Proposition 2.1(1), we can assert that *every affine function on M is harmonic.*

Now we prove an inequality which gives a relation between $A(M, \Gamma)$ and the curvature of M. For any $x \in M$, let $\mathfrak{P}_x$ denote the linear subspace of $T_x^*(M)$ consisting of all covectors ω such that $\omega(R(X, Y)Z) = 0$ for all $X, Y, Z \in T_x(M)$.

PROPOSITION 2.2. *If Γ is symmetric, then we have $a(M, \Gamma) \leqq \dim \mathfrak{P}_x$ for any $x \in M$.*

Proof. Let $F_x : A(M, \Gamma) \to T_x^*(M)$ denote the linear mapping given by $F_x(f) = (df)_x$ $(f \in A(M, \Gamma))$. Then, by Proposition 2.1(3) and Lemma 2.2, the image $F_x(A(M, \Gamma))$ is contained in $\mathfrak{P}_x$. Hence the sequence

$$0 \longrightarrow \boldsymbol{R} \xrightarrow{i} A(M, \Gamma) \xrightarrow{F_x} \mathfrak{P}_x$$

is exact. This proves Proposition 2.2.

PROPOSITION 2.3. *For a given affine connection Γ on M, there exists an affine connection $\tilde{\Gamma}$ on M such that $\tilde{\Gamma}$ is symmetric and $A(M, \tilde{\Gamma}) =$*

$A(M, \Gamma)$. Moreover, if Γ is complete, then $\tilde{\Gamma}$ can be taken so that it is complete.

Proof. For all vector fields X and Y on M, we set $\tilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2}T(X, Y)$. Then it is easy to see that $\tilde{\nabla}$ defines a desired affine connection on M (cf. [9] vol. 1, p. 146).

§ 3. The main theorems

Let M be a connected smooth manifold with an affine connection Γ, $A(M, \Gamma)$ the vector space of all affine functions on M and $P^1(M, \Gamma)$ the vector space of all parallel 1-form of M. As before, we set $a(M, \Gamma) = \dim A(M, \Gamma) - 1$. Let $\tilde{W}$ denote the set of all vector fields X on M such that, for every $f \in A(M, \Gamma)$, Xf is a constant function on M and that $\nabla_X X = 0$. We set

$$W(M, \Gamma) = \{X \in \tilde{W};\ \nabla_Y X = 0 \text{ for all } Y \in \tilde{W}\}$$

and

$$W_0(M, \Gamma) = \{X \in W(M, \Gamma);\ Xf = 0 \text{ for all } f \in A(M, \Gamma)\}.$$

Then it is easy to see that $W(M, \Gamma)$ is a linear subspace of the real vector space of all vector fields on M and that $W_0(M, \Gamma)$ is a linear subspace of $W(M, \Gamma)$. Hence we can define a real vector space $V(M, \Gamma)$ by $V(M, \Gamma) = W(M, \Gamma)/W_0(M, \Gamma)$.

PROPOSITION 3.1. *If Γ is symmetric, then every parallel vector field X on M belongs to $W(M, \Gamma)$.*

Proof. Let f be any element of $A(M, \Gamma)$. By Proposition 2.1(3), df is parallel. Hence Xf is a constant function on M. Since X is parallel, we have $\nabla_Y X = 0$ for *all* vector fields Y on M. Therefore, X belongs to $W(M, \Gamma)$.

PROPOSITION 3.2. *$V(M, \Gamma)$ is finite-dimensional and satisfies* dim $V(M, \Gamma) \leqq a(M, \Gamma)$.

Proof. We can assume that $n = a(M, \Gamma) > 0$. Let $1, f_1, \cdots, f_n$ be a basis of $A(M, \Gamma)$ and let F denote the linear mapping of $W(M, \Gamma)$ into $\boldsymbol{R}^n$ defined by $F(X) = (Xf_1, \cdots, Xf_n)(X \in W(M, \Gamma))$. Then it is easy to verify that the kernel of F coincides with $W_0(M, \Gamma)$. Hence we get $\dim V(M, \Gamma) \leqq n$.

PROPOSITION 3.3. *Let M be a connected Riemannian manifold with metric tensor g and Γ the Riemannian connection of M. Then we have* $\dim V(M, \Gamma) = a(M, \Gamma)$.

Proof. For any smooth function f on M, we denote by grad f the gradient of f. Namely, grad f is a unique vector field on M such that $g(\operatorname{grad} f, X) = df(X)$ for any vector field X on M. Let $f \in A(M, \Gamma)$. For all vector fileds X and Y on M, we have

$$\begin{aligned} g(\nabla_X \operatorname{grad} f, Y) &= Xg(\operatorname{grad} f, Y) - g(\operatorname{grad} f, \nabla_X Y) \\ &= X(df(Y)) - df(\nabla_X Y) \\ &= H_f(X, Y) \end{aligned}$$

and hence grad f is a parallel vector field of M. For any $X \in W(M, \Gamma)$, let $[X] \in V(M, \Gamma)$ denote the coset determined by X. Let $n = a(M, \Gamma)$ and let $1, f_1, \cdots, f_n$ be a basis of $A(M, \Gamma)$. To prove Proposition 3.3, it suffices to verify that $[\operatorname{grad} f_1], \cdots, [\operatorname{grad} f_n]$ are linearly independent in $V(M, \Gamma)$. Assume now that $\sum_{i=1}^n a_i [\operatorname{grad} f_i] = 0$ for real constants $a_1, \cdots, a_n$. If we set $f = \sum_{i=1}^n a_i f_i$, then $\operatorname{grad} f$ belongs to $W_0(M, \Gamma)$. Let $x \in M$ and let $\|v\|$ denote the norm of $v \in T_x(M)$. Then we have

$$\|(\operatorname{grad} f)_x\|^2 = (df(\operatorname{grad} f))(x) = 0$$

and hence $(\operatorname{grad} f)_x = 0$. It follows from Proposition 1.1 that there is a real constant b such that $\sum_{i=1}^n a_i f_i + b = 0$. Thus we get $a_i = 0$ for all $i = 1, \cdots, n$. This completes the proof of Proposition 3.3.

Let $H^1(M)$ be the first de Rham cohomology group of M and $H_*(M, Z_2)$ the singular homology group of M with coefficient group $Z_2 = Z/2Z$, Z being the module of all rational integers. We define a non-negative integer $k(M)$ by the following two conditions:

1) $H_i(M, Z_2) = 0 \quad$ for all $i > k(M)$;
2) $H_k(M, Z_2) \neq 0 \quad$ for $k = k(M)$.

We are now in a position to state our main theorems, which will be proved in the next section.

THEOREM 3.4. *Let M be a connected smooth manifold with a symmetric affine connection Γ. Then there exist natural linear homomorphisms $j : A(M, \Gamma) \to P^1(M, \Gamma)$ and $k : P^1(M, \Gamma) \to H^1(M)$ such that the sequence*

$$0 \longrightarrow \boldsymbol{R} \xrightarrow{i} A(M, \Gamma) \xrightarrow{j} P^1(M, \Gamma) \xrightarrow{k} H^1(M)$$

is exact. Hence,

$$0 \leqq \dim P^1(M, \Gamma) - a(M, \Gamma) \leqq \dim H^1(M).$$

In particular, if M is compact, then $\dim P^1(M, \Gamma) \leqq b_1(M)$. *Here* $b_1(M)$ *denotes the first Betti number of M.*

THEOREM 3.5. *Let M be a non-compact connected smooth manifold with a complete affine connection Γ. Assume that $n = a(M, \Gamma) > 0$. Then there exists a totally geodesic surjective submersion $\pi : M \to \boldsymbol{R}^n$ with the following properties:*

(1) *every fibre $N_a = \pi^{-1}(a)$ $(a \in \boldsymbol{R}^n)$ is a connected totally geodesic submanifold of M;*

(2) *for every $a \in \boldsymbol{R}^n$, the inclusion $i_a : N_a \to M$ is a homotopy equivalence;*

(3) *if N_b is compact for some $b \in \boldsymbol{R}^n$, then so is N_a for every $a \in \boldsymbol{R}^n$;*

(4) *if M is non-orientable, then so is N_a for every $a \in \boldsymbol{R}^n$. Moreover, if $\pi' : M \to \boldsymbol{R}^n$ is another totally geodesic surjective submersion, then there exists an affine transformation T of $\boldsymbol{R}^n$ such that $\pi' = T \circ \pi$.*

We remark that if Γ is symmetric then π is an affine mapping (see [11]). Let (x, y) be the canonical coordinate system on $\boldsymbol{R}^2$ and set $M = \boldsymbol{R}^2 - \{(-1, 0), (1, 0)\}$. Then we have $H_1(M, \boldsymbol{Z}) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}$. Let $p : M \to \boldsymbol{R}$ denote the smooth function given by

$$p(x, y) = \log((x-1)^2 + y^2) - \log((x+1)^2 + y^2).$$

Then p is a surjective submersion. The fibre $p^{-1}(0)$, 0 being the origin of $\boldsymbol{R}$, is a line, while any other fibre $p^{-1}(a)$ $(a \in \boldsymbol{R}, a \neq 0)$ is a circle. Therefore, this shows that the existence of surjective submersion does not always imply (2) or (3) of Theorem 3.5. It should be also remarked that M has no soul.

THEOREM 3.6. *Let M and Γ be as in Theorem* 3.5. *Then we have*

$$k(M) \leqq \dim M - a(M, \Gamma).$$

The equality holds if and only if there exists a compact connected totally geodesic submanifold N of M such that

1) $\dim N = \dim M - a(M, \Gamma)$;

2) *the inclusion $i : N \to M$ is a homotopy equivalence.*

THEOREM 3.7. *Let M and Γ be as in Theorem* 3.5 *and let* $n = \dim M$. *If $a(M, \Gamma) = n$, then M is diffeomorphic to $\boldsymbol{R}^n$. Assume further that Γ is* *symmetric. Then M is affinely isomorphic to $\boldsymbol{R}^n$ if and only if $a(M, \Gamma) = n$.*

THEOREM 3.8. *Let M and Γ be as in Theorem 3.5. Assume that $m = \dim V(M, \Gamma) > 0$. Then there exists a connected totally geodesic submanifold M' of M such that M is diffeomorphic to the product manifold $\boldsymbol{R}^m \times M'$. Moreover, M' is compact if and only if $k(M) = \dim M - m$.*

We remark that there is a connected manifold M with a complete and symmetric affine connection Γ satisfying the following inequalities:

$$0 < \dim V(M, \Gamma) < a(M, \Gamma) < \dim M .$$

Let M be a connected complete Riemannian manifold and Γ the Riemannian connection of M. From Proposition 3.3, we have $\dim V(M, \Gamma) = a(M, \Gamma)$. In this case, we can prove more: *There exists a connected Riemannian manifold M' such that M is isometric to the Riemannian product $\boldsymbol{R}^n \times M'$ of the standard Euclidean space $\boldsymbol{R}^n$ and M', where we put $n = a(M, \Gamma)$.* We shall prove this theorem in [7].

§ 4. Proof of the main theorems

We keep the notations in Section 3. First of all, we prove Theorem 3.4. Let Γ be a symmetric affine connection on M. Then, by Proposition 2.1(3), we can define a linear mapping $j: A(M, \Gamma) \to P^1(M, \Gamma)$ by $j(f) = df$ $(f \in A(M, \Gamma))$. Since every parallel 1-form ω is closed, it determines a cohomology class $k(\omega) \in H^1(M)$. Thus we get the linear mapping $k: P^1(M, \Gamma) \to H^1(M)$ and the sequence:

$$0 \longrightarrow \boldsymbol{R} \xrightarrow{\ i\ } A(M, \Gamma) \xrightarrow{\ j\ } P^1(M, \Gamma) \xrightarrow{\ k\ } H^1(M) .$$

To prove the exactness of the sequence, it suffices to verify the relation $\operatorname{Ker} k \subset \operatorname{Im} j$. Let ω be any element of $\operatorname{Ker} k$. Then there is a smooth function f on M such that $\omega = df$. By Proposition 2.1(2), f lies in $A(M, \Gamma)$ and hence $\omega = j(f) \in \operatorname{Im} f$. If M is compact, then every $f \in A(M, \Gamma)$ is constant on M (Proposition 1.2). This means that the sequence

$$0 \longrightarrow P^1(M, \Gamma) \xrightarrow{\ k\ } H^1(M)$$

is exact. Hence we have $\dim P^1(M, \Gamma) \leqq b_1(M)$. We have thereby proved Theorem 3.4.

To prove Theorem 3.5, we need some lemmas.

Let M be a non-compact connected manifold with a complete affine connection Γ. Assume that $n = a(M, \Gamma) > 0$. Let us fix a basis $1, f_1, \cdots, f_n$ of $A(M, \Gamma)$ and define a smooth mapping $\pi: M \to \boldsymbol{R}^n$ by $\pi(x) = (f_1(x), \cdots, f_n(x))$ $(x \in M)$. Then it is clear that, for every geodesic $c(t)$ $(t \in \boldsymbol{R})$, there are two elements a and b of $\boldsymbol{R}^n$ such that $\pi(c(t)) = at + b$ for all $t \in \boldsymbol{R}$. This shows that π is a totally geodesic mapping. Let us fix a Riemannian metric g^0 on M and consider the vector fields $\operatorname{grad} f_1, \cdots, \operatorname{grad} f_n$ on M. Let A_{ij} denote the function on M given by $A_{ij} = g^0\,(\operatorname{grad} f_i, \operatorname{grad} f_j)$ $(i, j = 1, \cdots, n)$. From Proposition 1.4, it is easy to see that $\operatorname{grad} f_1, \cdots, \operatorname{grad} f_n$ are linearly independent at each point of M. Hence the $n \times n$ matrix $(A_{ij}(x))$ is non-singular for every $x \in M$. Let (B_{ij}) be the inverse matrix of (A_{ij}). For any $a = (a_1, \cdots, a_n) \in \boldsymbol{R}^n$, we set

$$X(a) = \sum_{i,j=1}^{n} a_i B_{ij} \operatorname{grad} f_j \,.$$

Then $X(a)$ is a smooth vector field on M. As usual, we identify $\boldsymbol{R}^n$ with the tangent space $T_a(\boldsymbol{R}^n)$, $a \in \boldsymbol{R}^n$, by the canonical absolute parallelism on $\boldsymbol{R}^n$.

LEMMA 4.1. *For any* $a = (a_1, \cdots, a_n) \in \boldsymbol{R}^n$ *and any* $x \in M$, *we have* $df_i(X(a)_x) = a_i$, $i = 1, \cdots, n$ *and* $\pi_*(X(a)_x) = a$.

Proof. Let $(x_1, \cdots, x_n)$ be the canonical coordinate system on $\boldsymbol{R}^n$. Then we have

$$\begin{aligned}(\pi_*(X(a)_x))(x_i) &= df_i(X(a)_x)\\ &= df_i\left(\sum_{j,k=1}^{n} a_j B_{jk}(x)(\operatorname{grad} f_k)_x\right)\\ &= \sum_{j,k=1}^{n} a_j B_{jk}(x) A_{ki}(x)\\ &= a_i\end{aligned}$$

for all $i = 1, \cdots, n$, which proves the formulas.

Let TM be the tangent bundle of M and $\exp\colon TM \to M$ the exponential mapping of M. Let $G: \boldsymbol{R} \times \boldsymbol{R}^n \times M \to M$ denote the mapping given by $G(t, a, x) = \exp tX(a)_x$ $(t \in \boldsymbol{R}, a \in \boldsymbol{R}^n, x \in M)$.

LEMMA 4.2. $G: \boldsymbol{R} \times \boldsymbol{R}^n \times M \to M$ *is smooth and satisfies*

$$\pi(G(t, a, x)) = at + \pi(x)$$

for all $t \in \boldsymbol{R}$, $a \in \boldsymbol{R}^n$ *and* $x \in M$.

Proof. We define a mapping $G_0: \boldsymbol{R} \times \boldsymbol{R}^n \times M \to TM$ by $G_0(t, a, x) = tX(a)_x$ $(t \in \boldsymbol{R}, a \in \boldsymbol{R}^n, x \in M)$, so that $G = \exp \circ G_0$. Therefore, it suffices to prove that G_0 is smooth. But this can be easily checked by taking suitable local coordinate systems. Now we shall prove the second assertion. Since the curve $\boldsymbol{R} \ni t \mapsto G(t, a, x) \in M$ is a geodesic, there is an element $b = b(a, x)$ of $\boldsymbol{R}^n$ such that $\pi(G(t, a, x)) = bt + \pi(x)$. Differentiating this with respect to t at $t = 0$, we have $b = \pi_*(X(a)_x)$ and hence $a = b$ by Lemma 4.1.

LEMMA 4.3. *$\pi: M \to \boldsymbol{R}^n$ is a surjective submersion.*

Proof. It is clear from Proposition 1.4 that the rank of π is equal to n at each point of M. Let x_0 be a point of M and let a be any point of $\boldsymbol{R}^n$. Then, from Lemma 4.2, we have

$$\pi(G(1, a - \pi(x_0), x_0) = a ,$$

so π is surjective.

Proof of Theorem 3.5. Consider the totally geodesic surjective submersion $\pi: M \to \boldsymbol{R}^n$ and set $N_a = \pi^{-1}(a)$ $(a \in \boldsymbol{R}^n)$. For any $x \in N_a$ and any $v \in T_x(N_a)$, let $c(t)$ $(t \in \boldsymbol{R}^n)$ be the geodesic determined by (x, v). Then we can put $\pi(c(t)) = bt + a$ $(b \in \boldsymbol{R}^n)$. Now we have

$$b = \frac{d}{dt}\pi(c(t))|_{t=0} = \pi_*(v) = 0$$

and hence $\pi(c(t)) = a$ for all $t \in \boldsymbol{R}$. Thus N_a is a totally geodesic submanifold of M.

By Lemma 4.2, $\pi(G(1, a - \pi(x), x) = a$ holds for all $a \in \boldsymbol{R}^n$ and $x \in M$, so we can define a smooth mapping $r_a: M \to N_a$ by $r_a(x) = G(1, a - \pi(x), x)$ $(x \in M)$. We have easily $r_a \circ i_a(x) = x$ for any $x \in N_a$, where $i_a: N_a \to M$ is the inclusion. It follows that N_a is connected. Let $H_a: \boldsymbol{R} \times M \to M$ denote the smooth mapping given by $H_a(t, x) = G(t, a - \pi(x), x)$ $(t \in \boldsymbol{R}, x \in M)$. Then we have $H_a(0, x) = x$ and $H_a(1, x) = i_a \circ r_a(x)$ for all $x \in M$. Hence the mapping $i_a \circ r_a: M \to M$ is homotopic to the identity mapping of M. Therefore $i_a: N_a \to M$ is a homotopy equivalence. (More precisely, N_a is a strong deformation retract of M.)

Suppose now that N_a is orientable for some $a \in \boldsymbol{R}^n$. Let $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n$ be the canonical orthonormal basis of $\boldsymbol{R}^n$ and set $X_i = X(\boldsymbol{e}_i)$, $i = 1, \cdots, n$. Then, from Lemma 4.1, it is easy to see that, for any $x \in N_a$ and any basis $v_1, \cdots, v_p$ of $T_x(N_a)$ ($p = \dim N_a$), $v_1, \cdots, v_p, (X_1)_x, \cdots, (X_n)_x$ form a basis of $T_x(M)$. Let ω be a non-vanishing continuous p-form on N_a and let $m = \dim M$. For any $x \in N_a$, let Ω_x^0 denote the m-covector of $T_x(M)$ defined by

$$\Omega_x^0(v_1 \wedge \cdots \wedge v_p \wedge (X_1)_x \wedge \cdots \wedge (X_n)_x) = \omega_x(v_1 \wedge \cdots \wedge v_p)$$

for all vectors $v_1, \cdots, v_p$ of $T_x(N_a)$. Let E be the pull-back of TM by the inclusion $i_a : N_a \to M$, i.e., $E = i_a^{\#} TM$. Then $\Omega_x^0 (x \in N_a)$ defines a non-vanishing continuous cross section of $\Lambda^m E^*$, where E^* is the dual bundle of E and $\Lambda^m E^*$ the exterior product bundle of E^*. For any $y \in M$, we set $c_y(t) = H_a(t, y)$ $(t \in \boldsymbol{R})$, so that $c_y(0) = y$ and $c_x(1) = r_a(y) \in N_a$. Let $p(c_y)$ denote the parallel translation along the curve $c_y(t)$ $(0 \leqq t \leqq 1)$. Thus $p(c_y)$ is a linear isomorphism of $T_y(M)$ onto E_x $(x = r_a(y))$. Then $p(c_y)$ can be canonically extended to a smooth vector bundle homomorphism $p^m(c) : \Lambda^m TM \to \Lambda^m E$. Now we define the m-covector Ω_y on $T_y(M)$ by $\Omega_y(V) = \Omega_x^0\, (p^m(c)(V))$ $(V \in \Lambda^m T_y(M),\ x = r_a(y))$. Then it can be easily seen that Ω_y $(y \in M)$ defines a non-vanishing continuous m-form on M. Hence M is orientable. We have thereby proved (4) of Theorem 3.5.

Let $\pi' : M \to \boldsymbol{R}^n$ be another totally geodesic surjective submersion. Let $(x_1, \cdots, x_n)$ be the canonical coordinate system on $\boldsymbol{R}^n$ and set $f_i' = x_i \circ \pi'$, $i = 1, \cdots, n$. Then $f_1', \cdots, f_n'$ are affine functions on M and linearly independent in $A(M, \Gamma)$. Hence there are a non-singular $n \times n$ matrix (a_{ij}) and real numbers $b_1, \cdots, b_n$ such that $f_i' = \sum_{j=1}^n a_{ij} f_j + b_i$ for all $i = 1, \cdots, n$. This proves the last assertion of Theorem 3.5.

To prove (3) of Theorem 3.5, we need the following lemma.

LEMMA 4.4. *Let N be a connected smooth manifold. Then we have $0 \leqq k(N) \leqq \dim N$. Moreover, N is compact if and only if $k(N) = \dim N$.*

Proof of Lemma 4.4. It is well-known that the singular homology group $H_*(N, \boldsymbol{Z}_2)$ has the following properties:

1) $H_q(N, \boldsymbol{Z}_2) = 0$ for all $q > \dim N$;

2) If N is non-compact, then $H_p(N, \boldsymbol{Z}_2) = 0$ for $p = \dim N$;

3) If N is compact, then $H_p(N, \boldsymbol{Z}_2) \cong \boldsymbol{Z}_2$ for $p = \dim N$.

(For more details, see for example [6]). Now the lemma follows immediately from these properties.

We return to the proof of Theorem 3.5(3). Suppose that N_b $(b \in \boldsymbol{R}^n)$ is compact. Let a be any point of $\boldsymbol{R}^n$. Then N_a is homotopy equivalent to N_b. From Lemma 4.4, we have

$$k(N_a) = k(N_b) = \dim N_b = \dim N_a .$$

Hence N_a is compact. This completes the proof of Theorem 3.5.

Proof of Theorem 3.6. By Theorem 3.5, there exists a connected totally geodesic submanifold N of M such that a) the inclusion $i: N \to M$ is a homotopy equivalence and b) $\dim N = \dim M - a(M, \Gamma)$. From a), b) and Lemma 4.4, we have

$$k(M) = k(N) \leqq \dim N = \dim M - a(M, \Gamma) .$$

Moreover, N is compact if and only if $k(M) = \dim M - a(M, \Gamma)$. We have thereby proved Theorem 3.5.

Proof of Theorem 3.7. In view of Propositions 1.5 and 2.3, it will be sufficient to prove that if Γ is symmetric and $a(M, \Gamma) = \dim M$ then M is affinely isomorphic to $\boldsymbol{R}^n$. Accordingly, we assume that Γ is symmetric and $a(M, \Gamma) = \dim M$. Let $n = \dim M$ and consider the surjective submersion $\pi: M \to \boldsymbol{R}^n$ (Lemma 4.3). In this case, π is an immersion, so we can define a Riemannian metric g on M by $g = \pi^* ds^2$, where ds^2 denotes the standard Euclidean metric on $\boldsymbol{R}^n$ and π^* the codifferential of π. As before, let $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n$ be the canonical orthonormal basis of $\boldsymbol{R}^n$ and set $X_i = X(\boldsymbol{e}_i)$, $i = 1, \cdots, n$. By Lemma 4.1, $df_i(X_j)$ is constant on M $(i, j = 1, \cdots, n)$. On the other hand, by Proposition 2.1(3), df_i is a parallel 1-form of M $(i = 1, \cdots, n)$. Hence, for any vector field X on M, we have $df_i(\nabla_X X_j) = X(df_i(X_j)) = 0$ for all $i, j = 1, \cdots, n$. It follows that $X_1, \cdots, X_n$ are parallel vector fields. Hence,

$$[X_i, X_j] = \nabla_{X_i} X_j - \nabla_{X_j} X_i = 0 \qquad (i, j = 1, \cdots, n) .$$

Let $\tilde{\nabla}$ denote the covariant differentiation of the Riemannian connection of (M, g). Since $g(X_i, X_j)$ is constant on M for all $i, j = 1, \cdots, n$ we have $g(\tilde{\nabla}_{X_i} X_j, X_k) = 0, i, j, k = 1, \cdots, n$, and hence $\tilde{\nabla}_{X_i} X_j = 0$, $i, j = 1, \cdots, n$ (cf. [9] vol. 1 p. 160). This means that Γ coincides with the Riemannian connection of (M, g). As Γ is complete, (M, g) is a complete Riemannian manifold. It therefore follows from a well-known theorem in [9] (vol. 1 p. 176, Theorem 4.6) that π is an isometry of M onto $\boldsymbol{R}^n$. This completes the proof of Theorem 3.7.

Proof of Theorem 3.8. We begin with the following lemma.

LEMMA 4.5. *Let X be any element of $W(M, \Gamma)$. Then:*

1) *Every integral curve of X is a geodesic of M;*

2) *X is a complete vector field on M.*

Proof. 1) follows immediately from the condition $\nabla_X X = 0$.

2) Let $x(t)$ ($|t| < \varepsilon, \varepsilon > 0$) be an integral curve of X. Since Γ is complete, $x(t)$ can be extended to a geodesic $c(t)$ defined for all $t \in \boldsymbol{R}$. Let I denote the subset of $\boldsymbol{R}$ consisting of all points t such that $\dot{c}(t) = X_{c(t)}$. Clearly, I is non-empty and closed in $\boldsymbol{R}$. Let t_0 be any point of I and let $y(t)$ ($|t - t_0| < \varepsilon', \varepsilon' > 0$) be an integral curve of X with $y(t_0) = c(t_0)$. Then $y(t)$ is a geodesic with the initial condition $(c(t_0), X_{c(t_0)})$. Hence $c(t)$ must coincide with $y(t)$ on a small open neighborhood of t_0. This shows that I is open in $\boldsymbol{R}$ and hence $I = \boldsymbol{R}$. Therefore every integral curve of X can be extended to an integral curve defined for all $t \in \boldsymbol{R}$. Hence X is complete.

Let us set $m = \dim V(M, \Gamma)$ and $n = a(M, \Gamma)$. Then we have $m \leqq n$ (Proposition 3.2).

LEMMA 4.6. *We can choose $Y_1, \cdots, Y_m \in W(M, \Gamma)$ and $f_1, \cdots, f_m \in A(M, \Gamma)$ in such a way that*

1) *$1, f_1, \cdots, f_m$ are linearly independent in $A(M, \Gamma)$;*

2) *$df_i(Y_j) = \delta_{ij}$ for all $i, j = 1, \cdots, m$, where δ_{ij} denotes Kronecker's delta.*

Proof. For any $Y \in W(M, \Gamma)$, let $[Y] \in V(M, \Gamma)$ denote the coset determined by Y. Then we can choose $Y_1, \cdots, Y_m \in W(M, \Gamma)$ so that $[Y_1], \cdots, [Y_m]$ form a basis of $V(M, \Gamma)$. Let $1, g_1, \cdots, g_n$ be a basis of $A(M, \Gamma)$ and set $A_{ij} = dg_i(Y_j)$, $i = 1, \cdots, n$, $j = 1, \cdots, m$. Then, by definition, A_{ij}'s are constants. Assume that $\sum_{j=1}^m a_j A_{ij} = 0$ for real constants $a_1, \cdots, a_m$ $(i = 1, \cdots, n)$. Then we have easily $\sum_{j=1}^m a_j [Y_j] = 0$ and hence $a_j = 0$ for all $j = 1, \cdots, m$. This means that the rank of the $n \times m$ matrix (A_{ij}) is equal to m. We can assume that the $m \times m$ matrix $(A_{ij})_{1 \leq i,j \leq m}$ is non-singular. Let $(B_{ij})_{1 \leq i,j \leq m}$ be the inverse matrix of (A_{ij}) and set $f_i = \sum_{j=1}^m B_{ij} g_j$, $i = 1, \cdots, m$. Then $1, f_1, \cdots, f_m$ are linearly independent. Moreover, we have

$$df_i(Y_j) = \sum_{k=1}^m B_{ik}\, dg_k(Y_j) = \sum_{k=1}^m B_{ij}\, A_{kj} = \delta_{ij}$$

for all $i, j = 1, \cdots, m$. This proves Lemma 4.6.

From now on, we fix $Y_1, \cdots, Y_m \in W(M, \Gamma)$ and $f_1, \cdots, f_m \in A(M, \Gamma)$ with the properties listed in Lemma 4.6. Let $p : M \to \boldsymbol{R}^m$ denote the smooth mapping given by $p(x) = (f_1(x), \cdots, f_m(x))$ $(x \in M)$. For any $a = (a_1, \cdots, a_m) \in \boldsymbol{R}^m$, let Y^a denote the vector field given by $Y^a = \sum_{i=1}^m a_i Y_i$. By Lemma 4.5, Y^a is a complete vector field on M. We denote by F_t^a the 1-parameter family of diffeomorphisms generated by Y^a. Let $F : \boldsymbol{R} \times \boldsymbol{R}^m \times M \to M$ denote the mapping defined by $F(t, a, x) = F_t^a(x)$ $(t \in \boldsymbol{R}, a \in \boldsymbol{R}^m, x \in M)$.

LEMMA 4.7. *$F : \boldsymbol{R} \times \boldsymbol{R}^m \times M \to M$ is smooth and satisfies*

$$p(F(t, a, x)) = at + p(x)$$

for all $t \in \boldsymbol{R}$, $a \in \boldsymbol{R}^m$ and $x \in M$.

Proof. By Lemma 4.5, the curve $\boldsymbol{R} \ni t \mapsto F(t, a, x) \in M$ is a geodesic, so we can write $F(t, a, x) = \exp tY_x^a$. Therefore, we can prove Lemma 4.7 by the same reasoning as in Lemma 4.2.

Now we set $M' = p^{-1}(0)$, 0 being the origin of $\boldsymbol{R}^m$. Then M' is a closed totally geodesic submanifold of M. Let h denote the smooth mapping $\boldsymbol{R}^m \times M' \to M$ given by $h(a, x) = F(1, a, x) = F_1^a(x)$ $(a \in \boldsymbol{R}^m, x \in M')$. From Lemma 4.7, we have $p(F(-1, p(x), x)) = 0$ for any $x \in M$, so we can define the smooth mapping $q : M \to M'$ by $q(x) = F(-1, p(x), x)$ $(x \in M)$. For any $a \in \boldsymbol{R}^m$ and any $x \in M'$, we have $p(h(a, x)) = a$ and hence

$$\begin{aligned}(p \times q) \circ h(a, x) &= (p(h(a, x)), q(h(a, x))) \\ &= (a, F_{-1}^a \circ F_1^a(x)) \\ &= (a, x) .\end{aligned}$$

On the other hand, we have for any $y \in M$

$$\begin{aligned}h \circ (p \times q)(y) &= h(p(y), q(y)) \\ &= F_1^a \circ F_{-1}^a(y) \\ &= y ,\end{aligned}$$

where we put $a = p(y)$. These results show that h is a diffeomorphism of $\boldsymbol{R}^m \times M'$ onto M. The last assertion follows easily from Lemma 4.4. This completes the proof of Theorem 3.8.

§ 5. Affine symmetric spaces

A *symmetric space* is a triple (G, H, s) consisting of a connected Lie group G, a closed subgroup H of G and an involutive automorphism s of G

such that H lies between the closed subgroup G_s of all fixed points of s and the identity component of G_s. Let us consider the coset space G/H and let s_0 denote the diffeomorphism $gH \to s(g)H$ of G/H onto itself. Then G/H has a unique affine connection Γ invariant under s_0 and under the natural left action of G (see [10] § 15). Γ is called the *canonical affine connection* on G/H. Then G/H turns out to be an affine symmetric space with respect to the canonical affine connection. Conversely, every affine symmetric space is expressed in this form. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively. Let $\mathfrak{m}$ denote the (-1)-eigenspace of the differential of s. Then we have the canonical decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} \qquad \text{(direct sum)}.$$

THEOREM 5.1. *Let (G, H, s) be a symmetric space, $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ the canonical decomposition of the Lie algebra of G and Γ the canonical affine connection on $M = G/H$. Then:*

1) $a(M, \Gamma) \leqq \dim \mathfrak{m} - \dim [\mathfrak{m}, [\mathfrak{m}, \mathfrak{m}]]$;
2) *The equality holds if M is simply connected.*

Proof. We denote by $\mathfrak{gl}(\mathfrak{m})$ the Lie algebra of all linear endomorphisms of $\mathfrak{m}$ and by $\mathfrak{m}^*$ the dual vector space of $\mathfrak{m}$. Let $\rho : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{m})$ denote the linear isotropy representation given by $\rho(X)(Y) = [X, Y]$ $(X \in \mathfrak{h}, Y \in \mathfrak{m})$. If we set $\mathfrak{h}' = [\mathfrak{m}, \mathfrak{m}]$, then $\mathfrak{h}'$ is an ideal of $\mathfrak{h}$. We remark here that the Lie subalgebra $\rho(\mathfrak{h}')$ of $\mathfrak{gl}(\mathfrak{m})$ can be identified with the Lie algebra of the linear holonomy group L_0 at the origin $0 = H \in G/H$ (see [9] vol. 2 p. 232). Moreover, if $M = G/H$ is simply connected, L_0 is connected. Let $\rho^* : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{m}^*)$ denote the representation defined by

$$(\rho^*(X)\omega)(Y) = -\,\omega(\rho(X)(Y)) \qquad (X \in \mathfrak{h},\ Y \in \mathfrak{m},\ \omega \in \mathfrak{m}^*).$$

We set

$$\tilde{\mathfrak{a}} = \{\omega \in \mathfrak{m}^*;\ \rho^*(X)\omega = 0 \quad \text{for all } X \in \mathfrak{h}'\}.$$

Let $P^1(M, \Gamma)$ be the vector space of all parallel 1-form of M. From the above remarks, it can be easily seen that $\dim P^1(M, \Gamma) \leqq \dim \tilde{\mathfrak{a}}$ and that if M is simply connected then $\dim P^1(M, \Gamma) = \dim \tilde{\mathfrak{a}}$. Let $\mathfrak{a}$ denote the linear subspace of $\mathfrak{m}$ consisting of all vectors Y such that $\omega(Y) = 0$ for all $\omega \in \tilde{\mathfrak{a}}$. Then we have $\dim \tilde{\mathfrak{a}} = \dim \mathfrak{m} - \dim \mathfrak{a}$. For simplicity, we write $\mathfrak{b} = \rho(\mathfrak{h}')(\mathfrak{m})$. Let $X \in \mathfrak{h}'$ and $Y \in \mathfrak{m}$. For any $\omega \in \tilde{\mathfrak{a}}$, we have

$$\omega(\rho(X)(Y)) = -\,(\rho^*(X)\omega)(Y) = 0,$$

which implies that $\mathfrak{b}$ is a linear subspace of $\mathfrak{a}$. Hence,

$$\dim \tilde{\mathfrak{a}} \leqq \dim \mathfrak{m} - \dim \mathfrak{b}\,.$$

On the other hand, let $\tilde{\mathfrak{b}}$ denote the linear subspace of $\mathfrak{m}^*$ consisting of all $\omega \in \mathfrak{m}^*$ such that $\omega(Z) = 0$ for any $Z \in \mathfrak{b}$. Then, as in the above case, we have $\tilde{\mathfrak{b}} \subset \tilde{\mathfrak{a}}$. Hence,

$$\dim \tilde{\mathfrak{a}} \geqq \dim \mathfrak{m} - \dim \mathfrak{b}\,.$$

We have thereby proved the formula: $\dim \tilde{\mathfrak{a}} = \dim \mathfrak{m} - \dim \mathfrak{b}$. Now Theorem 5.1 follows easily from Theorem 3.4.

COROLLARY 5.2. *Let (G, H, s) be a symmetric space and Γ the canonical affine connection on $M = G/H$.*

1) *If G is semisimple, then $a(M, \Gamma) = 0$.*
2) *If G is solvable and if M is simply connected, then $a(M, \Gamma) > 0$.*

Proof. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the canonical decomposition. If we set $\mathfrak{g}' = [\mathfrak{m}, \mathfrak{m}] + \mathfrak{m}$, then $\mathfrak{g}'$ is an ideal of $\mathfrak{g}$. We have easily

$$[\mathfrak{g}', \mathfrak{g}'] = [\mathfrak{m}, \mathfrak{m}] + [\mathfrak{m}, [\mathfrak{m}, \mathfrak{m}]] \qquad \text{(direct sum)}\,.$$

Suppose first that $\mathfrak{g}$ is semisimple. Then $\mathfrak{g}'$ is also semisimple. Thus we get $[\mathfrak{g}', \mathfrak{g}'] = \mathfrak{g}'$ and hence $[\mathfrak{m}, [\mathfrak{m}, \mathfrak{m}]] = \mathfrak{m}$. Suppose now that $\mathfrak{g}$ is solvable. Then $\mathfrak{g}'$ is also solvable. Thus $[\mathfrak{g}', \mathfrak{g}'] \subsetneqq \mathfrak{g}'$ and hence $[\mathfrak{m}, [\mathfrak{m}, \mathfrak{m}]] \subsetneqq \mathfrak{m}$. Therefore the assertions 1) and 2) follow from Theorem 5.1.

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Department of Mathematics
Faculty of Science
Rikkyo University
Tokyo 171
Japan