H. Kaneta
Nagoya Math. J.
Vol. 94 (1984), 43-59

THE INVARIANT POLYNOMIAL ALGEBRAS FOR THE GROUPS $IU(n)$ AND $ISO(n)$

HITOSHI KANETA*

§ 1. Introduction

By the coadjoint representation of a connected Lie group G with the Lie algebra $\mathfrak{g}$ we mean the representation $\mathrm{CoAd}(g) = {}^t\mathrm{Ad}\,(g^{-1})$ in the dual space $\mathfrak{g}^*$. Imitating Chevalley's argument for complex semi-simple Lie algebras, we shall show that the $\mathrm{CoAd}\,(G)$-invariant polynomial algebra on $\mathfrak{g}^*$ is finitely generated by algebraically independent polynomials when G is the inhomogeneous linear group $IU(n)$ or $ISO(n)$. In view of a well-known theorem [8, p. 183] our results imply that the centers of the enveloping algebras for the (or the complexified) Lie algebras of these groups are also finitely generated. Recently much more inhomogeneous groups have been studied in a similar context [2]. Our results, however, are further reaching as far as the groups $IU(n)$ and $ISO(n)$ are concerned [cf. 3, 4, 6, 7, 9].

We shall state our results.

(i) $IU(n)$ $(n \geqq 2)$.

Let G_n and $\mathfrak{g}_n$ be the group $IU(n)$ and its Lie algebra respectively, namely

$$G_n = \left\{\begin{pmatrix} u & a \\ 0 & 1\end{pmatrix};\ u \in U(n),\ a \in C^n\right\},$$

$$\mathfrak{g}_n = \left\{\begin{pmatrix} X & x \\ 0 & 0\end{pmatrix};\ X \in u(n),\ x \in C^n\right\}.$$

The dual space $\mathfrak{g}_n^*$ of $\mathfrak{g}_n$ can be identified with $\mathfrak{g}_n$ by the following non-degenerate bilinear form $\langle\,,\rangle_n$ on $\mathfrak{g}_n \times \mathfrak{g}_n$;

$$\left\langle \begin{pmatrix} X & x \\ 0 & 0\end{pmatrix}, \begin{pmatrix} Y & y \\ 0 & 0\end{pmatrix}\right\rangle_n = \left\langle \begin{pmatrix} X & 0 \\ 0 & -\mathrm{tr}\,X\end{pmatrix}, \begin{pmatrix} Y & 0 \\ 0 & -\mathrm{tr}\,Y\end{pmatrix}\right\rangle_{su(n+1)} + \langle x, y\rangle\,,$$

Received September 2, 1982.

* Grant-in-Aid for Encouragement of Young Scientist 56740065.

where $\langle\ ,\ \rangle_{su(n+1)}$ = the Killing form of the Lie algebra $su(n+1)$ and $\langle x, y\rangle = \mathrm{Re}\, x^* y$. Thus we realize the coadjoint representation CoAd of G_n in $\mathfrak{g}_n^* = \mathfrak{g}_n$ as follows.

$$\left\langle g^{-1}\begin{pmatrix} X & x \\ 0 & 0\end{pmatrix} g,\ \begin{pmatrix} Y & y \\ 0 & 0\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix} X & x \\ 0 & 0\end{pmatrix},\ \mathrm{CoAd}\,(g)\begin{pmatrix} Y & y \\ 0 & 0\end{pmatrix}\right\rangle, \qquad g \in G_n .$$

Let $\mathfrak{H}_n$ = be the n-dimensional subspace

$$\left\{\begin{bmatrix} Y_1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & Y_{n-1} & 0 \\ 0 & \cdots & 0 & y_n \\ 0 & \cdots & 0 & 0 \end{bmatrix};\ Y_1, \cdots, Y_{n-1}, y_n \in \sqrt{-1}\boldsymbol{R}\right\}$$

of $\mathfrak{g}_n^*$. Set $Z_j = Y_j + \sum_{k=1}^{n-1} Y_k$ $(1 \leqq j \leqq n-1)$, s_i = the i-th fundamental polynomial $(0 \leqq i \leqq n-1)$ in Z_j, and $t_i = s_i y_n^2$.

THEOREM 1. *The $\boldsymbol{C}$-algebra of* CoAd $(IU(n))$*-invariant polynomial functions on $\mathfrak{g}_n^*$ is isomorphic, via the restriction map $f \to f|\mathfrak{H}_n$, to the $\boldsymbol{C}$-algebra $\boldsymbol{C}[t_0, \cdots, t_{n-1}]$. The polynomials $t_0, \cdots, t_{n-1}$ are algebraically independent over $\boldsymbol{C}$.*

(ii) $ISO(n)$ $(n \geqq 2)$.

Let G_n and $\mathfrak{g}_n$ be the group $ISO(n)$ and its Lie algebra respectively, namely

$$G_n = \left\{\begin{pmatrix} u & a \\ 0 & 1\end{pmatrix};\ u \in SO(n),\ a \in \boldsymbol{R}^n\right\},$$

$$\mathfrak{g}_n = \left\{\begin{pmatrix} X & x \\ 0 & 0\end{pmatrix};\ X \in so(n),\ x \in \boldsymbol{R}^n\right\}.$$

Denote by $\langle\ ,\ \rangle_n$ a non-degenerate bilinear form on $\mathfrak{g}_n \times \mathfrak{g}_n$;

$$\left\langle \begin{pmatrix} X & x \\ 0 & 0\end{pmatrix},\ \begin{pmatrix} Y & y \\ 0 & 0\end{pmatrix}\right\rangle_n = \left\langle \begin{pmatrix} X & 0 \\ 0 & 0\end{pmatrix},\ \begin{pmatrix} Y & 0 \\ 0 & 0\end{pmatrix}\right\rangle_{so(n+1)} + \langle x, y\rangle,$$

where $\langle\ ,\ \rangle_{so(n+1)}$ = the Killing form of the Lie algebra $so(n+1)$ and $\langle x, y\rangle = x^* y$. Identifying the dual space $\mathfrak{g}_n^*$ of $\mathfrak{g}_n$ with $\mathfrak{g}_n$ by this form, we define the coadjoint representation CoAd of G_n in $\mathfrak{g}_n^* = \mathfrak{g}_n$ as in the case (i). If $n = 2l+1$ $(l \geqq 1)$, let $\mathfrak{H}_n$ be a $(n+1)/2$-dimensional subspace

$$\left\{\begin{pmatrix} 0 & Y_1 & 0 & & & 0 & 0 \\ -Y_1 & 0 & & & & \vdots & \vdots \\ & & \ddots & & & \vdots & \vdots \\ & & & 0 & Y_l & 0 & 0 \\ & & & -Y_l & 0 & 0 & 0 \\ 0 & \cdots & & 0 & 0 & 0 & y_{l+1} \\ 0 & \cdots & & 0 & 0 & 0 & 0 \end{pmatrix} ; \; Y_1, \cdots, Y_l, \; y_{l+1} \in \boldsymbol{R}\right\}$$

of $\mathfrak{g}_n^*$. If $n = 2(l+1)$ $(l \geqq 0)$, let $\mathfrak{H}_n$ be a $n/2$-dimensional subspace

$$\left\{\begin{pmatrix} 0 & Y_1 & 0 & & & 0 & 0 \\ -Y_1 & 0 & & & & \vdots & \vdots \\ & & \ddots & & & \vdots & \vdots \\ & & & 0 & Y_l & 0 & \vdots \\ & & & -Y_l & 0 & 0 & \vdots \\ 0 & \cdots & & 0 & 0 & 0 & 0 \\ 0 & \cdots & & 0 & 0 & 0 & y_{l+1} \\ 0 & \cdots & & 0 & 0 & 0 & 0 \end{pmatrix} ; \; Y_1, \cdots, Y_l, \; y_{l+1} \in \boldsymbol{R}\right\}$$

of $\mathfrak{g}_n^*$. Set $s_i =$ the i-th fundamental symmetric polynomial in $Y_1^2, \cdots, Y_l^2$, and $t_i = s_i y_{l+1}^2$ $(0 \leqq i < l)$. Set, further,

$$s_l = Y_1 \cdots Y_l\,, \quad t_l = s_l y_{l+1} \quad \text{for } n = 2l+1\,,$$
$$s_l = (Y_1 \cdots Y_l)^2\,, \quad t_l = s_l y_{l+1}^2 \quad \text{for } n = 2(l+1).$$

THEOREM 2. *The $\boldsymbol{C}$-algebra of* CoAd $(ISO(n))$*-invariant polynomial functions on $\mathfrak{g}_n^*$ is isomorphic, via the restriction map $f \to f \mid \mathfrak{H}_n$, to the $\boldsymbol{C}$-algebra $\boldsymbol{C}[t_0, \cdots, t_l]$. The polynomials $t_0, \cdots, t_l$ are algebraically independent over $\boldsymbol{C}$.*

We shall prove Theorems 1 and 2 in Section 2 and Section 3 respectively.

The author would like to express his sincere thanks to Dr. G. Sekiguchi and Dr. M. Hashizume for their valuable suggestions.

At the end of this section, we shall explain our basic notation. $\boldsymbol{R}$ and $\boldsymbol{C}$ stand for real and complex number fields respectively. Imaginary unit will be denoted by $\sqrt{-1}$. $\sqrt{-1}\boldsymbol{R}$ means the set of pure imaginary numbers. For nonnegative integers m and n, $M_{mn}(\boldsymbol{R})$ and $M_{mn}(\boldsymbol{C})$ stand for the set of real and complex $m \times n$-matrices respectively, $X^*(X \in M_{mn}(\boldsymbol{C}))$ means the complex conjugate of the transposed matrix X^t. As usual $\operatorname{tr} X$ $(X \in M_{nn}(\boldsymbol{C}))$ means the trace of X. $M_{n1}(\boldsymbol{R})$ and $M_{n1}(\boldsymbol{C})$ will be denoted by $\boldsymbol{R}^n$ and $\boldsymbol{C}^n$ respectively. I_n is the unit matrix in $M_{nn}(\boldsymbol{C})$. For positive integers i and j, E_{ij} means the matrix whose (i, j)-component alone is non-vanishing and is equal to 1. Set

$$U(n)=\{u\in M_{nn}(\boldsymbol{C});\ u^*u=I_n\}\,,$$
$$SO(n)=\{v\in M_{nn}(\boldsymbol{R});\ v^*v=I_n,\ \det v=1\}\,.$$

Their Lie algebras will be denoted by $u(n)$ and $so(n)$ respectively. Finally $S_{\varepsilon,n}$ is a transformation of $M_{n+1,n+1}(\boldsymbol{C})$ sending X to $\begin{pmatrix} I_n & 0\\ 0 & \varepsilon\end{pmatrix} X \begin{pmatrix} I_n & 0\\ 0 & \varepsilon\end{pmatrix}^{-1}$.

§2. The proof of Theorem 1

Let $G_{n,0}$ and $G_{n,1}$ be the matrix group $U(1)\times IU(n)$ and $U(n+1)$ respectively. Denote by $\mathfrak{g}_{n,0}$ and $\mathfrak{g}_{n,1}$ the corresponding Lie algebras;

$$\mathfrak{g}_{n,0}=\left\{cI_{n+1}+\begin{pmatrix} X & x\\ 0 & 0\end{pmatrix};\ c\in\sqrt{-1}\boldsymbol{R},\ X\in u(n),\ x\in \boldsymbol{C}^n\right\}\,,$$
$$\mathfrak{g}_{n,1}=\left\{cI_{n+1}+\begin{pmatrix} X & x\\ -x^* & -\operatorname{tr}X\end{pmatrix};\ c\in\sqrt{-1}\boldsymbol{R},\ X\in u(n),\ x\in \boldsymbol{C}^n\right\}\,.$$

Let for $0<\varepsilon\leqq 1$

$$G_{n,\varepsilon}=S_{\varepsilon,n}(G_{n,1}),\qquad \mathfrak{g}_{n,\varepsilon}=S_{\varepsilon,n}(\mathfrak{g}_{n,1}),$$

where $S_{\varepsilon,n}$ is the transformation of $M_{n+1,n+1}(\boldsymbol{C})$ defined at the end of Section 1. For $\underline{X}=\begin{pmatrix} X & x\\ 0 & 0\end{pmatrix}\in\mathfrak{g}_{n,0}$, $\underline{X}_\delta$ $(0\leqq\delta\leqq 1)$ means the matrix either

$$\begin{pmatrix} X & x\\ -\delta^2x^* & -\operatorname{tr}X\end{pmatrix}\quad\text{for } 0<\delta\leqq 1\quad\text{or}\quad\begin{pmatrix} X & x\\ 0 & 0\end{pmatrix}\quad\text{for } \delta=0\,.$$

In addition, $(c, \underline{X}_\delta)$ $(c\in\sqrt{-1}\boldsymbol{R})$ means the matrix $cI_{n+1}+\underline{X}_\delta$ in $\mathfrak{g}_{n,\delta}$. Now the bilinear form $\langle\,,\rangle_{\delta,n}$ on $\mathfrak{g}_{n,\delta}\times\mathfrak{g}_{n,\delta}$ will be defined as follows.

$$\langle(a, \underline{X}_\delta), (b, \underline{Y}_\delta)\rangle_{\delta,n}=ab+\langle S_{\delta,n}^{-1}\underline{X}_\delta, S_{\delta,n}^{-1}\underline{Y}_\delta\rangle_{su(n+1)}\qquad(0<\delta\leqq 1)\,,$$
$$\langle(a, \underline{X}_0), (b, \underline{Y}_0)\rangle_{0,n}=ab+\langle\iota X, \iota Y\rangle_{su(n+1)}+\langle x, y\rangle\,.$$

In the above $\langle\,,\rangle_{su(n+1)}=$ the Killing form of the Lie algebra $su(n+1)$, $\langle x, y\rangle=\operatorname{Re}x^*y$ and $\iota X=\begin{pmatrix} X & 0\\ 0 & -\operatorname{tr}X\end{pmatrix}$. Since the bilinear forms $\langle\,,\rangle_{\delta,n}$ are non-degenerate, we can identify the dual space $\mathfrak{g}_{n,\delta}^*$ with $\mathfrak{g}_{n,\delta}$. Let $\langle\,,\rangle_{\varepsilon,0,n}$ be the non-degenerate bilinear form on $\mathfrak{g}_{n,\varepsilon}\times\mathfrak{g}_{n,0}$ $(0<\varepsilon\leqq 1)$ defined by

$$\langle(a, \underline{X}_\varepsilon), (b, \underline{Y}_0)\rangle_{\varepsilon,0,n}=(a-\operatorname{tr}X)b+\langle x, y\rangle+\langle\iota(X+\operatorname{tr}X), \iota Y\rangle_{su(n+1)}\,.$$

Thus we have isomorphisms $J_{\varepsilon,n}$: $\mathfrak{g}_{n,0}^*\to\mathfrak{g}_{n,0}^*$ and $J_{\varepsilon,n}^*$: $\mathfrak{g}_{n,\varepsilon}\to\mathfrak{g}_{n,0}$ satisfying

$$\langle (a, \underline{X}_\varepsilon), (b, \underline{Y}_0)\rangle_{\varepsilon,0,n} = \langle (a, \underline{X}_\varepsilon), J_{\varepsilon,n}(b, \underline{Y}_0)\rangle_{\varepsilon,n}$$
$$= \langle J^*_{\varepsilon,n}(a, \underline{X}_\varepsilon), (b, \underline{Y}_0)\rangle_{0,n} .$$

Let $\mathrm{Ad}_\delta(g)$ be the representation of $G_{n,\delta}$ in $\mathfrak{g}_{n,\delta}$ defined by the formula

$$\mathrm{Ad}_\delta(g)(c, \underline{X}_\delta) = g(c, \underline{X}_\delta)g^{-1} \qquad \text{for } g \in G_{n,\delta}\ (0 \leqq \delta \leqq 1) ,$$

while CoAd_δ means the representation of $G_{n,\delta}$ in $\mathfrak{g}^*_{n,\delta}$ such that

$$\langle \mathrm{Ad}_\delta(g^{-1})(c, \underline{X}_\delta), (b, \underline{Y}_\delta)\rangle_{\delta,n} = \langle (a, \underline{X}_\delta), \mathrm{CoAd}_\delta(g)(b, \underline{Y}_\delta)\rangle_{\delta,n} .$$

Denote by $\mathrm{coAd}_\varepsilon (0 < \varepsilon \leqq 1)$ the representation $J^{-1}_{\varepsilon,n}\,\mathrm{CoAd}_\varepsilon\, J_{\varepsilon,n}$ of $G_{n,\varepsilon}$ in $\mathfrak{g}^*_{n,0}$. In accordance with this notation we write coAd_0 for CoAd_0. To describe the transformation $\mathrm{coAd}_0(g)$ explicitly, let $\{\lambda_0, \lambda_i, \lambda_{ij}, \omega_{ij}, e_i, \sqrt{-1}e_i;\ 1 \leqq i \leqq n,\ 1 \leqq i \leqq j \leqq n\}$ be a basis of $\mathfrak{g}_{n,0}$ such that

$$\lambda_0 = \sqrt{-1}\,I_{n+1} , \quad \lambda_i = \sqrt{-1}\,E_{ii} , \quad \lambda_{ij} = \sqrt{-1}(E_{ij} + E_{ji}) ,$$
$$\omega_{ij} = E_{ij} - E_{ji} , \qquad e_i = E_{i,n+1} .$$

The dual basis of $\mathfrak{g}^*_{n,0}$ will be denoted by $\{\lambda^0, \lambda^i, \lambda^{ij}, \omega^{ij}, e^i, \sqrt{-1}e^i\}$.

LEMMA 2.1. *For* $g = \begin{pmatrix} u & a \\ 0 & 1\end{pmatrix} \in G_{n,0}$ *and* $(c, \underline{Y}_0) \in \mathfrak{g}^*_{n,0}$ *it holds that* $\mathrm{coAd}_0(g)(c, \underline{Y}_0) = (c, \underline{Y}'_0)$ *with* $y' = uy$ *and*

$$Y' = uYu^{-1} + \textstyle\sum_{i=1}^n \langle \lambda_i a, uy\rangle \lambda^i$$
$$+ \textstyle\sum_{i<j} (\langle \lambda_{ij} a, uy\rangle \lambda^{ij} + \langle \omega_{ij} a, uy\rangle \omega^{ij}) .$$

Proof. It suffices to note that $\langle Xa, uy\rangle$ $(X \in u(n))$ is equal to $\langle \iota X, \iota(Y' - uYu^{-1})\rangle_{su(n+1)}$.

Remark 2.2. Bearing in mind that the complexification of the Lie algebra $su(n+1)$ is isomorphic to the Lie algebra $sl(n+1, \boldsymbol{C})$, we can easily verify that

$$\lambda^i = \sqrt{-1}(\textstyle\sum_{j\neq i} E_{jj} - nE_{ii})/d_n , \quad \lambda^{ij} = \lambda_{ij}/c_n , \quad \omega^{ij} = \omega_{ij}/c_n$$

with $c_n = -4(n+1)$ and $d_n = 2(n+1)^2$ [8, p. 295 or p. 390].

LEMMA 2.3. (1) $J_{\varepsilon,n}(c, Y_0) = (c, Y'_\varepsilon)$ *for* $(c, Y_0) \in \mathfrak{g}^*_{n,0}$, *where* $y' = y/\varepsilon^2 c_n$ *and* $Y' = Y + \mathrm{tr}\, Y - c/d_n$.

(2) $J^*_{\varepsilon,n}(c, \underline{X}_\varepsilon) = (c', \underline{X}'_0)$ *for* $(c, \underline{X}_\varepsilon) \in \mathfrak{g}^*_{n,\varepsilon}$, *where* $c' = c - \mathrm{tr}\, X$, $x' = x$ *and* $X' = X + \mathrm{tr}\, X$.

Proof. Note that $\mathrm{tr}\, X = \sqrt{-1}\langle \iota X, \sum_{i=1}^n \iota\lambda^i\rangle_{su(n+1)}$. Then (1) follows at once. The second assertion is almost obvious. Observe that $(c, \underline{X}_\varepsilon) \in \mathfrak{g}_{n,\varepsilon}$

tends to $(c, \underline{X}_0) \in \mathfrak{g}_{n,0}$ as $\varepsilon \to 0$. In other words, contracting the Lie algebra $\mathfrak{g}_{n,1}$, we obtain the Lie algebra $\mathfrak{g}_{n,0}$ [4]. Consequently $\mathrm{coAd}_\varepsilon$ converge to coAd_0 in the following sense.

LEMMA 2.4. *Assume $t \in \boldsymbol{R}$ and $(b, \underline{Y}_0) \in \mathfrak{g}_{n,0}^*$, and let $\underline{Z}_\varepsilon \in \mathfrak{g}_{n,\varepsilon}$ $(\varepsilon > 0)$ be the element corresponding to $\underline{Z} = \begin{pmatrix} Z & z \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_{n,0}$. Then*

$$\lim_{\varepsilon \to 0} \mathrm{coAd}_\varepsilon (\exp tZ_\varepsilon)(b, Y_0) = \mathrm{coAd}_0 (\exp t\begin{pmatrix} Z & z \\ 0 & -\mathrm{tr}\, Z \end{pmatrix})(b, Y_0) .$$

Proof. Evidently it suffices to show that the generators of the semigroups $\mathrm{coAd}_\varepsilon (\exp tZ_\varepsilon)$ converge to the generator of the semigroup $\mathrm{coAd}_0 (\exp t\begin{pmatrix} Z & z \\ 0 & -\mathrm{tr}\, Z \end{pmatrix})$. To this end, we can verify easily that

$$\frac{d}{dt}\Big|_{t=0} \Big\{\langle \mathrm{Ad}_\varepsilon (\exp (-tZ_\varepsilon)) J_{\varepsilon,n}^{*-1}(a, X_0),\ J_{\varepsilon,n}(b, Y_0)\rangle_{\varepsilon,n}$$
$$-\Big\langle \mathrm{Ad}_0 \Big(\exp -t\begin{pmatrix} Z & z \\ 0 & -\mathrm{tr}\, Z \end{pmatrix}\Big)(a, X_0),\ (b, Y_0)\Big\rangle_{0,n}\Big\} \longrightarrow 0 \quad (\varepsilon \longrightarrow 0)$$

for any $(a, X_0) \in \mathfrak{g}_{n,0}$.

Remark 2.5. It is clear that $\mathrm{coAd}_\delta (\exp cI_{n+1})$ is the identity operator for any $cI_{n+1} \in \mathfrak{g}_{n,\delta}$.

Let $F(b, Y, y)$ be the value at $(b, Y_1) \in \mathfrak{g}_{n,1}^*$ of a $\mathrm{CoAd}_1 (G_{n,1})$-invariant polynomial F. Then the polynomial function $f_\varepsilon = F \circ S_{\varepsilon,n}^{-1} \circ J_{\varepsilon,n}$ on $\mathfrak{g}_{n,0}^*$ is clearly $\mathrm{coAd}_\varepsilon (G_{n,\varepsilon})$-invariant. By Lemma 2.3 we have

$$f_\varepsilon(b, Y, y) = F(b, Y + (\mathrm{tr}\, Y - b/d_n)I_n,\ y/\varepsilon c_n) ,$$

where the left-hand side stands for the value of f_ε at $(b, Y_0) \in \mathfrak{g}_{n,0}^*$. Regarding f_ε as polynomial in ε^{-1}, let $f(b, Y, y)$ be the coefficient of the ε^{-d} for the highest degree d.

LEMMA 2.6. *The polynomial function f on $\mathfrak{g}_{n,0}^*$ defined above is $\mathrm{coAd}_0(G_{n,0})$-invariant.*

Proof. In view of Lemma 2.4 the assertion is an immediate consequence of the fact that $f(b, Y, y)$ coincides with the limit of $\varepsilon^d f_\varepsilon(b, Y, y)$ as $\varepsilon \to 0$.

We shall now clarify the relation between a $\mathrm{CoAd}(G_n)$-invariant polynomial function on $\mathfrak{g}_n^*$ and a $\mathrm{coAd}_0 (G_{n,0})$-invariant polynomial function on $\mathfrak{g}_{n,0}^*$ (recall the notation in Section 1). Note first that $\mathfrak{g}_{n,0} = \sqrt{-1}\boldsymbol{R} + \mathfrak{g}_n$, the sum being orthogonal direct sum of $\mathrm{Ad}_0(G_{n,0})$-invariant subspaces. Therefore, $\mathfrak{g}_{n,0}^*$ is a direct sum of $\mathrm{coAd}_0(G_{n,0})$-invariant subspaces $\sqrt{-1}\boldsymbol{R}$ and $\mathfrak{g}_n^* = \mathfrak{g}_n$. Moreover, the restriction $\mathrm{coAd}_0(g)|\sqrt{-1}\boldsymbol{R}$ is the identity operator for $g \in G_{n,0}$, because so is the restriction $\mathrm{Ad}_0(g)|\sqrt{-1}\boldsymbol{R}$. The bilinear form $\langle\, ,\rangle_n$ on $\mathfrak{g}_n \times \mathfrak{g}_n$ is nothing but the restriction $\langle\, ,\rangle_{0,n}|\mathfrak{g}_n \times \mathfrak{g}_n$. It now follows that $\mathrm{coAd}_0(g) = \mathrm{CoAd}(g)$ in $\mathfrak{g}_n^* \subset \mathfrak{g}_{n,0}^*$ for any $g \in G_n$. On account of Remark 2.5 the set of operators $\{\mathrm{coAd}_0(g)|\mathfrak{g}_n;\, g \in G_{n,0}\}$ coincides with $\{\mathrm{CoAd}(g);\, g \in G_n\}$. Bearing these observations in mind, we can easily verify the validity of

LEMMA 2.7. *A polynomial function $f(b, Y, y)$ on $\mathfrak{g}_{n,0}^*$ is $\mathrm{coAd}_0(G_{n,0})$-invariant iff the polynomial function $f(b, Y, y)$ on $\mathfrak{g}_n^*$ is $\mathrm{CoAd}(G_n)$-invariant for any b.*

Let $I(\mathfrak{g}_n^*)$ be the set of all $\mathrm{CoAd}(G_n)$-invariant polynomial functions on $\mathfrak{g}_n^* = \mathfrak{g}_n$, and let r_n be the restriction map sending $f \in I(\mathfrak{g}_n^*)$ to $f|\mathfrak{H}_n$ whose image will be denoted by $I(\mathfrak{H}_n)$. For the definition of the subspace $\mathfrak{H}_n$ of $\mathfrak{g}_n$, see Section 1.

LEMMA 2.8. *The union of the orbits $\{\mathrm{CoAd}(G_n)Y_0;\, Y_0 \in \mathfrak{H}_n\}$ is dense in $\mathfrak{g}_n^*$. In particular, the map r_n is an algebraic isomorphism of $I(\mathfrak{g}_n^*)$ onto $I(\mathfrak{H}_n)$.*

Proof. For a $Y_0 = \begin{pmatrix} Y & y \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_n^*$ with $y \in \boldsymbol{C}^n \backslash \{0\}$ we shall show that there exists a $g \in G_n$ such that $\mathrm{CoAd}(g)Y_0 \in \mathfrak{H}_n$. Take, first, a $u \in U(n)$ satisfying $uy = (0, \cdots, -\sqrt{-1}|y|)^*$ with $|y| = (y^*y)^{1/2}$. We can find an $a \in \boldsymbol{C}^n$ $(\subset G_n)$ for which $\underline{Y}_0' = \mathrm{CoAd}(a\cdot u)\underline{Y}_0$ satisfies $Y_{in}' = 0$ $(1 \leqq i \leqq n)$. Recall the well-known fact that any $Y'' \in u(n-1)$ can be diagonalized by an element of $SU(n-1)$. Now we can find a $v = \begin{pmatrix} v' & 0 \\ 0 & 1 \end{pmatrix} \in SU(n)$ such that $\mathrm{CoAd}(v\cdot a\cdot u)Y_0 \in \mathfrak{H}_n$.

Let $\tilde{A}_n$ be the subgroup of G_n consisting of all $g \in G_n$ such that $\mathrm{CoAd}(g)\mathfrak{H}_n \subset \mathfrak{H}_n$, and let A_n be the subgroup of $\tilde{A}_n$ consisting of all $g \in \tilde{A}_n$ such that the restriction $\mathrm{CoAd}(g)|\mathfrak{H}_n$ is the identity operator. The group $\mathrm{CoAd}(\tilde{A}_n)|\mathfrak{H}_n$ turns out not to differ so much from the Weyl group of $U(n-1)$ [5, p. 305]. To be more precise, let σ_i $(1 \leqq i < n-1)$ and τ be the linear transformations on $\mathfrak{H}_n$ such that

$$\sigma_i\colon (Y_i, Y_{i+1}) \longrightarrow (Y_{i+1}, Y_i)\,, \qquad \tau\colon y_n \longrightarrow -y_n\,.$$

Denote by W_{n-1} the group generated by σ_i $(1 \leqq i < n-1)$. Then we can verify easily that $\mathrm{CoAd}(\tilde{A}_n)|\mathfrak{H}_n$, which is isomorphic to the quotient group $\tilde{A}_n/A_n$, is generated by W_{n-1} and τ. Note that W_{n-1} is the Weyl group of $U(n-1)$. We shall now show that the polynomial functions $t_0, \cdots, t_{n-1}$ on $\mathfrak{H}_n$ (see Section 1) belong to $I(\mathfrak{H}_n)$. Indeed, there exist $\mathrm{CoAd}_1(G_{n,1})$-invariant polynomial functions $F_i(b, Y)$ on $\mathfrak{g}_{n,1}^*$ such that

$$\det(t+Y) = t^{n+1} + \sum_{i=1}^{n} F_i(b, Y)t^{n-i} \qquad (t;\ \text{an indeterminate})\ .$$

Applying Lemmas 2.6 and 2.7 to F_i, we deduce that $s_{i-1}y_n^2$ $(1 \leqq i < n)$ lie in $I(\mathfrak{H}_n)$. Denote by T_n the set of all polynomial functions on $\mathfrak{H}_n$ generated by $t_0, \cdots, t_{n-1}$. We shall show that T_n coincides with $I(\mathfrak{H}_n)$. For this purpose it is convenient to introduce another subspace $\tilde{\mathfrak{H}}_n$ of $\mathfrak{g}_n^*$.

$$\tilde{\mathfrak{H}}_n = \left\{\begin{pmatrix} Y & y \\ 0 & 0 \end{pmatrix};\ Y \text{ is a diagonal matrix } [Y_1, \cdots, Y_n] \text{ and } y_i = 0 \text{ except for } i = n\right\}.$$

Let $\tilde{r}_n$ be the restriction map sending an $f \in I(\mathfrak{g}_n^*)$ to $f|\tilde{\mathfrak{H}}_n$, whose image will be denoted by $I(\tilde{\mathfrak{H}}_n)$. Set $\tilde{t}_i = \tilde{r}_n r_n^{-1}(t_i)$ $(0 \leqq i \leqq n-1)$ and denote by $\tilde{T}_n$ the set of all polynomial functions on $\tilde{\mathfrak{H}}_n$ generated by $\tilde{t}_0, \cdots, \tilde{t}_{n-1}$. Note that an $f(Y_1, \cdots, Y_n, y_n) \in I(\tilde{\mathfrak{H}}_n)$ is invariant under the transformation $y_n \to -y_n$ and the permutations $Y_i \leftrightarrow Y_{i+1}$ $(1 \leqq i < n-1)$. For a $\begin{pmatrix} Y & y \\ 0 & 0 \end{pmatrix} \in \tilde{\mathfrak{H}}_n$ set $\tilde{Z}_i = Y_{ii} + \operatorname{tr} Y$ $(1 \leqq i \leqq n)$, $\tilde{s}_{i-1}$ = the $(i-1)$-th fundamental symmetric polynomial in $\tilde{Z}_1, \cdots, \tilde{Z}_{n-1}$. Evidently $\tilde{t}_{i-1} = \tilde{s}_{i-1}y_n^2$. We claim that $\tilde{T}_n$ consists of all polynomials of the form

$$\sum_{l \geqq 0} f_l(\tilde{Z}_1, \cdots, \tilde{Z}_{n-1})y_n^{2l}\ ,$$

where f_l are symmetric polynomials whose degrees in $\tilde{Z}_i$ do not exceed l. In fact, $f_l y_n^{2l}$ can be rewritten as

$$\begin{aligned}&\sum_{\alpha_1, \cdots, \alpha_{n-1} \geqq 0} a_{l,\alpha_1,\cdots,\alpha_{n-1}} \tilde{s}_1^{\alpha_1} \cdots \tilde{s}_{n-1}^{\alpha_{n-1}} y_n^{2l} \\ &\quad = \sum_{\alpha_1, \cdots, \alpha_{n-1} \geqq 0} a_{l,\alpha_1,\cdots,\alpha_{n-1}} \tilde{t}_0^{\alpha_0} \tilde{t}_1^{\alpha_1} \cdots \tilde{t}_{n-1}^{\alpha_{n-1}}\end{aligned}$$

with $\alpha_0 = -\alpha_1 - \cdots - \alpha_{n-1} + l \geqq 0$.

LEMMA 2.9. *$T_n = I(\mathfrak{H}_n)$ or, equivalently, $\tilde{T}_n = I(\tilde{\mathfrak{H}}_n)$.*

Proof. We shall proceed by induction on $n \geqq 2$. The case $n = 2$ will be discussed in Lemma 2.10. So assume that the lemma holds up to $n - 1 \geqq 2$. Define subspaces $\mathfrak{g}_n^\circ$, $\tilde{\mathfrak{g}}_{n-1}$, $\mathfrak{g}_{n-1}$ and $\mathscr{W}_n$ of $\mathfrak{g}_n$ as follows.

$$\mathfrak{g}_n^\circ = \left\{\begin{pmatrix} X & x \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_n;\ X_{11} = 0\right\},$$

$$\tilde{\mathfrak{g}}_{n-1} = \left\{ M(\mu, X', x') = \begin{pmatrix} \mu & 0 & 0 \\ 0 & X' & x' \\ 0 & 0 & 0 \end{pmatrix};\ \mu \in \sqrt{-1}\boldsymbol{R},\ X' \in u(n-1),\ x' \in \boldsymbol{C}^{n-1} \right\},$$

$$\mathfrak{g}_{n-1} = \{M(0, X', x') \in \tilde{\mathfrak{g}}_{n-1}\},$$

$$\mathscr{W}_n = \left\{ \begin{pmatrix} \mu & x & c \\ -x^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\ \mu \in \sqrt{-1}\boldsymbol{R},\ x^* \in \boldsymbol{C}^{n-1},\ c \in \boldsymbol{C} \right\}.$$

We can identify the group G_{n-1} with the connected subgroup of G_n corresponding to the subalgebra $\mathfrak{g}_{n-1}$. $\mathfrak{g}_n$ is a direct sum of the $\operatorname{Ad}(G_{n-1})$-invariant subspaces $\mathfrak{g}_{n-1}$ and $\mathscr{W}_n$. Moreover, since

$$\operatorname{Ad}(g)\begin{pmatrix} X & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X' & x' \\ 0 & 0 \end{pmatrix} \text{ with } X_{11} = X'_{11} \text{ for } g \in G_{n-1} \text{ and } \begin{pmatrix} X & x \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_n,$$

it follows that the representation of G_{n-1} in the quotient space $\mathfrak{g}_n/\mathfrak{g}_n^\circ$ is the trivial one. Denote by $\mathscr{W}^\perp$ the orthogonal complement of a subspace $\mathscr{W}$ of $\mathfrak{g}_n$. It is easy to verify that

$$\mathscr{W}_n^\perp = \left\{ \begin{pmatrix} Y'_1 & 0 & 0 \\ 0 & Y' & y' \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}_n;\ Y'_1 + \operatorname{tr} Y' = 0 \right\}.$$

Observe that $\mathscr{W}_n^\perp$ is $\operatorname{CoAd}(G_{n-1})$-invariant and that $\begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_n$ with $Y \in u(n)$ lies in $\mathscr{W}_n^\perp$ iff Y is a linear combination of $\lambda^2, \cdots, \lambda^n$ (see Remark 2.2). Besides, $\operatorname{CoAd}(g)\lambda^1 = \lambda^1$ for $g \in G_{n-1}$, because the representation of G_{n-1} in $\mathfrak{g}_n^{\circ\perp} \subset \mathfrak{g}_n^*$ is the trivial one as the representation of G_{n-1} in the quotient space $\mathfrak{g}_n/\mathfrak{g}_n^\circ$. Note also that $\tilde{\mathfrak{g}}_{n-1}$, which we may regard as a subspace of $\mathfrak{g}_n^*$, is a direct sum of the $\operatorname{CoAd}(G_{n-1})$-invariant subspaces $\mathscr{W}_n^\perp$ and $\left\{ \begin{pmatrix} \mu\lambda^1 & 0 \\ 0 & 0 \end{pmatrix}; \mu \in \boldsymbol{R} \right\}$. In particular, $\tilde{\mathfrak{g}}_{n-1}$ is $\operatorname{CoAd}(G_{n-1})$-invariant. For $\begin{pmatrix} Y & y \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_n^*$ set $\tilde{Z}_i = Y_{ii} + \operatorname{tr} Y$ $(1 \leqq i \leqq n)$. Then simple computaion yields that

$$\text{the diagonal part of } Y = -2\sqrt{-1} \textstyle\sum_{i=1}^n \tilde{Z}_i \lambda^i / c_n$$

with $c_n = -4(n+1)$. Define a basis $\{\lambda_i, \lambda_{ij}, \omega_{ij};\ 2 \leqq i \leqq n,\ 2 \leqq i < j \leqq n\}$ of $\mathfrak{g}_{n-1}$ as for $\mathfrak{g}_{n,0}$, and denote by $\{\underline{\lambda}^i, \underline{\lambda}^{ij}, \underline{\omega}^{ij},\ 2 \leqq i \leqq n,\ 2 \leqq i < j \leqq n\}$ the dual basis. Now let L_n be a linear isomorphism of $\mathscr{W}_n^\perp$ onto $\mathfrak{g}_{n-1}^* = \mathfrak{g}_{n-1}$ defined by

$$\left\langle \begin{pmatrix} X' & x' \\ 0 & 0 \end{pmatrix}, L_n \begin{pmatrix} Y'_1 & 0 & 0 \\ 0 & Y' & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle_{0,n-1} = \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & X' & x' \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} Y'_1 & 0 & 0 \\ 0 & Y' & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle_{0,n}.$$

It turns out that

$$(*)\qquad L_n\lambda^i = \lambda^i \qquad (2 \leqq i \leqq n)\,.$$

Assume that the restriction $F|\tilde{\mathfrak{g}}_{n-1}$ of an $F \in I(\mathfrak{g}_n^*)$ takes the form

$$\textstyle\sum_{k\geqq 1} \tilde{Z}_1^k F_k(\tilde{Z}_2, \cdots, \tilde{Z}_n, y_n, y_2 \cdots, y_{n-1}, Y_{ij})$$

at $\begin{pmatrix} Y & y \\ 0 & 0 \end{pmatrix} \in \tilde{\mathfrak{g}}_{n-1} \subset \mathfrak{g}_n^*$. In the above Y_{ij} stands for all the off-diagonal components of Y. Since the function $\tilde{Z}_1$ on $\tilde{\mathfrak{g}}_{n-1}$ is $\mathrm{CoAd}\,(G_{n-1})$-invariant, so are F_k. On account of $(*)$, together with the induction hypothesis, it follows that $F_k(\tilde{Z}_2, \cdots, y_n, 0, \cdots, 0)$ take the form

$$\textstyle\sum_{l\geqq 0} f_{kl}(\tilde{Z}_2, \cdots, \tilde{Z}_{n-1}) y_n^{2l}\,,$$

where f_{kl} are symmetric polynomials whose degrees in $\tilde{Z}_i$ are not greater than l. Consequently the restriction $f = F|\tilde{\mathfrak{H}}_n$ is of the form

$$\textstyle\sum_{l\geqq 0} f_l(\tilde{Z}_1, \cdots, \tilde{Z}_{n-1}) y_n^{2l}\,,$$

where

$$f_l(\tilde{Z}_1, \cdots, \tilde{Z}_{n-1}) = \textstyle\sum_{k\geqq 0} \tilde{Z}_1^k f_{kl}(\tilde{Z}_2, \cdots, \tilde{Z}_{n-1})\,.$$

As we observed before, f_l are symmetric polynomials in $\tilde{Z}_1, \cdots, \tilde{Z}_{n-1}$. Thus the degrees of f_l in $\tilde{Z}_i$ $(1 \leqq i \leqq n-1)$ do not exceed l, which proves $F|\tilde{\mathfrak{H}}_n \in \tilde{T}_n$.

LEMMA 2.10. $T_2 = I(\tilde{\mathfrak{H}}_2)$.

Proof. Throughout the proof let $n = 2$. Assume an F in $I(\mathfrak{g}_n^*)$ to be a homogeneous polynomial such that the restriction $f = F|\tilde{\mathfrak{H}}_n$ takes the form

$$f(Y, y_2) = \textstyle\sum_{k=0}^{m} b_k Y^{m-2k} y_2^{2k}$$

for some positive integer m. We shall show that $m - 2k \leqq k$ if $b_k \neq 0$. For Y, z_1, $z_2 \in \sqrt{-1}\,\boldsymbol{R}\backslash\{0\}$ let α be a real number satisfying

$$\sqrt{-1}\,(\cos\alpha, \sin\alpha) = (z_2, z_1)/|z| \quad \text{with} \quad |z| = (z_1 z_1^* + z_2 z_2^*)^{1/2}.$$

Set

$$a^* = (a_1, a_2) = (\sqrt{-1}\,c_n \cos\alpha \sin\alpha,\ -\sqrt{-1}\,n^{-1} d_n \sin^2\alpha) Y/|z|.$$

See Remark 2.2 for the definition of c_n and d_n. Denote by g the matrix $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$. Then simple calculation yields

$$\mathrm{CoAd}\,(a\cdot g)\begin{pmatrix} Y & 0 & z_1 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} Y_\alpha & 0 & 0 \\ 0 & 0 & \sqrt{-1}|z| \\ 0 & 0 & 0 \end{pmatrix},$$

where $Y_\alpha = Y(\cos^2\alpha + 1)/n$. Consequently $f(Y_\alpha, \sqrt{-1}|z|)$ must be a polynomial in Y, z_1, z_2, which implies the desired inequality $2(m-2k) \leqq 2k$, provided $b_k \neq 0$.

Lemmas 2.9 and 2.10 prove the first assertion of Theorem 1, while the second one is obvious in view of the well-known fact; polynomials $f_i(X_1, \cdots, X_n)$ $(1 \leqq i \leqq n)$ in X_i $(1 \leqq i \leqq n)$ are algebraically independent if the determinant of the Jacobian $(\partial f_i/\partial X_j)$ is a non-zero polynomial.

§ 3. The proof of Theorem 2

Let $G_{n,0}$ and $G_{n,1}$ be the matrix group $ISO(n)$ and $SO(n+1)$ respectively. Denote by $\mathfrak{g}_{n,0}$ and $\mathfrak{g}_{n,1}$ the corresponding Lie algebras;

$$\mathfrak{g}_{n,0} = \left\{\begin{pmatrix} X & x \\ 0 & 0 \end{pmatrix};\ X \in so(n),\ x \in \boldsymbol{R}^n\right\},$$

$$\mathfrak{g}_{n,1} = \left\{\begin{pmatrix} X & x \\ -x^* & 0 \end{pmatrix};\ X \in so(n),\ x \in \boldsymbol{R}^n\right\}.$$

We define $G_{n,\delta}$ and $\mathfrak{g}_{n,\delta}$ $(0 \leqq \delta \leqq 1)$ in the same manner as in Section 2. For $\underline{X} = \begin{pmatrix} X & x \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_{n,0}$ (i.e. $X \in so(n)$, $x \in \boldsymbol{R}^n$), $\underline{X}_\delta$ stands for the matrix

$$\begin{pmatrix} X & x \\ -\delta^2 x^* & 0 \end{pmatrix} \in \mathfrak{g}_{n,\delta}\,.$$

Let $\langle\ ,\ \rangle_{\delta,n}$ denote the non-degenerate bilinear form on $\mathfrak{g}_{n,\delta} \times \mathfrak{g}_{n,\delta}$ such that

$$\langle X_\varepsilon,\ Y_\varepsilon\rangle_{\varepsilon,n} = \langle S_{\varepsilon,n}^{-1}X_\varepsilon,\ S_{\varepsilon,n}^{-1}Y_\varepsilon\rangle_{so(n+1)} \qquad (0 < \varepsilon \leqq 1)\,,$$
$$\langle X_0,\ Y_0\rangle_{0,n} = \langle \iota X,\ \iota Y\rangle_{so(n+1)} + \langle x, y\rangle\,,$$

where $\langle x, y\rangle = x^*y$ and $\iota X = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \in so(n+1)$. Thanks to the bilinear form we can identify the dual space $\mathfrak{g}_{n,\delta}^*$ of $\mathfrak{g}_{n,\delta}$ with $\mathfrak{g}_{n,\delta}$. Since the bilinear form $\langle\ ,\ \rangle_{\varepsilon,n}$ is the Killing form of $\mathfrak{g}_{n,\varepsilon}$, the coadjoint representation $\mathrm{CoAd}_\varepsilon$ of $G_{n,\varepsilon}$ in $\mathfrak{g}_{n,\varepsilon}^*$ is nothing but the adjoint representation Ad_ε of $G_{n,\varepsilon}$. Another bilinear form $\langle\ ,\ \rangle_{\varepsilon,0,n}$ on $\mathfrak{g}_{n,\varepsilon} \times \mathfrak{g}_{n,0}$ is defined by the relation

$$\langle X_\varepsilon,\ Y_0\rangle_{\varepsilon,0,n} = \langle \iota X,\ \iota Y\rangle_{so(n+1)} + \langle x, y\rangle\,.$$

Let $J_{\varepsilon,n}$ (resp. $J_{\varepsilon,n}^*$) be the linear isomorphism of $\mathfrak{g}_{n,0}^*$ (resp. $\mathfrak{g}_{n,\varepsilon}$) onto $\mathfrak{g}_{n,\varepsilon}^*$

(resp. $\mathfrak{g}_{n,0}$) satisfying the following equalities.

$$\langle X_\varepsilon, Y_0\rangle_{\varepsilon,0,n} = \langle X_\varepsilon, J_{n,\varepsilon} Y_0\rangle_{\varepsilon,n} = \langle J^*_{n,\varepsilon} X_\varepsilon, Y_0\rangle_{0,n} .$$

Denote by $\mathrm{coAd}_\varepsilon$ the representation $J^{-1}_{\varepsilon,n}\,\mathrm{CoAd}_\varepsilon\, J_{\varepsilon,n}$ of $G_{n,\varepsilon}$ in $\mathfrak{g}^*_{n,0}$. We write coAd_0 for CoAd_0. Set $\omega_{ij} = E_{ii} - E_{jj}$ $(1 \leqq i < j \leqq n)$ and $\omega^{ij} = \omega_{ij}/c_n$ with $c_n = \langle \omega_{ij}, \omega_{ij}\rangle_{0,n}$ [cf. 8, p. 390]. Now we can easily verify the following Lemmas 3.1–3.3.

LEMMA 3.1. *For*

$$g = \begin{pmatrix} u & a \\ 0 & 1 \end{pmatrix} \in G_{n,0} \quad \text{and} \quad \begin{pmatrix} Y & y \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}^*_{n,0}$$

we have

$$\mathrm{coAd}_0 \begin{pmatrix} u & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Y & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} Y' & uy \\ 0 & 0 \end{pmatrix},$$

where $Y' = uYu^{-1} + \sum_{i<j} \langle \omega_{ij} a, uy\rangle \omega^{ij}$.

LEMMA 3.2.

$$J_{\varepsilon,n} \begin{pmatrix} Y & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} Y & y/\varepsilon^2 c_n \\ -y^*/c_n & 0 \end{pmatrix},$$

$$J^*_{\varepsilon,n} \begin{pmatrix} X & x \\ -\varepsilon^2 x^* & 0 \end{pmatrix} = \begin{pmatrix} X & x \\ 0 & 0 \end{pmatrix},$$

with $c_n = \langle \omega_{ij}, \omega_{ij}\rangle_{0,n}$.

LEMMA 3.3. *Assume* $t \in \boldsymbol{R}$ *and* $\underline{Y}_0 \in \mathfrak{g}_{n,0}$. *and let* $\underline{Z}_\varepsilon \in \mathfrak{g}_{n,\varepsilon}$ $(0 < \varepsilon \leqq 1)$ *be the element corresponding to* $\underline{Z} = \begin{pmatrix} Z & z \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_{n,0}$. *Then*

$$\lim_{\varepsilon \to 0} \mathrm{coAd}_\varepsilon (\exp t\underline{Z}_\varepsilon) Y_0 = \mathrm{coAd}_0 (\exp t\underline{Z}) Y_0 .$$

Let $F(Y, y)$ be the value at $\begin{pmatrix} Y & y \\ -y^* & 0 \end{pmatrix} \in \mathfrak{g}^*_{n,1}$ of a $\mathrm{CoAd}_1 (G_{n,1})$-invariant polynomial function F. The polynomial function $f_\varepsilon = F \circ S^{-1}_{\varepsilon,n} \circ J_{\varepsilon,n}$ on $\mathfrak{g}^*_{n,0}$ is obviously $\mathrm{coAd}_\varepsilon (G_{n,\varepsilon})$-invariant. Note that the value $f_\varepsilon(Y, y)$ of f_ε at $\begin{pmatrix} Y & y \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}^*_{n,0}$ is equal to $F(Y, y/\varepsilon c_n)$. Considering f_ε as a polynomial in ε^{-1}, let $f(Y, y)$ be the coefficient of $(\varepsilon^{-1})^d$ for the highest degree d. On account of Lemma 3.3 we obtain

LEMMA 3.4. *The polynomial function f defined above is* $\mathrm{coAd}_0 (G_{n,0})$-*invariant.*

In what follows, we shall freely use the notation introduced in Section 1 for the group $ISO(n)$. Denote by $I(\mathfrak{g}_n^*)$ the set of $\mathrm{CoAd}(G_n)$-invariant polynomial functions on $\mathfrak{g}_n^*$. Let r be the restriction map sending $f \in I(\mathfrak{g}_n^*)$ to $f|\mathfrak{H}_n$, whose image will be denoted by $I(\mathfrak{H}_n)$. To describe the symmetry shared by elements of $I(\mathfrak{H}_n)$, set $\tilde{A}_n = \{g \in G_n;\ \mathrm{CoAd}(g)(\mathfrak{H}_n) \subset \mathfrak{H}_n\}$ and $A_n = \{g \in \tilde{A}_n;\ \mathrm{CoAd}(g)|\mathfrak{H}_n = \text{the identity}\}$. In case $n = 2l + 1$ $(l \geqq 1)$, denote by W_{n-1} the group of the linear transformations on $\mathfrak{H}_n$ generated by the following σ_i $(1 \leqq i \leqq l)$ and τ_j $(1 \leqq j \leqq l)$;

$$\sigma_i\colon (Y_i, Y_{i+1}) \longrightarrow (Y_{i+1}, Y_i), \qquad \tau_j\colon Y_j \longrightarrow -Y_j\,.$$

If $n = 2(l + 1)$ $(l \geqq 1)$, let W_{n-1} be the group of the linear transformations on $\mathfrak{H}_n$ generated by the above σ_i $(1 \leqq i < l)$ and the following σ_l;

$$\sigma_l\colon (Y_{l-1}, Y_l) \longrightarrow (-Y_l, -Y_{l-1})\,.$$

Note that the group W_{n-1} is isomorphic to the Weyl group of $SO(n-1)$. In view of Lemma 3.1 we can easily verify that the group $\mathrm{CoAd}(\tilde{A}_n)|\mathfrak{H}_n$ which is isomorphic to the quotient group $\tilde{A}_n/A_n$, is generated by W_{n-1} and $\tau = \mathrm{CoAd}(I_{n-1,2})$. Here $I_{n-1,2} = I_{n+1} - 2(E_{n-1,n-1} + E_{n,n})$. In view of Lemma 3.1 τ is a linear transformation of $\mathfrak{H}_n$ such that $\tau\colon (Y_l, y_{l+1}) \to ((-1)^\varepsilon Y_l, -y_{l+1})$ with $\varepsilon = 0$ or 1 according as n is even or odd. We have seen that any element of $I(\mathfrak{H}_n)$ is $\mathrm{CoAd}(\tilde{A}_n)$-invariant. As is well known, a polynomial f in $Y_1, \cdots, Y_l$ is W_{n-1}-invariant iff f lies in the algebra $C[s_1, \cdots, s_l]$ [5, p. 302]. Note also that $t_i \in I(\mathfrak{H}_n)$ for $0 \leqq i \leqq l$. To see this, in case $n = 2l + 1$, let p_i $(0 \leqq i \leqq l)$ be the $\mathrm{CoAd}_1(G_{n,1})$-invariant polynomial functions on $\mathfrak{g}_{n,1}^*$ such that

$$\det\left(t + \begin{pmatrix} Y & y \\ -y^* & 0 \end{pmatrix}\right) = t^{2l+2} + p_l^2(Y, y) + \textstyle\sum_{1 \leqq i \leqq l} p_{l-i}(Y, y) t^{2i}\,,$$

while in case $n = 2(l + 1)$, let q_i $(0 \leqq i \leqq l)$ be the $\mathrm{CoAd}_1(G_{n,1})$-invariant polynomial functions on $\mathfrak{g}_{n,1}^*$ such that

$$\det\left(t + \begin{pmatrix} Y & y \\ -y^* & 0 \end{pmatrix}\right) = t^{2l+3} + \textstyle\sum_{1 \leqq i \leqq l+1} q_{l+1-i}(Y, y) t^{2i-1}$$

[8, pp. 410–411]. Evidently p_i and q_i $(0 \leqq i \leqq l)$ are of degree 2 in $y_1, \cdots, y_n$ except for p_l, which is of degree 1 in $y_1, \cdots, y_n$. Applying Lemma 3.4 to polynomials p_i and q_i, we conclude that $t_i \in I(\mathfrak{H}_n)$. The determinant of the Jacobian matrix $(\partial(t_0, \cdots, t_l)/\partial(Y_1, \cdots, Y_l, y_{l+1}))$ does not vanish in the polynomial ring, which can be verified by all means. Consequently the polynomials t_i $(0 \leqq i \leqq l)$ are algebraically independent over C. Denote by T_n the C-algebra generated by t_i $(0 \leqq i \leqq l)$. In what follows we shall show that $T_n = I(\mathfrak{H}_n)$. Observe that a $\mathrm{CoAd}(\tilde{A}_n)$-invariant polynomial f on $\mathfrak{H}_n$ belongs to T_n iff it takes the form

$$f(Y_1, \cdots, Y_l, y_{l+1}) = \sum_{k \geqq 0} f_k(Y_1, \cdots, Y_l) y_{l+1}^k ,$$

where the degree of f_k in Y_l does not exceed k. Indeed, f_k being invariant under W_{n-1}, $f_k y_{l+1}^k$ can be rewritten as

$$\begin{aligned} &\sum_{\alpha_i \geqq 0} a_{\alpha_1 \cdots \alpha_l} s_1^{\alpha_1} \cdots s^{\alpha_l} y_{l+1}^k \ (a_{\alpha_1 \cdots \alpha_l} \in C) \\ &= \sum_{\alpha_i \geqq 0} a_{\alpha_1 \cdots \alpha_l} t_0^{\alpha_0} t_1^{\alpha_1} \cdots t^{\alpha_l} \end{aligned}$$

with $2\alpha_0 = k - 2\alpha_1 - \cdots - 2\alpha_{l-1} - (2 - \varepsilon)\alpha_l \geqq 0$, where $\varepsilon = 0$ or 1 according as n is even or odd.

LEMMA 3.5. *The union of the orbits* $\left\{\mathrm{CoAd}(G_n)\begin{pmatrix} Y & y \\ 0 & 0 \end{pmatrix}; \begin{pmatrix} Y & y \\ 0 & 0 \end{pmatrix} \in \tilde{\mathfrak{H}}_n\right\}$ *is dense in* $\mathfrak{g}_n^*$. *In particular, the restriction map* $r\colon I(\mathfrak{g}_n^*) \to I(\mathfrak{H}_n)$ *is injective.*

Proof. It suffices to show that for $Y_0 = \begin{pmatrix} Y & y \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_n^*$ with $y \neq 0$ there exists a $g \in G_n$ such that $\mathrm{Coad}(g) Y_0 \in \mathfrak{H}_n$. Take a $u \in SO(n)$ such that $uy = (0, \cdots, 0, |y|)^*$ with $|y| = (y^* y)^{1/2}$. Set $Y_0' = \mathrm{CoAd}(u) Y_0 = \begin{pmatrix} Y' & y' \\ 0 & 0 \end{pmatrix}$, where $y' = uy$. We can find an $a = (***, 0)^* \in R^n$ such that $Y_0'' = \mathrm{CoAd}(a) Y_0' = \begin{pmatrix} Y'' & y' \\ 0 & 0 \end{pmatrix}$, where $Y'' \in so(n)$ takes the form $\begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix}$ for some $Z \in SO(n-1)$. As is well known, the set $B = \left\{X \in so(n-1); \begin{pmatrix} \tilde{X} & x \\ 0 & 0 \end{pmatrix} \in \mathfrak{H}_n, \text{ where } \tilde{X} = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \in so(n)\right\}$ is a maximal abelian subalgebra of $so(n-1)$. Hence there exists a $v \in SO(n-1)$ such that $vZv^{-1} \in B$. Thus $\mathrm{CoAd}(v \cdot a \cdot u) Y_0$ belongs to $\mathfrak{H}_n$.

LEMMA 3.6. $T_n = I(\mathfrak{H}_n)$ *for* $n \geqq 2$.

Proof. Since $\mathfrak{H}_n = \left\{\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}; y = (0, y_2)^*\right\}$ for $n = 2$, the assertion is valid. The cases $n = 3, 4$ will be separately discussed in the following Lemma 3.7. We thus proceed by induction on n assuming that $T_{n-2} = I_{n-2}(\mathfrak{H}_n)$. Define subspaces $\mathscr{W}_n$, $\mathscr{W}_n^0$ and $\mathfrak{g}_{n-2}$ of $\mathfrak{g}_n$ as follows.

$$\mathscr{W}_n = \left\{ M(\mu, x, c) = \begin{bmatrix} 0 & \mu & x & c \\ -\mu & 0 & & \\ -x^* & & 0 & 0 \\ 0 & & 0 & 0 \end{bmatrix}; \ \mu \in R, \ x \in M_{2, n-2}(R), \ c \in R^2 \right\}$$

$$\mathscr{W}_n^0 = \{M(0, x, c) \in \mathscr{W}_n\} ,$$
$$\mathfrak{g}_{n-2} = \{X \in \mathfrak{g}_n ;\ X_{ij} = 0 \text{ for } 1 \leqq i \leqq 2,\ 1 \leqq j \leqq n+1\} .$$

We can naturally identify the Lie algebra $\mathfrak{g}_{n-2}$ and the group G_{n-2} with $\underline{\mathfrak{g}}_{n-2}$ and the connected subgroup $\underline{G}_{n-2}$ of G_n associated with the subalgebra $\underline{\mathfrak{g}}_{n-2}$ respectively. For $M(\mu, x, c) \in \mathscr{W}_n$ and $g \in \underline{G}_{n-2}$ we have

$$\mathrm{Ad}\,(g)M(\mu, x, c) = M(\mu, \tilde{x}, \tilde{c}) \in \mathscr{W}_n . \tag{3.1}$$

Therefore $\mathrm{Ad}\,(\underline{G}_{n-2})$ leaves $\mathscr{W}_n$, $\mathscr{W}_n^0$ and $\mathfrak{g}_{n-2}$ invariant. $\mathfrak{g}_n$ being a direct sum of $\mathrm{Ad}\,(\underline{G}_{n-2})$-invariant subspaces $\mathscr{W}_n$ and $\mathfrak{g}_{n-2}$, it follows that $\mathrm{CoAd}\,(\underline{G}_{n-2})$ leaves the orthogonal complements $\mathscr{W}_n^{\perp}$ and $\mathfrak{g}_{n-2}^{\perp}$ invariant. Since $\mathscr{W}_n^0$ is $\mathrm{Ad}\,(\underline{G}_{n-2})$-invariant, $\mathscr{W}_n^{0\perp}$ is $\mathrm{CoAd}\,(\underline{G}_{n-2})$-invariant. We can easily verify that $\mathscr{W}_n^{\perp} = \mathfrak{g}_{n-2}$ and

$$\mathscr{W}_n^{0\perp} = \{Y_1(E_{12} - E_{21}) + X ;\ X \in \underline{\mathfrak{g}}_{n-2},\ Y_1 \in \boldsymbol{R}\} .$$

Since the representation of $\underline{G}_{n-2}$ in the quotient space $\mathfrak{g}_n/\mathscr{W}_n^0$ is trivial due to (3.1), so is the representation of $\underline{G}_{n-2}$ in $\mathscr{W}_n^{0\perp} = \mathscr{W}_n^{0\perp}/\mathfrak{g}_n^{\perp}$. This implies that the function Y_1 on $\mathscr{W}_n^{0\perp}$ is $\mathrm{CoAd}\,(\underline{G}_{n-2})$-invariant. Identifying the dual space $\mathfrak{g}_{n-2}^*$ of $\mathfrak{g}_{n-2}$ with $\mathscr{W}_n^{\perp}$, define a linear isomorphism L_n of $\underline{\mathfrak{g}}_{n-2}^* = \mathscr{W}_n^{\perp}$ onto $\mathfrak{g}_{n-2}^* = \mathfrak{g}_{n-2}$ by requiring

$$\left\langle \begin{pmatrix} X' & x' \\ 0 & 0 \end{pmatrix}, L_n \begin{pmatrix} Y & y \\ 0 & 0 \end{pmatrix} \right\rangle_{0,n-2} = \left\langle \iota \begin{pmatrix} 0 & 0 \\ 0 & X' \end{pmatrix}, \iota Y \right\rangle_{so(n+1)} + \langle x', y' \rangle$$

for $\begin{pmatrix} X' & x' \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_{n-2}$ and $\begin{pmatrix} Y & y \\ 0 & 0 \end{pmatrix} \in \mathscr{W}_n^{\perp}$ with $y = (0, y'^*)^* \in \boldsymbol{R}^n$. Then $L_n \begin{pmatrix} Y & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} Z & y \\ 0 & 0 \end{pmatrix}$ with $Z = c_n Y/c_{n-2}$. The constants c_n are defined just before Lemma 3.1. Now let $\tilde{f}$ be the restriction $F|\mathscr{W}_n^{0\perp}$ of an $F \in I(\mathfrak{g}_n^*)$ assuming the value

$$\tilde{f}(Y, y) = \textstyle\sum_{k \geqq 0} Y_1^k \tilde{f}_k(Y', y)$$

at $\begin{pmatrix} Y & y \\ 0 & 0 \end{pmatrix} \in \mathscr{W}_n^{0\perp}$, where $Y = \begin{pmatrix} Y_1 & 0 \\ 0 & Y' \end{pmatrix}$. Since the function Y_1 and the subspace $\mathscr{W}_n^{\perp}$ are $\mathrm{CoAd}\,(\underline{G}_{n-2})$-invariant, so are $\tilde{f}_k$. In particular, $\tilde{f}_k$, as functions on $\mathscr{W}_n^{\perp}$, are $\mathrm{CoAd}\,(\underline{G}_{n-2})$-invariant. Denote by f_k the restriction $\tilde{f}_k \circ L_n^{-1} | \mathfrak{H}_{n-2}$. By the induction hypothesis f_k take the form

$$\textstyle\sum_{m \geqq 0} f_{km}(Y_2, \cdots, Y_l) y_{l+1}^m ,$$

where the degree of f_{km} in Y_l does not exceed m. Consequently the restriction $\hat{f} = \tilde{f} | \mathfrak{H}_n$ takes the form

$$\sum_{m\geqq 0}\hat{f}_m(Y_1, \cdots, Y_l)y_{l+1}^m$$

with

$$\hat{f}_m(Y_1, \cdots, Y_l) = \sum_{k\geqq 0} Y_1^k f_{km}(c'Y_2, \cdots, c'Y_l)\ ,$$

where $c' = c_{n-2}/c_n$. Therefore the degree of $\hat{f}_m$ in Y_l does not exceed m. Since $\hat{f}_m y_{l+1}^m$ is $\mathrm{CoAd}\,(\tilde{A}_n)$-invariant, $\hat{f}_m y_{l+1}^m$ (hence, $\hat{f}$ as well) belongs to T_n. This concludes the proof of Lemma 3.6.

LEMMA 3.7. *$T_n = I(\mathfrak{H}_n)$ for $n = 3, 4$.*

Proof. It suffices to show that $T_n \supset I(\mathfrak{H}_n)$. The case $n = 4$ alone will be discussed in detail, for the another one can be dealt with similarly. Let F be a homogeneous polynomial in $I(\mathfrak{g}_n^*)$, whose restriction $F|\mathfrak{H}_n$ may be assumed to be of even degree $2m$ of the form

$$f(Y, y_{l+1}) = \sum_{k=0}^m b_k Y^{2m-2k} y_{l+1}^{2k} \qquad (l = 1)\ .$$

We must show that $2m - 2k \leqq 2k$, provided $b_k \neq 0$. For $(z_2, z_4)^* \in \boldsymbol{R}^2\backslash\{0\}$, let α be a real satisfying

$$(\cos\alpha, \sin\alpha) = (z_4, z_2)/|z| \text{ with } |z| = (z_2^2 + z_4^2)^{1/2}\ .$$

Moreover, set

$$u_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & 0 & -\sin\alpha \\ 0 & 0 & 1 & 0 \\ 0 & \sin\alpha & 0 & \cos\alpha \end{bmatrix}, \qquad a = -\frac{c_n \sin\alpha}{|z|}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then simple calculation yields the following equality;

$$\mathrm{CoAd}\,(a\cdot u_\alpha)\begin{bmatrix} 0 & Y & 0 & 0 & 0 \\ -Y & 0 & 0 & 0 & z_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & Y_\alpha & 0 & 0 & 0 \\ -Y_\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & |z| \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $Y_\alpha = Y\cos\alpha$. Since $f(Y_\alpha, |z|)$ can be rewritten as

$$\sum_{k=0}^m b_k Y^{2m-2k} z_4^{2m-2k}(z_2^2 + z_4^2)^{2k-m}\ ,$$

it follows that $m \leqq 2k$, provided $b_k \neq 0$, because $f(Y_\alpha, |z|)$ must be a polymomial in Y, z_2, z_4.

Added in proof. After this paper had been accepted for publication, [10] appeared. [2] is now published (Comm. Math. Phys., 90 (1983), 353–372).

References

[1] L. Abellanas and L. Martinez Alonso, A general setting for Casimir invariants, J. Math. Phys., **16** (1975), 1580–1584.

[2] M. Chaichian, A. P. Demichev and N. F. Nelipa, The Casimir operators of inhomogeneous groups, preprint.

[3] R. Gilmore, Rank 1 expansions, J. Math. Phys., **13** (1972), 883–886.

[4] R. Gilmore, Lie groups, Lie algebras, and some of their applications, John Wiley & Sons, New York 1974.

[5] S. Kobayashi and K. Nomizu, Foundations of differential geometry, Vol. 2, John Wiley & Sons, New York 1969.

[6] J. Rosen and P. Roman, Some observations on enveloping algebras of noncompact groups, J. Math. Phys., **7** (1966), 2072–2078.

[7] J. Rosen, Construction of invariants for Lie algebras of inhomogeneous pseudo-orthogonal and pseudo- unitary groups, J. Math. Phys., **9** (1968), 1305–1307.

[8] V. S. Varadarajan, Lie groups, Lie algebras, and their representations, Prenticc-Hall, Inc., London 1974.

[9] M. L. F. Wong and Hsin-Yang Yeh, Invariant operators of $IU(n)$ and $IO(n)$ and their eigenvalues, J. Math. Phys., **20** (1979), 247–250.

[10] M. Perroud, The fundamental invariants of inhomogeneous classical groups, J. Math. Phys., **24** (1983), 1381–1391.

Faculty of Education
Tokushima University
Tokushima 770
Japan