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TRANSIENT MARKOV CONVOLUTION SEMI-GROUPS AND THE ASSOCIATED NEGATIVE DEFINITE FUNCTIONS

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Dedicated to Professor Makoto Ohtsuka on the occasion of his 60th birthday

§1. Let X be a locally compact and σ-compact abelian group and $\hat{X}$ be the dual group of X[1]. We denote by ξ a fixed Haar measure on X and by $\hat{\xi}$ the Haar measure on $\hat{X}$ associated with ξ. It is well-known that (see, for example, [1]):

(A) For a sub-Markov convolution semi-group $(\alpha_t)_{t\geqq 0}$ on X, there exists a uniquely determined negative definite function ψ on $\hat{X}$ such that

$$(1.1) \qquad \hat{\alpha}_t(\hat{x}) = \exp(-t\psi(\hat{x})) \qquad \text{for any } \hat{x}\in\hat{X}\ (t\geqq 0)\,,$$

where $\hat{\alpha}_t$ denotes the Fourier transform of α_t.

(B) For a negative definite function ψ on $\hat{X}$, there exists a uniquely determined sub-Markov convolution semi-group $(\alpha_t)_{t\geqq 0}$ on X satisfying (1.1).

In this case, ψ is called the negative definite function associated with $(\alpha_t)_{t\geqq 0}$.

There is an interesting characterization of the transience of sub-Markov convolution semi-groups.

THEOREM. *Let $(\alpha_t)_{t\geqq 0}$ be a sub-Markov convolution semi-group on X and ψ be the negative definite function associated with $(\alpha_t)_{t\geqq 0}$. Then $(\alpha_t)_{t\geqq 0}$ is transient if and only if* $\mathrm{Re}(1/\psi)$ *is locally $\hat{\xi}$-summable, where* $\mathrm{Re}(1/\psi)$ *denotes the real part of* $1/\psi$.

The "only if" part is easily seen (see, for example, [1]). But it is known only to show the "if" part by probabilistic methods (see [3]).

The purpose of this note is to give a simple and non-probabilistic proof of the "if" part.

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1) *We denote by + the product of X and that of $\hat{X}$.*

§ **2.** We denote by:

$C_K(X)$ the usual topological vector space of all real-valued continuous functions on X with compact support;

$M(X)$ the topological vector space of all real Radon measures on X with the vague (weak*) topology;

$M_K(X)$ the subspace of $M(X)$ constituted by real Radon measures on X with compact support;

$C_K^+(X)$, $M^+(X)$ and $M_K^+(X)$ their subsets of all non-negative elements.

A family $(\alpha_t)_{t \geqq 0}$ in $M^+(X)$ is called a convolution semi-group on X if $\alpha_0 =$ the unit measure ε at the origin 0, $\alpha_t * \alpha_s = \alpha_{t+s}$ for all $t \geqq 0$, $s \geqq 0$ and the mapping $R^+ \ni t \to \alpha_t \in M^+(X)$ is continuous, where R^+ denotes the totality of non-negative numbers.

It is said to be transient if $\int_0^\infty \alpha_t dt \in M^+(X)$, which results from $\int_0^\infty dt \int f d\alpha_t < \infty$ for all $f \in C_K^+(X)$. Put

$$N = \int_0^\infty \alpha_t dt \,.$$

We call it the Hunt convolution kernel on X defined by $(\alpha_t)_{t \geqq 0}$.

A sub-Markov (resp. Markov) convolution semi-group $(\alpha_t)_{t \geqq 0}$ on X is, by definition, a convolution semi-group on X which satisfies $\int d\alpha_t \leqq 1$ $\left(\text{resp. } \int d\alpha_t = 1\right)$ for all $t \geqq 0$. In this case, we see that, for any $0 < p \in R^+$, $(\exp(-pt)\alpha_t)_{t \geqq 0}$ is a transient sub-Markov convolution semi-group on X. Put

$$N_p = \int_0^\infty \exp(-pt)\alpha_t dt \qquad (p > 0)\,;$$

$(N_p)_{p>0}$ is called the resolvent defined by $(\alpha_t)_{t \geqq 0}$, and it satisfies the resolvent equation:

$$N_p - N_q = (q - p) N_p * N_q \qquad \text{for all } p > 0 \text{ and } q > 0\,.$$

LEMMA 1. *Let $(\alpha_t)_{t \geqq 0}$ be a sub-Markov convolution semi-group on X and let $(N_p)_{p>0}$ be the resolvent defined by $(\alpha_t)_{t \geqq 0}$. Then, for any $p \geqq q > 0$, $N_p \ll N_q$, that is, for any $f, g \in C_K^+(X)$ and any $a \in R^+$, $N_p * f \leqq N_q * g + a$ on* supp(f) *implies that the same inequality holds on X, where* supp(f) *denotes the support of f.*

It is well-known that $N_p \ll N_p$ (the complete maximum principle of N_p)

(see, for example, [1]). This and the resolvent equation show that $N_p \ll N_q$.

LEMMA 2. *Let $(\alpha_t)_{t\geqq 0}$ and $(N_p)_{p>0}$ be the same as in Lemma 1. If there exist $p>0$ and $\eta \in M^+(X)$ such that $N_p * \eta$ is defined in $M^+(X)$, $\eta \geqq pN_p*\eta$ in X and $\eta \neq pN_p*\eta$, then $(\alpha_t)_{t\geqq 0}$ is transient.*

Proof. We write inductively $(pN_p)^1 = pN_p$ and $(pN_p)^n = (pN_p)^{n-1}*(pN_p)$ $(n = 2, 3, \cdots)$. Then, for any integer $n \geqq 1$,

$$\eta \geqq \left(\varepsilon + \sum_{k=1}^{n} (pN_p)^k\right) * (\eta - pN_p*\eta) .$$

Since $\eta - pN_p*\eta \in M^+(X)$ and $\eta - pN_p*\eta \neq 0$, $\sum_{n=1}^{\infty} (pN_p)^n$ converges vaguely. We see easily that

$$\int_0^\infty \alpha_t dt = \frac{1}{p} \sum_{n=1}^{\infty} (pN_p)^n ,$$

which shows Lemma 2.

LEMMA 3. *Let $(\alpha_t)_{t\geqq 0}$ be a Markov convolution semi-group on X and assume that the closed subgroup generated by $\bigcup_{t\geqq 0} \operatorname{supp}(\alpha_t)$ is equal to X. If $(\alpha_t)_{t\geqq 0}$ is not transient, then X is generated by some compact neighborhood of the origin.*

Proof. Let V be a compact neighborhood of the origin and let X_V denote the closed subgroup generated by V. We denote by $\alpha_{t,V}$ the restriction of α_t to X_V. Then we see easily that $(\alpha_{t,V})_{t\geqq 0}$ is a sub-Markov convolution semi-group on X_V and that $(\alpha_t)_{t\geqq 0}$ is transient if and only if, for any compact neighborhood V of the origin, $(\alpha_{t,V})_{t\geqq 0}$ is transient. Hence there exists a compact neighborhood V_0 of the origin such that $(\alpha_{t,V_0})_{t\geqq 0}$ is not transient, that is, $(\alpha_{t,V_0})_{t\geqq 0}$ is a Markov convolution semi-group on X_{V_0}. Consequently $\alpha_t = \alpha_{t,V_0}$ for all $t \geqq 0$. This implies that $X = X_{V_0}$. Thus Lemma 3 is shown.

LEMMA 4 (see, for example, [1], p. 156). *Let $(\alpha_t)_{t\geqq 0}$ be a transient sub-Markov convolution semi-group on X. Put $N = \int_0^\infty \alpha_t dt$. Then N satisfies the equilibrium principle, that is, for any relatively compact open set ω in X, there exists $\gamma \in M_K^+(X)$ such that $\operatorname{supp}(\gamma) \subset \bar{\omega}$, $N*\gamma = \xi$ in ω and $N*\gamma \leqq \xi$ in X.*

Here $\operatorname{supp}(\gamma)$ denotes also the support of γ. We say that γ is an N-equilibrium measure of ω.

LEMMA 5. *Let* $(\alpha_t)_{t\geqq 0}$ *and* N *be the same as in Lemma* 4, ω *a relatively compact open set in* X, γ *an N-equilibrium measure of* ω. *Then, for any* $\sigma \in M^+(X)$ *with* $\int d\sigma \leqq 1$, *any* $a \in R^+$ *and any* $f \in C_K^+(X)$ *with* $\mathrm{supp}(\check{f}) \subset \omega$,

$$N*(a\gamma)*(\varepsilon - \sigma)*f(0) \geqq 0 .$$

Here we denote by $\check{f}$ the function defined by $\check{f}(x) = f(-x)$ for all $x \in X$.

In fact, this follows from

$$N*(a\gamma)*(\varepsilon - \sigma)*f(0) = a\Big(\int \check{f} d\xi - \int \check{f} dN*\gamma*\sigma\Big) \geqq a\Big(1 - \int d\sigma\Big)\int \check{f} d\xi \geqq 0 .$$

There exists a very useful result concerning the convolution equation:

LEMMA 6 (see [2]). *Let* $\sigma \in M^+(X)$ *with* $\int d\sigma = 1$ *and let* $\mu \in M(X)$. *Assume that* μ *is shift-bounded, that is, for any* $f \in C_K(X)$, $\mu*f$ *is bounded on* X. *If* $\mu*\sigma = \mu$, *then every point* x *in the closed subgroup generated by* $\mathrm{supp}(\sigma)$ *is a period of* μ, *that is* $\mu = \mu*\varepsilon_x$, *where* ε_x *denotes the unit measure at* x.

LEMMA 7. *Let* $(\alpha_t)_{t\geqq 0}$ *and* $(N_p)_{p>0}$ *be the same as in Lemma* 1. *If* $\overline{\bigcup_{t\geqq 0} \mathrm{supp}(\alpha_t)}$ *is non-compact, then* $\lim_{p\to 0} pN_p = 0$[2].

Proof. Since $p\int dN_p \leqq 1$, $(pN_p)_{p>0}$ is vaguely bounded. Let λ be an arbitrary vaguely accumulation point of $(pN_p)_{p>0}$ as $p \to 0$. Then $\int d\lambda \leqq 1$. Choose a net $(p_i N_{p_i})_{i\in I}$ with $p_i \to 0$ such that $\lim_{i\in I} p_i N_{p_i} = \lambda$. Then, for any $0 < p \in R^+$, the resolvent equation and $p\int dN_p \leqq 1$ give

$$\lambda*(pN_p) = \lim_{i\in I}(p_i N_{p_i})*(pN_p) = \lim_{i\in I}(p_i(N_{p_i} - N_p) + p_i^2 N_{p_i}*N_p) = \lambda .$$

If $p\int dN_p < 1$, this and $\int d\lambda \leqq 1$ give $\lambda = 0$. Assume that $p\int dN_p = 1$. Then the above lemma shows that for any $x \in \overline{\bigcup_{t\geqq 0}\mathrm{supp}(\alpha_t)} = \mathrm{supp}(pN_p)$, $\lambda = \lambda*\varepsilon_x$. Since $\int d\lambda \leqq 1$ and $\overline{\bigcup_{t\geqq 0}\mathrm{supp}(\alpha_t)}$ is non-compact, we have $\lambda = 0$. Thus we obtain that $\lim_{p\to 0} pN_p = 0$.

In the case that $\overline{\bigcup_{t\geqq 0}\mathrm{supp}(\alpha_t)}$ is compact, the similar argument shows that $\lim_{p\to 0} pN_p$ exists and it is equal to 0 or a Haar measure on the compact subgroup generated by $\bigcup_{t\geqq 0}\mathrm{supp}(\alpha_t)$.

2) For a net $(\mu_i)_{i\in I} \subset M(X)$ and $\mu \in M(X)$, we write $\lim_{i\in I} \mu_i = \mu$ if $(\mu_i)_{i\in I}$ converges vaguely to μ along I.

For a real Radon measure μ on X, we denote by $\check{\mu}$ the real Radon measure on X defined by $\int f d\check{\mu} = \int \check{f} d\mu$.

LEMMA 8. *Let $(\alpha_t)_{t\geqq 0}$ and $(N_p)_{p>0}$ be the same as above and let $(a_p)_{p>0}$ be a family of positive numbers such that $(a_p N_p * \check{N}_p)_{p>0}$ is vaguely bounded. Assume that the closed subgroup generated by $\bigcup_{t\geqq 0} \operatorname{supp}(\alpha_t)$ is equal to X. Take a vaguely accumulation point η of $(a_p N_p * \check{N}_p)_{p>0}$ as $p \to 0$ and a net $(p_i)_{i\in I}$ of positive numbers with $p_i \to 0$ and $\lim_{i\in I} a_{p_i} N_{p_i} * \check{N}_{p_i} = \eta$. If, for any $q > 0$, $\lim_{i\in I} a_{p_i} N_{p_i} * \check{N}_q = 0$, then $\eta = 0$ or η is proportional to ξ.*

Proof. Since $N_{p_i} * \check{N}_{p_i}$ is of positive type, for any $f \in C_K(X)$,

$$(a_{p_i} N_{p_i} * \check{N}_{p_i} * f * \check{f})_{i\in I}$$

is uniformly bounded. Let $0 < q \in R^+$. Since $q \int dN_q \leqq 1$, we have

$$\lim_{i\in I} a_{p_i} q^2 N_{p_i} * \check{N}_{p_i} * N_q * \check{N}_q * f * \check{f}(x) = q^2 \eta * N_q * \check{N}_q * f * \check{f}(x)$$

for all $f \in C_K(X)$ and $x \in X$, which implies that

$$\lim_{i\in I} a_{p_i} q^2 N_{p_i} * \check{N}_{p_i} * N_q * \check{N}_q = q^2 \eta * N_q * \check{N}_q \ .$$

On the other hand, we have, by our assumption,

$$\lim_{i\in I} a_{p_i} q^2 N_{p_i} * \check{N}_{p_i} * N_q * \check{N}_q = \lim_{i\in I} a_{p_i} (N_{p_i} - N_q) * (\check{N}_{p_i} - \check{N}_q) = \eta \ .$$

Thus we have

$$\eta = q^2 \eta * N_q * \check{N}_q \ .$$

Assume that $\eta \neq 0$. Since η is of positive type, η is shift-bounded. Hence $q^2 \int dN_q * \check{N}_q = 1$. Evidently $\operatorname{supp}(N_q) = \overline{\bigcup_{t\geqq 0} \operatorname{supp}(\alpha_t)}$ and $\operatorname{supp}(N_q)$ is a closed semi-group. Hence $\operatorname{supp}(N_q * \check{N}_q) = X$, and Lemma 6 gives $\eta = c\xi$ with some constant $c > 0$. Thus Lemma 8 is shown.

§ 3. A complex valued continuous function $\psi(\hat{x})$ on $\hat{X}$ is, by definition, negative definite if $\psi(\hat{0}) \geqq 0$, $\psi(-\hat{x}) = \overline{\psi(\hat{x})}$ and for any integer $m \geqq 1$, any $(\hat{x}_j)_{j=1}^m \subset \hat{X}$ and any $(\rho_j)_{j=1}^m \subset C$ with $\sum_{j=1}^m \rho_j = 0$,

$$\sum_{k=1}^m \sum_{j=1}^m \psi(\hat{x}_j - \hat{x}_k) \rho_j \bar{\rho}_k \leqq 0 \ .$$

Here $\hat{0}$ denotes the origin of $\hat{X}$ and C denotes the totality of complex numbers.

Remark 9 (see, for example, [1]). Let ψ be a negative definite function on $\hat{X}$. Then we have:

(1) Re ψ is also negative definite.

(2) Re $\psi(\hat{x}) \geqq \psi(\hat{0})$ for all $\hat{x} \in \hat{X}$, that is, Re $\psi(\hat{x}) \geqq 0$. So we can write $\psi(\hat{x}) = |\psi(\hat{x})| \exp(i\theta_{\hat{x}})$ with $|\theta_{\hat{x}}| \leqq \pi/2$.

(3) Let $\alpha \in R^+$ with $0 < \alpha \leqq 1$ and put

$$\psi^{\alpha}(\hat{x}) = \begin{cases} |\psi(\hat{x})|^{\alpha} \exp(i\alpha\theta_{\hat{x}}) & \text{if } \psi(\hat{x}) \neq 0 \\ 0 & \text{if } \psi(\hat{x}) = 0 \end{cases},$$

where $\theta_{\hat{x}} = \arg \psi(\hat{x})$ with $|\theta_{\hat{x}}| \leqq \pi/2$. Then ψ^{α} is negative definite.

Evidently we have the following

Remark 10. Let $(\alpha_t)_{t\geqq 0}$ and ψ be a sub-Markov convolution semi-group on X and the negative definite function associated with $(\alpha_t)_{t\geqq 0}$. Then we have:

(1) $\psi(\hat{0}) = 0$ if and only if $\int d\alpha_t = 1$ for all $t \geqq 0$.

(2) $p(1 - p\hat{N}_p)$ converges uniformly to ψ on any compact set as $p \to \infty$, where $(N_p)_{p>0}$ is the resolvent defined by $(\alpha_t)_{t\geqq 0}$.

Consequently, if $\psi(\hat{0}) \neq 0$, then $(\alpha_t)_{t\geqq 0}$ is always transient. We remark here that $\hat{N}_p(\hat{x}) = 1/(p + \psi(\hat{x}))$.

§4. In this paragraph, we shall show the "if" part of Theorem.

PROPOSITION 11. *Let $(\alpha_t)_{t\geqq 0}$ and ψ be a sub-Markov convolution semi-group on X and the negative definite function associated with $(\alpha_t)_{t\geqq 0}$. If* Re $(1/\psi)$ *is locally $\hat{\xi}$-summable, then $(\alpha_t)_{t\geqq 0}$ is transient.*

Proof. Evidently we may assume that $(\alpha_t)_{t\geqq 0}$ is a Markov convolution semi-group, that is, $\psi(\hat{0}) = 0$. Furthermore, we may assume also that the closed subgroup generated by $\bigcup_{t\geqq 0} \operatorname{supp}(\alpha_t)$ is equal to X (see, [1], p. 105). For any $0 < p \in R^+$, we put $\psi_p(\hat{x}) = p(1 - p\hat{N}_p(\hat{x}))$ on $\hat{X}$. Then $\psi_p(\hat{x}) = p\psi(\hat{x})/(p + \psi(\hat{x}))$, so that Re $(1/\psi_p)$ is locally $\hat{\xi}$-summable. Furthermore, we remark that $(\alpha_t)_{t\geqq 0}$ is transient if and only if $\sum_{n=1}^{\infty} (pN_p)^n$ converges vaguely.

Consequently, we may assume that $\psi(\hat{x}) = 1 - \hat{\sigma}(\hat{x})$ on $\hat{X}$, where $\sigma \in M^+(X)$ with $\int d\sigma = 1$ and $\operatorname{supp}(\sigma) - \operatorname{supp}(\sigma) = X$[3]. Then $|\psi(\hat{x})| \leqq 2$ and $\psi(\hat{x}) \neq 0$ if $\hat{x} \neq \hat{0}$.

3) For a subsets A, B of X, $A - B = \{x - y;\ x \in A,\ y \in B\}$.

Assume that $(\alpha_t)_{t\geqq 0}$ is not transient. Then X is non-compact, and Lemma 3 shows that X is generated by a certain compact neighborhood of the origin. Hence we may assume that $X = R^n \times Z^m \times F$, where n, m are integers $\geqq 0$, R is the additive group of real numbers, Z is the additive group of integers and where F is a compact abelian group (see, for example, [4], p. 109). Let ξ_F be the normalised Haar measure on F. By considering the canonical projection of $\alpha_t * \xi_F$ on $R^n \times Z^m$ for all $t \geqq 0$, we may assume that $X = R^n \times Z^m$. Then $\hat{X} = R^n \times T^m$, where T^m is the m-dimensional torus.

Assume that $n \geqq 1$. First we shall show that Re $(1/\psi)\hat{\xi}$ is temperate. Since $|\psi(\hat{x})| \geqq a|\hat{x}|^2$ in a certain neighborhood of $\hat{0}$ with some constant $a > 0$, there exists an integer $m \geqq 1$ such that $(1/|\psi|^2)^{1/m}$ is locally $\hat{\xi}$-summable. Here $|\hat{x}|$ denotes the distance between $\hat{x}$ and $\hat{0}$ in $R^n \times T^m$. Let $(\alpha_{t,m})_{t\geqq 0}$ be the Markov convolution semi-group on X satisfying $\widehat{\alpha_{t,m}} = \exp(-t\psi^{1/m})$ for all $t \geqq 0$ and let $(N_{p,m})_{p>0}$ be the resolvent defined by $(\alpha_{t,m})_{t\geqq 0}$. Since, for any $p > 0$,

$$\widehat{N_{p,m} * \check{N}_{p,m}}(\hat{x}) = \frac{1}{|p + \psi^{1/m}(\hat{x})|^2} \qquad \text{on } \hat{X},$$

$(N_{p,m} * \check{N}_{p,m})_{p>0}$ is vaguely bounded. This implies that $(\alpha_{t,m})_{t\geqq 0}$ is transient. Put $N_{0,m} = \int_0^\infty \alpha_{t,m} dt$. Then $N_{0,m} * \check{N}_{0,m}$ is defined and

$$\widehat{N_{0,m} * \check{N}_{0,m}} = \left(\frac{1}{|\psi|^2}\right)^{1/m} \hat{\xi}.$$

Since $(\mathrm{Re}\,\psi)^{1/m}$ is bounded, $(\mathrm{Re}\,\psi/|\psi|^2)^{1/m}\hat{\xi}$ is temperate. Consequently, $(\mathrm{Re}\,\psi/|\psi|^2)\hat{\xi} = \mathrm{Re}\,(1/\psi)\hat{\xi}$ is temperate. Since, for any $p > 0$.

$$\frac{1}{2}(\hat{N}_p(\hat{x}) + \hat{\check{N}}_p(\hat{x})) - p\widehat{N_p * \check{N}_p}(\hat{x}) = \frac{\mathrm{Re}\,\psi}{|p + \psi(\hat{x})|^2} \leqq \mathrm{Re}\left(\frac{1}{\psi(\hat{x})}\right) \qquad \text{on } \hat{X},$$

we see that for any $f \in C_K^\infty(X)$, $((\frac{1}{2}(N_p + \check{N}_p) - pN_p * \check{N}_p) * f * \check{f}(0))_{p>0}$ is bounded. Here $C_K^\infty(X)$ denotes the totality of functions $f \in C_K(X)$ such that for any $y \in Z^m$, the function $f(x, y)$ of x is infinitely differentiable on R^n.

Assume that $n = 0$. Then $\hat{X}$ is compact. Hence, similarly as above, we see that for any $f \in C_K(X)$, $((\frac{1}{2}(N_p + \check{N}_p) - pN_p * \check{N}_p) * f * \check{f}(0))_{p>0}$ is bounded.

Thus, in general, there exists $f_0 \in C_K^+(X)$ with $f_0 \neq 0$ such that $((\frac{1}{2}(N_p + \check{N}_p) - pN_p * \check{N}_p) * f_0 * \check{f}_0(0))_{p>0}$ is bounded. Furthermore, $(pN_p * \check{N}_p)_{p>0}$ is not vaguely bounded. Hence $(pN_p * \check{N}_p * f_0 * \check{f}_0(0))_{p>0}$ is not bounded. Put

$a_p = (1/pN_p*\check{N}_p*f_0*\check{f}_0(0))(p > 0)$. Since $a_p pN_p*\check{N}_p$ is of positive type, $(a_p pN_p*\check{N}_p)_{p>0}$ is vaguely bounded. We choose a decreasing sequence $(p_k)_{k=1}^{\infty}$ such that $\lim_{k\to\infty} p_k = 0$, $(a_{p_k} p_k N_{p_k}*\check{N}_{p_k})_{k=1}^{\infty}$ converges vaguely and that $(a_{p_k})_{k=1}^{\infty}$ converges decreasingly to 0 as $k \uparrow \infty$ (Remark that $X = R^n \times Z^m$). Put $\eta = \lim_{k\to\infty} a_{p_k} p_k N_{p_k}*\check{N}_{p_k}$. Since $\int f_0*\check{f}_0 d\eta = 1$, Lemma 6 shows that $\eta = c\xi$ with some constant $c > 0$. Since

$$((\tfrac{1}{2}(N_{p_k} + \check{N}_{p_k}) - p_k N_{p_k}*\check{N}_{p_k})*f_0*\check{f}_0(0))_{k=1}^{\infty}$$

is bounded, we have also

$$\lim_{k\to\infty} a_{p_k}(N_{p_k} + \check{N}_{p_k}) = 2c\xi \,.$$

We may assume that $(a_{p_k} N_{p_k})_{k=1}^{\infty}$ converges vaguely. Put $\lambda = \lim_{k\to\infty} a_{p_k} N_{p_k}$; then $\lim_{k\to\infty} a_{p_k}\check{N}_{p_k} = \check{\lambda}$. Hence $\lambda \neq 0$. By Lemma 1, we see easily that for any $0 < p \in R^+$, $N_p \ll \lambda$ and $\lambda \ll \lambda$. This implies that λ is shift-bounded and $\lambda \geqq p\lambda*N_p$ for all $p > 0$. By Lemma 2, we have $\lambda = p\lambda*N_p$ for all $p > 0$. This and Lemma 6 show that λ is proportional to ξ, which implies $\lambda = c\xi$. Thus $\lim_{k\to\infty} a_{p_k} N_{p_k} = \lim_{k\to\infty} a_{p_k}\check{N}_{p_k} = c\xi$. We choose a relatively compact open set ω in X such that $\omega \supset \mathrm{supp}\,(f_0*\check{f}_0)$. Let γ_{p_k} be an $\check{N}_p$-equilibrium measure of ω and put $\nu_k = (1/a_{p_k})\gamma_{p_k}$ $(k = 1, 2, \cdots)$. Then $(\nu_k)_{k=1}^{\infty}$ is vaguely bounded, and hence we may assume that it converges vaguely. Put $\nu = \lim_{k\to\infty} \nu_k$. Then $\int d\nu = 1/c$, that is, $\nu \neq 0$. Let $0 < p \in R^+$. Then the resolvent equation and Lemma 7 give

$$\lim_{k\to\infty} p_k N_{p_k}*\check{N}_{p_k}*(\varepsilon - (p - p_k)N_p)*\nu_k = \lim_{k\to\infty} p_k N_p*\check{N}_{p_k}*\nu_k = 0 \,.$$

Lemma 5 gives

$$\check{N}_{p_k}*(\varepsilon - (p - p_k)N_p)*\nu_k*f_0*\check{f}_0(0) \geqq 0$$

provided with $p \geqq p_k$. Hence, by putting

$$A = \sup_{q>0}\,(\tfrac{1}{2}(N_q + \check{N}_q) - qN_q*\check{N}_q)*f_0*\check{f}_0(0) \,,$$

we have, for $p \geqq p_k$,

$$(\tfrac{1}{2}(N_{p_k} + \check{N}_{p_k}) - p_k N_{p_k}*\check{N}_{p_k}))*(\varepsilon - (p - p_k)N_p)*\nu_k*f_0*\check{f}_0(0)$$
$$\leqq 2A \sup_{1\leqq k<\infty} \int d\nu_k \,,$$

because $(\tfrac{1}{2}(N_{p_k} + \check{N}_{p_k}) - p_k N_{p_k}*\check{N}_{p_k})*f_0*\check{f}_0$ is of positive type. Letting $k \to \infty$, we obtain that

$$N_p * \nu * f_0 * \check{f}_0(0) \leqq 4A \sup_{1 \leqq k < \infty} \int d\nu_k .$$

This implies that $\left(\int \check{\nu} * f_0 * \check{f}_0 dN_{p_k}\right)_{k=1}^{\infty}$ is bounded, which contradicts

$$\lim_{k \to \infty} a_{p_k} N_{p_k} = c\xi \quad \text{and} \quad \lim_{k \to \infty} a_{p_k} = 0 .$$

Thus we see that $(\alpha_t)_{t \geqq 0}$ is transient. This completes the proof.

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