Y. Kitaoka
Nagoya Math. J.
Vol. 92 (1983), 145-152

A NOTE ON LOCAL DENSITIES OF QUADRATIC FORMS

YOSHIYUKI KITAOKA

Let L, M be regular quadratic lattices over $\boldsymbol{Z}_p$. The local density $\alpha_p(L, M)$ is an important invariant in the theory of representation of quadratic forms and they appear naturally in Fourier coefficients of Eisenstein series. In spite of the importance we knew little except the case when either $L = M$ or $\operatorname{rk} L = 1$ and M is unimodular. Evaluating them is a laborious task. Recently M. Ozeki gave an formula of local densities $\alpha_p(L, M)$, where $\operatorname{rk} L = 2$, M is unimodular and $p \neq 2$, by calculating generalized Gauss sums. Here we give a general induction formula and a simpler proof of his results as an application and interesting relations which are not referred in [4], [5].

For a prime p, we denote by $\boldsymbol{Q}_p, \boldsymbol{Z}_p$ the p-adic completion of the rational number field $\boldsymbol{Q}$, the rational integer ring $\boldsymbol{Z}$ respectively.

Let $G = GL_n(\boldsymbol{Q}_p) \cap M_n(\boldsymbol{Z}_p)$, $U = GL_n(\boldsymbol{Z}_p)$. By $H_n(\boldsymbol{Q}_p)$ we denote the vector space over $\boldsymbol{Q}$ consisting of all formal finite linear combinations with coefficients in $\boldsymbol{Q}$ of the double cosets UgU, $g \in G$. Let $g, h \in G$ and $UgU = Ug_1 \cup \cdots \cup Ug_s$, $UhU = Uh_1 \cup \cdots \cup Uh_t$ be left cosets decompositions of UgU, UhU respectively. For every $k \in G$, the number $\mu(UkU; UgU, UhU)$ of pairs (i, j) such that $Uk = Ug_ih_j$ is determined uniquely by UgU, UhU and UkU. Then we define the product $UgU \cdot UhU$ by $\sum_{UkU} \mu(UkU; UgU, UhU)\, UkU$ $(= \sum_{i,j} Ug_ih_j$ formally). Let $\chi(r_1, \cdots, r_n)$ denote the matrix $\begin{pmatrix} p^{r_1} & & 0 \\ & \ddots & \\ 0 & & p^{r_n} \end{pmatrix}$ where $r_1, \cdots, r_n$ are integers with $r_1 \geq \cdots \geq r_n$. Put

$$\begin{aligned} \pi_0 &= U\chi(0, \cdots, 0)U \\ \pi_1 &= U\chi(1, 0, \cdots, 0)U \\ \pi_2 &= U\chi(1, 1, 0, \cdots, 0)U \\ &\vdots \\ \pi_n &= U\chi(1, \cdots, 1)U\,, \end{aligned}$$

Received August 26, 1982.

and $T(k) = \sum_{\substack{r_1+\cdots+r_n=k \\ r_n \geq 0}} U\chi(r_1, \cdots, r_n)U$ for $k \geq 0$ and $T(k) = 0$ for $k < 0$. Then we know, [10], the following

THEOREM. *$H_n(\boldsymbol{Q}_p)$ is a commutative ring and*

$$\sum_{h=0}^{n} (-1)^h p^{h(h-1)/2} T(k-h)\pi_h = 0 \qquad \text{for } k \geq 1 .$$

Applying this, we give an induction formula of local densities.

Let $\tilde{S}$ be the set of non-singular symmetric $n \times n$ matrices over $\boldsymbol{Q}_p$. For $S_1, S_2 \in \tilde{S}$ we define an equivalence relation $S_1 \sim S_2$ by $S_1 = {}^tTS_2T$ for some $T \in U$. By $\mathscr{L}(n)$ we denote the vector space over $\boldsymbol{Q}$ spanned by $[S]$, $S \in \tilde{S}$, where $[S]$ stands for the equivalence class containing S. Let g be an element of G and $UgU = Ug_1 \cup \cdots \cup Ug_s$ a left cosets decomposition. Then we define the action of UgU by

$$[T] \mid UgU = p^{(\mathrm{ord}_p (\det g))(n+1-m)} \sum_i [{}^tg_i^{-1}Tg_i^{-1}] \qquad \text{for } T \in \tilde{S} ,$$

where m is a fixed integer. It is easy to see that this is well-defined.

LEMMA 1. *Let g, h be elements of G. Then we have*

$$([T] \mid UgU) \mid UhU = [T] \mid (UgU \cdot UhU) \qquad \text{for } T \in \tilde{S} .$$

Proof. Let $UgU = Ug_1 \cup \cdots \cup Ug_s$, $UhU = Uh_1 \cup \cdots \cup Uh_t$. Then we have

$$\begin{aligned} ([T] \mid UgU) \mid UhU &= p^{(\mathrm{ord}_p (\det g))(n+1-m)} \sum_i [{}^tg_i^{-1}Tg_i^{-1}] \mid UhU \\ &= p^{(\mathrm{ord}_p (\det gh))(n+1-m)} \sum_{i,j} [{}^th_j^{-1}\,{}^tg_i^{-1}Tg_i^{-1}h_j^{-1}] \\ &= [T] \mid (UhU \cdot UgU) = [T] \mid (UgU \cdot UhU) . \end{aligned}$$

Thus we get a homomorphism from $H_n(\boldsymbol{Q}_p)$ to $\mathrm{End}(\mathscr{L}(n))$ and the above theorem implies

LEMMA 2. *For $T \in \tilde{S}$ we have*

$$\sum_{h=0}^{n} (-1)^h p^{h(h-1)/2} [T] \mid T(k-h)\pi_h = 0 \qquad \text{for } k \geq 1 .$$

Let S be a non-singular symmetric $m \times m$ matrix and even, that is, entries of S are in $\boldsymbol{Z}_p$ and diagonals are in $2\boldsymbol{Z}_p$. Suppose $m \geq n$. For $T \in \tilde{S} \cap M_n(\boldsymbol{Z}_p)$ we put

$$A_{p^t}(T, S) = \#\{X \in M_{m,n}(\boldsymbol{Z}_p/(p^t)) \mid {}^tXSX \equiv T \bmod p^t\} .$$

Then $2^{-\delta_{m,n}}(p^t)^{n(n+1)/2-mn}A_{p^t}(T, S)$ is independent of t if t is sufficiently large and we denote the value by $\alpha_p(T, S)$ ($\delta_{m,n}$ is the Kronecker's delta). Let $N = Z_p[u_1, \cdots, u_n]$, $M = Z_p[v_1, \cdots, v_m]$ be lattices over Z_p with $\operatorname{rk} N = n$, $\operatorname{rk} M = m$ respectively. We define inner products $(\ ,\ )$ by $((u_i, u_j)) = T$, $((v_i, v_j)) = S$. We denote by $C_{p^t}(T, S)$

$$\#\left\{u: N \to M/p^t M^\# \,\middle|\, \begin{array}{l}(ux, ux) \equiv (x, x) \bmod 2p^t Z_p \text{ for } x \in N \text{ and } u \\ \text{induces an injective mapping from } N/pN \text{ to } M/pM\end{array}\right\}$$

and by $d_p(T, S)$ $2^{-\delta_{n,m}}p^{n \operatorname{ord}_p \det S} \times (p^s)^{n(n+1)/2-mn}C_{p^s}(T, S)$ where s is any natural number such that $p^{s-1}S^{-1}$ is even and $M^\#$ is the dual lattice of M. Then the following is proved in [1].

LEMMA 3. *Keeping the above, we have*

$$\alpha_p(T, S) = 2^{n\delta_{2,p}} \sum_{U \backslash UGU \ni g} p^{(\operatorname{ord}_p (\det g))(n+1-m)} d_p({}^t g^{-1} T g^{-1}, S)\ .$$

We define linear mappings d, α from $\mathscr{L}(n)$ to $\boldsymbol{Q}$ by $d([T]) = 2^{n\delta_{2,p}}d_p(T, S)$, $\alpha([T]) = \alpha_p(T, S)$. Then Lemma 3 means

$$\alpha([T]) = \sum_{UgU \in UGU} d([T] \mid UgU) = \sum_{k=0}^{\infty} d([T] \mid T(k))$$

(for sufficiently large k, $d([T] \mid T(k)) = 0$).

THEOREM 1. *Let $T^{(n)}$, $S^{(m)}$ $(m \geq n)$ be even non-singular symmetric matrices. Then we have*

$$2^{n\delta_{2,p}}d_p(T, S) = \sum_{h=0}^{n} (-1)^h p^{h(h-1)/2 + h(n+1-m)} \sum_{U \backslash \pi_h \ni g} \alpha_p({}^t g^{-1} T g^{-1}, S)\ .$$

Proof. Lemmas 2, 3 yield

$$\begin{aligned}\alpha\Big(\sum_{h=0}^{n} (-1)^h p^{h(h-1)/2}[T] \mid \pi_h\Big) &= \sum_{h=0}^{n} (-1)^h p^{h(h-1)/2} \sum_{i \geq 0} d([T] \mid \pi_h T(i-h)) \\ &= \sum_{i \geq 0} d\Big(\sum_{h=0}^{n} (-1)^h p^{h(h-1)/2}[T] \mid \pi_h T(i-h)\Big) \\ &= d([T])\ . \end{aligned}$$

Q.E.D.

Hereafter we assume that p is an odd prime and $n = 2$. Let $U\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}U = Ug_1 \cup \cdots \cup Ug_s$ be a left cosets decomposition of $U\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}U$. Then $U\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}U = pg_1^{-1}U \cup \cdots \cup pg_s^{-1}U$ is a right cosets decomposition and we can take $\{g_i\}$ as $\{g_i^{-1}\} = \left\{\begin{pmatrix} 1/p & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & a/p \\ 0 & 1/p \end{pmatrix} a \bmod p\right\}$. Put

$$T = \begin{pmatrix} \varepsilon_1 p^{a_1} & 0 \\ 0 & \varepsilon_2 p^{a_2} \end{pmatrix}, \quad \varepsilon_i \in Z_p^\times, \quad 0 \leq a_1 \leq a_2 .$$

If $a \not\equiv 0 \bmod p$, $a_1 < a_2$, then

$$T\left[\begin{pmatrix} 1 & a/p \\ 0 & 1/p \end{pmatrix}\right] \sim \begin{pmatrix} \varepsilon_1 p^{a_1-2} & 0 \\ 0 & \varepsilon_2 p^{a_2} \end{pmatrix}.$$

If $a \not\equiv 0 \bmod p$, $a^2\varepsilon_1 + \varepsilon_2 \not\equiv 0 \bmod p$ and $a_1 = a_2$, then

$$T\left[\begin{pmatrix} 1 & a/p \\ 0 & 1/p \end{pmatrix}\right] \sim (a^2\varepsilon_1 + \varepsilon_2)\begin{pmatrix} p^{a_1-2} & 0 \\ 0 & \varepsilon_1\varepsilon_2 p^{a_1} \end{pmatrix}.$$

If $a \not\equiv 0 \bmod p$, $a^2\varepsilon_1 + \varepsilon_2 \equiv 0 \bmod p$ and $a_1 = \alpha_2$, then

$$T\left[\begin{pmatrix} 1 & a/p \\ 0 & 1/p \end{pmatrix}\right] \sim \begin{pmatrix} p^{a_1-1} & 0 \\ 0 & \varepsilon_1\varepsilon_2 p^{a_1-1} \end{pmatrix}.$$

Applying Theorem 1 and writing as $\alpha_p(\varepsilon_1 p^{a_1}, \varepsilon_2 p^{a_2}, S)$ instead of $\alpha_p(T, S)$, we have the following:

if $0 \leq a_1 < a_2$, then

$$\begin{aligned} \alpha_p(T, S) = d_p(T, S) &+ p^{4-m}\alpha_p(\varepsilon_1 p^{a_1-2}, \varepsilon_2 p^{a_2}, S) + p^{3-m}\alpha_p(\varepsilon_1 p^{a_1}, \varepsilon_2 p^{a_2-2}, S) \\ &- p^{7-2m}\alpha_p(\varepsilon_1 p^{a_1-2}, \varepsilon_2 p^{a_2-2}, S) . \end{aligned}$$

if $0 \leq a_1 = a_2$, then

$$\begin{aligned} \alpha_p(T, S) = d_p(T, S) &+ p^{3-m}\alpha_p(\varepsilon_1 p^{a_1-2}, \varepsilon_2 p^{a_1}, S) + p^{3-m}\alpha_p(\varepsilon_1 p^{a_1}, \varepsilon_2 p^{a_1-2}, S) \\ &+ p^{3-m} \sum_{\substack{a \bmod p \\ a(a^2\varepsilon_1+\varepsilon_2)\not\equiv 0(p)}} \alpha_p((a^2\varepsilon_1 + \varepsilon_2)p^{a_1-2}, (a^2\varepsilon_1 + \varepsilon_2)\varepsilon_1\varepsilon_2 p^{a_1}, S) \\ &+ p^{3-m}(1 + \chi(-\varepsilon_1\varepsilon_2))\alpha_p(p^{a_1-1}, \varepsilon_1\varepsilon_2 p^{a_1-1}, S) \\ &- p^{7-2m}\alpha_p(\varepsilon_1 p^{a_1-2}, \varepsilon_2 p^{a_1-2}, S) , \end{aligned}$$

where χ is the quadratic residue symbol defined modulo p.

THEOREM 2. *Let $\varepsilon_1, \varepsilon_2$ be p-adic units in Z_p and $0 \leq a_1 \leq a_2$ $(a_i \in Z)$ and S a unimodular symmetric $m \times m$ matrix in $M_m(Z_p)$. Suppose that $m > 2$ and p is an odd prime. Let*

$$\chi(S) = \chi((-1)^{m/2} \det S) \quad or \quad \chi((-1)^{(m-1)/2} \det S)$$

according to $2|m$ or $2\nmid m$ where χ is the quadratic residue symbol defined modulo p.

Put

$$\beta_0(a, b) = (1 - \chi(S)p^{-m/2})(1 - p^{2-m})(1 - p^{3-m})^{-1}$$
$$\times \left\{\sum_{k=0}^{a} (\chi(S)p^{2-m/2})^k - p^{(3-m)b/2} \sum_{k=0}^{a} (\chi(S)p^{m/2-1})^k\right\},$$
$$\beta_1(a, b) = p^{a/2+(3-m)b/2+1-m/2}(1 - p^{1-m})(1 - p^{(2-m)(a+1)/2})(1 - p^{2-m})^{-1}.$$

Then we have, for

$$T = \begin{pmatrix} \varepsilon_1 p^{a_1} & 0 \\ 0 & \varepsilon_2 p^{a_2} \end{pmatrix},$$

(1) *if* $m \equiv 0 \bmod 2,\ a_1 + a_2 \equiv 0 \bmod 2,$ *then*

$$\alpha_p(T, S) = \beta_0(a_1, a_1 + a_2) + p^{(3-m)(a_1+a_2)/2}(1 - \chi(S)p^{-m/2})$$
$$\times (1 + \chi(S)\chi(-\varepsilon_1\varepsilon_2)p^{1-m/2}) \sum_{k=0}^{a_1} (\chi(S)p^{m/2-1})^k,$$

(2) *if* $m \equiv 0 \bmod 2,\ a_1 + a_2 \equiv 1 \bmod 2,$ *then*

$$\alpha_p(T, S) = \beta_0(a_1, a_1 + a_2 + 1),$$

(3) *if* $m \equiv 1 \bmod 2,\ a_1 \equiv a_2 \equiv 1 \bmod 2,$ *then*

$$\alpha_p(T, S) = \chi(-\varepsilon_1\varepsilon_2)\beta_1(a_1, a_2)$$
$$+ (1 - p^{1-m})(1 - p^{(4-m)(a_1+1)/2})(1 - p^{4-m})^{-1},$$

(4) *if* $m \equiv 1 \bmod 2,\ a_1 \equiv 1,\ a_2 \equiv 0 \bmod 2,$ *then*

$$\alpha_p(T, S) = \chi(S)\chi(\varepsilon_2)\beta_1(a_1, a_2)$$
$$+ (1 - p^{1-m})(1 - p^{(4-m)(a_1+1)/2})(1 - p^{4-m})^{-1},$$

(5) *if* $m \equiv 1 \bmod 2,\ a_1 \equiv 0,\ a_2 \equiv 1 \bmod 2,$ *then*

$$\alpha_p(T, S) = \chi(S)\chi(\varepsilon_1)\beta_1(a_1 - 1, a_2 + 1)$$
$$+ (1 - p^{1-m})(1 - p^{(4-m)a_1/2})(1 - p^{4-m})^{-1}$$
$$+ p^{(4-m)a_1/2}(1 - p^{1-m}) \sum_{k=0}^{a_2-a_1} (\chi(S)\chi(\varepsilon_1)p^{(3-m)/2})^k,$$

(6) *if* $m \equiv 1 \bmod 2,\ a_1 \equiv a_2 \equiv 0 \bmod 2,$ *then*

$$\alpha_p(T, S) = p^{(3-m)/2}\beta_1(a_1 - 1, a_2) + (1 - p^{1-m})(1 - p^{(4-m)a_1/2})(1 - p^{4-m})^{-1}$$
$$+ p^{(4-m)a_1/2}(1 - p^{1-m}) \sum_{k=0}^{a_2-a_1} (\chi(S)\chi(\varepsilon_1)p^{(3-m)/2})^k.$$

Proof. First we note that if N is a unimodular quadratic lattice and L, M are quadratic lattices with integral scale, then $\alpha_p(N \perp L, N \perp M) = \alpha_p(N, N \perp M)\alpha_p(L, M)$. If, hence, $a_1 = 0$, then

$$\alpha_p(T, S) = \alpha_p(\varepsilon_1, S)\alpha_p(\varepsilon_2 p^{a_2}, S_1) \qquad \text{where } S \sim \begin{pmatrix} \varepsilon_1 & \\ & S_1 \end{pmatrix},$$

and the theorem follows from Hilfssatz 16 in [7]. It is easy to prove the theorem by the induction on $a_1 + a_2$ in each case, using the induction formula before the theorem and $d_p(T, S) = (1 - \chi(S)p^{-m/2})(1 - p^{2-m})\cdot(1 + \chi(S)p^{2-m/2})$ (resp. $(1 - p^{3-m})(1 - p^{1-m})$) for $2|m$ (resp. $2\nmid m$), $1 \leq a_1 \leq a_2$.

THEOREM 3. *Let S be a unimodular symmetric $m \times m$ matrix ($m \geq 3$) and $p \neq 2$ put*

$$\alpha_p\left(\begin{pmatrix} \varepsilon_1 p^{a_1} & \\ & \varepsilon_2 p^{a_2} \end{pmatrix}, S\right) = \alpha_p(\varepsilon_1 p^{a_1}, \varepsilon_2 p^{a_2}, S).$$

Then we have

$$\begin{aligned} &\alpha_p(\varepsilon_1 p^{a_1+2}, \varepsilon_2 p^{a_2+2}, S) - p^{4-m}\alpha_p(\varepsilon_1 p^{a_1}, \varepsilon_2 p^{a_2}, S) \\ &\quad = \alpha_p(\varepsilon_1 p^{a_1}, \varepsilon_2 p^{a_2+4}, S) - p^{4-m}\alpha_p(\varepsilon_1 p^{a_1-2}, \varepsilon_2 p^{a_2+2}, S) \\ &\qquad\qquad \textit{for } 2 \leq a_1 \leq a_2,\ \varepsilon_i \in \boldsymbol{Z}_p^\times. \end{aligned}$$

If m is even, then we have a detailed relation:

$$\begin{aligned} &\alpha_p(\varepsilon_1 p^{a_1+1}, \varepsilon_2 p^{a_2+1}, S) - \chi(S)p^{2-m/2}\alpha_p(\varepsilon_1 p^{a_1}, \varepsilon_2 p^{a_2}, S) \\ &\quad = \alpha_p(\varepsilon_1, \varepsilon_2 p^{a_1+a_2+2}, S) \qquad \textit{for } 0 \leq a_1 \leq a_2,\ \varepsilon_i \in \boldsymbol{Z}_p^\times. \end{aligned}$$

Proof. This follows immediately from Theorem 2.

Remark. By the Siegel formula relations in Theorem 3 imply relations among Fourier coefficients of Eisenstein series of degree 2. Especially detailed relations for even m and $\chi(S) = 1$ are equivalent to Resnikoff-Saldaña-Maass relations. We note that $\alpha_p(\varepsilon T, S) = \alpha_p(T, S)$ ($\varepsilon \in \boldsymbol{Z}_p^\times$) if $S^{(m)}$ is unimodular and m is even, because we have $S \sim \varepsilon S$ in this case.

Remark. We conjecture that the first relations in Theorem 3 hold for any regular symmetric matrix S with $2 + s \leqq a_1 \leqq a_2$ where s is the smallest integer such that $p^s S^{-1}$ is integral. This is the case for any binary matrix S, evaluating densities by reduction formulas. Note that if $n = m$, then $p^{-(\mathrm{ord}_p(|T||S|^{-1}))/2}\alpha_p(T, S)/\alpha_p(S, S)$ is the number of quadratic lattices corresponding to S which contain a fixed lattice corresponding to T ($\mathrm{rk}\, T = n$, $\mathrm{rk}\, S = m$).

Theorem 1 is not so useful to evaluate local densities in general. At first the author got Theorem 2 in another way. (The formulas in [4] were not correct.) But at any rate Theorem 1 suggests that there are many relations among local densities since $d_p(T, S)$ is constant if $T \equiv 0 \bmod p^s$ such that $p^{s-1}S^{-1}$ is even.

We can evaluate local densities for $n = 3$ by using Theorem 1. But formulas are complicated. We give them for $n = 3$, $m = 3, 4$.

Let p be an odd prime and S ($\mathrm{rk}\, S = m$) a symmetric unimodular matrix. Put

$$T(\varepsilon_1 p^{a_1}, \varepsilon_2 p^{a_2}, \varepsilon_3 p^{a_3}) = \begin{pmatrix} \varepsilon_1 p^{a_1} & & \\ & \varepsilon_2 p^{a_2} & \\ & & \varepsilon_3 p^{a_3} \end{pmatrix},$$

where $\varepsilon_i \in \boldsymbol{Z}_p^\times$, $0 \leqq a_1 \leqq a_2 \leqq a_3$.

1) *Case of* $m = 3$.

It is obvious that $\alpha_p(T(\varepsilon_1 p^{a_1}, \varepsilon_2 p^{a_2}, \varepsilon_3 p^{a_3}), S) = 0$ unless $\chi(\varepsilon_1\varepsilon_2\varepsilon_3 |S|) = 1$ and $a_1 + a_2 + a_3 \equiv 0 \bmod 2$. Suppose that $\chi(\varepsilon_1\varepsilon_2\varepsilon_3 |S|) = 1$ and $a_1 + a_2 + a_3 \equiv 0 \bmod 2$. Then we have

$$\alpha_p(T(\varepsilon_1 p^{a_1}, \varepsilon_2 p^{a_2}, \varepsilon_3 p^{a_3}), S) = p^{(a_1+a_2+a_3)/2} \cdot \alpha_p(S, S)$$
$$\times \begin{cases} (1 + \chi(-\varepsilon_1\varepsilon_3)) \sum_{0 \leqq k \leqq (a_1-1)/2} p^k & \text{if } 2 \nmid a_1,\ 2 \mid a_2\,, \\ (1 + \chi(-\varepsilon_1\varepsilon_2)) \sum_{0 \leqq k \leqq (a_1-1)/2} p^k & \text{if } 2 \nmid a_1,\ 2 \nmid a_2\,, \\ (2 + \chi(-\varepsilon_2\varepsilon_3) - \chi(-\varepsilon_2\varepsilon_3)^{a_2+1}) \sum_{0 \leqq k \leqq a_1/2-1} p^k & \\ \quad + \sum_{0 \leqq k \leqq a_2 - a_1} \chi(-\varepsilon_2\varepsilon_3)^k \cdot p^{a_1/2} & \text{if } 2 \mid a_1\,. \end{cases}$$

2) *Case of* $m = 4$.

The value $\alpha_p(T(\varepsilon_1 p^{a_1}, \varepsilon_2 p^{a_2}, \varepsilon_3 p^{a_3}), S)(1 - \chi(S)p^{-2})^{-1}(1 - p^{-2})^{-1}$ is equal to

2.1)
$$\Big(\sum_{0 \leqq k \leqq a_1} \chi(S)^k\Big)\Big(\sum_{0 \leqq k \leqq a_3 - a_2} (\chi(-\varepsilon_1\varepsilon_2)\chi(S))^k\Big) p^{(a_1+a_2)/2}$$
$$+ 2\Big(\sum_{0 \leqq k \leqq a_1} \chi(S)^k\Big) \sum_{a_1 \leqq j \leqq (a_1+a_2)/2-1} p^j$$
$$+ \sum_{0 \leqq j \leqq a_1 - 1} \Big(\sum_{0 \leqq k \leqq j} \chi(S)^k\Big) p^j \begin{cases} 2 & \text{if } 2 \mid a_1,\ 2 \mid a_2,\ 2 \mid a_3\,, \\ 1 + \chi(S) & \text{if } 2 \nmid a_1,\ 2 \nmid a_2,\ 2 \mid a_3\,, \end{cases}$$

2.2)
$$(1 + \chi(-\varepsilon_2\varepsilon_3))\Big(\sum_{0 \leqq k \leqq a_1} \chi(S)^k\Big) \sum_{a_1+1 \leqq j \leqq (a_1+a_2-1)/2} p^j$$
$$+ \sum_{0 \leqq j \leqq a_1} \Big(\sum_{0 \leqq k \leqq j} \chi(S)^k\Big) p^j \begin{cases} 1 + \chi(-\varepsilon_2\varepsilon_3) & \text{if } 2 \mid a_1,\ 2 \nmid a_2,\ 2 \nmid a_3\,, \\ 1 + \chi(-\varepsilon_2\varepsilon_3)\chi(S) & \text{if } 2 \nmid a_1,\ 2 \mid a_2,\ 2 \mid a_3\,, \end{cases}$$

2.3)
$$\Big(\sum_{0 \leqq k \leqq a_1} \chi(S)^k\Big)\Big(\sum_{0 \leqq k \leqq a_3 - a_2} (\chi(-\varepsilon_1\varepsilon_2)\chi(S))^k\Big) p^{(a_1+a_2)/2}$$
$$+ (1 + \chi(-\varepsilon_1\varepsilon_2)\chi(S))\Big(\sum_{0 \leqq k \leqq a_1} \chi(S)^k\Big) \sum_{a_1 \leqq j \leqq (a_1+a_2)/2-1} p^j$$

$$+\sum_{0\leqq j\leqq a_1-1}\Big(\sum_{0\leqq k\leqq j}\chi(S)^k\Big)p^j\begin{cases}1+\chi(-\varepsilon_1\varepsilon_2)\chi(S) & \text{if } 2\,|\,a_1,\ 2\,|\,a_2,\ 2\nmid a_3\,,\\ 1+\chi(-\varepsilon_1\varepsilon_2) & \text{if } 2\nmid a_1,\ 2\nmid a_2,\ 2\,|\,a_3\,,\end{cases}$$

$$\text{2.4)}\quad (1+\chi(-\varepsilon_1\varepsilon_3)\chi(S))\Big(\sum_{0\leqq k\leqq a_1}\chi(S)^k\Big)\sum_{a_1\leqq j\leqq (a_1+a_2-1)/2}p^j$$

$$+\sum_{0\leqq j\leqq a_1-1}\Big(\sum_{0\leqq k\leqq j}\chi(S)^k\Big)p^j\begin{cases}1+\chi(-\varepsilon_1\varepsilon_3)\chi(S) & \text{if } 2\,|\,a_1,\ 2\nmid a_2,\ 2\,|\,a_3\,,\\ 1+\chi(-\varepsilon_1\varepsilon_3) & \text{if } 2\nmid a_1,\ 2\,|\,a_2,\ 2\nmid a_3\,.\end{cases}$$

Hence we see that

$$\alpha_p(T(\varepsilon_1p^{a_1+2},\varepsilon_2p^{a_2+2},\varepsilon_3p^{a_3+2}),S)-p^2\alpha_p(T(\varepsilon_1p^{a_1},\varepsilon_2p^{a_2},\varepsilon_3p^{a_3}),S)$$
$$=\begin{cases}\alpha_p(T(\varepsilon_1,\varepsilon_2p^{a_1+a_2+4},\varepsilon_3p^{a_1+a_3+4}),S) & \\ \qquad +\chi(S)p\alpha_p(T(\varepsilon_1,\varepsilon_2p^{a_1+a_2+2},\varepsilon_3p^{a_1+a_3+2}),S) & \text{if } 2\,|\,a_1\,,\\ \alpha_p(T(\varepsilon_1p,\varepsilon_2p^{a_1+a_2+3},\varepsilon_3p^{a_1+a_3+3}),S) & \text{if } 2\nmid a_1\,.\end{cases}$$

References

[1] Y. Kitaoka, Modular forms of degree n and representation by quadratic forms II, Nagoya Math. J., **87** (1982), 127–146.

[2] H. Maass, Die Fourier koeffizienten der Eisensteinreihen zweiten Grades, Mat.-Fys. Medd. Danske Vid. Selsk., **34**, Nr. 7 (1964).

[3] ——, Über die Fourier koeffizienten der Eisensteinreihen zweiten Grades, ibid. **38**, Nr. 14 (1972).

[4] M. Ozeki, On the evaluation of certain generalized Gauss sums in non-dyadic case, preprint, (1981).

[5] ——, On certain generalized Gaussian sums, Proc. Japan Acad., **58** (1982), 223–226.

[6] H. L. Resnikoff, R. L. Saldaña, Some properties of Fourier coefficients of Eisenstein series of degree two, J. reine angew. Math., **265** (1974), 90–109.

[7] C. L. Siegel, Über die analytische Theorie der quadratischen Formen, Ann. of Math., **36** (1935), 527–606.

[8] ——, Einführung in die Theorie der Modulfunktionen n-ten Grades, Math. Ann., **116** (1939), 617–657.

[9] ——, Über die Fourierschen Koeffizienten der Eisensteinschen Reihen, Mat.-Fys. Medd. Danske Vid. Selsk. **34**, Nr. 6 (1964).

[10] T. Tamagawa, On the ζ-functions of a division algebra, Ann. of Math., **77** (1963), 387–405.

Department of Mathematics
Faculty of Science
Nagoya University
Chikusa-ku, Nagoya 464
Japan