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THE EXTENDED PLUS-ONE HYPOTHESIS—A RELATIVE CONSISTENCY RESULT

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§ 1. Introduction

This paper includes a proof, relative to the consistency of *ZFC*, of the consistency of *ZFC*, the continuum has singular cardinality and the extended plus-one hypothesis.

The extended plus-one hypothesis. Suppose $n > k \geqslant 1$ and $\mathscr{F}$ is a normal type n object. Then there exists a normal type $k+1$ object $\mathscr{D}$ whose $(k-1, k)$-section is equal to that of $\mathscr{F}$.

Here $Tp(0)$ is ω and $Tp(n+1)$ is the power set of $Tp(n)$. $\mathscr{F}$ is a normal element of $Tp(n)$ if $\mathscr{F}$ can compute the equality relation between sets in $Tp(n-1)$. The $(k-1, k)$-section of a normal element of $Tp(n)$ consists of those elements of $Tp(k)$ which are computable from $\mathscr{F}$ using a parameter from $Tp(k-1)$. The $(k-1, k)$-section of $\mathscr{F}$ is denoted by ${}^{k-1}_{k}\mathrm{sc}\,\mathscr{F}$.

The notions of recursion in higher types are due to Kleene [8]; the extended plus-one hypothesis goes back to Sacks (see [11]), who proved the plus-one theorem:

Plus-One Theorem. *Suppose $n > k \geqslant 1$ and $\mathscr{F}$ is a normal type n object. Then there exists a normal type $k+1$ object $\mathscr{D}$ whose k-section is equal to that of $\mathscr{F}$.*

The k-section of $\mathscr{F}$, the set of elements of $Tp(k)$ which are recursive in $\mathscr{F}$, is the parameter free (lightface) version of the extended k-section of $\mathscr{F}$. Both of these plus-one principles imply that a restricted section

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of a normal object contains little information about the type of that object.

Sacks noted that the extended plus-one hypothesis follows from the generalized continuum hypothesis. Recently, Griffor and Normann [2] have shown that it also follows, for fixed k, from the existence of a regular well-ordering of $Tp(k)$ which is recursive in ${}^{k+3}E$. On the other hand, Harrington has shown it is false for $k = 2$ if the axiom of determinateness is true. The result of this paper implies that it cannot be proven false, unless ZFC is inconsistent, by assuming only ZFC and the continuum is singular.

Section 2 reformulates recursion in a normal object of finite type in the more set theoretic context of E-recursion. In Section 3, the basic facts about E-recursively closed structures and their generic extensions are reviewed.

Section 4 is devoted to a proof of the main theorem. The model of ZFC which is constructed satisfies that the continuum has a well-ordering of height ω_{ω_1} which is recursive in $Tp(1)$. Suppose $\mathscr{F}$ is a given normal element of $Tp(n)$ where $n > 3$. Then $\frac{1}{2}\mathrm{sc}\,\mathscr{F}$ naturally breaks into ω_1 many pieces.

The type 3 object $\mathscr{H}$ which is to have $\frac{1}{2}\mathrm{sc}\,\mathscr{H} = \frac{1}{2}\mathrm{sc}\,\mathscr{F}$ is constructed in ω_1 many stages. At stage α, the α^{th} piece of $\frac{1}{2}\mathrm{sc}\,\mathscr{F}$ is coded into $\mathscr{H}$ so that it can be computed for some real a, and $\mathscr{H}$. Thus $\frac{1}{2}\mathrm{sc}\,\mathscr{F} \subseteq \frac{1}{2}\mathrm{sc}\,\mathscr{H}$. To show that $\frac{1}{2}\mathrm{sc}\,\mathscr{H} \subseteq \frac{1}{2}\mathrm{sc}\,\mathscr{F}$ it will be shown that the amount of $\mathscr{H}$ constructed at stage α is recursive in $\mathscr{F}$ and some real and that it completely determines the values of all computations using the first ω_α many reals and $\mathscr{H}$. This will be made possible by regarding the $(\alpha + 1)^{\text{st}}$ stage of the construction as a generic extension via the continuum of a sufficiently well behaved (E-closed) initial segment of L. The result will be that every $\mathscr{H}$ computation using a real will be able to be duplicated by $\mathscr{F}$ using some other real so $\frac{1}{2}\mathrm{sc}\,\mathscr{H} \subset \frac{1}{2}\mathrm{sc}\,\mathscr{F}$.

§ 2. *E*-recursion

2.1. *The basics.* The notions of computability found in Kleene's recursion in a normal object of finite type were adapted to the universe of sets by Normann [10] and later by Moschovakis. The reader may wish to consult Slaman [17] as a general reference.

DEFINITION 2.2. Let $\mathscr{R}$ be a predicate on sets. The partial recursive function which is recursive in $\mathscr{R}$ with index e is denoted by $\{e\}^{\mathscr{R}}$ and defined by the following schemes.

(i) $\{e\}^{\mathscr{R}}(x_1, \cdots, x_n) = x_i$ $\qquad e = \langle 1, n, i\rangle$

(ii) $\{e\}^{\mathscr{R}}(x_1, \cdots, x_n) = x_i \backslash x_j$ $\qquad e = \langle 2, n, i, j\rangle$

(iii) $\{e\}^{\mathscr{R}}(x_1, \cdots, x_n) = \{x_i, x_j\}$ $\qquad e = \langle 3, n, i, j\rangle$

(iv) $\{e\}^{\mathscr{R}}(x_1, \cdots, x_n) \cong \bigcup_{y \in x_1} \{e'\}^{\mathscr{R}}(y, x_2, \cdots, x_n)$ $\qquad e = \langle 4, n, e'\rangle$

(v) $\{e\}^{\mathscr{R}}(x_1, \cdots, x_n) \cong \{e'\}^{\mathscr{R}}(\{e_1\}^{\mathscr{R}}(x_1, \cdots, x_n), \cdots, \{e_m\}^{\mathscr{R}}(x_1, \cdots, x_n))$
$\qquad e = \langle 5, n, m, e', e_1, \cdots, e_m\rangle$

(vi) $\{e\}^{\mathscr{R}}(x_1, \cdots, x_n) = x_i \cap \mathscr{R}$ $\qquad e = \langle 6, n, i\rangle$

(vii) $\{e\}^{\mathscr{R}}(e_1, x_1, \cdots, x_n, y_1, \cdots, y_m) \cong \{e_1\}^{\mathscr{R}}(x_1, \cdots, x_n)$
$\qquad e = \langle 7, n, m\rangle$.

The E-recursive schemes are the rudimentary ones (i)–(v), intersection with a predicate (vi) and a universal machine scheme (vii). There are several conventions in notation: $\{e\}^{\mathscr{R}}(x_1, \cdots, x_n)\downarrow$ if there is a y so that $\{e\}^{\mathscr{R}}(x_1, \cdots, x_n) \cong y$; $\{e\}^{\mathscr{R}}(x_1, \cdots, x_n)\uparrow$ otherwise; $y \leqslant_E \langle x_1, \cdots, x_n; \mathscr{R}\rangle$ if there is an index e so that $\{e\}^{\mathscr{R}}(x_1, \cdots, x_n) \cong y$.

DEFINITION 2.3. A predicate p is E-recursively enumerable in the parameters $a_1, \cdots, a_n$ relative to $\mathscr{R}$ if there is an index e so that p is the domain of the partial function $\lambda y | \{e\}^{\mathscr{R}}(y, a_1, \cdots, a_n)$.

DEFINITION 2.4. (i) A transitive set is E-closed relative to $\mathscr{R}$ if it is closed under application of those functions which are E-recursive in $\mathscr{R}$.

(ii) If x is a set then the E-closure of x relative to $\mathscr{R}$, denoted $E(x; \mathscr{R})$, is the smallest transitive set A so that $x \in A$ and A is E-closed relative to $\mathscr{R}$.

2.5. *Connections with recursion in higher types.*

THEOREM 2.6 (Normann [10]). (i) *Let $\mathscr{F}$ be a normal element of $Tp(n+2)$. Let $\mathscr{R}^{\mathscr{F}}$ be the predicate $\mathscr{R}^{\mathscr{F}}(x)$ iff $x \in \mathscr{F}$. There is a recursive function t so that the e^{th} (Kleene) partial recursive function relative to $\mathscr{F}$ with parameters $a_1, \cdots, a_n$ from $Tp(n)$ is equal to $\lambda x | \{t(e)\}^{\mathscr{R}^{\mathscr{F}}}(x, a_1, \cdots, a_n)$ on $Tp(n)$.*

(ii) *Let $\mathscr{R}$ be a predicate on sets and n be an integer. Then there is a normal type $n+2$ object $\mathscr{F}^{\mathscr{R}}$ and a recursive function t so that if $a_1, \cdots, a_n$ are parameters from $Tp(n)$ then the $t(e)^{\text{th}}$ (Kleene) partial recursive function relative to $\mathscr{F}^{\mathscr{R}}$ is equal to $\lambda x | \{e\}^{\mathscr{R}}(x, a_1, \cdots, a_n)$ on $Tp(n)$.*

COROLLARY 2.7. (i) *Let $\mathscr{F}$ be a normal type $n+2$ object. ${}^{n}_{n+1}\mathrm{sc}\,\mathscr{F}$*

is equal to $E(Tp(n); \mathscr{R}^{\mathscr{F}}) \cap Tp(n+1)$.

(ii) *If* $\mathscr{R}$ *is a predicate then there is a normal type* $n+2$ *object* $\mathscr{F}^{\mathscr{R}}$ *so that* $E(Tp(n); \mathscr{R}) \cap Tp(n+1)$ *is equal to* ${}^{\,n}_{n+1}\mathrm{sc}\,\mathscr{F}^{\mathscr{R}}$.

Normann's theorem and its corollary make precise the statement that E-recursion generalizes the original notions of recursion in normal objects. In what follows, the notions of E-recursion will be used exclusively; it is a consequence of Theorem 2.6 that the arguments could be reformulated strictly in terms of finite types.

As a notational point, let ${}^{k-1}_{\;k}\mathrm{sc}\,\mathscr{R}$ be defined for predicates exactly as it was for objects of finite type: $z \in {}^{k-1}_{\;k}\mathrm{sc}\langle Tp(n); \mathscr{R}\rangle$ if $z \in Tp(k)$ and there is an a in $Tp(k-1)$ so that $z \leqslant_E \langle a, Tp(n); \mathscr{R}\rangle$.

2.8. *The Moschovakis phenomenon.* The definition of E-recursive function includes, implicitly, the notions of subcomputation, computation tree and height of a computation, $\|\ \|$. $\{e\}^{\mathscr{R}}(x_1, \cdots, x_n)\downarrow$ iff the computation tree, $T^{\mathscr{R}}_{\langle e, x_1, \cdots, x_n\rangle}$, associated with the index e relative to $\mathscr{R}$ and arguments $x_1, \cdots, x_n$ is well-founded. If $\{e\}^{\mathscr{R}}(x_1, \cdots, x_n)\downarrow$ then $\|\langle e, x_1, \cdots, x_n; \mathscr{R}\rangle\|$ is the same as the height of the tree $T^{\mathscr{R}}_{\langle e, x_1, \cdots, x_n\rangle}$ as a well-founded relation.

DEFINITION 2.9. (i) If $\{e\}^{\mathscr{R}}(x_1, \cdots, x_n)\uparrow$ then an infinite descending path in $T^{\mathscr{R}}_{\langle e, x_1, \cdots, x_n\rangle}$ is called a Moschovakis witness to the divergence of $\{e\}^{\mathscr{R}}$ at $\langle x_1, \cdots, x_n\rangle$.

(ii) A set A which is E-closed relative to $\mathscr{R}$ satisfies the Moschovakis phenomenon relative to $\mathscr{R}$ if whenever $a_1, \cdots, a_n$ are elements of A and $\{e\}^{\mathscr{R}}(a_1, \cdots, a_n)\uparrow$ there is a Moschovakis witness to the divergence which is an element of A.

These witnesses to divergence were introduced by Moschovakis [9] to show that $E(Tp(1))$ is not the same as the least admissible set over $Tp(1)$ and that the set of indicies for divergent computations is Σ_1-definable over $E(Tp(1))$. When $n \geqslant 1$, $E(Tp(n))$ satisfies the Moschovakis phenomenon since any countable sequence in $Tp(n)$ is coded by an element of $Tp(n)$. An arbitrary E-closed structure may not satisfy the Moschovakis phenomenon.

2.10. *L.* The E-recursive functions are defined from below by recursion, hence are absolute. Any set which is E-recursive in x relative to $\mathscr{R}$ belongs to $L[x; \mathscr{R}]$, the constructible universe built over $\mathrm{TC}(x)$ (the transitive closure of $\{x\}$) using $\mathscr{R}$. Moreover, scheme (vii), the universal machine scheme in the definition of E-recursive, can be used to prove the fixed point theorem for E-recursion and hence show that functions defined by effective transfinite recursion in $\mathscr{R}$ are E-recursive relative to $\mathscr{R}$. This implies that $E(x;\mathscr{R})$ is an initial segment of $L[x;\mathscr{R}]$.

DEFINITION 2.11.
(i) $\kappa_0^{x;\mathscr{R}} = \sup\{\|\langle e, x; \mathscr{R}\rangle\| \mid e$ is an index & $\{e\}^{\mathscr{R}}(x)\downarrow\}$.
(ii) $\kappa^{x;\mathscr{R}} = \sup\{\|\langle e, x, y; \mathscr{R}\rangle\| \mid e$ is an index & $y \in \mathrm{TC}(x)$ & $\{e\}^{\mathscr{R}}(x, y)\downarrow\}$.

$\kappa_0^{x;\mathscr{R}}$ is the supremum of the ordinals which are recursive in x relative to $\mathscr{R}$; $\kappa^{x;\mathscr{R}}$ is the ordinal height of $E(x;\mathscr{R})$. There is a uniform correspondence $e \Leftrightarrow \phi_e$ between indicies and a certain set of Σ_1 formulas so that

$$\{e\}^{\mathscr{R}}(x_1, \cdots, x_n)\downarrow \qquad \text{iff } L_{\kappa_0^{\langle x_1,\cdots,x_n\rangle;\mathscr{R}}}[\langle x_1, \cdots, x_n\rangle; \mathscr{R}] \models \phi_e(x_1, \cdots, x_n) .$$

The informal definitions of E-recursive functions which follow are implicitly appealing to this characterization of E-recursion.

DEFINITION 2.12. (i) An ordinal $\alpha < \kappa^{x;\mathscr{R}}$ is $(x;\mathscr{R})$-reflecting if given any Σ_1 formula ϕ with only parameter x

$$L_\alpha[x;\mathscr{R}] \models \phi \qquad \text{iff } L_{\kappa_0^{x;\mathscr{R}}}[x;\mathscr{R}] \models \phi .$$

(ii) The greatest $(x;\mathscr{R})$-reflecting ordinal is denoted $\kappa_r^{x;\mathscr{R}}$.

Harrington [5] characterized the κ_r function in higher types by showing that if $\mathscr{R}$ is a predicate, n is a positive integer and a is an element of $Tp(n)$ then $\kappa_r^{a,Tp(n);\mathscr{R}}$ is the least ordinal γ so that a complete set of Moschovakis witnesses for $\langle a, Tp(n); \mathscr{R}\rangle$ is recursive in every ordinal greater than γ relative to $\langle a, Tp(n); \mathscr{R}\rangle$. That is to say that if $\{e\}^{\mathscr{R}}(a, Tp(n))\downarrow$ then

$$\|\langle e, a, Tp(n); \mathscr{R}\rangle\| < \kappa_r^{a,Tp(n);\mathscr{R}}$$

and if $\{e\}^{\mathscr{R}}(a, Tp(n))\uparrow$ then the ordinal $\kappa_r^{a,Tp(n);\mathscr{R}}$ is large enough to enumerate all of the points from some Moschovakis witness into $T_{\langle e,a,Tp(n)\rangle}$.

Sacks [13] showed that if x is a set of ordinals then κ_r^x ($\kappa_r^x = \kappa_r^{x;\phi}$) is the least ordinal γ so that a complete set of Moschovakis witnesses is available in the same sense as above for all the x computations at $\gamma + 1$. If $T^{\mathscr{R}}_{\langle e,x\rangle}$ is not well-founded and x is a set of ordinals then $T^{\mathscr{R}}_{\langle e,x\rangle}$ to the left of its leftmost path (in the natural well-ordering) has height less than or equal to $\kappa_r^{x;\mathscr{R}}$; its leftmost path is an element of $L_{\kappa_r^{x;\mathscr{R}}+1}[x;\mathscr{R}]$. In fact, for initial segments of L the global structure of reflection and so of the Moschovakis phenomenon has been understood.

DEFINITION 2.13. Let L_κ be E-closed. Define ρ^κ to be the least $\gamma < \kappa$ so that there is a parameter a in L_κ and an index e so that $\lambda x|\{e\}(x, a)$ maps a subset of γ onto L_κ.

ρ^κ is the least ordinal so that there is a parameter a in L_κ so that $E(\rho^\kappa \cup \{a\}) = L_\kappa$. Sacks showed in [14], for those L_κ satisfying the Moschovakis phenomenon, that if $\gamma < \rho^\kappa$ and a is an element of L_κ then

$$\sup\{\kappa_r^{\gamma',a} \mid \gamma' < \gamma\} < \kappa .$$

This implies that all the Moschovakis witnesses for a "small" set of parameters in L_κ are simultaneously available at a bounded point in L_κ.

2.14. *Selection.*

DEFINITION 2.15. If a and x are sets and $\mathscr{R}$ is a predicate then a selects from x relative to $\mathscr{R}$ if any non-empty predicate on x which is E-recursively enumerable in $\langle a, x\rangle$ relative to $\mathscr{R}$ has a non-empty subset which is E-recursive in $\langle a, x\rangle$ relative to $\mathscr{R}$.

Selection and reflection are two facets of the same phenomenon: they measure the degree to which the E-recursively enumerable predicates are closed under existential quantification. a selects from x relative to $\mathscr{R}$ exactly when the predicates which are E-recursively enumerable in $\langle a, x\rangle$ relative to $\mathscr{R}$ are closed under the quantifier $\exists z \in x$. In terms of reflection, this is exactly when for all b in x, $\kappa_r^{a,x;\mathscr{R}} \geqslant \kappa_r^{a,x,b;\mathscr{R}}$. The relevant selection theorems are

THEOREM 2.16. (i) (Gandy [1]) *Every set selects uniformly from ω relative to every predicate. (The index for the E-recursive subset of ω is a recursive function of the index for the E-recursively enumerable predicate on ω.)*

(ii) (Grilliot-Harrington-MacQueen [3, 4]) *If $a \in Tp(n)$ then $\langle a, Tp(n)\rangle$ selects from $Tp(n-1)$ relative to every predicate.*

§ 3. Forcing extensions of E-closed sets

The basic facts concerning forcing and E-recursion can be found in Sacks [15] or Sacks-Slaman [16]. In general, a set generic extension of an E-closed structure may not be E-closed. However, many interesting partial orders do preserve E-closure. If $\boldsymbol{P}$ is a partial order satisfying the countable chain condition (c.c.c.) the $\boldsymbol{P}$-generically extending an E-closed set preserves not only the E-closure of the ground model but also the reflection structure:

THEOREM 3.1 (Sacks [15]). *Suppose A is E-closed, if $x \in A$ then there is a well-ordering of x in A and $\boldsymbol{P}$ is a partial order with the countable chain condition in A. (Assume that each of the parameters $\boldsymbol{P}, \tau$ and a can E-recursively compute a well-ordering of its transitive closure which has smallest possible height in A.)*

(i) *If $p \in \boldsymbol{P}$, τ is a term in A and $p \Vdash \|\langle e, \tau\rangle\| = \gamma$ then γ is E-recursive in $\langle \tau, \boldsymbol{P}\rangle$.*

(ii) *If G is $\boldsymbol{P}$-generic over A and a is an element of A then $\kappa_\gamma^{a,G} = \kappa_\gamma^a$.*

Part (ii) is actually a consequence of part (i).

§ 4. The forcing construction

4.1. P. This section describes a forcing extension of L in which the continuum has singular cardinality and the extended plus-one hypothesis is true. In this model, if $n \geqslant 2$ then $Tp(n)$ has a regular well-ordering which is E-recursive in $Tp(n)$ and a fixed real number. By results of Griffor-Normann [2], only (1, 2)-sections need to be considered.

In short, begin with L and expand the cardinality of the continuum to ω_{ω_1} using a c.c.c. partial order so that the generic $\boldsymbol{G}$ is E-recursive in $Tp(1) \cap L[\boldsymbol{G}]$ and some real in $L[\boldsymbol{G}]$. If $\mathscr{R}$ is a predicate and n an integer in $L[\boldsymbol{G}]$, build $\mathscr{H}$ so that $\frac{1}{2}\mathrm{sc}\langle Tp(n); \mathscr{R}\rangle = Tp(2) \cap E(Tp(1), \mathscr{H})$. $\mathscr{H}$ is constructed in ω_1 many steps representing each step as adding $\boldsymbol{G}$ to some E-closed structure.

The forcing notion, $\boldsymbol{P}$, was developed by Harrington [6] and is also described in Jech [7]. It has two steps: the first is to use Cohen forcing to extend L to $L[G]$ where the continuum is ω_{ω_1}, the second is to use a version of almost disjoint forcing to add a real a so that the Cohen generic is Π_2^1 in a in $L[\langle G, a\rangle]$. The generic $\boldsymbol{G}$ is the pair $\langle G, a\rangle$. For the present, the actual definition of $\boldsymbol{P}$ is not important. Only the following facts are needed about a generic object $\langle G, a\rangle$:

(1) $P \leqslant_E \omega_{\omega_1}$.

(2) P has the c.c.c.

(3) $\langle G, a\rangle \leqslant_E \langle a, Tp(1) \cap L[\langle G, a\rangle]\rangle$.

4.2. *Canonical Terms.* With any notion of forcing Q over L there is a class of canonical terms for sets of ordinals in the generic extension. If τ is a term in the forcing language and $\|_{\bar{Q}}$ "$\tau \subseteq \lambda$" then there is a canonical term τ^* so that $\|_{\bar{Q}}$ "$\tau^* = \tau$". τ^* is defined from τ and Q as follows. For $\alpha < \lambda$, let A_α be the L-least antichain in Q so that if $p \in A_\alpha$ then $p \|_{\bar{Q}}$ "$\alpha \in \tau$" and also so that A_α is maximal with respect to this property. Define τ^* from the indexed set $A = \{A_\alpha \mid \alpha < \lambda\}$ by

$$\alpha \in \tau^* \Longleftrightarrow (\exists p \in A_\alpha)[p \in \underline{G}]$$

$\underline{G}$ is the term for the Q-generic object.

In the particular case of $\boldsymbol{P}$, each A_α will be countable since $\boldsymbol{P}$ has the c.c.c. There is a set R in $E(\omega_{\omega_1})$ of canonical terms for reals so that every real in $L[\langle G, a\rangle]$ is the denotation of some term in R. This follows from the proof of the G.C.H. in L.

Fix $\boldsymbol{G} = \langle G, a\rangle$ to be $\boldsymbol{P}$-generic over L. Since G is E-recursive in a and $Tp(1) \cap L[\langle G, a\rangle]$ the ordinal ω_{ω_1} is also. Thus, there is a well-ordering W of all the reals in $L[\langle G, a\rangle]$ which has height ω_{ω_1} and is E-recursive in a and the set of reals in $L[\langle G, a\rangle]$. Using W to code sets of reals by sets of ordinals, there are canonical terms for sets of reals in $L[\langle G, a\rangle]$ as well as for sets of ordinals.

In what follows, $Tp(n)$ will mean the $Tp(n)$ of $L[\langle G, a\rangle]$.

LEMMA 4.3 ($V = L[\langle G, a\rangle]$). *If X is a set of reals then there is a canonical term τ_X in L so that X is denoted by τ_X and*

(i) *X is E-recursive in τ_X, a and $Tp(1)$;*

(ii) *τ_X is E-recursive in X, a and $Tp(2)$.*

Proof. (i) Let τ_X be any canonical term for X. Both W and G are E-recursive in a and $Tp(1)$. X is first order definable using the parameters $\langle G, a\rangle$, W and τ_X since the α^{th} real in W is in X exactly when the α^{th} antichain in τ_X meets the generic, $\langle G, a\rangle$.

(ii) First, note that ω_{ω_1+1} is E-recursive in $Tp(2)$:

$$\omega_{\omega_1+1} = \left\{ |W| \,\middle|\, \begin{array}{l} W \text{ is a well-ordering of } Tp(1) \\ \text{and } |W| \text{ is its height} \end{array} \right\}.$$

Let X be a set of reals. By an effective transfinite recursion of length

ω_{ω_1+1}, there is a well-ordering of all canonical terms in L for sets of reals in $L[\langle G, a\rangle]$ which is E-recursive in $Tp(2)$. This relies on the fact that $\boldsymbol{P}$ has the countable chain condition. W and G are E-recursive in a and $Tp(2)$; whether or not a term τ denotes X in $L[\langle G, a\rangle]$ is the E-recursive in τ, a, X and $Tp(2)$. Then the least term τ_X which denotes X is E-recursive in X, a and $Tp(2)$.

4.4. (1, 2)-*sections of higher type objects in* $L[\langle G, a\rangle]$. There is one additional structural fact necessary to the proof of the main theorem: If $\mathscr{R}$ is a predicate and n is greater than 1 the $\frac{1}{2}\mathrm{sc}\langle Tp(n); \mathscr{R}\rangle$ has cofinality ω_1.

LEMMA 4.5 ($V = L[\langle G, a\rangle]$). *Let $\mathscr{R}$ be a predicate and n be an integer greater than* 1. *There is a sequence of sets $\langle X_\delta \mid \delta < \omega_1\rangle$ so that*

(i) $(\forall \gamma < \omega_1)(\exists b \in Tp(1))[\langle X_\delta \mid \delta < \gamma\rangle \leqslant_E \langle b, Tp(n); \mathscr{R}\rangle]$.

(ii) *If X is an element of $\frac{1}{2}\mathrm{sc}\langle Tp(n); \mathscr{R}\rangle$ then there is a real b and a δ less than ω_1 so that $X \leqslant_E \langle b, X_\delta\rangle$.*

Proof. By the preceding remarks W and ω_{ω_1} are both E-recursive in $\langle a, Tp(n); \mathscr{R}\rangle$. Moreover, the cofinal function $f\colon \omega_1 \to \omega_{\omega_1}$ defined by $f\colon \alpha \to \omega_\alpha$ is also E-recursive in $\langle a, Tp(n); \mathscr{R}\rangle$. The set X_δ is defined by

$$X_\delta = \left\{\langle X, e, b\rangle \,\middle|\, \begin{array}{l} X \in Tp(2) \text{ and } b \in Tp(1) \text{ and } |b|_W < \omega_\delta \\ \text{and } X = \{e\}^{\mathscr{R}}(b, a, Tp(n)) \end{array}\right\}.$$

$|b|_W$ is the ordinal height of b in the well-ordering W. Clearly, (ii) is satisfied by this sequence.

In order to show that any initial segment of the sequence $\langle X_\delta \mid \delta < \omega_1\rangle$ is recursive in $Tp(n)$ and some real relative to $\mathscr{R}$ it is sufficient to show that if $\gamma < \omega_1$ then the ordinal $\kappa_0(\gamma)$, defined to be equal to the supremum of $\{\kappa_0^{b,a,Tp(n);\mathscr{R}} \mid |b|_W < \omega_\gamma\}$, is E-recursive in some real and $Tp(n)$ relative to $\mathscr{R}$.

Define the partial E-recursive function g on ω_{ω_1} by effective transfinite recursion:

$$\begin{aligned} g(0) &= 0 \\ g(\alpha+1) &= (\text{the least } \gamma') \left[\begin{array}{l} \gamma' > g(\alpha) \text{ and } \exists b \in Tp(1) \\ \qquad \left[\begin{array}{l} |b|_W < \omega_\gamma \text{ and} \\ (\exists e \in \omega)[\|\langle e, b, a, Tp(n); \mathscr{R}\rangle\| = \gamma'] \end{array}\right] \end{array}\right] \\ g(\lambda) &= \sup_{\alpha<\lambda} g(\alpha) \qquad \text{if } \lambda \text{ is a limit ordinal}. \end{aligned}$$

The Gandy and Grilliot-Harrington-MacQueen Selection Theorems 2.16 together imply that the recursion step in defining $g(\alpha+1)$ from $g(\alpha)$ is E-recursive. Hence, g is also E-recursive.

If g happened to be total then it would induce a surjective function $h: \omega \times \omega_\gamma \to \omega_{\omega_1}$ defined by $h(e, \beta)$ is equal to α when $\{e\}^a (b_\beta, a, Tp(n)) = g(\alpha)$ (b_β is the β^{th} real in W). This is impossible since ω_{ω_1} is a cardinal and $\omega_\gamma < \omega_{\omega_1}$. Let β^* be the least ordinal so that g is undefined at β^*. Let b^* be the real so that $|b^*|_W = \beta^*$.

The supremum of $\{g(\beta) \mid \beta < \beta^*\}$ is E-recursive in $\langle b^*, a, Tp(n); \mathscr{R}\rangle$. This supremum must be $\kappa_0(\gamma)$ otherwise g would be defined at β^*. Its value would be the next ordinal which is the height of a computation using some parameter which is below ω_γ in W together with $a, Tp(n)$ and $\mathscr{R}$.

THEOREM 4.6 ($V = L[\langle G, a\rangle]$). *Suppose $\mathscr{R}$ is a predicate and n is a positive integer greater than* 1. *There is a predicate $\mathscr{H}$ so that* $\frac{1}{2}\mathrm{sc}\langle Tp(1); \mathscr{H}\rangle = \frac{1}{2}\mathrm{sc}\langle Tp(n); \mathscr{R}\rangle$.

Proof. Let $\langle X_\delta \mid \delta < \omega_1\rangle$ be the sequence exhausting $\frac{1}{2}\mathrm{sc}\langle Tp(n); \mathscr{R}\rangle$ constructed in Lemma 4.5. It is necessary to construct $\mathscr{H}$ so that $\frac{1}{2}\mathrm{sc}\langle Tp(n); \mathscr{R}\rangle$ consists of exactly those sets of reals in $E(Tp(1); \mathscr{H})$.

$\mathscr{H}$ is constructed in ω_1 many steps along with an auxillary function γ which has domain ω_1. At step δ, both $\gamma(\delta)$ and $\mathscr{H} \cap L_{\gamma(\delta)}[Tp(1); \mathscr{H}]$ will be defined to satisfy the inductive hypotheses:

(1) $\gamma(\delta) = \sup\{\kappa_\gamma^{b,a,Tp(1);\mathscr{H}} \mid |b|_W < \omega_\delta\}$;

(2) $L_{\gamma(\delta)}[Tp(1); \mathscr{H}]$ is not E-closed relative to $\mathscr{H}$;

(3) $X_\delta \in L_{\gamma(\delta)+1}[Tp(1); \mathscr{H}]$ and is uniformly defined in terms of δ and $\mathscr{H}$;

(4) $L_{\gamma(\delta)+1}[Tp(1); \mathscr{H}]$ is uniformly E-recursive in a, X_δ and $Tp(n)$.

The construction of $\mathscr{H}$ is simply described. Suppose that the function γ has been defined at all arguments less than δ and that $\mathscr{H}$ has been defined on all the sets in $\bigcup_{\delta'<\delta} L_{\gamma(\delta')}[Tp(1); \mathscr{H}]$. If δ is a limit ordinal let $\gamma(\delta)$ be the supremum of $\{\gamma(\delta') \mid \delta' < \delta\}$. X_δ will automatically be an element of $L_{\gamma(\delta)+1}[Tp(1); \mathscr{H}]$. Otherwise, δ is equal to $\sigma + 1$. Let τ_δ be the L-least canonical term for X_δ. Let β_δ be the least ordinal so that τ_δ is an element of L_{β_δ} and let W_{β_δ} be the L-least well-ordering of ω_{ω_1} of height β_δ. W_{β_δ} is recursive in some real, $Tp(n)$ and $\mathscr{R}$ by Lemma 4.3. Code W_{β_δ} and τ_δ into $\mathscr{H}$ at $\gamma(\sigma) + 1$ by

$$\mathscr{H}(X) = \begin{cases} 2 & \text{if } X = \langle \gamma(\sigma) + 1, \delta' \rangle,\ \delta' < \omega_{\omega_1} \text{ and } \delta' \in W_{\beta_\delta}\,. \\ 1 & \text{if } X = \langle \gamma(\sigma) + 1, \delta', 0 \rangle,\ \delta' < \omega_{\omega_1} \text{ and } \tau_\delta \text{ is the } \delta'^{\text{th}} \text{ element} \\ & \text{of } L_{\beta_\delta} \text{ in the } L\text{-least well-ordering of } L_{\beta_\delta}. \text{ (This well-} \\ & \text{ordering is an element of } L_{\beta_\delta + 1})\,. \\ 0 & \text{if } X \text{ is not covered by the above and} \\ & X \in L_{\gamma(\sigma)+2}[Tp(1); \mathscr{H}] - L_{\gamma(\sigma)}[Tp(1); \mathscr{H}]\,. \end{cases}$$

This defines $\mathscr{H}$, regarded as a function from sets to $\{0, 1, 2\}$, on $L_{\gamma(\sigma)+2}[Tp(1); \mathscr{H}]$. Set $\mathscr{H}(X)$ equal to 0 inductively for each X and $\beta > \gamma(\sigma) + 2$ so that X is an element of $L_\beta[Tp(1); \mathscr{H}] - L_{\gamma(\sigma)+2}[Tp(1); \mathscr{H}]$ until β is equal to $\gamma(\delta)$:

$$\gamma(\delta) = \sup \{\kappa_r^{b,a,Tp(1);\mathscr{H}} \,|\, |b|_W < \omega_\delta\}\,.$$

First, if the induction hypotheses can be verified then the construction is successful in making $\frac{1}{2}\mathrm{sc}\langle Tp(1); \mathscr{H}\rangle = \frac{1}{2}\mathrm{sc}\langle Tp(n); \mathscr{R}\rangle$. Let γ be the supremum of $\gamma(\delta)$ as δ varies over ω_1. $L_\gamma[Tp(1); \mathscr{H}]$ satisfies the Moschovakis phenomenon by hypothesis (1) and the remarks in Section 2.10. So $L_\gamma[Tp(1); \mathscr{H}]$ is E-closed relative to $\mathscr{H}$; hypothesis (2) implies that no proper initial segment is E-closed. Thus, $L_\gamma[Tp(1); \mathscr{H}]$ is equal to $E(Tp(1); \mathscr{H})$. By hypothesis (3), each X_δ is an element of $E(Tp(1); \mathscr{H})$ so $\frac{1}{2}\mathrm{sc}\langle Tp(n); \mathscr{R}\rangle \subseteq \frac{1}{2}\mathrm{sc}\langle Tp(1); \mathscr{H}\rangle$. Finally, hypothesis (4) implies that $\frac{1}{2}\mathrm{sc}\langle Tp(1); \mathscr{H}\rangle \subseteq \frac{1}{2}\mathrm{sc}\langle Tp(n); \mathscr{R}\rangle$ since every initial segment of $L_\gamma[Tp(1); \mathscr{H}]$ is E-recursive in $Tp(n)$ and some real relative to $\mathscr{R}$.

It remains to verify the inductive hypotheses.

The limit case in the definition of γ and $\mathscr{H}$ is the easier one to analyze. Suppose that λ is a countable limit ordinal and the inductive hypotheses are satisfied for each δ below λ. Hypothesis (1) is automatically true. For each δ less than λ, let b_δ be a real so that $\gamma(\delta) \leqslant_E \langle b_\delta, a, Tp(1); \mathscr{H}\rangle$. λ is countable, so there is a real b_λ which computes $\{\langle e, b_\delta\rangle \,|\, \{e\}^{\mathscr{H}}(b_\delta, a, Tp(1)) = \gamma(\delta)\}$. By the union scheme of E-recursion, $\gamma(\lambda) \leqslant_E \langle b_\delta, a, Tp(1); \mathscr{H}\rangle$. This establishes hypothesis (2). Hypotheses (3) and (4) follow from the uniformity of the construction, the continuity of $\langle X_\delta \,|\, \delta < \omega_1\rangle$ and the fact that $\mathscr{H}$ is defined to be 0 for all X in $L_{\gamma(\lambda)+1}[Tp(1); \mathscr{H}] - L_{\gamma(\lambda)}[Tp(1); \mathscr{H}]$.

The case when δ is a successor, say $\delta = \sigma + 1$, is more subtle. Suppose the hypotheses are true at level σ. Hypothesis (3) is true for $\sigma + 1$ as $X_{\sigma+1}$ is uniformly coded into $\mathscr{H}$ and $\gamma(\sigma)$ via $W_{\beta_{\sigma+1}}$ and $\tau_{\sigma+1}$ (see

Lemma 4.3). But $\gamma(\delta)$ is easily defined from $\sigma + 1$ and $\mathscr{H}$ (not E-recursively though!) using the characterization of κ_r of 2.10. Hypothesis (4) is seen true since $L_{\gamma(\sigma)+1}[Tp(1); \mathscr{H}]$ can be built from X_δ and $Tp(n)$ using an effective transfinite recursion of shorter length than ω_{ω_1}. But $\omega_{\omega_1} \leqslant_E Tp(n)$ and being $L_{\gamma(\sigma+1)}[Tp(1); \mathscr{H}]$ is recursive in X_δ and $Tp(1)$ as a predicate so this recursion can be done recursively in X_δ and $Tp(n)$.

The value of $\gamma(\sigma + 1)$ is designed specifically to insure that hypotheses (1) is true so it remains to verify hypothesis (2). Namely, it must be shown that $L_{\gamma(\sigma+1)}[Tp(1); \mathscr{H}]$ is not E-closed relative to $\mathscr{H}$. Assuming hypothesis (2) at level σ, let b_σ be the W-least real so that there is an integer e so that $\|\langle e, b_\sigma, a, Tp(1); \mathscr{H}\rangle\| = \gamma(\sigma)$.

The characterization of $\kappa_r^{x, Tp(1); \mathscr{H}} + 1$ as the least ordinal where all the Moschovakis witnesses for x and $Tp(1)$ relative to $\mathscr{H}$ can be E-recursively recognized implies that if b_1 and b_2 are reals then $\kappa_r^{b_1, a, Tp(1); \mathscr{H}} \leqslant \kappa_r^{b_1, b_2, a, Tp(1); \mathscr{H}}$. Define α by

$$\alpha = \sup \{\kappa_r^{b, b_\sigma, a, Tp(1); \mathscr{H}} \mid |b|_W < \omega_{\sigma+1}\} .$$

By the increasing nature of the κ_r function α is greater than or equal to $\gamma(\sigma + 1)$. It is sufficient to show that there is a real which, together with $Tp(1)$, E-recursively computes α relative to $\mathscr{H}$.

Define the sequence S by

$$S = \left\{ \delta_{\sigma'} \middle| \begin{array}{l} \sigma' < \delta \text{ and the } \delta_{\sigma'}^{\text{th}} \text{ element in the } L\text{-least} \\ \text{well-ordering of } L_{\beta_\sigma} \text{ of height } \omega_{\omega_1} \text{ is } \tau_{\sigma'} \end{array} \right\} .$$

The parameters S, a, $W_{\beta\delta}$, X_δ and $Tp(1)$ are E-recursive in $\gamma(\sigma)$, a and $Tp(1)$ relative to $\mathscr{H}$ (see Lemma 4.3). These parameters are all that is needed to compute $\gamma(\sigma)$, a, $Tp(1)$ and $\mathscr{H} \cap L_{\gamma(\sigma+1)}[Tp(1); \mathscr{H}]$. S is a countable subset of ω_{ω_1} in $L[\langle G, a\rangle]$. Since P has the countable chain condition there is a term τ_S in $L_{\omega_{\omega_1}}$ which denotes S in $L[\langle G, a\rangle]$. Consider the structure $E(W_{\beta\delta}, X_\delta, S, Tp(1))$ which is equal to $L_\kappa[W_{\beta\delta}, X_\delta, S, Tp(1)]$ for some E-closed ordinal κ. This structure can be, alternatively, produced by starting with the ground model L_κ, which includes $W_{\beta\delta}$, P and the canonical terms τ_δ and τ_S for X_δ and S, and then P-generically adding $\langle G, a\rangle$. Since P has the countable chain condition, Theorem 3.1 implies that the addition of $\langle G, a\rangle$ to L_κ does not change the reflection structure of L_κ: If τ is an element of L_κ and τ is a set of ordinals then $\kappa_r^{\tau, P, \langle G, a\rangle} = \kappa_r^{\tau, P}$.

L_κ must be $E(W_{\beta\delta})$ since this structure remains E-closed when generically extended by $\langle G, a\rangle$. Then $\rho^{\kappa} = \omega_{\omega_1}$ and by the remarks after 2.13 if p is an element of L_{κ} then $\lambda x|\kappa_r^{x,P}$ is uniformly bounded below κ on proper initial segments of ω_{ω_1}.

Let ν_{σ} be the height of b_{σ} in W_{β_δ}. (b_{σ} is the real which computes $\gamma(\sigma)$ relative to $\mathscr{H}$. Then define α^* by

$$\alpha^* = \sup \{\kappa_r^{\nu,\nu_\sigma,\tau_S,W_{\beta_\delta},\tau_\delta,\boldsymbol{P}} | \nu < \omega_{\sigma+1}\} .$$

Since $\omega_{\sigma+1}$ is less than ω_{ω_1}, α^* is less than κ. But then forcing with $\boldsymbol{P}$ preserves the values of κ_r^x so

$$\alpha^* = \sup \{\kappa_r^{\nu,\nu_\sigma,\tau_S,W_{\beta_\delta},\tau_\delta,\boldsymbol{P},\langle G,a\rangle} | \nu < \omega_{\sigma+1}\} .$$

Also, $\kappa_r^{b,b_\sigma,S,W_{\beta_\delta},X_\delta,Tp(1)} \leqslant \kappa_r^{\nu,\nu_\sigma,\tau_S,W_{\beta_\delta},\tau_\delta,\boldsymbol{P},\langle G,a\rangle}$ if b is the ν^{th} real in W. Thus α^* is greater than or equal to α. α^* is E-recursive in some ordinal less than ω_{ω_1} and W_{β_δ} since it is less than κ; thus α is E-recursive in some real, b_σ, S, W_{β_σ}, X_δ and $Tp(1)$; or, in other words, α is E-recursive in some real, b_σ and $Tp(1)$ relative to $\mathscr{H}$. This verifies hypothesis (2) in the successor case and completes the proof of the theorem.

4.7. *Remarks and open questions.* The proof of the Theorem 4.6 can be easily adapted to find a model where the continuum is ω_α and α is any ordinal of uncountable cofinality. The arguments which were special to ω_{ω_1} can be replaced by invoking condensation arguments in L. Secondly, each of the structures $E(Tp(1); \mathscr{H})$ constructed during the course of the proof had the feature that $\lambda x|\kappa_r^{x,Tp(1);\mathscr{H}}$ is bounded on initial segments of ω_{ω_1} $(=\rho^{\mathscr{H}})$. Implicitly, it was shown that this is also true for $E(Tp(1))$ in $L[\langle G, a\rangle]$. This feature of $E(Tp(1))$ is enough to guarantee that various other constructions can be executed in $E(Tp(1))$ (i.e. for ${}^3E$) in $L[\langle G, a\rangle]$ which would usually require that the continuum be a regular cardinal. (see Sacks [12]).

QUESTION 4.8. Does the consistency of *ZFC* imply the consistency of *ZFC* together with the failure of the extended plus-one hypothesis?

The solution of this question would certainly involve the solution of the following one.

QUESTION 4.9. Is there a predicate $\mathscr{R}$ and an ordinal γ so that $\lambda x|\kappa_r^{x,\gamma,\mathscr{R}}$ is not bounded (in $E(\gamma; \mathscr{R})$) on initial segments of $\rho^{\gamma;\mathscr{R}}$ (relativize definition 2.13)?

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