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THE BERGMAN METRIC ON A THULLEN DOMAIN

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§1. Introduction

In this paper we shall study the holomorphic sectional curvature of the Bergman metric on a domain

$$D_p := \{(z, w) \in C^2 \,|\, |z| < 1,\ |w|^2 < (1 - |z|^2)^p\}$$

in C^2, where $0 \leqq p \leqq 1$. (If $p \neq 0$ then

$$D_p = \{(z, w) \in C^2 \,|\, |z|^2 + |w|^{2/p} < 1\}.)$$

If $0 < p < 1$ then D_p is called a Thullen domain. (D_0 is the unit bidisc and D_1 the unit ball.)

We shall determine the maximum and the minimum of the curvature at an arbitrary point of D_p (Theorem 1), and examine the boundary behavior of the curvature (Corollary of Theorem 2).

We shall have the maximum and the minimum of the curvature on D_p, which are negative and given by simple rational functions of p (Theorem 3).

§2. Bergman metric on a complete Reinhardt bounded domain in C^2

Let D be a bounded domain in C^n with the natural coordinate $(z^1, \cdots, z^n)$ and $K(z^1, \cdots, z^n)$ be the Bergman kernel function of D. The Bergman metric on D is defined by

$$h := 2 \sum_{a,b} h_{a\bar{b}} dz^a \cdot d\bar{z}^b ,$$

where $h_{a\bar{b}} := \partial^2 \log K / \partial z^a \partial \bar{z}^b$. The Riemann curvature tensor of the metric is given by

$$R_{a\bar{b}c\bar{d}} := \frac{\partial^2 h_{a\bar{b}}}{\partial z^c \partial \bar{z}^d} - \sum_{e,f} h^{e\bar{f}} \frac{\partial h_{a\bar{f}}}{\partial z^c} \frac{\partial h_{e\bar{b}}}{\partial \bar{z}^d} ,$$

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where $(h^{e\bar{f}})$ is the inverse matrix of $(h_{a\bar{b}})$ in the sense that $\sum_b h_{a\bar{b}}h^{c\bar{b}} = \delta^c_a$. The holomorphic sectional curvature of the Bergman metric in a direction u at $q \in D$, which is a holomorphic tangent vector at q (i.e. $u \in T_q(D)$) such that $h(q)(u, \bar{u}) = 1$, is given by

$$H(q; u) := -\sum_{a,b,c,d} R_{a\bar{b}c\bar{d}}(q)u^a\bar{u}^b u^c\bar{u}^d ,$$

where $u = \sum_a u^a(\partial/\partial z^a)_q$. We use the following notations:

$$K_a := \partial K/\partial z^a , \quad K_{\bar{a}} := \partial K/\partial \bar{z}^a \qquad \text{for } a = 1, \cdots, n ;$$
$$K_{a_1\cdots a_s a} := \partial K_{a_1\cdots a_s}/\partial z^a , \qquad K_{a_1\cdots a_s\bar{a}} := \partial K_{a_1\cdots a_s}/\partial \bar{z}^a$$
$$\text{for } a_j = 1, \cdots, n, \bar{1}, \cdots, \bar{n} \quad \text{and} \quad a = 1, \cdots, n .$$

Then the following formulas hold (cf. Kobayashi [4], p. 275):

$$h_{a\bar{b}} = \frac{KK_{a\bar{b}} - K_aK_{\bar{b}}}{K^2} .$$
$$R_{a\bar{b}c\bar{d}} = -(h_{a\bar{b}}h_{c\bar{d}} + h_{a\bar{d}}h_{c\bar{b}}) + \hat{R}_{a\bar{b}c\bar{d}} ,$$

where

$$\hat{R}_{a\bar{b}c\bar{d}} := \frac{K_{a\bar{b}c\bar{d}}}{K} - \frac{K_{ac}K_{\bar{b}\bar{d}}}{K^2}$$
$$(2.1) \qquad - \frac{1}{K^4}\sum_{e,f} h^{e\bar{f}}(KK_{ac\bar{f}} - K_{ac}K_{\bar{f}})(KK_{\bar{b}\bar{d}e} - K_{\bar{b}\bar{d}}K_e) .$$

Suppose D is a complete Reinhardt bounded domain. Since then K is a C^∞-function of the variables $|z^j|^2$ $(j = 1, \cdots, n)$, making use of (2.1), we have

$$(2.2) \qquad \hat{R}_{a\bar{b}c\bar{d}} = \hat{R}_{c\bar{b}a\bar{d}} , \quad \hat{R}_{a\bar{b}c\bar{d}} = \hat{R}_{a\bar{d}c\bar{b}} , \quad \hat{R}_{a\bar{b}c\bar{d}} = \overline{\hat{R}_{b\bar{a}d\bar{c}}} .$$

If $n = 2$, making use of (2.2), we obtain the following:

LEMMA 1. *If D is a complete Reinhardt bounded domain in C^2 then*

$$2 - H(q; u(\partial/\partial z^1)_q + v(\partial/\partial z^2)_q)$$
$$= \hat{R}_{1\bar{1}1\bar{1}}(q)|u|^4 + 4\hat{R}_{1\bar{1}2\bar{2}}(q)|u|^2|v|^2 + \hat{R}_{2\bar{2}2\bar{2}}(q)|v|^4$$
$$+ 2\mathrm{Re}(2\hat{R}_{1\bar{1}1\bar{2}}(q)u^2\bar{u}\bar{v} + \hat{R}_{1\bar{2}1\bar{2}}(q)u^2\bar{v}^2 + 2\hat{R}_{1\bar{2}2\bar{2}}(q)uv\bar{v}^2) ,$$

where $q \in D$, $(u, v) \in C^2$ with

$$h_{1\bar{1}}(q)|u|^2 + 2\,\mathrm{Re}\,(h_{1\bar{2}}(q)u\bar{v}) + h_{2\bar{2}}(q)|v|^2 = 1$$

and $\hat{R}_{a\bar{b}c\bar{d}}$ is the tensor defined by (2.1).

§ 3. Upper and lower curvatures of a bounded domain

Let D be an arbitrary bounded domain in C^n. Let h be the Bergman metric on D and $H(q; u)$ the holomorphic sectional curvature of h in a direction u at $q \in D$. We shall use the following:

DEFINITION. Set

$$\begin{aligned} U_D(q) &:= \max\{H(q; u) \mid u \in T_q(D),\ h(q)(u, \bar{u}) = 1\}\,, \\ L_D(q) &:= \min\{H(q; u) \mid u \in T_q(D),\ h(q)(u, \bar{u}) = 1\}\,, \qquad q \in D\,; \\ u_D &:= \sup\{U_D(q) \mid q \in D\}\,, \\ \ell_D &:= \inf\{L_D(q) \mid q \in D\}\,. \end{aligned}$$

We call $U_D(q)$, $L_D(q)$, u_D and ℓ_D the upper, the lower curvature at q, the upper and the lower curvature of D respectively.

The upper and the lower curvatures are biholomorphically invariant quantities on the bounded domains in a fixed C^n:

PROPOSITION. *Let f be a biholomorphic mapping of D to $\hat{D}$, where D and $\hat{D}$ are bounded domains in C^n. Then $U_D = U_{\hat{D}} \circ f$, $L_D = L_{\hat{D}} \circ f$, $u_D = u_{\hat{D}}$ and $\ell_D = \ell_{\hat{D}}$.*

Proof. Let h and $\hat{h}$ be the Bergman metrics on D and $\hat{D}$ respectively, and H_h, $H_{\hat{h}}$ and $H_{f^*\hat{h}}$ the holomorphic sectional curvatures of h, $\hat{h}$ and $f^*\hat{h}$ respectively. Then $h = f^*\hat{h}$. If $u \in T_q(D)$ $(q \in D)$ and $h(q)(u, \bar{u}) = 1$ then $\hat{h}(f(q))(f_* u, \overline{f_* u}) = (f^*\hat{h})(q)(u, \bar{u}) = h(q)(u, \bar{u}) = 1$. Hence the fact $H_h(q; u) = H_{f^*\hat{h}}(q; u) = H_{\hat{h}}(f(q); f_* u)$ implies our assertion. Q.E.D.

§ 4. Upper and lower curvatures at a point of D_p

We now return to our domain D_p defined in the section 1. The Bergman kernel function of D_p is given by

$$K(z, w) = c\frac{(1 - |z|^2)^p - r|w|^2}{((1 - |z|^2)^p - |w|^2)^3(1 - |z|^2)^{2-p}}\,, \qquad (z, w) \in D_p\,, \tag{4.1}$$

where $1/c$ $(= \pi^2/(1 + p))$ is the volume of D_p with respect to the euclidean metric on C^2 and

$$r = r(p) := (1 - p)/(1 + p) \tag{4.2}$$

(cf. Ise[2]). The group of all biholomorphic transformations of D_p includes the group of the mappings

$$\begin{cases} z' = \lambda(z+\alpha)/(1+\bar{\alpha}z)\,, \\ w' = \mu(1-|\alpha|^2)^{p/2}(1+\bar{\alpha}z)^{-p}w\,, \end{cases} \tag{4.3}$$

where λ, μ, $\alpha \in \boldsymbol{C}$; $|\lambda| = |\mu| = 1$, $|\alpha| < 1$ (cf. Ise[2], p. 517). Now we set

$$U_p := U_{D_p}, \quad L_p := L_{D_p}, \quad u_p := u_{D_p}, \quad \ell_p := \ell_{D_p}\,.$$

LEMMA 2. *If* $(z, w) \in D_p$ *then*

$$U_p(z, w) = U_p(0, |w|(1-|z|^2)^{-p/2})\,,$$
$$L_p(z, w) = L_p(0, |w|(1-|z|^2)^{-p/2})\,.$$

Proof. Let $(z_0, w_0) \in D_p$. Set

$$f(z, w) := ((z - z_0)/(1 - \bar{z}_0 z), \mu(1-|z_0|^2)^{p/2}(1-\bar{z}_0 z)^{-p}w)\,,$$

where $\mu := |w_0|/w_0$ if $w_0 \neq 0$, or $\mu := 1$ if $w_0 = 0$. Then f satisfies the condition (4.3) and maps (z_0, w_0) to $(0, |w_0|(1-|z_0|^2)^{-p/2})$. Therefore, Proposition in the previous section implies our assertion. Q.E.D.

By virtue of Lemma 2, for the purpose of finding the values $U_p(z, w)$ and $L_p(z, w)$, it is enough to examine U_p and L_p at $(0, w)$ with $|w| < 1$. For the convenience of calculations we introduce a new variable

$$t = t(w) := (1-|w|^2)/(1-r|w|^2)\,, \qquad |w| < 1\,, \tag{4.5}$$

where $r = (1-p)/(1+p)$ as (4.2).

LEMMA 3. *Let* $0 < p \leqq 1$ *and* $|w| < 1$. *If* r *and* t *are as* (4.2) *and* (4.5) *then*

$$2 - U_p(0, w) = 4\min\{Ax^2 + 2Bxy + Cy^2 \,|\, x, y \geqq 0, \alpha x + \beta y = 1\}\,,$$
$$2 - L_p(0, w) = 4\max\{Ax^2 + 2Bxy + Cy^2 \,|\, x, y \geqq 0, \alpha x + \beta y = 1\}\,,$$

where

$$\begin{cases} \alpha = 3 + rt^2, \qquad \beta = 3 - rt^2\,; \\ A = 6 + 4rt^2 + (1+r)rt^3\,, \\ B = 2(9 + 3rt^2 - 3(1+r)rt^3 + 2r^2t^4)/(3+rt^2)\,, \\ C = 3(6 - 6rt^2 + (1+r)rt^3)/(3-rt^2)\,. \end{cases} \tag{4.6}$$

Proof. We note $0 \leqq r < 1$, because $p > 0$. Then $0 < t \leqq 1$ and $|w|^2 = (1-t)/(1-rt)$. It follows that

$$\begin{cases} h_{1\bar{1}}(0, w) = \alpha/(1+r)t\,, \\ h_{2\bar{2}}(0, w) = \beta(1-rt)^2/(1-r)^2t^2\,, \\ h_{1\bar{2}}(0, w) = 0\,; \end{cases}$$

$$\begin{cases} \hat{R}_{1\bar{1}1\bar{1}}(0, w) = 4A/(1+r)^2t^2\,, \\ \hat{R}_{1\bar{1}2\bar{2}}(0, w) = 2(1-rt)^2B/(1+r)(1-r)^2t^3\,, \\ \hat{R}_{2\bar{2}2\bar{2}}(0, w) = 4(1-rt)^4C/(1-r)^4t^4\,, \\ \hat{R}_{1\bar{1}1\bar{2}}(0, w) = 0\,, \\ \hat{R}_{1\bar{2}1\bar{2}}(0, w) = 0\,, \\ \hat{R}_{1\bar{2}2\bar{2}}(0, w) = 0\,. \end{cases}$$

Setting $x := |u|^2/(1+r)t$, $y := |v|^2(1-rt)^2/(1-r)^2t^2$, we obtain the desired formulas by Lemma 1. Q.E.D.

Now our key theorem is the following:

THEOREM 1. *Let $0 \leqq p \leqq 1$ and $|w| < 1$. If r and t are as (4.2) and (4.5) then*

$$\begin{aligned} U_p(0, w) &= 2 - 4F/(3+rt^2)^2E\,, \\ L_p(0, w) &= 2 - 4\max\{3(6 - 6rt^2 + (1+r)rt^3)/(3-rt^2)^3\,, \\ &\qquad (6 + 4rt^2 + (1+r)rt^3)/(3+rt^2)^2\}\,, \end{aligned}$$

where

$$\begin{aligned} E &= 162(1+r) - 180rt - 81(1+r)rt^2 + 48r^2t^3 + 24(1+r)r^2t^4 \\ &\quad - 12r^3t^5 - (1+r)r^3t^6 > 0\,, \\ F &= 972(1+r) - 1080rt + 162(1+r)rt^2 - 27(3(1+r)^2 + 16r)rt^3 \\ &\quad + 72(1+r)r^2t^4 + 18(3(1+r)^2 - 4r)r^2t^5 - 54(1+r)r^3t^6 \\ &\quad + (3(1+r)^2 + 16r)r^3t^7\,. \end{aligned}$$

To prove Theorem 1, we prepare the following:

LEMMA 4. *Let α, β, A, B and C be real numbers such that α, β, $C\alpha - B\beta$ and $A\beta - B\alpha$ are all positive. Set $f(x, y) := Ax^2 + 2Bxy + Cy^2$, $g(x, y) := \alpha x + \beta y$. Then we have*

$$\begin{aligned} \max\{f(x, y)\,|\,x, y \geqq 0, g(x, y) = 1\} &= \max\{A/\alpha^2, C/\beta^2\}\,, \\ \min\{f(x, y)\,|\,x, y \geqq 0, g(x, y) = 1\} &= \frac{AC - B^2}{A\beta^2 - 2B\beta\alpha + C\alpha^2}\,. \end{aligned}$$

Proof. Using A/α^2, $B/\beta^2 \geqq (AC - B^2)/(A\beta^2 - 2B\beta\alpha + C\alpha^2)$, we obtain our assertion by the Lagrange's method. Q.E.D.

Proof of Theorem 1. Suppose $0 < p \leqq 1$. Let α, β, A, B and C be as (4.6). It follows that

$$\begin{aligned}
&C\alpha - B\beta = rt^3E_1/(3 - rt^2)(3 + rt^2)\,, \qquad A\beta - B\alpha = rt^3E_2\,,\\
&A\beta^2 - 2B\beta\alpha + C\alpha^2 = \beta(A\beta - B\alpha) + \alpha(C\alpha - B\beta) = rt^3E/(3 - rt^2)\,,\\
&E = (3 - rt^2)^2E_2 + E_1\,,\\
&AC - B^2 = rt^3F/(3 - rt^2)(3 + rt^2)^2\,,
\end{aligned}$$

where

$$\begin{cases}
E_1 := 9E_{11} + E_{12}r^2t^4\,, \quad E_{11} := 9(1 + r) - 12rt - 9(1 + r)rt^2\,,\\
\qquad\qquad\qquad\qquad\quad\;\; E_{12} := 9(1+r) - 4rt\,,\\
E_2 := 9(1 + r) - 8rt - (1 + r)rt^2\,.
\end{cases}$$

If $0 < p < 1$ then $0 < r < 1$ and $E_{11}, E_{12}, E_2 > 0$ $(0 < t \leqq 1)$. Moreover $C\alpha - B\beta > 0$, $A\beta - B\alpha > 0$. Applying Lemma 4 to the above values, we obtain the desired formulas in the case $0 < p < 1$.

If $p = 1$ then $r = 0$. In this case we can prove our assertion directly from Lemma 3.

Suppose $p = 0$. Then $t = 1$ identically. But we know that $U_0(0, w) = -1/2$, $L_0(0, w) = -1$ (cf. Kobayashi [5], p. 40). Hence our assertion is valid also for $p = 0$. Q.E.D.

§ 5. Upper and lower curvatures of D_p

From Theorem 1 we induce some consequences.

THEOREM 2. *Let* $0 < p < 1$. *Then:*

(i) $\lim_{|w|\to 1} L_p(0, w) = \lim_{|w|\to 1} U_p(0, w) = -2/3$.

(ii) $L_p(0, w)$ *is strictly increasing with respect to* $|w|$.

(iii) $U_p(0, w)$ *is strictly decreasing with respect to* $|w|$.

Proof. (i): Obvious by Theorem 1.

(ii): If $0 < r < 1$ and $0 < t \leqq 1$ then

$$\frac{\partial}{\partial t}\left(\frac{6 - 6rt^2 + (1 + r)rt^3}{(3 - rt^2)^3}\right) = \frac{3rt^2(3(1 + r) - 8rt + (1 + r)rt^2)}{(3 - rt^2)^4} > 0\,,$$

$$\frac{\partial}{\partial t}\left(\frac{6 + 4rt^2 + (1 + r)rt^3}{(3 + rt^2)^2}\right) = \frac{rt^2(9(1 + r) - 8rt - (1 + r)rt^2)}{(3 + rt^2)^3} > 0\,.$$

It follows that $L_p(0, w)$ is strictly decreasing with respect to t.

(iii): If $0 < r < 1$ and $0 < t \leqq 1$ then

$$\frac{\partial}{\partial t}(F/(3+rt^2)^2E) = \left(\left(\frac{\partial F}{\partial t}E - F\frac{\partial E}{\partial t}\right)(3+rt^2) - 4rtEF\right)\Big/(3+rt^2)^3E^2$$
$$= rt^2M/(3+rt^2)^3E^2\,,$$

where

$$(5.1)\quad \begin{cases} M := 9^2M_1 + 9r^4t^7M_2 + 3r^6t^{11}M_3\,, \\ M_1 := -2\cdot 9^3(1+3r+3r^2+r^3) + 56\cdot 9^3(r+2r^2+r^3)t \\ \qquad + 5\cdot 9(45r+23r^2+23r^3+45r^4)t^2 - 48(123r^2+206r^3+123r^4)t^3 \\ \qquad - 3(141r^2-1385r^3-1385r^4+141r^5)t^5 + 48(13r^3-6r^4+13r^5)t^5 \\ \qquad + 6(31r^3+109r^4+109r^5+31r^6)t^6 - 32(9r^4+26r^5+5r^6)t^7\,, \\ M_2 := -32\cdot 9\cdot 4r^2 - 16\cdot 9(3-r-r^2+3r^3)t + 8(75+86r^2+75r^3)t^2 \\ \qquad + (21r-241r^2-241r^3+21r^4)t^3 - r^2t^4\,, \\ M_3 := (-45+32r-48r^2) + (3+25r+25r^2+3r^3)t\,. \end{cases}$$

But it can be proved that $M_1, M_2, M_3 < 0$ for $0 < r < 1$ and $0 < t \leqq 1$. As the authors' proof is tedious, we leave it in Appendices (Proposition A3 and Proposition A4). Admitting the above facts, we conclude our assertion by a similar proof to (ii). Q.E.D.

Instead of (iii) the following is more easily proved:

(iii′) $U_p(0, w) > -2/3$ for $|w| < 1$;

which we shall use in the following:

COROLLARY. *Let $0 < p < 1$. Let $H_p(z, w; u)$ be the holomorphic sectional curvature of the Bergman metric on D_p in a direction u at $(z, w) \in D_p$. Let $(\zeta, \omega) \in \partial D_p$. Then:*

(i) *If $\omega \neq 0$ then $\lim_{(z,w)\to(\zeta,\omega)} H_p(z, w; u) = -2/3$ uniformly in the directions u.*

(ii) *If $\omega = 0$ then there does not exist the uniform limit of $H_p(z, w; u)$ as $(z, w) \to (\zeta, 0)$.*

Proof. By Lemma 2 the image of the mapping $u \mapsto H_p(z, w; u)$ is the closed interval $[L_p(0, |w|(1-|z|^2)^{-p/2}),\ U_p(0, |w|(1-|z|^2)^{-p/2})]$.

(i): If $\omega \neq 0$ then $|w|(1-|z|^2)^{-p/2} \to 0$ as $(z, w) \to (\zeta, \omega)$, hence $\operatorname{Im} H_p(z, w; \cdot) \to \{-2/3\}$ as $(z, w) \to (\zeta, \omega)$ by (i), (ii) in Theorem 2 and (iii′).

(ii): If $\omega = 0$ and a complex sequence (z_j) satisfies $|z_j| < 1$, $z_j \to \zeta$ then $\operatorname{Im} H_p(z_j, 0; \cdot) = [L_p(0, 0), U_p(0, 0)] \supsetneqq \{-2/3\}$ by (i), (ii) in Theorem 2 and (iii′). Q.E.D.

Remark. If $D \subset C^n$ is a strongly pseudoconvex bounded domain with

C^∞ boundary and if $\hat{q} \in \partial D$ then $\lim_{q \to \hat{q}} H(q, u) = -2/(n+1)$ uniformly in the directions u (cf. Theorem 1 in Klembeck [3]). In our domain D_p, suppose $1/p$ be a positive integer. Then D_p is with C^∞ boundary and is strongly pseudoconvex at $(\zeta, \omega) \in \partial D_p$ if and only if $\omega \neq 0$. Corollary gives a counter example to the question whether the above theorem is valid under the assumption that D is pseudoconvex instead of strongly pseudoconvex.

As an immediate consequence of Theorem 2, we obtain:

THEOREM 3. *Let* $0 \leqq p \leqq 1$. *Then:*
(i) $\ell_p = L_p(0, 0) = -(1 + 4p + p^2)/(1 + 2p)^2$.
(ii) $u_p = U_p(0, 0) = -2(2 + 11p + 15p^2 + 8p^3)/(2+p)(1+3p)(4+5p)$.

Instead of (ii) the following is more easily proved:
(ii′) $u_p = \max\{U_p(0, w) \,|\, |w| < 1\} < 0$.

According to Proposition in the section 3, we obtain:

COROLLARY 1. *If* $0 \leqq p_1 < p_2 \leqq 1$, *then* $\ell_{p_1} < \ell_{p_2}$, *hence* D_{p_1} *is not biholomorphically equivalent to* D_{p_2}.

From (ii′) we have the following:

COROLLARY 2. *Let* $0 \leqq p \leqq 1$. *The holomorphic sectional curvature of the Bergman metric on* D_p *is strictly negative.*

Appendices

A1. Fourier's theorem concerning to the zeros of a polynomial

Set $\operatorname{sgn} c := c/|c|$, $c \in \boldsymbol{R} - \{0\}$. Let q be the number of the non-zero terms in a real finite sequence $(c_j)_{j=0}^{p}$. We define the number of changes of sign in (c_j) as follows:

$$V(c_0, \cdots, c_p) := \begin{cases} \sum_{j=1}^{q-1} (1 - \operatorname{sgn} c_{n_{j-1}} c_{n_j})/2\,, & q \geqq 2\,, \\ 0\,, & q = 0 \text{ or } 1\,, \end{cases}$$

where if $q \geqq 1$, $(c_{n_j})_{j=0}^{q-1}$ is the subsequence deleted the terms c_j with $c_j = 0$ (i.e. $n_0 = \min\{k \,|\, c_k \neq 0\}$, $n_j = \min\{k > n_{j-1} \,|\, c_k \neq 0\}$ $(1 \leqq j \leqq q-1)$).

Let $f \in \boldsymbol{R}[t] - \{0\}$, $c \in \boldsymbol{R}$ and $I \subset \boldsymbol{R}$ be an interval. We denote

$$V(c) := V_f(c) := V(f(c), f^{(1)}(c), \cdots, f^{(n)}(c)),\ n := \deg f\,;$$
$$NI := N_f I := \sum_{t \in I} (\text{the order of zero to } f \text{ at } t)\,.$$

The following theorem is well known:

FOURIER'S THEOREM ([1]). *Let $f \in \boldsymbol{R}[t] - \{0\}$ and a, $b \in \boldsymbol{R}$ with $a < b$. Then there is a non-negative integer ν such that*

$$N(a, b] = V(a) - V(b) - 2\nu .$$

As an immediate consequence of Fourier's Theorem we have:

PROPOSITION A1. *Let f, a and b be as in Fourier's Theorem. Then:*
(i) *If $V(a) = V(b)$, then f has no zero in $(a, b]$.*
(ii) *If $V(a) = V(b) + 1$, then f has only one simple zero in $(a, b]$.*

We shall use Proposition A1 in the following section.

A2. Negativity of M_j in the proof of Theorem 3

In this section we shall show that the functions M_j of the variables r and t defined by (5.1) are negative for $(r, t) \in (0, 1]^2$. First we can write

$$\frac{\partial M_1}{\partial t} = 6rN_1 + 4r^4t^5N_2 ,$$

where

$$\begin{cases} N_1 := 756(1 + 2r + r^2) + 15(45 + 23r + 23r^2 + 45r^3)t \\ \qquad - 24(123r + 206r^2 + 123r^3)t^2 - 2(141r - 1385r^2 - 1385r^3 + 141r^4)t^3 \\ \qquad + 40(13r^2 - 6r^3 + 13r^4)t^4 + 186r^2t^5 , \\ N_2 := 9(109 + 109r + 31r^2) - 56(9 + 26r + 5r^2)t . \end{cases}$$

PROPOSITION A2. *$N_1(r, t) > 0$ for $(r, t) \in (0, 1]^2$.*

Proof. Set $f_r(t) := N_1(r, t)$, $(r, t) \in (0, 1]^2$. We shall apply Proposition A1 to f_r and the interval $(0, 1]$. It follows that

$$\begin{aligned} f_r^{(j)}(0) &= j!\,(\text{the coefficient of } t^j \text{ in } f_r) ; \\ f_r(1) &= 1431 - 1377r - 367r^2 + 253r^3 + 238r^4 , \\ f_r^{(1)}(1) &= 675 - 6405r + 1777r^2 + 2121r^3 + 1234r^4 , \\ f_r^{(2)}(1) &= 12r(-633 + 1391r + 653r^2 + 379r^3) , \\ f_r^{(3)}(1) &= 12r(-141 + 3355r + 905r^2 + 899r^3) , \\ f_r^{(4)}(1) &= 240r^2(145 - 24r + 52r^2) , \\ f_r^{(5)}(1) &= 240 \cdot 93r^2 . \end{aligned}$$

Applying Proposition A1 to the polynomials $f_r^{(j)}(0)$, $f_r^{(j)}(1)$ of variable r and the interval (0, 1], we can see that $f_r(0)$, $f_r^{(1)}(0)$, $f_r^{(2)}(0)$, $f_r^{(4)}(0)$, $f_r^{(5)}(0)$, $f_r(1)$, $f_r^{(4)}(1)$ and $f_r^{(5)}(1)$ have no zero in (0, 1], while each of $f_r^{(3)}(0)$, $f_r^{(1)}(1)$, $f_r^{(2)}(1)$ and $f_r^{(3)}(1)$ has only one simple zero in (0, 1], say r_1, r_2, r_3 and r_4 respectively. Moreover we have

$$0 < r_4 < \frac{1}{20} < r_1 < \frac{1}{10} < r_2 < \frac{1}{5} < r_3 < 1$$

and the following tables of signs:

r	$0 \quad r_1 \quad 1$
$f_r(0)$	$: + : + :$
$f_r^{(1)}(0)$	$: + : + :$
$f_r^{(2)}(0)$	$0 - : - :$
$f_r^{(3)}(0)$	$0 - 0 + :$
$f_r^{(4)}(0)$	$0 + : + :$
$f_r^{(5)}(0)$	$0 + : + :$

Table 1.

r	$0 \quad r_4 \quad r_2 \quad r_3 \quad 1$
$f_r(1)$	$: + : + : + : + :$
$f_r^{(1)}(1)$	$: + : + 0 - : - :$
$f_r^{(2)}(1)$	$0 - : - : - 0 + :$
$f_r^{(3)}(1)$	$0 - 0 + : + : + :$
$f_r^{(4)}(1)$	$0 + : + : + : + :$
$f_r^{(5)}(1)$	$0 + : + : + : + :$

Table 2.

It follows from the tables that $V_{f_r}(0) = V_{f_r}(1) = 2$, $r \in (0, 1]$. Therefore f_r has no zero in (0, 1] for any $r \in (0, 1]$. Q.E.D.

PROPOSITION A3. *$M_1 < 0$ for $(r, t) \in (0, 1]^2$.*

Proof. It is easily seen that

$$N_2 \geqq N_2(r, 1) \geqq N_2(1, 1) = 1 . \tag{A2.1}$$

Proposition A2 and (A2.1) show that $M_1(r, t) \leqq M_1(r, 1)$, $(r, t) \in (0, 1]^2$. But we have

$$M_1(r, 1) = -2\cdot 9^3 + 3\cdot 9^3 r - 66\cdot 9r^2 - 10\cdot 9^2 r^3 + 354r^4 + 23r^5 + 26r^6 ;$$

therefore using Proposition A1, we obtain $M_1(r, 1) < 0$, $r \in (0, 1]$. Q.E.D.

Finally we consider M_2 and M_3. Set $g_r(t) := M_2(r, t)$, $(r, t) \in (0, 1]^2$. Then $V_{g_r}(0) = V_{g_r}(1)$, $r \in (0, 1]$. On the other hand, $M_3 \leqq M_3(r, 1) \leqq M_3(1, 1) = -5$. Therefore we have proved the following:

PROPOSITION A4. *M_2, $M_3 < 0$ for $(r, t) \in (0, 1]^2$.*

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