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HOLOMORPHIC AUTOMORPHISMS AND CANCELLATION THEOREMS

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§ 1. Statement of the result

In this note, complex analytic spaces are always assumed to be reduced and connected.

Let X be a complex analytic space of positive dimension and A a complex analytic subvariety of X. We call A a *direct factor* of X if there exist a complex analytic space B and a biholomorphic mapping $f: A \times B \to X$ such that, for some $b \in B$, $f(a, b) = a$ on A, and a complex analytic space X to be *primary* if X has no direct factor, not equal to X itself, of positive dimension. By a *primary decomposition* of X, we mean a cartesian product $X_1 \times X_2 \times \cdots \times X_n$ of primary complex analytic spaces $X_1, X_2, \cdots, X_n$ of positive dimension, such that $X_1 \times X_2 \times \cdots \times X_n$ is biholomorphic to X. We shall give examples of primary decomposition in § 7.

Now, consider the following condition (C) for an arbitrary complex analytic space X:

(C) Given arbitrarily a complex analytic space Y and a holomorphic mapping ϕ: $Y \times X \to X$, if $\phi(y_0, \cdot)$: $X \to X$ is a biholomorphic mapping for some $y_0 \in Y$ then

$$\phi(y, \cdot) = \phi(y_0, \cdot) \qquad \text{on } X \text{ for every } y \in Y\,.$$

We denote by C the collection of all complex analytic spaces which satisfy the condition (C).

Throughout this note we are concerned with two classes of complex analytic spaces, (1) hyperbolic complex analytic spaces in the sense of Kobayashi [4] and (2) compact complex analytic spaces of general type (see the following, extracted from Ueno [10]).

Let M be a compact complex analytic manifold with the canonical line bundle $K(M)$. The Kodaira dimension $\kappa(M)$ of M is equal to an in-

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teger $\kappa(0 \leqslant \kappa \leqslant \dim_C M)$ provided that

$$0 < \limsup_{m \to +\infty} \frac{\dim_C H^0(M, mK(M))}{m^\kappa} < \infty ,$$

or $-\infty$ otherwise. Note that the Kodaira dimension is a bimeromorphic invariant of compact complex analytic manifolds. For a compact irreducible complex analytic space X, the Kodaira dimension $\kappa(X)$ of X is defined as the Kodaira dimension of any desingularization of X. A compact complex analytic space X is called of general type (or of hyperbolic type in the terminology of [10]), if X is irreducible and $\kappa(X) = \dim_C X$.

We shall show the following in § 2.

PROPOSITION 1 (cf. Royden [9]). *Every hyperbolic complex analytic space is contained in C.*

PROPOSITION 2. *Every compact complex analytic space of general type is contained in C.*

PROPOSITION 3. *Let X and Y be complex analytic spaces. Then the product $X \times Y$ is contained in C if and only if X, Y are contained in C.*

And, we shall prove in § 4

THEOREM 1. *Take an arbitrary complex analytic space X of positive dimension, which is a member of C. Then X possesses a unique primary decomposition $X_1 \times \cdots \times X_n$. Here, the uniqueness means that if $Y_1 \times \cdots \times Y_m$ is another primary decomposition of X then* (1) $m = n$ *and* (2) *for any biholomorphic mapping f of $X_1 \times \cdots \times X_n$ to $Y_1 \times \cdots \times Y_n$ there are biholomorphic mappings $f_i: X_i \to Y_i$ $(i = 1, \cdots, n)$ such that $f = f_1 \times \cdots \times f_n$ on $X_1 \times \cdots \times X_n$ after a suitable reordering of $Y_1, \cdots, Y_n$.*

COROLLARY 1. *Let X, Y and V be complex analytic spaces of the collection C. If $V \times X$ is biholomorphic to $V \times Y$, then X is biholomorphic to Y.*

For an arbitrary complex analytic space X, let $\mathrm{Aut}(X)$ denote the group of all biholomorphic mappings of X to X itself.

Let X be a complex analytic space of the collection C and assume that

$$X = \underbrace{X_1 \times \cdots \times X_1}_{n_1 \text{ terms}} \times \underbrace{X_2 \times \cdots \times X_2}_{n_2 \text{ terms}} \times \cdots \times \underbrace{X_p \times \cdots \times X_p}_{n_p \text{ terms}}$$

$(n_1 + n_2 + \cdots + n_p = n)$ for primary complex analytic spaces $X_1, X_2, \cdots, X_p$ which are not biholomorphic to each other. Consider the subgroup G of $\mathrm{Aut}(X)$ which is induced naturally by permutations, namely, $\sigma \in G$ if and only if

$$\sigma(x_1, x_2, \cdots, x_n) = (x_{\sigma_1}, x_{\sigma_2}, \cdots, x_{\sigma_n}) \qquad \text{on } X$$

for some permutation $(\sigma_1, \sigma_2, \cdots, \sigma_n)$ of $(1, 2, \cdots, n)$ such that

$$\{\sigma_{n_1+\cdots+n_{i-1}+1}, \cdots, \sigma_{n_1+\cdots+n_i}\} = \{n_1 + \cdots + n_{i-1} + 1, \cdots, n_1 + \cdots + n_i\} \qquad (1 < i \leqslant p)\,.$$

Obviously G is group-isomorphic to a direct product of some symmetric groups. Immediately from Theorem 1, we obtain

COROLLARY 2. *For each* $f \in \mathrm{Aut}(X)$, *there exist uniquely* $f_{n_1+\cdots+n_{i-1}+1}, \cdots, f_{n_1+\cdots+n_i} \in \mathrm{Aut}(X_i)$ $(i = 1, \cdots, p)$ *and* $\sigma \in G$ *such that*

$$f = \sigma \circ (f_1 \times \cdots \times f_n) \qquad \text{on } X\,.$$

This means that $\mathrm{Aut}(X)$ *is a semidirect product of* G *and a normal subgroup*

$$\underbrace{\mathrm{Aut}(X_1) \times \cdots \times \mathrm{Aut}(X_1)}_{n_1\ terms} \times \underbrace{\mathrm{Aut}(X_2) \times \cdots \times \mathrm{Aut}(X_2)}_{n_2\ terms} \times \cdots \times \underbrace{\mathrm{Aut}(X_p) \times \cdots \times \mathrm{Aut}(X_p)}_{n_p\ terms}$$

of $\mathrm{Aut}(X)$.

Note that the above Corollary 2 implies Satz 3.4 of Peters [8], which is a generalization of a theorem of H. Cartan. We shall prove the following cancellation theorems in § 5 and § 6.

THEOREM 2. *Let* X, Y *and* V *be complex analytic spaces such that* $V \times X$ *is biholomorphic to* $V \times Y$. *If* V *is hyperbolic, then* X *is biholomorphic to* Y.

THEOREM 3. *Let* V *be a compact complex analytic space of general type. Suppose that* X *and* Y *are compact irreducible complex analytic spaces such that* $V \times X$ *is biholomorphic to* $V \times Y$. *Then* X *is biholomorphic to* Y.

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§ 2. Proofs of Propositions

Proof of Proposition 1. Let X be a hyperbolic complex analytic space. Take arbitrarily a complex analytic space Y and a holomorphic mapping $\phi\colon Y \times X \to X$ such that $\phi(y_0, \cdot) \in \mathrm{Aut}(X)$ for some $y_0 \in Y$. We have to prove that $\phi(y, \cdot) = \phi(y_0, \cdot)$ on X for every $y \in Y$. Since Y is connected, every two points of Y can be connected by a chain of holomorphic mappings of the unit open disc D (in the complex line $\boldsymbol{C}$) into Y (cf. Kobayashi [4], p. 97). Hence it suffices to prove the above for the case $Y = D$. Suppose that $Y = D$. Without loss of generality, we may assume that $\phi(z_0, \cdot) = \mathit{identity}$ on X for some $z_0 \in D$. Let S be the set of all singular points of X and take an arbitrary point x_0 of $X - S$. Define the holomorphic mappings ϕ_n $(n = 1, 2, \cdots)$ of D into X as follows:

$$\phi_1(z) = \phi(z, x_0) \quad \text{and} \quad \phi_{n+1}(z) = \phi(z, \phi_n(z)) \qquad \text{on } D \ (n = 1, 2, \cdots)\,.$$

Obviously, $\phi_n(z_0) = x_0$ for $n = 1, 2, \cdots$. Now, fix holomorphic local coordinates at x_0 in X. Since X is hyperbolic, the family $\{\phi_n\}_{n>0}$ is equicontinuous on D. This implies that, for every positive integer ℓ, there exists a positive constant A_ℓ such that

$$\left\| \frac{d^\ell \phi_n}{dz^\ell}(z_0) \right\| \leqslant A_\ell \qquad (n = 1, 2, \cdots)\,.$$

On the other hand, since $\phi(z_0, x) = x$ on X, the mapping ϕ has a power series expansion of the form

$$\phi(z, x) = x + a(x)(z - z_0)^\ell + O((z - z_0)^{\ell+1})$$

near the point (z_0, x_0) in $D \times X$. Then ϕ_n has the expansion

$$\phi_n(z) = x_0 + na(x_0)(z - z_0)^\ell + O((z - z_0)^{\ell+1})$$

near z_0 in D $(n = 1, 2, \cdots)$. Thus we see that

$$\ell!\, n \,\| a(x_0) \| = \left\| \frac{d^\ell \phi_n}{dz^\ell}(z_0) \right\| \leqslant A_\ell \qquad (n = 1, 2, \cdots)\,.$$

Hence $a(x_0) = 0$. This means that $\phi(z, x) = x$ locally at (z_0, x_0) in $D \times X$. Since D is connected and S is nowhere dense in X, we see that $\phi(z, x) = x$ on $D \times X$. This completes the proof.

Proof of Proposition 2. Let X be a compact complex analytic space of general type. Take arbitrarily a complex analytic space Y and a holomorphic mapping $\phi: Y \times X \to X$ such that $\phi(y_0, \cdot) \in \mathrm{Aut}(X)$ for some $y_0 \in Y$. We have to prove that $\phi(y, \cdot) = \phi(y_0, \cdot)$ on X for every $y \in Y$. Let $\mathrm{Hol}(X, X)$ be the set of all holomorphic mappings of X into X and consider the mapping

$$\tilde{\phi}: Y \to \mathrm{Hol}(X, X)$$

defined by the formula $\tilde{\phi}(y) = \phi(y, \cdot) \in \mathrm{Hol}(X, X)$ for each $y \in Y$. Obviously, $\tilde{\phi}$ is a continuous mapping into the space $\mathrm{Hol}(X, X)$ equipped with the compact-open topology. It is well known that $\mathrm{Aut}(X)$ is open in $\mathrm{Hol}(X, X)$. On the other hand, by Corollary 14.3 of Ueno [10], $\mathrm{Aut}(X)$ is finite. Hence each member of $\mathrm{Aut}(X)$ is isolated in $\mathrm{Hol}(X, X)$. This implies that $\tilde{\phi}$ is constant, i.e., $\phi(y, \cdot) = \phi(y_0, \cdot)$ on X for every $y \in Y$. This completes the proof.

Proof of Proposition 3. Suppose that X and Y are contained in C. Take arbitrarily a complex analytic space S and a holomorphic mapping $\phi: S \times X \times Y \to X \times Y$ such that $\phi(s_0, \cdot, \cdot) \in \mathrm{Aut}(X \times Y)$ for some $s_0 \in S$. Then we have the holomorphic mapping

$$\psi = \phi(s_0, \cdot, \cdot)^{-1} \circ \phi: S \times X \times Y \to X \times Y .$$

Write $\psi = (\alpha, \beta): S \times X \times Y \to X \times Y$, where α and β are mappings from $S \times X \times Y$ into X and Y respectively. Since $\psi(s_0, \cdot, \cdot) = \textit{identity}$ on $X \times Y$,

$$\begin{aligned} &\alpha(s_0, \cdot, y) = \textit{identity} \text{ on } X \qquad \text{for any } y \in Y \text{ and} \\ &\beta(s_0, x, \cdot) = \textit{identity} \text{ on } Y \qquad \text{for any } x \in X . \end{aligned}$$

This implies that $\alpha(s, x, y) = \alpha(s_0, x, y)$ and $\beta(s, x, y) = \beta(s_0, x, y)$ for any $(s, x, y) \in S \times X \times Y$, because X and Y are contained in C. Thus we see that $\psi(s, x, y) = \psi(s_0, x, y) = (x, y)$ and hence $\phi(s, x, y) = \phi(s_0, x, y)$ for any $(s, x, y) \in S \times X \times Y$. This means that $X \times Y$ is also contained in C.

Conversely, suppose that $X \times Y$ is contained in C. Take arbitrarily a complex analytic space S and a holomorphic mapping $\phi: S \times X \to X$ such that $\phi(s_0, \cdot) \in \mathrm{Aut}(X)$ for some $s_0 \in S$. Let us define the holomorphic mapping $\psi: S \times X \times Y \to X \times Y$ by the formula

$$\psi(s, x, y) = (\phi(s, x), y) \qquad \text{for any } (s, x, y) \in S \times X \times Y .$$

Obviously, $\psi(s_0, \cdot, \cdot) \in \mathrm{Aut}(X \times Y)$. By the assumption, $\psi(s, x, y) = \psi(s_0, x, y)$ for any $(s, x, y) \in S \times X \times Y$. Hence $\phi(s, x) = \phi(s_0, x)$ for any $(s, x) \in S \times X$. This means that X is contained in C. Similarly we see that Y is contained in C. This completes the proof.

§ 3. Lemmas

In this section we fix the situation as follows:

Let X, Y, V and W be complex analytic spaces. Let $h=(\phi,\psi): X\times V\to Y\times W$ be a biholomorphic mapping and write $h^{-1}=(\alpha,\beta): Y\times W\to X\times V$. Take an arbitrary point x_0 of X and put $Y'=\phi(x_0, V):=\{\phi(x_0, v); v\in V\}$ $(\subset Y)$ and $W'=\psi(x_0, V)$ $(\subset W)$.

LEMMA 1. *Suppose that X is contained in C. Then we have the following:*
(1) *Y' (resp. W') is a complex analytic subvariety of Y (resp. W) and $h(x_0, V)=Y'\times W'$ in $Y\times W$; hence $h(x_0,\cdot): V\to Y'\times W'$ is biholomorphic.*
(2) *$\phi(x_0, \beta(y, w))=y$ for any $(y, w)\in Y'\times W'$.*
(3) *$\psi(x_0, \beta(y, w))=w$ for any $(y, w)\in Y'\times W'$.*
(4) *If Y (resp. W) is contained in C, then W' (resp. Y') is a direct factor of W (resp. Y).*

Proof. (1) Obviously,

$$h(x_0, V)=\{(\phi(x_0, v), \psi(x_0, v)); v\in V\}\subset Y'\times W' .$$

Consider the holomorphic mapping $\pi: (V\times V)\times X\to X$ defined as $\pi((v, w), x)=\alpha(\phi(x, v), \psi(x, w))$ for each $((v, w), x)\in(V\times V)\times X$. Evidently, $\pi((v, v), \cdot): X\to X$ is the identity mapping on X for any $v\in V$. Since X is contained in C, we see that $\alpha(\phi(x, v), \psi(x, w))=x$ for all $x\in X$ and $v, w\in V$. This means that $h^{-1}(Y'\times W')\subset\{x_0\}\times V$ in $X\times V$. Hence $h(x_0, V)=Y'\times W'$. Since $h: X\times V\to Y\times W$ is biholomorphic, $Y'\times W'$ is a complex analytic subvariety of $Y\times W$ and then Y' (resp. W') is a complex analytic subvariety of Y (resp. W). Clearly, $h(x_0,\cdot): V\to Y'\times W'$ is biholomorphic.
(2) Take any $(y, w)\in Y'\times W'$. Then $\alpha(y, w)=x_0$ and so

$$\phi(x_0, \beta(y, w))=\phi(\alpha(y, w), \beta(y, w))=y .$$

(3) Similarly, for any $(y, w)\in Y'\times W'$,

$$\psi(x_0, \beta(y, w))=w .$$

(4) Suppose that Y is contained in C. Consider the complex analytic space $X'=X\times Y'$ and the biholomorphic mapping γ of $X\times V$ to $X'\times W'$ defined by the formula

$$\gamma(x, v)=((x, \bar{\phi}(v)), \bar{\psi}(v))\in X'\times W' \qquad \text{for each } (x, v)\in X\times V ,$$

where $\bar{\phi} = \phi(x_0, \cdot): V \to Y'$ and $\bar{\psi} = \psi(x_0, \cdot): V \to W'$. Then $h' = \gamma \circ h^{-1}: Y \times W \to X' \times W'$ is biholomorphic. Write $h' = (\phi', \psi'): Y \times W \to X' \times W'$. Then $\psi'(y, W) = \bar{\psi}(\beta(y, W)) = W'$ for any $y \in Y'$ by (3) above. By (1) applied to $h': Y \times W \to X' \times W'$, $h'(y, \cdot): W \to W'' \times W'$ $(\subset X' \times W')$ is biholomorphic for an arbitrarily fixed point y of Y', where $W'' = \phi'(y, W)$. Furthermore

$$h'(y, w) = ((x_0, y), w) \qquad \text{for all } w \in W' ,$$

because

$$\phi'(y, w) = (\alpha(y, w), \phi(x_0, \beta(y, w))) = (x_0, y)$$

and

$$\psi'(y, w) = \psi(x_0, \beta(y, w)) = w \qquad \text{for all } w \in W'$$

by (2), (3) above. This means that W' is a direct factor of W. If W is contained in C, then we see that Y' is a direct factor of Y by applying the above argument to the biholomorphic mapping $(\psi, \phi): X \times V \to W \times Y$. This completes the proof.

LEMMA 2. *Suppose that X, Y, V and W are contained in C and that $\psi(x_0, \cdot): V \to W$ is biholomorphic. Then* (1) $\psi(x, \cdot) = \psi(x_0, \cdot)$ *on V for all $x \in X$ and* (2) $\phi(\cdot, v) = \phi(\cdot, w)$ *on X for all $v, w \in V$ and, moreover, $\phi(\cdot, v)$: $X \to Y\,(v \in V)$ is biholomorphic.*

Proof. Consider the holomorphic mapping $\psi(x_0, \cdot)^{-1} \circ \psi: X \times V \to V$. Then we see easily that $\psi(x, \cdot) = \psi(x_0, \cdot)$ on V for all $x \in X$, because V is contained in C. Hence

$$h(x, v) = (\phi(x, v), \psi(x_0, v)) \qquad \text{for all } (x, v) \in X \times V .$$

This implies that $h(\cdot, v): X \to Y \times \{\psi(x_0, v)\}$ $(\subset Y \times W)$ is biholomorphic for every $v \in V$, i.e., $\phi(\cdot, v): X \to Y$ is biholomorphic for every $v \in V$. Since X and Y are contained in C, we see easily that $\phi(\cdot, v) = \phi(\cdot, w)$ on X for all $v, w \in V$.

LEMMA 3. *Suppose that X and Y are contained in C and that V and W are primary. If $\psi(x_0, \cdot): V \to W$ is non-constant, then $\psi(x_0, \cdot): V \to W$ is biholomorphic.*

Proof. By Lemma 1, $h(x_0, \cdot): V \to Y' \times W'$ is biholomorphic, where $Y' = \phi(x_0, V)$ $(\subset Y)$ and $W' = \psi(x_0, V)$ $(\subset W)$. Note that $\dim_C W' > 0$, because $\psi(x_0, \cdot)\colon V \to W$ is non-constant. Since V is primary and $\dim_C W' > 0$, necessarily $\dim_C Y' = 0$. This means that $\phi(x_0, \cdot)\colon V \to Y$ is constant and $\psi(x_0, \cdot)\colon V \to W'$ is biholomorphic. Now, by Lemma 1, W' is a direct factor of W. Since W is primary, we conclude that $W' = W$. Hence $\psi(x_0, \cdot)\colon V \to W$ is biholomorphic.

§4. Proofs of Theorem 1 and Corollary 1

Proof of Theorem 1. Let X be a complex analytic space of the collection C and let $X_1 \times \cdots \times X_n$ be a primary decomposition of X. Take another primary decomposition $Y_1 \times \cdots \times Y_m$ of X. Since $X_1 \times \cdots \times X_n$ and $Y_1 \times \cdots \times Y_m$ are biholomorphic to X, they are contained in C. Hence, by Proposition 3, $X_1, \cdots, X_n$, $Y_1, \cdots, Y_m$, $X_1 \times \cdots \times X_i$ $(i = 2, \cdots, n)$ and $Y_1 \times \cdots \times Y_j$ $(j = 2, \cdots, m)$ are also contained in C. Take an arbitrary biholomorphic mapping

$$f = (f_1, \cdots, f_m)\colon X_1 \times \cdots \times X_n \to Y_1 \times \cdots \times Y_m .$$

Then, after a suitable reordering of $Y_1, \cdots, Y_m$, we may assume that $f_m(x_1, \cdots, x_{n-1}, \cdot)\colon X_n \to Y_m$ is non-constant on X_n for some $(x_1, \cdots, x_{n-1})$ of $X_1 \times \cdots \times X_{n-1}$. By Lemma 3, we see that $f_m(x_1, \cdots, x_{n-1}, \cdot)\colon X_n \to Y_m$ is biholomorphic. Furthermore, by Lemma 2, $f_m(x_1, \cdots, x_{n-1}, \cdot)\colon X_n \to Y_m$ is independent of $(x_1, \cdots, x_{n-1})$ in $X_1 \times \cdots \times X_{n-1}$ and

$$(f_1(\cdot, x_n), \cdots, f_{m-1}(\cdot, x_n))\colon X_1 \times \cdots \times X_{n-1} \to Y_1 \times \cdots \times Y_{m-1}$$

is a biholomorphic mapping which is independent of x_n in X_n. By these argument, Theorem 1 is easily proved by induction on n.

Proof of Corollary 1. Suppose that $V \times X$ is biholomorphic to $V \times Y$ for complex analytic spaces X, Y and V contained in C. Let $X_1 \times \cdots \times X_\ell$, $Y_1 \times \cdots \times Y_m$ and $V_1 \times \cdots \times V_n$ be the primary decompositions of X, Y and V, respectively. Obviously, $V_1 \times \cdots \times V_n \times X_1 \times \cdots \times X_\ell$ and $V_1 \times \cdots \times V_n \times Y_1 \times \cdots \times Y_m$ are primary decompositions of $V \times X$. Note that $V \times X$ is contained in C by Proposition 3. Hence, by Theorem 1, $\ell = m$ and $\{X_1, \cdots, X_\ell\}$ is equal to $\{Y_1, \cdots, Y_\ell\}$ as sets of biholomorphic classes of complex analytic spaces. Then $X_1 \times \cdots \times X_\ell$ is biholomorphic to $Y_1 \times \cdots \times Y_\ell$, i.e., X is biholomorphic to Y.

§ 5. Proof of Theorem 2

Let V be a hyperbolic complex analytic space and let X, Y be complex analytic spaces such that $V \times X$ is biholomorphic to $V \times Y$. Decompose X and Y by complex analytic spaces V', V'', A and B so that the following hold:

(1) X is biholomorphic to $V' \times A$ and Y is biholomorphic to $V'' \times B$.

(2) V' and V'' are hyperbolic.

(3) A and B have no direct factor of positive dimension which is hyperbolic.

Consider the hyperbolic complex analytic spaces $C = V \times V'$ and $D = V \times V''$. Then we have a biholomorphic mapping $h = (\phi, \psi): C \times A \to D \times B$. Take any point x of C. By Lemma 1, we have the following:

(4) $D' = \phi(x, A)$ and $B' = \psi(x, A)$ are complex analytic subvarieties of D and B, respectively.

(5) $h(x, \cdot): A \to D' \times B'$ is biholomorphic.

Since the subvariety D' of the hyperbolic complex analytic space D is hyperbolic, A has a direct factor which is hyperbolic. Hence it is necessary that $\dim_C D' = 0$. This means that $\phi(x, \cdot): A \to D$ is constant for any $x \in C$. Now, write $h^{-1} = (\alpha, \beta)$ on $D \times B$ by holomorphic mappings $\alpha: D \times B \to C$ and $\beta: D \times B \to A$. Then, by the similar argument above, $\alpha(z, \cdot): B \to C$ is constant for any $z \in D$. Thus we have, for arbitrarily fixed points $y_0 \in A$ and $w_0 \in B$, $\alpha(\phi(x, y_0), w_0) = x$ on C and $\phi(\alpha(z, w_0), y_0) = z$ on D. This means that C is biholomorphic to D. Hence V' is biholomorphic to V'' by Corollary 1. Take any $x_0 \in C$ and put $\{z_1\} = \phi(x_0, A) \subset D$. Then $\beta(z_1, \psi(x_0, y)) = y$ on A and $\psi(x_0, \beta(z_1, w)) = w$ on B, because $x_0 = \alpha(\phi(x_0, y_0), w_0) = \alpha(z_1, w_0) = \alpha(z_1, w)$ for any $w \in B$. This means that A is biholomorphic to B. Hence $V' \times A$ is biholomorphic to $V'' \times B$, i.e., X is biholomorphic to Y.

§ 6. Proof of Theorem 3

Decompose X and Y by compact irreducible complex analytic spaces V', V'', X_1 and Y_1 so that the following hold:

(1) X is biholomorphic to $V' \times X_1$ and Y is biholomorphic to $V'' \times Y_1$.

(2) V' and V'' are of general type.

(3) X_1 and Y_1 have no direct factor of positive dimension which is of general type.

Put $V_1 = V \times V'$ and $V_2 = V \times V''$. Then V_1 and V_2 are of general type (cf. Ueno [10], p. 69). Since $V \times X$ is biholomorphic to $V \times Y$, we have a biholomorphic mapping $h = (\phi, \psi)\colon V_1 \times X_1 \to V_2 \times Y_1$. Put $V_v = \phi(v, X_1)$ $(\subset V_2)$ and $Y_v = \psi(v, X_1)$ $(\subset Y_1)$ for each $v \in V_1$. By Lemma 1, $h(v, \cdot)\colon X_1 \to V_v \times Y_v$ is biholomorphic for every $v \in V_1$. Write $h^{-1} = (\alpha, \beta)\colon V_2 \times Y_1 \to V_1 \times X_1$. Now, take any $y_0 \in Y_1$ and put $S = \alpha(V_2, y_0)$. Then S is a compact irreducible complex analytic subvariety of V_1. Consider the holomorphic surjection $\bar{\alpha} = \alpha(\cdot, y_0)\colon V_2 \to S$. Then we have the following assertion:

$$\bar{\alpha}^{-1}(v) = V_v \qquad \text{for any } v \in S\,.$$

Proof of the above assertion. Take any $v' \in \bar{\alpha}^{-1}(v)$, where $v \in S$. Then $\alpha(v', y_0) = v$ and hence $(v', y_0) \in h(v, X_1) = V_v \times Y_v$. Hence $v' \in V_v$. Conversely assume that $v' \in V_v$ for some $v \in S$. Since $v = \alpha(v'', y_0)$ for some $v'' \in V_2$, $y_0 = \psi(v, \beta(v'', y_0)) \in Y_v$. Hence $(v', y_0) \in V_v \times Y_v = h(v, X_1)$. Then $\alpha(v', y_0) = v$, i.e., $v' \in \bar{\alpha}^{-1}(v)$. This completes the proof of the assertion.

Now, put $n = \operatorname{Min}\{\dim_C \bar{\alpha}^{-1}(v); v \in S\}$. Since $\bar{\alpha}^{-1}(v) = V_v = \phi(v, X_1)$ is irreducible for every $v \in S$, we see easily that $U = \{v \in S; \dim_C V_v = n\}$ is open dense in S (cf. Proposition 7 of Holmann [2]). Furthermore, $\dim_C V_2 = n + \dim_C S$ by the maximal rank theorem (cf. Narasimhan [12], Chapter VII). On the other hand, by Theorem 6.12 of Ueno [10], there exists an open dense set U' of S such that

$$\kappa(V_2) \leqslant \kappa(\bar{\alpha}^{-1}(v)) + \dim_C S = \kappa(V_v) + \dim_C S$$

for any $v \in U'$. Note that $U \cap U'$ is also open dense in S. Since V_2 is of general type, i.e., $\kappa(V_2) = \dim_C V_2$, we obtain

$$\kappa(V_v) = n = \dim_C V_v$$

for any $v \in U \cap U'$ because of the general fact that $\kappa(V_v) \leqslant \dim_C V_v$ $(v \in V_1)$. Hence V_v is of general type for any $v \in U \cap U'$. Since X_1 is biholomorphic to $V_v \times Y_v$ $(v \in V_1)$ and it has no direct factor of positive dimension which is of general type, we obtain $n = 0$ necessarily. This means that $\phi(v, \cdot)\colon X_1 \to V_2$ is constant for every $v \in U$. In fact $\phi(v, \cdot)\colon X_1 \to V_2$ is constant for every $v \in S$, because U is dense in S. Furthermore, $\dim_C V_2 = \dim_C S \leqslant \dim_C V_1$. By applying the above argument to $h^{-1}\colon V_2 \times Y_1 \to V_1 \times X_1$, we have $\dim_C V_1 \leqslant \dim_C V_2$. Hence $\dim_C V_1 = \dim_C V_2 = \dim_C S$. By the irreducibility of V_1, we see that $S = V_1$. Consequently $\phi(v, \cdot)\colon X_1 \to V_2$ is constant for every $v \in V_1$. Hence the biholomorphic mapping h has a representation

$$h(v, x) = (\phi_1(v), \psi(v, x))$$

for each $(v, x) \in V_1 \times X_1$, where $\phi_1\colon V_1 \to V_2$. Similarly, h^{-1} has a representation

$$h^{-1}(w, z) = (\alpha_1(w), \beta(w, z))$$

for each $(w, z) \in V_2 \times Y_1$, where $\alpha_1\colon V_2 \to V_1$. We see easily that V_1 is biholomorphic to V_2 and X_1 is biholomorphic to Y_1. Then, by Corollary 1, V' is biholomorphic to V''. Hence $V' \times X_1$ is biholomorphic to $V'' \times Y_1$, i.e., X is biholomorphic to Y.

§ 7. Example

In this section complex analytic manifolds are always paracompact and connected. Let M be a complex analytic manifold of complex dimension n. Consider the complex vector space H of all holomorphic n-forms f on M such that

$$\int_M (\sqrt{-1})^{n^2} f \wedge \bar{f} < \infty .$$

Then H is a Hilbert space with the inner product

$$(f, g) = (\sqrt{-1})^{n^2} \int_M f \wedge \bar{g} \qquad \text{for } f, g \in H .$$

Further we know that H is a separable Hilbert space. Hence we have the Bergman kernel form K_M defined on $M \times M$ (cf. Kobayashi [6], Lichnerowicz [11]). Suppose that

(A.1) Given any point z of M, there exists an $f \in H$ such that $f(z) \neq 0$. Then, associated with the Bergman kernel form K_M on $M \times M$, we have the positive semidefinite quadratic form ds_M^2 on M which is invariant under the holomorphic automorphism group $\mathrm{Aut}(M)$ of M (see [6]). The complex analytic manifold M is called a Kobayashi manifold if the form ds_M^2 is positive definite on M; then ds_M^2 is called the Bergman metric on M. It has been proved in [6] that, for any complex analytic manifolds M, N of complex dimension m, n respectively,

$$K_{M \times N} = (-1)^{mn} K_M \wedge K_N \qquad \text{on } (M \times N) \times (M \times N) .$$

Take any complex analytic manifolds M and N such that $M \times N$ satisfies the condition (A.1) above. Using the Fubini's theorem we see that M and N satisfy the condition (A.1). Then we have

$$ds^2_{M\times N} = ds^2_M + ds^2_N \qquad \text{on } M \times N\,.$$

Hence M and N are Kobayashi manifolds, if $M \times N$ is a Kobayashi manifold. In fact, the converse is also true (cf. [6]). We obtained that if M is a Kobayashi manifold with a primary decomposition $M_1 \times \cdots \times M_m$ then

$$(M, ds^2_M) = (M_1, ds^2_{M_1}) \times \cdots \times (M_m, ds^2_{M_m})\,.$$

By the uniqueness of the (Kähler) de Rham decomposition of Kähler manifolds (cf. Kobayashi and Nomizu [7]), we can conclude that, for any simply connected complete Kobayashi manifold (M, ds^2_M), primary decompositions of M are given by the (Kähler) de Rham decompositions of the Kähler manifold (M, ds^2_M).

EXAMPLE 1. Let D be a homogeneous bounded domain of C^m (the cartesian product of the complex line C). Then the Bergman metric of D is complete, and D is simply connected (cf. [3]). Kaneyuki [3] proved that the (Kähler) de Rham decomposition of D is given uniquely by the product of irreducible homogeneous bounded domains $D_1, \cdots, D_n$ up to isometries. Since D is hyperbolic (cf. Kobayashi [4]), by Theorem 1 we see that $D_1 \times \cdots \times D_n$ is the unique primary decomposition of D. Furthermore, if each D_i is not biholomorphic to D_j ($i \neq j = 1, \cdots, n$), then $\mathrm{Aut}(D) = \mathrm{Aut}(D_1) \times \cdots \times \mathrm{Aut}(D_n)$ by Corollary 2.

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