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IRREDUCIBILITY OF SOME UNITARY REPRESENTATIONS OF THE POINCARÉ GROUP WITH RESPECT TO THE POINCARÉ SUBSEMIGROUP, I

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§ 1. Introduction

Since E. Wigner set up a framework of the relativistically covariant quantum mechanics, several aspects of unitary representations of the Poincaré group have been investigated (see [8], [16]). In this paper it will be shown that some unitary representations of the Poincaré group are irreducible, even if they are restricted to the Poincaré semigroup (Theorem 1, 2 and 3). As a result of the argument we shall also give the irreducible decomposition of induced representations $\operatorname{Ind}_{SU(1,1)\uparrow SL(2,C)} \pi$ (see § 3, cf. [3]). Here the Poincaré group P means a semi-direct product between R_4 and $SL(2,C)$ with the multiplication

$$(x,g)(x',g') = (x + g^{-1*}x'g^{-1}, gg') \qquad \text{for } x, x' \in \boldsymbol{R}_4 \text{ and } g, g' \in SL(2,\boldsymbol{C}),$$

where $x = (x_0, x_1, x_2, x_3)$ is identified with the matrix $\begin{pmatrix} x_0 - x_3 & x_2 - ix_1 \\ x_2 + ix_1 & x_0 + x_3 \end{pmatrix}$ and g^* shows the adjoint of the matrix g. The Poincaré semigroup P_+ is the subsemigroup $\{(x,g) \in P : x_0^2 - x_1^2 - x_2^2 - x_3^2 \geqslant 0, x_0 \geqslant 0\}$.

We have not yet succeeded in proving that any irreducible unitary representations of P are irreducible with respect to P_+, but in a lower dimensional case we have the following.

THEOREM 1. *Every irreducible unitary representation of the 2-dimensional space-time Poincaré group $P(2)$ is irreducible too as the representation restricted to its Poincaré subsemigroup. Here $P(2)$ is the semi-direct product between $\boldsymbol{R}_2$ and $\left\{\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \boldsymbol{R}\right\}$ with the same multiplication as P under the identification $(x_0, x_3) \to \begin{pmatrix} x_0 - x_3 & 0 \\ 0 & x_0 + x_3 \end{pmatrix}$.*

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The semigroup is just $\{(x, g): x_0^2 - x_3^2 \geqslant 0, x_0 \geqslant 0\}$.

§ 2. Main theorems

Let us define a bilinear form $\langle\ ,\ \rangle$ between $\boldsymbol{R}_4$ and $\hat{\boldsymbol{R}}_4$ by $\langle x, \hat{x}\rangle = x_0\hat{x}_0 - x_1\hat{x}_1 - x_2\hat{x}_2 - x_3\hat{x}_3$. By abuse of symbol, $\langle\ ,\ \rangle$ stands also for the similar bilinear form on $\boldsymbol{R}_4$ or $\hat{\boldsymbol{R}}_4$. Defining the action of $G = SL(2, \boldsymbol{C})$ on $\hat{\boldsymbol{R}}_4$ by $x\cdot g = g^*xg$ (recall the identification), we obtain the well known diagram:

G-orbits	fixed points	little groups
$V_M^{\pm} = \{\langle\hat{x}, \hat{x}\rangle = M^2, \hat{x}_0 \gtrless 0\}$	$\pm M\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$SU(2)$
$V_0^{\pm} = \{\langle\hat{x}, \hat{x}\rangle = 0, x_0 \gtrless 0\}$	$\pm\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$	$E(2) = \left\{\begin{pmatrix}e^{i\theta} & 0\\ \zeta & e^{-i\theta}\end{pmatrix}\right\}$
$V_{iM} = \{\langle\hat{x}, \hat{x}\rangle = -M^2\}$	$M\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$	$SU(1,1) = \left\{\begin{pmatrix}\beta & \alpha\\ \bar{\beta} & \bar{\alpha}\end{pmatrix} : \lvert\alpha\rvert^2 - \lvert\beta\rvert^2 = 1\right\}$
$V_0 = \{\langle\hat{x}, \hat{x}\rangle = 0, \hat{x}_0 = 0\}$	$\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$	$SL(2, \boldsymbol{C})$

M: positive number.

Furthermore there exists a well known correspondence between an irreducible unitary representation of P and a triplet (ω, G_0, π), where ω stands for one of G-orbits and π denotes an irreducible unitary representation of the little group G_0. More precisely, denote $\mathfrak{H}_\pi$ the representation space of π and ν_ω the G-invariant measure on the homogeneous space $\omega = G_0\backslash G$ and let $\mathfrak{H}^{\omega,\pi}$ be a Hilbert space consisting of $\mathfrak{H}_\pi$-valued measurable functions on P such that

$$(1)\qquad f((x, g_0)(x', g')) = e^{i\langle x, \hat{x}\rangle}\pi(g_0)f(x', g') \qquad \text{for } g_0 \in G_0$$

where $\hat{x}$ is a fixed point with the little group G_0,

$$(2)\qquad \int_\omega \lVert f(x, g)\rVert_{\mathfrak{H}_\pi}^2 d\nu_\omega < \infty\ .$$

Then the irreducible unitary representation of P corresponding to the triplet (ω, G_0, π) say $U^{\omega,\pi}$ is realized on $\mathfrak{H}^{\omega,\pi}$ by the formula

$$(3)\qquad U^{\omega,\pi}(x, g)f(x', g') = f((x', g')(x, g))\ .$$

THEOREM 2. *Irreducible unitary representations of the Poincaré group corresponding to one of the orbits $V_M^{\pm}$, $V_0^{\pm}$ and V_0 are irreducible as the representation of the Poincaré subsemigroup.*

Proof. Let $(U, \mathfrak{H})$ be an irreducible unitary representation of P. If it is reducible with respect to P_+, there exists a non-trivial closed subspace $D \subset \mathfrak{H}$ such that $U_t D \subsetneq D$ for any $t > 0$, where U_t denotes $U((t, 0, 0, 0), e)$. Put $D_+ = D \ominus \bigcap_{t>0} U_t D$ and $\mathfrak{H}_+ = \overline{\bigcup_t U_t D_+}$. Then D_+ is an outgoing subspace of $\mathfrak{H}_+$ in the sense that

(i) $U_t D_+ \subset D_+$ for all $t > 0$,

(ii) $\bigcap_t U_t D_+ = 0$,

(iii) $\overline{\bigcup_t U_t D_+} = \mathfrak{H}_+ \neq \{0\}$.

In view of Sinai's theorem (Theorem 3.1 in chap. 2 [11]) the restriction $(U_t, \mathfrak{H}_+)$, which is a unitary representation of $\boldsymbol{R}$, is unitarily equivalent to some multiple of the regular representation of $\boldsymbol{R}$. Consequently the representation $(U_t, \mathfrak{H})$ of $\boldsymbol{R}$ must contain at least one regular representation of R. On the other hand, making use of (1) and (3) and putting $g' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, we can verify easily that

$$U_t f(x', g') = e^{it\varepsilon M(|\alpha|^2 + |\beta|^2)/2} f(x', g') ,$$

where ε denotes one of constants ± 1, $\pm M^{-1}$ and 0. This implies that the spectrum of the selfadjoint operator $iU_t'|_{t=0}$ has either upper or lower bounds. In particular the representation U_t never contains the regular representation. Q.E.D.

We turn now to the representations corresponding to the orbit V_{iM}. Since each of them is specified by an irreducible unitary representation of the little group $G_0 = SU(1, 1)$, we summarize those representations after Vilenkin (§ 2 in chap. VI [17]). All of them can be obtained from algebraic representations on closed subspaces D of C^∞-functions $C^\infty(T)$ on the 1-dimensional torus T. We denote the inner product by (,).

THEOREM 3. *Irreducible unitary representations of the Poincaré group P given by the so-called discrete series representations $\pi^{\pm}(\ell, 0)$ and $\pi^{\pm}(\ell, 1/2)$ of $G_0 = SU(1,1)$ and the orbit V_{iM} are also irreducible even if they are restricted to the subsemigroup P_+.*

We shall give the proof of Theorem 3 as well as Theorem 1 in the following § 5.

representations π	$\pi(g_0)f(e^{i\psi})$ for $g_0 = \begin{pmatrix}\alpha & \beta \\ \bar{\beta} & \bar{\alpha}\end{pmatrix}$	D	the values of $(e^{i\nu\psi}, e^{i\nu\psi})$ or $(e^{-i\nu\psi}, e^{-i\nu\psi})$
$\pi_{(\ell,0)}$ $\ell = -1/2 + i\rho, \rho \geqslant 0$	$I_0 = \lvert\beta e^{i\psi} + \bar{\alpha}\rvert^{2\ell} f\left(\dfrac{\alpha e^{i\psi} + \bar{\beta}}{\beta e^{i\psi} + \bar{\alpha}}\right)$	$C^\infty(T)$	1
$\pi_{(\ell,1/2)}$ $\ell = -1/2 + i\rho, \rho > 0$	$I_{1/2} = \lvert\beta e^{i\psi} + \bar{\alpha}\rvert^{2\ell-1}(\beta e^{i\psi} + \bar{\alpha}) f\left(\dfrac{\alpha e^{i\psi} + \bar{\beta}}{\beta e^{i\psi} + \bar{\alpha}}\right)$	$C^\infty(T)$	1
$\pi_{(\ell,0)}$ $-1 < \ell < -1/2$	I_0	$C^\infty(T)$	$\dfrac{\Gamma(\ell - \nu + 1)}{\Gamma(-\ell - \nu)}$
$\pi^+_{(\ell,0)}$ $\ell = -1, -2, \cdots$	I_0	$\sum_{\nu \geqslant -\ell} a_\nu e^{i\nu\psi}$	$\dfrac{\Gamma(\ell + \nu + 1)}{\Gamma(-\ell + \nu)}$
$\pi^+_{(\ell,1/2)}$ $\ell = -1/2, -3/2, \cdots$	$I_{1/2}$	$\sum_{\nu \geqslant -\ell+1/2} a_\nu e^{i\nu\psi}$	$\dfrac{\Gamma(\ell + \nu + 1/2)}{\Gamma(-\ell + \nu - 1/2)}$
$\pi^-_{(\ell,0)}$ $\ell = -1, -2, \cdots$	I_0	$\sum_{\nu \geqslant -\ell} a_\nu e^{-i\nu\psi}$	$\dfrac{\Gamma(\ell + \nu + 1)}{\Gamma(-\ell + \nu)}$
$\pi^-_{(\ell,1/2)}$ $\ell = -1/2, -3/2, \cdots$	$I_{1/2}$	$\sum_{\nu \geqslant -\ell-1/2} a_\nu e^{-i\nu\psi}$	$\dfrac{\Gamma(\ell + \nu + 3/2)}{\Gamma(-\ell + \nu + 1/2)}$

§ 3. Decomposition of unitary representations of $SL(2, \boldsymbol{C})$

We begin with reviewing the irreducible unitary representations of $SL(2, \boldsymbol{C})$ after Naimark [12]. Throughout this section G stands for $SL(2, \boldsymbol{C})$. For an integer m denote by $\boldsymbol{L}^2_m(SU(2))$ a subspace of $\boldsymbol{L}^2(SU(2))$ consisting of functions φ satisfying

$$\varphi(\gamma u) = e^{-imt}\varphi(u) \qquad \text{for } \gamma = \begin{pmatrix} e^{+it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix}.$$

The irreducible representations $S_{m,\rho}(m \in \boldsymbol{Z}, \rho \in \boldsymbol{R})$ has a realization on $\boldsymbol{L}^2_m(SU(2))$:

$$V(g)\varphi(u) = -\frac{\alpha(ug)}{\alpha(u\bar{g})}\varphi(u\bar{g}),$$

where $\alpha(g) = |g_{22}|^{i\rho-m-2}g_{22}{}^m$ and $u\bar{g}$ denotes a unitary representative of the coset Kug with $K = \left\{\begin{pmatrix} \lambda^{-1} & \mu \\ 0 & \lambda \end{pmatrix} : \lambda > 0, \mu \in \boldsymbol{C}\right\}$. Meanwhile the irreducible representation D_σ $(0 < \sigma < 2)$ has a realization on the Hilbert space $\mathfrak{H}_\sigma$ in which a subspace B_0 of bounded functions belonging to $\boldsymbol{L}^2_0(SU(2))$ is dense:

$$V(g)\varphi(u) = -\frac{\alpha(ug)}{\alpha(u\bar{g})}\varphi(u\bar{g}) \qquad \text{for } \varphi \in B_0,$$

where $\alpha(g) = |g_{22}|^{-\sigma-2}$. We put

$$\omega_1(t) = \begin{pmatrix} \cos t/2 & i\sin t/2 \\ i\sin t/2 & \cos t/2 \end{pmatrix} \qquad \omega_2(t) = \begin{pmatrix} \cos t/2 & -\sin t/2 \\ \sin t/2 & \cos t/2 \end{pmatrix}$$
$$\omega_3(t) = \begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix} \qquad \omega_4(t) = \begin{pmatrix} \mathrm{ch}\, t/2 & \mathrm{sh}\, t/2 \\ \mathrm{sh}\, t/2 & \mathrm{ch}\, t/2 \end{pmatrix}$$
$$\omega_5(t) = \begin{pmatrix} \mathrm{ch}\, t/2 & i\,\mathrm{sh}\, t/2 \\ -i\,\mathrm{sh}\, t/2 & \mathrm{ch}\, t/2 \end{pmatrix} \qquad \omega_6(t) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} .$$

We now introduce linear operators associated with a unitary representation $(T, \mathfrak{H})$ of G. Define

$$\omega_j = \frac{d}{dt}\Big|_{t=0} T(\omega_j(t)) \qquad \text{for } j = 1, 2, \cdots, 6 ,$$
$$H_\pm = i\omega_2 \pm \omega_1 , \quad H_3 = i\omega_3 , \quad F_\pm = i\omega_5 \pm \omega_4 , \quad F_3 = i\omega_6 ,$$
$$\Delta_o = -(H_+H_- + H_-H_+ + 2H_3^2)/2 ,$$
$$\Delta = (F_+F_- + F_-F_+ + 2F_3^2)/2 + \Delta_0 - 1 ,$$
$$\Delta' = (H_+F_- + H_-F_+ + F_+H_- + F_-H_+ + 4H_3F_3)/2 .$$

More precisely, since the operator Δ_o (resp. Δ and Δ') is essentially selfadjoint with domain $\left\{\text{finite sum of } \int_{SU(2)} \varphi_i(u)T(u)f_i du \colon \varphi_i \in C^\infty(SU(2)),\right.$ $\left. f_i \in \mathfrak{H}\right\}$ $\left(\text{resp. } \left\{\text{finite sum of } \int_G \varphi_i(g)T(g)f_i dg \colon \varphi_i \in C_0^\infty(G),\ f_i \in \mathfrak{H}\right\}\right)$ ([14]), we shall use the same letters for their selfadjoint extensions. We denote the domain of an operator A by D_A. Then $D_{H\pm}$ (resp. $D_{F\pm}$) is the intersection $D_{\omega_1} \cap D_{\omega_2}$ (resp. $D_{\omega_4} \cap D_{\omega_5}$). Clearly $i\omega_j$ is a selfadjoint operator with domain $D\omega_j$.

Remark. A homomomorphism Λ from G onto the proper Lorentz group defined by $\Lambda(g)x = g^{*-1}xg^{-1}$ for $x \in \boldsymbol{R}_4$ (recall the identification in § 1) satisfies

$$\Lambda(\omega_1(t)) = a_2(-t) , \quad \Lambda(\omega_2(t)) = a_1(t) , \quad \Lambda(\omega_3(t)) = a_3(t) ,$$
$$\Lambda(\omega_4(t)) = b_2(-t) , \quad \Lambda(\omega_5(t)) = b_1(t) , \quad \Lambda(\omega_6(t)) = b_3(t) .$$

We refer subgroups $a_i(t)$ and $b_i(t)$ to [12] where a homomorphism $\tilde{\Lambda}(g)x = gxg^*$ is used.

We write down explicitly a canonical basis of the representations $S_{m,\rho}$ and D_σ.

LEMMA 1. *A canonical basis of the representation* $S_{m,\rho}$ *is given by* $\{\varphi^k_{p,m,\rho} \colon p = -k, -k+1, \cdots, k$ *and* $k = m/2, m/2+1, \cdots\}$, *where*

$$\varphi^k_{p,m,\rho}(u) = \sqrt{2k+1}\left(\prod_{\nu=m/2}^{k} \frac{(2i\nu+\rho)}{\sqrt{4\nu^2+\rho^2}}\right) C^k_{m/2,p}(u) .$$

A canonical basis of the representation D_σ is given by $\{\varphi^k_{p,\sigma} : p = -k, -k+1, \cdots, k$ and $k = 0, 1, \cdots\}$, where

$$\varphi^k_{p,\sigma}(u) = \sqrt{2k+1}\left(\prod_{\nu=1}^{k} \frac{i(2\nu+\sigma)}{\sqrt{4\nu^2-\sigma^2}}\right)\sqrt{\frac{\sigma}{2\pi}}\, C^k_{0,p}(u) .$$

The function $C^k_{\mu,\nu}$ on $SU(2)$ is defined by

$$C^k_{\mu,\nu}(u) = (-1)^{2k-\mu-\nu}\sqrt{\frac{(k-\mu)!\,(k+\mu)!}{(k-\nu)!\,(k+\nu)!}} \sum_\alpha \binom{k-\alpha}{\alpha}\binom{k+\nu}{k-\mu-\alpha} \times u_{11}^{\alpha} u_{12}^{k-\mu-\alpha} u_{21}^{k-\nu-\alpha} u_{22}^{\mu+\nu+\alpha} ,$$

where α ranges from max $(0, -\mu-\nu)$ up to min $(k-\mu, k-\nu)$.

Proof. See § 11 and § 12 of [12]. Since we use the homomophism Λ, the canonical basis above differs a little from the one cited in [12].

It seems convenient to reparametrize these representations of G as follows:

$$(T_{m,\lambda}, \mathfrak{H}_{m,\lambda}) = \begin{cases} S_{m,\lambda} & \text{for } m \geqslant 1 \\ S_{0,2\sqrt{\lambda}} & \text{for } m = 0\,,\ \lambda \geqslant 0 \\ D_{2\sqrt{-\lambda}} & \text{for } m = 0\,,\ -1 < \lambda < 0 \\ \text{unit representation} & \text{for } m = 0,\ \lambda = -1\,. \end{cases}$$

Thus the representation $(T_{m,\lambda}, \mathfrak{H}_{m,\lambda})$ has the canonical basis $f^k_{\nu,m,\lambda}$ in accordance with Lemma 1 and it holds that

$$\Delta = -\left(\frac{m}{2}\right)^2 + \lambda\,, \qquad \Delta' = -\frac{m}{2}\lambda\,.$$

Furthermore, putting $\ell_0 = \{(0, \lambda) : -1 \leqslant \lambda\}$ and $\ell_m = \{(m, \lambda) : \lambda \in \boldsymbol{R}\}$ for positive integer m, we can identify the dual space $\hat{G}$ with a Borel subset $\sum_{m\geqq 0} \ell_m$ in $\boldsymbol{R}_2$ (18. 9. 13 [4]).

LEMMA 2. *Denote $\{f^k_{\nu,m,\lambda}\}$ the canonical basis of the representation $(T_{m,\lambda}, \mathfrak{H}_{m,\lambda})$ then it holds that*

(i) $\Delta_0 f^k_{\nu,m,\lambda} = -k(k+1) f^k_{\nu,m,\lambda}$

(ii) $H_3 f^k_{\nu,m,\lambda} = \nu f^k_{\nu,m,\lambda}$

(iii) $F_+ f^k_{k,m,\lambda} = \sqrt{(2k+1)(2k+2)}\, C_{k+1,m}\, f^{k+1}_{k+1,m}$, *where*

$$C_{k+1,m} = \begin{cases} i\sqrt{\left\{(k+1)^2 - \left(\frac{m}{2}\right)^2\right\}\left\{(k+1)^2 + \frac{\lambda^2}{4}\right\}\Big/\{4(k+1)^2 - 1\}}/(k+1) & \text{for } m \geqslant 1 \\ i\sqrt{\{(k+1)^2 + \lambda\}/\{4(k+1)^2 - 1\}} & \text{for } m = 0 \end{cases}$$

(iv) *Put* $f^k_{\nu,m,\lambda} = 0$ *for* $k = 0, 1/2, 1, 3/2, \cdots$ *and* $|\nu| = 0, 1/2, 1, \cdots$ *unless* $\nu = -k, -k+1, \cdots, k$ *and* $k = m/2, m/2+1, \cdots$. *Then the function* $(T_{m,\lambda}(g)f^k_{\nu,m,\lambda}, f^{k'}_{\nu',m,\lambda})_{m,\lambda}$ *on* $G \times \hat{G}$ *is measurable.*

(v) *As* $t \to 0$, *the norm*

$$\left\| \frac{T_{m,\lambda}(\omega_j(t))f^k_{\nu,m,\lambda} - f^k_{\nu,m,\lambda}}{t} - \omega_j f^k_{\nu,m,\lambda} \right\|_{m,\lambda}$$

converges to zero uniformly on any compact set of $\{(0,\lambda): -1 < \lambda < 0\}$, $\{(0,\lambda): \lambda \geqslant 0\}$ *and* ℓ_m *with positive integer m.*

Proof. A canonical basis has properties (i), (ii) and (iii). Assume that $g = (g_{ij}) \in G$, $u \in SU(2)$, $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(2)$, $\begin{pmatrix} \delta^{-1} & \mu \\ 0 & \delta \end{pmatrix} \in K$ and that $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} g = \begin{pmatrix} \delta^{-1} & \mu \\ 0 & \delta \end{pmatrix} u$, then we have (see § 11.1 in [12])

$$u_{22} = (-\bar{\beta}g_{12} + \bar{\alpha}g_{22})\{|-\bar{\beta}g_{11} + \bar{\alpha}g_{21}|^2 + |-\bar{\beta}g_{12} + \bar{\alpha}g_{22}|^2\}^{-1/2}.$$

Hence $\alpha(ug)/\alpha(u\bar{g})$ is given by

$$\{|-\bar{\beta}g_{11} + \bar{\alpha}g_{21}|^2 + |-\bar{\beta}g_{12} + \bar{\alpha}g_{22}|^2\}^{-1+(i\rho-m)/2} \quad \text{for } S_{m,\rho},$$
$$\{|-\bar{\beta}g_{11} + \bar{\alpha}g_{21}|^2 + |-\bar{\beta}g_{12} + \bar{\alpha}g_{22}|^2\}^{-1-\sigma/2} \quad \text{for } D_\sigma.$$

Consequently $V(g)\varphi^k_{p,m,\rho}(u)$ and $V(g)\varphi^k_{p,\sigma}(u)$ are C^∞-functions on $G \times SU(2) \times \boldsymbol{R}$ and $G \times SU(2) \times (0,2)$ respectively. Recalling that the inner products of the representation space of $S_{m,\rho}$ and D_σ are of the form

$$(\varphi,\varphi)_{m,\rho} = \int_{SU(2)} |\varphi(u)|^2\, du$$
$$(\varphi,\varphi)_\sigma = \pi \iint_{SU(2)\times SU(2)} \Phi(u'u''^{-1})\varphi(u')\overline{\varphi(u'')}du'du''$$

respectively, where $\Phi(u) = |u_{21}|^{-2+\sigma}$, we easily verify (iv). Since $V(g)\varphi(u)$ is smooth, (v) is clear. Q.E.D.

Thanks to Lemma 2 (especially to (iv)), for a σ-finite measure on G we can define a unitary representation $\int^{\oplus}_{\hat{G}} T_{m,\lambda} d\sigma$ on the Hilbert space

$\int_{\hat{G}}^{\oplus} \mathfrak{H}_{m,\lambda} d\sigma$. To decompose a unitary representation of G is, by definition, to determine a sequence of mutually singular σ-finite measures $\{\sigma_1, \sigma_2, \cdots, \sigma_\infty\}$ on the measurable space $\hat{G}$ so that the representation is unitarily equivalent to the representation (T, H) defined by

$$T = \int_{\hat{G}}^{\oplus} T_{m,\lambda} d\sigma_1 \oplus [2] \int_{\hat{G}}^{\oplus} T_{m,\lambda} d\sigma_2 \oplus \cdots \oplus [\aleph_0] \int_{\hat{G}}^{\oplus} T_{m,\lambda} d\sigma_\infty$$

on the Hilbert space

$$\mathfrak{H} = \int_{\hat{G}}^{\oplus} \mathfrak{H}_{m,\lambda} d\sigma_1 \oplus [2] \int_{\hat{G}}^{\oplus} \mathfrak{H}_{m,\lambda} d\sigma_2 \oplus \cdots \oplus [\aleph_0] \int_{\hat{G}}^{\oplus} \mathfrak{H}_{m,\lambda} d\sigma_\infty ,$$

where the cardinal number in the bracket indicates the multiplicity. We shall search for a procedure to determine the measure σ_i up to the usual equivalence.

LEMMA 3. *For $k = 0, 1/2, 1, \cdots$, let W_k be the space of solutions of the equations*

$$(4) \qquad H_3 f = kf , \qquad \Delta_0 f = -k(k+1)f$$

with respect to the representation $(T, \mathfrak{H})$ above. Denote $\sigma_i^{(m)}$ the restriction $\sigma_i | \ell_m$. Then we have unitary equivalences among selfadjoint operators:

$$\Delta | W_0 \simeq \int_{[-1,\infty)}^{\oplus} \lambda d\sigma_1^{(0)} \oplus [2] \int_{[-1,\infty)}^{\oplus} \lambda d\sigma_2^{(0)} \oplus \cdots \oplus [\aleph_0] \int_{[-1,\infty)}^{\oplus} \lambda d\sigma_\infty^{(0)} ,$$

$$\Delta' | W_k \ominus F_+ W_{k-1} \simeq \int_{R}^{\oplus} (-k)\lambda d\sigma_1^{(2k)} \oplus [2] \int_{R}^{\oplus} (-k)\lambda d\sigma_2^{(2k)} \oplus \cdots \oplus [\aleph_0] \int_{R}^{\oplus} (-k)\lambda d\sigma_\infty^{(2k)} .$$

Proof. Without loss of generality we may assume that all measures except for σ_1 are zero measures. Rewrite $\sigma_1 = \sigma$. We claim

$$1^\circ \qquad W_k = \left\{ \int_{\hat{G}}^{\oplus} a(2k, \lambda) f_{k,m,\lambda}^k d\sigma : \int_{\hat{G}} |a|^2 \, d\sigma < \infty \right\} .$$

Indeed, set

$$\tilde{W}_k = \left\{ \int_{\hat{G}}^{\oplus} \sum_{\nu=-k}^{k} a_\nu(m, \lambda) f_{\nu,m,\lambda}^k d\sigma : \int_{\hat{G}} |a_\nu|^2 \, d\sigma < \infty \text{ for each } \nu \right\} .$$

We will show that the restriction $\Delta_0 | \tilde{W}_k$ is equal to $-k(k+1)$. To this end define $f(\varphi)$ for $f = \int_{\hat{G}}^{\oplus} f_{m,\lambda} d\sigma \in \tilde{W}_k$ and φ in $C^\infty(SU(2))$ by $f(\varphi) =$

$\int_{SU(2)} \varphi(u)T(u)fdu \in \tilde{W}_k$. Denoting Δ_o^r and $\Delta_o^{m,\lambda}$ the operator Δ_o corresponding to the left regular representation of $SU(2)$ and the restriction $T_{m,\lambda}|SU(2)$ respectively, for $h = \int_{\hat{G}}^{\oplus} h_{m,\lambda} d\sigma$ we have

$$\begin{aligned}(\Delta_o f(\varphi), h) &= \int_{SU(2)} du(\Delta_o^r \varphi(u))(T(u)f, h) \\ &= \int_{\hat{G}} d\sigma \int_{SU(2)} du(\Delta_o^r \varphi(u))(T_{m,\lambda}(u)f_{m,\lambda}, h_{m,\lambda})_{m,\lambda} \\ &= \int_{\hat{G}} d\sigma(\Delta_o^{m,\lambda} f_{m,\lambda}(\varphi), h_{m,\lambda})_{m,\lambda} \\ &= -k(k+1)(f(\varphi), h) ,\end{aligned}$$

as desired. Since the set $\{f_{\nu,m,\lambda}^k : \nu = -k, -k+1, \cdots, k \text{ and } k = m/2, m/2 + 1, \cdots\}$ is an orthonormal basis in the Hilbert space $\mathfrak{H}_{m,\lambda}$, $\mathfrak{H}$ is a direct sum of $\tilde{W}_k$'s. Thus W_k is a subspace of $\tilde{W}_k$. From (v) of Lemma 2 $f = \int_{\hat{G}}^{\oplus} \sum_{\nu=-k}^{k} a_\nu(m, \lambda) f_{\nu,m,\lambda}^k d\sigma$ in $\tilde{W}_k$ satisfies

$$H_3 f = \int_{\hat{G}}^{\oplus} \sum_{\nu=-k}^{k} \nu a_\nu f_{\nu,m,\lambda}^k d\sigma = kf ,$$

which implies that a_ν is equal to zero a.e. unless $\nu = k$, proving 1°. Next step is to show

2° $$W_k \ominus F_+ W_{k-1} = \left\{ \int_{\ell_{2k}}^{\oplus} a(2k, \lambda) f_{k,2k,\lambda}^k d\sigma : \int_{\ell_{2k}} |a|^2 d\sigma < \infty \right\} .$$

To see this, define $W_{k,m} = \left\{ \int_{\ell_m}^{\oplus} a(m, \lambda) f_{k,m,\lambda}^k d\sigma : \int_{\ell_m} |a|^2 d\sigma < \infty \right\}$. Since W_k is a direct sum of $W_{k,m}$'s with non-negative integers $m = 2k, 2k-2, \cdots$ and since the closure $\overline{F_+ W_{k-1,m}}$ coincides with $W_{k,m}$ due to (iii) and (v) of Lemma 2, 2° is now clear. Finally we verify

3° $$\Delta \int_{\ell_0}^{\oplus} a(0, \lambda) f_{0,0,\lambda}^0 d\sigma = \int_{\ell_0}^{\oplus} \lambda a(0, \lambda) f_{0,0,\lambda}^0 d\sigma ,$$

$$\Delta' \int_{\ell_{2k}}^{\oplus} a(2k, \lambda) f_{k,2k,\lambda}^k d\sigma = \int_{\ell_{2k}}^{\oplus} (-k)\lambda a(2k, \lambda) f_{k,2k,\lambda}^k d\sigma ,$$

provided the members on the right side belong to $\mathfrak{H}$. Indeed we can argue as we showed that $\Delta_o | \tilde{W}_k = -k(k+1)$ in 1°. Now 1°, 2° and 3° yield the Lemma. Q.E.D.

The following lemma is also useful.

LEMMA 4. *The restriction $\Delta'|W_k$ and $\Delta'|\overline{F_+W_k}$ are unitarily equivalent selfadjoint operators.*

Proof. As mentioned in the proof of Lemma 3, the closure $\overline{F_+W_k}$ is a direct sum of $W_{k+1,m}$'s with non-negative integers $m=2k, 2k-2, \cdots$. The following isometry from W_k onto $\overline{F_+W_k}$ transforms the first operator to the second one:

$$\sum_{m=2k,2k-2,\cdots}\int_{\ell_m}^{\oplus} a(m,\lambda)f_{k,m,\lambda}^{k}d\sigma \to \sum_{m=2k,2k-2,\cdots}\int_{\ell_m}^{\oplus} a(m,\lambda)f_{k+1,m,\lambda}^{k+1}d\sigma .$$

Q.E.D.

To sum up, given a unitary representation of $SL(2,\boldsymbol{C})$, one can decompose it into irreducible ones if one could specify the space W_k (call it the space of *the k-th heighest weight vectors*) and carry out the spectral decomposition of selfadjoint operators $\Delta|W_0$ and $\Delta'|W_k\ominus F_+W_{k-1}$.

§4. The space of the k-th heighest weight vectors W_k

Let $U^{iM,\pi}$ denote an irreducible unitary representation of the Poincaré group P associated with the hyperboloid of one sheet V_{iM} and an irreducible unitary representation π of $SU(1,1)$ (see §2). In this section we shall first solve the equation (4), then determine the spectral type of selfadjoint operators $\Delta|W_0$ and $\Delta'|W_k$ of the restriction $U^{iM,\pi}|SL(2,\boldsymbol{C})$. From now on G and G_0 stand for $SL(2,\boldsymbol{C})$ and $SU(1,1)$ respectively.

We begin with specifying the representation $U^{iM,\pi}$ of P. $V_{iM}=\left\{y=\begin{pmatrix} y_0-y_3 & y_2-iy_1 \\ y_2+iy_1 & y_0+y_3\end{pmatrix}: \det y=-M^2\right\}$ in $\hat{\boldsymbol{R}}_4$ is a G-homogeneous space with the invariant measure $d\mu(y)=dy_1dy_2dy_3/|y_0|$. Let p be the projection from G onto V_{iM} defined by $p(g)=g^*\hat{x}g$, where $\hat{x}$ denotes the fixed point $M\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. For u in $SU(2)$ let s_u be a measurable section from V_{iM} into G such that $p\circ s_u=$ identity and that

(5) $\quad s_u\circ p(\langle\tau,\theta,\varphi\rangle)=\langle\tau,\theta,\varphi\rangle u \qquad \text{for } (\tau,\theta,\varphi)\in \boldsymbol{R}\times(0,\pi)\times(0,2\pi)$,

where $\langle\tau,\theta,\varphi\rangle$ stands for the matrix $\omega_6(\tau)\omega_2(\theta)\omega_3(\varphi)$. We fix s_u once for all. Then the representation $U^{iM,\pi}$ has the following realization $U^{\pi,u}$ on the Hilbert space $\mathfrak{H}^{\pi}=L^2(V_{iM},\mathfrak{H}_{\pi},\mu)$ for each $u\in SU(2)$:

(6) $\quad U^{\pi,u}(x,g)f(y)=e^{i\langle x',\hat{x}\rangle}\pi(g_0)f(y\cdot g)$,

(7) $\quad s_u(y)(x,g)=(x',g_0)s_u(y\cdot g) \qquad \text{with } g_0\in G_0$.

By the aid of the isometry $I_u: \tilde{\mathfrak{H}}^\pi(G) = \{\tilde{f} \in L^2(G, \mathfrak{H}_\pi, \mu): \tilde{f}(g_0 g) = \pi(g_0)\tilde{f}(g)$ for $g_0 \in G_0\} \to \mathfrak{H}^\pi$ such that $\tilde{f}(s_u(y)) = I_u \tilde{f}(y)$, $U^{\pi,u}$ is transformed to $U^{\pi,v}$ by $I_v I_u^{-1}$.

We proceed, assuming the representation π to be $\pi^+_{(\ell,0)}$. Other cases can be treated in the same way. Setting

$$Y = \{p(\omega_6(\tau)\omega_2(\theta)\omega_3(\varphi)): (\tau, \theta, \varphi) \in R \times (0, \pi) \times (0, 2\pi)\} \subset V_{iM} ,$$

for $u \in SU(2)$ define a dense subspace $\mathfrak{H}_0^{\pi,u}$ of $\mathfrak{H}^\pi$:

$$\mathfrak{H}_0^{\pi,u} = \left\{f \in C_0^\infty(Y \cdot u \times T): f(y, e^{i\psi}) = \sum_{\nu \geq -\ell} f_\nu(y) e^{i\nu\psi}\right\} .$$

We note that for f in $\mathfrak{H}_0^{\pi,u}$ (6) takes the form

$$(6)' \qquad U^{\pi,u}(0, g) f(y, e^{i\psi}) = |\beta e^{i\psi} + \bar{\alpha}|^{2\ell} f\left(y \cdot g, \frac{\alpha e^{i\psi} + \bar{\beta}}{\beta e^{i\psi} + \bar{\alpha}}\right)$$

provided $s_u(y) g = g_0 s_u(y \cdot g)$ with $g_0 = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in G_0$. Since the section s_u is smooth on $Y \cdot u$ as well as the map $(y, g) \to y \cdot g$, there exists a relatively compact neighborhood U of the unit element of G such that for $f \in \mathfrak{H}_0^{\pi,u}$, the function $U^{\pi,u}(0, g) f(y, e^{i\psi})$ belong to $C^\infty(U \times Y \cdot u \times T)$. This observation leads to

LEMMA 5. *The domain of $\omega_j^{\pi,u}$ includes $\mathfrak{H}_0^{\pi,u}$ for all j and the restriction $\omega_j^{\pi,u} | \mathfrak{H}_0^{\pi,u}$ is a differential operator with C^∞-coefficients.*

Now that $\omega_j^{\pi,u}$ is a continuous transformation of $\mathfrak{H}_0^{\pi,u}$ with the relative topology of $C_0^\infty(Y \cdot u \times T)$, we define the dual operator $\hat{\omega}_j^{\pi,u}$ by the following

$$\langle \hat{\omega}_j^{\pi,u} \hat{f}, f \rangle = \langle \hat{f}, \omega_j^{\pi,u} f \rangle$$

where $\hat{f} \in (\mathfrak{H}_0^{\pi,u})'$ and $f \in \mathfrak{H}_0^{\pi,u}$. Regarding $\mathfrak{H}^\pi$ as a subspace of the dual space $(\mathfrak{H}_0^{\pi,u})'$, we claim

LEMMA 6.

(i) $\omega_j^{\pi,u} \subset -\hat{\omega}_j^{\pi,u}$.

(ii) *Assume that f belongs to $\mathfrak{H}_0^{\pi,u}$ and* Supp $f \subset Y \cdot v$ *for some $v \in SU(2)$. Then $f^v = I_v I_u^{-1} f$ belongs to $\mathfrak{H}_0^{\pi,v}$ and satisfies*

$$(\omega_j^{\pi,u} f, h) = (\omega_j^{\pi,v} f^v, h^v) \qquad \textit{for any } h \in \mathfrak{H}^\pi .$$

(iii) *The intersection $D_{\Delta_0^\pi,u} \cap D_{\Delta^\pi,u} \cap D_{\Delta'^\pi,u}$ includes $\mathfrak{H}_0^{\pi,u}$. Further-*

more, it holds that (*the indexes* π *and* u *are omitted*)

$$\Delta_o \subset \sum_{i=1}^{3} (\hat{\omega}_j)^2 , \qquad \Delta \subset \sum_{i=1}^{3} (\hat{\omega}_i)^2 - \sum_{j=4}^{6} (\hat{\omega}_j)^2 - 1 ,$$
$$\Delta' \subset -(\hat{\omega}_1\hat{\omega}_4 + \hat{\omega}_4\hat{\omega}_1 + \hat{\omega}_2\hat{\omega}_5 + \hat{\omega}_5\hat{\omega}_2 + 2\hat{\omega}_3\hat{\omega}_6) .$$

Proof. Since $\omega_j^{\pi,u}$ is antihermitian, (i) follows. We note that $f^v(y) = \pi(g_0)f(y)$ provided $s_v(y) = g_0 s_u(y)$ with $g_0 = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in G_0$, namely

$$f^v(y, e^{i\psi}) = |\beta e^{i\psi} + \bar{\alpha}|^{2\ell} f\left(y, \frac{\alpha e^{i\psi} + \bar{\beta}}{\beta e^{i\psi} + \bar{\alpha}}\right) . \tag{8}$$

Since g_0 is smooth on $Y\cdot u \cap Y\cdot v$, f^v has a representative in $\mathfrak{H}_0^{\pi,v}$. Now (ii) is evident. As to (iii) we deal only with $\Delta^{\pi,u}$. It suffices to prove

$$\Delta^{\pi,u} \int_G \varphi(g) U^{\pi,u}(0, g) f dg$$
$$= \int_G \varphi(g) U^{\pi,u}(0, g) \Big[\sum_i (\omega_i^{\pi,u})^2 - \sum_j (\omega_j^{\pi,u})^2 - 1 \Big] f dg$$

for $\varphi \in C_0^\infty(G)$ and $f \in \mathfrak{H}_0^{\pi,u}$ [14]. To this end we will show that for $\psi \in C_0^\infty(G)$ and $h \in \mathfrak{H}_0^{\pi,u}$

$$\begin{aligned} &\left(\Delta^{\pi,u} \int \varphi(g) U^{\pi,u}(0, g) f dg, \int \psi(g') U^{\pi,u}(0, g') h dg'\right) \\ &= \left(\int \varphi(g) U^{\pi,u}(g) \Big[\sum_i (\omega_i^{\pi,u})^2 - \sum_j (\omega_j^{\pi,u})^2 - 1 \Big] f dg, \right. \\ &\qquad \left. \int \psi(g') U^{\pi,u}(0, g') h dg' \right) . \end{aligned} \tag{9}$$

A diffeomorphism $q\colon V_{iM} \to \boldsymbol{R} \times S_2$ defined by

$$q(y) = (y_0, y_1/(\sqrt{y_1^2 + y_2^2 + y_3^2}, \ y_2/\sqrt{y_1^2 + y_2^2 + y_3^2}, \ y_3/\sqrt{y_1^2 + y_2^2 + y_3^2})) \tag{10}$$

maps $Y\cdot u$ onto $\boldsymbol{R} \times S_2^u$. We note that each S_2^u is dense and open in the unit sphere S_2 and that the union $\bigcup_{u \in SU(2)} S_2^u$ covers the sphere. Observing that for given $a, a' \in G$ and $y, y' \in V_{iM}$ there exists $w \in SU(2)$ such that $\{y, y', y'\cdot a'^{-1}a\} \subset Y\cdot w$, we can show inductively that there exist a finite covering $\{U_\alpha\}$ of $\operatorname{Supp}\varphi$, finite covering $\{U_{\alpha\beta}\}$ of $\operatorname{Supp}\psi$, finite covering $\{Y_{\alpha\beta\gamma}\}$ of $\operatorname{Supp} f$, finite covering $\{Y_{\alpha\beta\gamma\delta}\}$ of $\operatorname{Supp} h$ and $w_{\alpha\beta\gamma\delta} \in SU(2)$ such that each member is relatively compact and that

$$Y_{\alpha\beta\gamma} \cup Y_{\alpha\beta\gamma\delta} \cup Y_{\alpha\beta\gamma\delta}\cdot U_{\alpha\beta}^{-1} U_\alpha \subset Y\cdot w .$$

Denote $\chi_\alpha, \chi_{\alpha\beta}, \chi_{\alpha\beta\gamma}$ and $\chi_{\alpha\beta\gamma\delta}$ the partition of unity associated with the coverings above. Now the left side of (9) is equal to

$$\begin{aligned}&\int dg\varphi(g)\Big(f,\, U^{\pi,u}(g^{-1})\Delta^{\pi,u}\int \psi(g')U^{\pi,u}(g')hdg'\Big)\\ &\quad=\int dg\varphi(g)\Big(f,\, \Delta^{\pi,u}U^{\pi,u}(g^{-1})\int \psi(g')U^{\pi,u}(g')hdg'\Big)\\ &\quad=\int dg\varphi(g)\Big(f,\, \Delta^{\pi,u}\int \psi(g')U^{\pi,u}(g^{-1}g')dg'\Big)\\ &\quad=\int \sum_{\alpha,\beta,\gamma,\delta}\int dg\varphi\chi_\alpha\Big(f\chi_{\alpha\beta\gamma},\, \Delta^{\pi,u}\int \psi\chi_{\alpha\beta}U^{\pi,u}(g^{-1}g')h\chi_{\alpha\beta\gamma\delta}dg'\Big)\,.\end{aligned}$$

Putting $w = w_{\alpha\beta\gamma\delta}$ we rewrite the $\alpha\beta\gamma\delta$-term above as

$$\int dg\varphi\chi_\alpha\Big((f\chi_{\alpha\beta\gamma})^w,\, \Delta^{\pi,w}\int \psi\chi_{\alpha\beta}U^{\pi,w}(g^{-1}g')(h\chi_{\alpha\beta\gamma\delta})^w dg'\Big)\,.$$

Since $\chi_\alpha(g)\int \psi\chi_{\alpha\beta}U^{\pi,w}(h\chi_{\alpha\beta\gamma\delta})^w dg'$ belongs to $\mathfrak{H}_0^{\pi,w}$, it holds that

$$\begin{aligned}&\Delta^{\pi,w}\chi_\alpha(g)\int \psi\chi_{\alpha\beta}U^{\pi,w}(h\chi_{\alpha\beta\gamma\delta})^w dg'\\ &\quad=\chi_\alpha(g)\Big[\sum_i(\omega_i^{\pi,w})^2-\sum_j(\omega_j^{\pi,w})^2-1\Big]\int \psi\chi_{\alpha\beta}U^{\pi,w}(h\chi_{\alpha\beta\gamma\delta})^w dg'\,.\end{aligned}$$

On account of Lemma 5 and (ii) of Lemma 6 the $\alpha\beta\gamma\delta$-term is equal to

$$\int dg\varphi\chi_\alpha\Big(\Big[\sum_i(\omega_i^{\pi,u})^2-\sum_j(\omega_j^{\pi,u})^2-1\Big]f\chi_{\alpha\beta\gamma},\, \int \psi\chi_{\alpha\beta}U^{\pi,u}(h\chi_{\alpha\beta\gamma\delta})dg'\Big)\,,$$

from which (9) follows. Q.E.D.

We now derive the concrete forms of the restrictions to $\mathfrak{H}_0^{\pi,e}$ of $\omega_i, H_i, F_i, \Delta_o, \Delta$ and Δ' with respect to the representation $(U^{\pi,e}, \mathfrak{H}^\pi)$. After tedious computation we obtain the following. The underlined terms disappear for nonspinor irreducible unitary representations $\pi_{(\ell,0)}$ and $\pi^\pm_{(\ell,0)}$ of $SU(1,1)$.

$$p(\omega_6(\tau)\omega_2(\theta)\omega_3(\varphi)) = \begin{pmatrix}-e^\tau\cos^2\theta/2+e^{-\tau}\sin^2\theta/2 & \mathrm{ch}\,\tau\sin\theta\, e^{-i\varphi}\\ \mathrm{ch}\,\tau\sin\theta e^{i\varphi} & -e^\tau\sin^2\theta/2+e^{-\tau}\cos^2\theta/2\end{pmatrix},$$

$$(y_0, y_1, y_2, y_3) = (-\mathrm{sh}\,\tau,\ \mathrm{ch}\,\tau\sin\theta\sin\psi,\ \mathrm{ch}\,\tau\sin\theta\cos\varphi,\ \mathrm{ch}\,\tau\cos\theta)\,,$$

$$d\mu = \mathrm{ch}^2\tau\sin\theta\, d\tau d\theta d\varphi\,,$$

$$\omega_1 = \sin\varphi\partial_\theta + \cot\theta\cos\varphi\partial_\varphi - \frac{\cos\varphi}{\sin\theta}\partial_\psi + \underline{\frac{i\cos\varphi}{2\sin\theta}}\,,$$

$$\omega_2 = \cos\varphi\partial_\theta - \cot\theta\sin\varphi\partial_\varphi + \frac{\sin\varphi}{\sin\theta}\partial_\psi - \underline{\frac{i\sin\varphi}{2\sin\theta}}\,,$$

$$\omega_3 = \partial_\varphi\,,$$

$$\begin{aligned}\omega_4 = {} & -\sin\theta\cos\varphi\partial_\tau - \operatorname{th}\tau\cos\theta\cos\varphi\partial_\theta + \frac{\operatorname{th}\tau\sin\varphi}{\sin\theta}\partial_\varphi \\ & + \left(-\operatorname{th}\tau\cot\theta\sin\varphi - \frac{\cos\theta\cos\varphi\sin\psi + \sin\varphi\cos\psi}{\operatorname{ch}\tau}\right)\partial_\psi \\ & + \frac{\ell(\cos\theta\cos\varphi\cos\psi - \sin\varphi\sin\psi)}{\operatorname{ch}\tau} \\ & + \underline{\frac{i(\cos\theta\cos\varphi\sin\psi + \sin\varphi\cos\psi)}{2\operatorname{ch}\tau}} + \underline{\frac{\operatorname{th}\tau\cot\theta\sin\varphi}{2}}\,,\end{aligned}$$

$$\begin{aligned}\omega_5 = {} & \sin\theta\sin\varphi\partial_\tau + \operatorname{th}\tau\cos\theta\sin\varphi\partial_\theta + \frac{\operatorname{th}\tau\cos\varphi}{\sin\theta}\partial_\varphi \\ & + \left(-\operatorname{th}\tau\cot\theta\cos\varphi + \frac{\cos\theta\sin\varphi\sin\psi - \cos\varphi\cos\psi}{\operatorname{ch}\tau}\right)\partial_\psi \\ & + \frac{\ell(-\cos\theta\sin\varphi\cos\psi - \cos\varphi\sin\psi)}{\operatorname{ch}\tau} \\ & + \underline{\frac{i(-\cos\theta\sin\varphi\sin\psi + \cos\varphi\cos\psi)}{2\operatorname{ch}\tau}} + \underline{\frac{\operatorname{th}\tau\cot\theta\cos\varphi}{2}}\,,\end{aligned}$$

$$\begin{aligned}\omega_6 = {} & \cos\theta\partial_\tau - \operatorname{th}\tau\sin\theta\partial_\theta - \frac{\sin\theta\sin\psi}{\operatorname{ch}\tau}\partial_\psi + \frac{\ell\sin\theta\cos\psi}{\operatorname{ch}\tau} \\ & + \underline{\frac{i\sin\theta\sin\psi}{2\operatorname{ch}\tau}}\,,\end{aligned}$$

$$H_+ = e^{-i\varphi}\left(i\partial_\theta + \cot\theta\partial_\varphi - \frac{1}{\sin\theta}\partial_\psi + \underline{\frac{i}{2\sin\theta}}\right),$$

$$H_- = e^{+i\varphi}\left(i\partial_\theta - \cot\theta\partial_\varphi + \frac{1}{\sin\theta}\partial_\psi - \underline{\frac{i}{2\sin\theta}}\right),$$

$$H_3 = i\partial_\varphi\,,$$

$$\begin{aligned}F_+ = e^{-i\varphi}\Big[& -\sin\theta\partial_\tau - \operatorname{th}\tau\cos\theta\partial_\theta + \frac{i\operatorname{th}\tau}{\sin\theta}\partial_\varphi \\ & + \left(-i\operatorname{th}\tau\cot\theta - \frac{\cos\theta\sin\psi + i\cos\psi}{\operatorname{ch}\tau}\right)\partial_\psi \\ & + \frac{\ell(\cos\theta\cos\psi - i\sin\psi)}{\operatorname{ch}\tau} + \underline{\frac{i\cos\theta\sin\psi - \cos\psi}{2\operatorname{ch}\tau}} \\ & - \underline{\frac{\operatorname{th}\tau\cot\theta}{2}}\Big]\,,\end{aligned}$$

$$F_- = e^{i\varphi}\Big[\sin\theta\,\partial_\tau + \operatorname{th}\tau\cos\theta\partial_\theta + \frac{i\operatorname{th}\tau}{\sin\theta}\partial_\varphi + \Big(-i\operatorname{th}\tau\cot\theta + \frac{\cos\theta\sin\psi - i\cos\psi}{\operatorname{ch}\tau}\Big)\partial_\psi - \frac{i\cos\theta\sin\psi + \cos\psi}{2\operatorname{ch}\tau} - \frac{\operatorname{th}\tau\cot\theta}{2} + \frac{\ell(-\cos\theta\cos\psi - i\sin\psi)}{\operatorname{ch}\tau}\Big],$$

$$F_3 = i\Big[\cos\theta\partial_\tau - \operatorname{th}\tau\sin\theta\partial_\theta - \frac{\sin\theta\sin\psi}{\operatorname{ch}\tau}\partial_\psi + \frac{\ell\sin\theta\cos\psi}{\operatorname{ch}\tau} + \frac{i\sin\theta\sin\psi}{2\operatorname{ch}\tau}\Big],$$

$$\Delta_0 = \partial_\theta^2 + \frac{1}{\sin^2\theta}\partial_\varphi^2 - \frac{2\cot\theta}{\sin\theta}\partial_\varphi\partial_\psi + \frac{1}{\sin^2\theta}\partial_\psi^2 + \cot\theta\partial_\theta + \frac{i\cot\theta}{2\sin\theta}\partial_\varphi - \frac{i}{2\sin^2\theta}\partial_\psi - \frac{1}{4\sin^2\theta},$$

$$\Delta' = -2\partial_\tau\partial_\psi + \frac{2\cos\psi}{\operatorname{ch}\tau}\partial_\theta\partial_\psi + \frac{2\sin\psi}{\operatorname{ch}\tau\sin\theta}\partial_\varphi\partial_\psi - \frac{2\cot\theta\sin\psi}{\operatorname{ch}\tau}\partial_\psi^2 + i\partial_\tau + \Big(\frac{\ell\sin\psi}{\operatorname{ch}\tau} - \frac{i\cos\psi}{\operatorname{ch}\tau}\Big)\partial_\theta + \Big(-\frac{2\ell\cos\psi}{\operatorname{ch}\tau\sin\theta} - \frac{i\sin\psi}{\operatorname{ch}\tau\sin\theta}\Big)\partial_\varphi + 2\Big(\frac{\ell\cot\theta\cos\psi}{\operatorname{ch}\tau} - \operatorname{th}\tau + \frac{i\cot\theta\sin\psi}{\operatorname{ch}\tau}\Big)\partial_\psi + \Big(-\frac{i\ell\cot\theta\cos\psi}{\operatorname{ch}\tau} + \frac{\cot\theta\sin\psi}{\operatorname{ch}\tau} + i\operatorname{th}\tau\Big),$$

$$\Delta = -\Big(\partial_\tau^2 + 2\operatorname{th}\tau\partial_\tau + \frac{\ell(\ell+1)}{\operatorname{ch}^2\tau} + 1\Big) + S.$$

We remark that the differential operator S does not contain any terms of the form $S(\tau,\theta,\varphi,\psi)\partial_\tau^j$ $(j = 0, 1, 2)$.

We are ready to solve the equation (4). Consider the following equation

$$(11)\qquad -i\hat{\omega}_3 f = kf,\quad \sum_{i=1}^{3}\hat{\omega}_i^2 f = -k(k+1)f,\quad f\in\mathfrak{H}^\pi\ (k = -\ell, -\ell+1, \cdots)$$

and denote $\hat{W}_k$ the space of solutions (in (11) we omitted the indexes π and e for the sake of simplicity). Lemma 6 implies that W_k is the intersection of $\hat{W}_k$, D_{H_3} and D_{Δ_0}.

LEMMA 7. *An $\hat{f}$ belongs to $\hat{W}_k$ if and only if f is of the form*:

$$(12)\qquad \hat{f}(\tau,\theta,\varphi,e^{i\psi}) = \sum_{\nu\geq-\ell}^{k}\sum_{i=1,2} f_{\nu,i}(\tau)Q_{\nu,i}(\cos\theta)e^{-ik\varphi+i\nu\psi},$$

where $f_{\nu,i}$ *belongs to* $L^2(R, \mathrm{ch}^2\,\tau d\tau)$ *and* $\{Q_{\nu,i}(z): i = 1, 2\}$ *span the space of solutions in* $L^2((-1, 1))$ *of the equation*:

$$(13)\qquad \left[(1-z^2)\partial_z^2 - 2z\partial_z - \frac{k^2+\nu^2+2k\nu z}{1-z^2} + k(k+1)\right]Q(z) = 0 \ \ on\,(-1, 1)\,.$$

For the proof we need

LEMMA 8. *Assume that* k *ranges* $0, 1/2, 1, \cdots$ *and that* $k+\nu$ *is an integer. Then the equation* (13) *has no solutions in* L^2 *for* $|\nu| > k$, *while the bounded solution of* (13) *is proportional to* $P^k_{k,-\nu}(z)$ *for* $|\nu| \leqslant k$. $P^k_{k,\nu}$ *is defined by*

$$P^k_{k,\nu}(z) = \frac{i^{k-\nu}}{2^k}\sqrt{\frac{(2k)!}{(k-\nu)!\,(k+\nu)!}}(1-z)^{(k-\nu)/2}(1+z)^{(k+\nu)/2}\,.$$

Proof of Lemma 8. A similar statement can be found in chap. 3, sec. 4 [17]. That $P^k_{k,-\nu}$ is a bounded solution of (13) is known. By the change of variable $t = (z+1)/2$, the solution of (13) may be written as

$$P\begin{pmatrix} -1 & 1 & \infty & \\ -|k-\nu|/2 & -|k+\nu|/2 & -k & z \\ |k-\nu|/2 & |k+\nu|/2 & k+1 & \end{pmatrix} = P\begin{pmatrix} -1 & 1 & \infty & \\ \alpha & \gamma & \beta & z \\ \alpha' & \gamma' & \beta' & \end{pmatrix}$$

$$= P\begin{pmatrix} 0 & 1 & \infty & \\ \alpha & \gamma & \beta & t \\ \alpha' & \gamma' & \beta' & \end{pmatrix} = t^\alpha(1-t)^\gamma P\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha+\beta+\gamma & t \\ \alpha'-\alpha & \gamma'-\gamma & \alpha+\beta'+\gamma & \end{pmatrix}$$

$$= t^\alpha(1-t)^\gamma P\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & t \\ 1-c & c-a-b & b & \end{pmatrix}.$$

If $c < 1$, equivalently $k \neq \nu$, then $t^\alpha(1-t)^\gamma F(a, b, a+b-c, 1-t)$ and $t^\alpha(1-t)^{c-a-b}F(c-a, c-b, c-a-b+1, 1-t)$ are linearly independent solutions around $t = 1$, where $F(a, b, c, t)$ denotes the hypergeometric function. Checking the behavior of them around $t = 0$ and 1 [5], one verifies the lemma for $k \neq \nu$. If $c = 1$, $w_1 = P^k_{k,-k}$ is a solution. As is well known, a linearly independent solution w_2 has the form

$$c_{-1}w_1(z)\log(z+1) + \sum_{n=0} c_n(z+1)^n \qquad \text{with } c_{-1}c_0 \neq 0\,.$$

This function is unbounded around $z = -1$. Q.E.D.

Proof of Lemma 7. Expand $\hat{f}: \hat{f}(y, e^{i\psi}) = \sum_{\nu \geq -\ell} \hat{f}_\nu(y) e^{i\nu\psi}$. For $h(\tau, \theta, \varphi, \psi) = h_1(\tau) h_2(\theta) h_3(\varphi) e^{i\nu\psi}$ with $h_i \in C_0^\infty$ we have

$$(-i\hat{f}, \omega_3 h) = k(\hat{f}, h) ,$$

from which it follows that $\hat{f}_\nu(y)$ is of the form $f_\nu(\tau, \theta) e^{-ik\varphi}$ with $f_\nu \in L^2(R \times (0, \pi): \mathrm{ch}^2 \tau \sin\theta \, d\tau d\theta)$. Furthermore f satisfies

$$\begin{aligned} 0 = (f, [\Delta_o + k(k+1)]h) &= \left(f, \left[\partial_\theta^2 + \cot\theta \partial_\theta + \frac{1}{\sin^2\theta}\partial_\varphi^2 - \frac{2\nu\cot\theta}{\sin\theta}\partial_\varphi \right.\right. \\ &\quad \left.\left. - \frac{\nu^2}{\sin^2\theta} + k(k+1)\right]h\right) \\ &= \|e^{i\nu\psi}\|^2 (e^{-ik\varphi}, h_3)\left(f_\nu, \left[\partial_\theta^2 + \cot\theta\partial_\theta - \frac{k^2 + \nu^2 + 2k\nu\cos\theta}{\sin^2\theta}\right.\right. \\ &\quad \left.\left. + k(k+1)\right] h_1 h_2\right) . \end{aligned}$$

Putting $G_\nu(\tau, \cos\theta) = f_\nu(\tau, \theta)$, we conclude that $G_\nu(\tau, z)$ is a weak solution, consequently, a smooth solution of (13) for a.e.τ. Thus f must have the desired expression. Conversely if f is of the form (10), it satisfies (11) because h's finite linear combinations form a dense set in $\mathfrak{H}_0^{\pi, e}$. Q.E.D.

LEMMA 9. *Assume f in $\mathfrak{H}^\pi$ to be of the form*

(14) $$f(\tau, \theta, \varphi, e^{i\psi}) = \sum_{\nu \geq -\ell}^{k} f_\nu(\tau) P_{k,-\nu}^k(\cos\theta) e^{-ik\varphi + i\nu\psi}$$

for some integer k and f_ν in $C_0^\infty(R)$. Then f belongs to domains of ω_j, Δ_o, Δ and Δ' ($j = 1, 2, \cdots, 6$). F belongs to W_k, too.

Proof. We may suppose $f = f_\nu P_{k,-\nu}^k e^{-ik\varphi + i\nu\psi}$. We will show that there exists an $\tilde{f}$ in $\mathfrak{H}^\pi(G)$ such that

(15) $$\tilde{f}(\omega_6(\tau)u, e^{i\psi}) = f_\nu(\tau) t_{-\nu,-k}^k(u) e^{i\nu\psi} , \qquad I_e \tilde{f} = f$$

(see below (7) for the definition of $\mathfrak{H}^\pi(G)$ and I_e), where $t_{m,n}^t(u)$ is the (m, n) matrix element corresponding to an irreducible unitary representation of $SU(2)$ (chap. 3 [17]). It suffices to prove

(16) $$f_\nu(\tau') t_{-\nu,-k}^k(u') e^{i\nu\psi} = \pi(g_0)(f_\nu(\tau) t_{-\nu,-k}^k(u) e^{i\nu\psi})$$

assuming that $\omega_6(\tau')u' = g_0 \omega_6(\tau) u$. As one verifies easily, the condition implies that $\tau' = \tau$ and $g_0 = \omega_3(t)$ for some t. Thus it holds that

$$t_{-,-k}^k(u') = e^{i\nu t} t_{-\nu,-k}^k(u) , \qquad \pi(g_0) e^{i\nu\psi} = e^{i\nu(t+\psi)} ,$$

which proves (16). Take a compact set B of the hyperboloid V_{iM} so that any $f\circ s_u$ ($u\in SU(2)$) vanishes on the complement B^c, then find a finite covering $\{Y_\alpha\}$, the partition of unity and a finite set $\{u_\alpha\}\subset SU(2)$ satisfying $\mathrm{Supp}\,\chi_\alpha\subset Y\cdot u_\alpha$. Since $I_{u_\alpha}I_e^{-1}f\chi_\alpha=(\tilde f\cdot s_{u_\alpha})\chi_\alpha$ belongs to $\mathfrak{H}_0^{\pi,u_\alpha}$, $D_{\Delta^{\pi},u_\alpha}$, for example, contains it due to Lemma 6. This in turn implies that $f\chi_\alpha$, hence f itself, belongs to the domain of $\Delta^{\pi,e}$. Recalling $W_k=\hat W_k\cap D_{H_3}\cap D_{\Delta_0}$, we complete the proof. Q.E.D.

Finally we solve the equations (4).

PROPOSITION 1. *The space of k-th heighest weight vectors W_k for the representation $U^{\pi,e}|SL(2,C)$ with $\pi=\pi^+_{(\ell,0)}$ is as follows:*

$$W_k=\left\{\sum_{\nu\geqslant-\ell}^{k}f_\nu(\tau)P^k_{k,-\nu}(\cos\theta)e^{-ik\varphi+i\nu\psi}:f_\nu\in L^2(\boldsymbol{R},\mathrm{ch}^2\,d\tau)\right\}\qquad \textit{for } k=-\ell,-\ell+1,\cdots$$

$$=\{0\}\qquad \textit{otherwise}\,.$$

Proof. Since $U^{\pi,e}(0,-e)=I$, W_k is a null space provided k is a half integer. On account of Lemma 9 and closedness of H_3 and Δ_0, W_k includes the right side above. Keeping Lemma 7 in mind and assuming that

$$f(\tau,\theta,\varphi,e^{i\psi})=\sum_{\nu\geqslant-\ell}^{k}f_\nu(\tau)Q_\nu(\cos\theta)e^{-ik\varphi+i\nu\psi}\,,$$

where $Q_\nu(z)$ is a L^2-solution of (13) which is independent of $P^k_{k,-\nu}(z)$, we will show the opposite inclusion. By Lemma 8, Q_ν is either identically zero or unbounded arround -1 or 1. From (8) we see that $f^u=I_u\circ I_e^{-1}f$ has the form:

$$f^u(\tau,\theta,\varphi,e^{i\psi})=\sum_{\nu\geqslant-\ell}^{k}f_\nu(\tau)Q_\nu(\cos\theta')e^{-ik\varphi'+i\nu t+i\nu\psi}$$

provided $\omega_6(\tau)\omega_2(\theta)\omega_3(\varphi)u=\omega_6(\tau')\omega_3(t)\omega_2(\theta')\omega_3(\varphi')$. Since f^u belongs to $\hat W_k^{\pi,u}$, it satisfies

$$\sum_{i=1}^{3}(\hat\omega_i^{\pi,u})^2f^u=-k(k+1)f^u\,. \tag{17}$$

Put $Q_\nu^u(\theta,\varphi)=Q_\nu(\cos\theta')e^{ik\varphi'+i\nu t}$. Assume that $Q_\nu(z)$ is unbounded around 1 and that for a positive constant a $a^{-1}<|f_\nu(\tau)|<a$ on a non-null set B_ν. In other words we assume that $f_\nu(\tau)Q_\nu(\cos\theta)e^{-ik\varphi}$, as a function on Y, is not essentially bounded around $y=(-\mathrm{sh}\,\tau,0,0,1)$. Let $u\in SU(2)$ be so chosen that $q\circ p(\omega_6(\tau)\omega_1(\pi/2)\omega_3(\pi)u)=y$ (see (10) for q). By the assumption

$f_\nu(\tau)\ Q_\nu^u(\theta, \varphi)$ is not essentially bounded on $B_\nu \times (\pi/2 - \varepsilon, \pi/2 + \varepsilon) \times (\pi - \varepsilon, \pi + \varepsilon)$. We will conclude the proof showing that $\sin\theta Q_\nu^u(\theta, \varphi)$ must be a smooth function on $(\pi/2 - \varepsilon, \pi/2 + \varepsilon) \times (\pi - \varepsilon, \pi + \varepsilon)$. To this end choose an open neighborhood U_1 of a point of $(0, \pi) \times (0, 2\pi) \times T$ and an open neighborhood U_2 of the unit element of $SU(2)$ so that the map: $(\theta, \varphi, e^{i\psi}, u_2) \to (\theta, \varphi, e^{i2(\psi+t)})$ defined by $\omega_2(\theta)\omega_3(\varphi)u_2 = \omega_3(t)\omega_3(\theta')\omega_3(\varphi')$ is smooth on $U_1 \times U_2$ and that for each $(\theta, \varphi, e^{i\psi}) \in U_1$ the map: $u_2 \to (\theta', \varphi', e^{i(\psi+t)})$ from U_2 into $(0, \pi) \times (0, 2\pi) \times T$ is a diffeomorphism. It turns out that the restriction $\omega_i^{\pi,u} | \mathfrak{H}_0^{\pi,u}$ is of the form

$$\omega_i^{\pi,u} = (a_{i1}\partial_\theta + a_{i2}\partial_\varphi + a_{i3}\partial_\psi) ,$$

where a_{ij} $(i, j = 1, 2, 3)$ are real-valued C^∞-functions depending only on (θ, φ) with $\det(a_{ij}) \neq 0$. Now it is not difficult to see that $\sum_i (\hat{\omega}_i^{\pi,u})^2$ is an elliptic differential operator with C^∞-coefficient and that each $f_\nu Q_\nu^u e^{i\nu\psi}$ satisfies (17), from which the smoothness of $\sin\theta Q_\nu^u(\theta, \varphi)$ follows. Q.E.D.

We summarise the k-th heighest weight vectors W_k for the representations $U^{\pi,e}$.

π	ℓ	W_k $(\neq\{0\})$	k
$\pi_{(\ell,0)}$	$\ell = -1/2 + i\rho, \rho \geqslant 0$	$\sum_{\nu=-k}^{k} f_\nu P_{-\nu} e^{-ik\varphi+i\nu\psi}$	$0, 1, \cdots$
$\pi_{(\ell,1/2)}$	$\ell = -1/2 + i\rho, \rho > 0$	$\sum_{\nu=-k}^{k} f_\nu P_{-\nu} e^{-ik\varphi+i(\nu+1/2)\psi}$	$1/2, 3/2, \cdots$
$\pi_{(\ell,0)}$	$-1 < \ell < -1/2$	$\sum_{\nu=-k}^{k} f_\nu P_{-\nu} e^{-ik\varphi+i\nu\varphi}$	$0, 1, \cdots$
$\pi^+_{(\ell,0)}$	$\ell = -1, -2, \cdots$	$\sum_{\nu=-\ell}^{k} f_\nu P_{-\nu} e^{-ik\varphi+i\nu\psi}$	$-\ell, -\ell+1, \cdots$
$\pi^+_{(\ell,1/2)}$	$\ell = -1/2, -3/2, \cdots$	$\sum_{\nu=-\ell}^{k} f_\nu P_{-\nu} e^{-ik\varphi+i(\nu+1/2)\psi}$	as above
$\pi^-_{(\ell,0)}$	$\ell = -1, -2, \cdots$	$\sum_{\nu=\ell}^{-k} f_\nu P_{-\nu} e^{-ik\varphi+i\nu\psi}$	as above
$\pi^-_{(\ell,1/2)}$	$\ell = -1/2, -3/2, \cdots$	$\sum_{\nu=\ell}^{-k} f_\nu P_{-\nu} e^{-ik\varphi+i(\nu+1/2)\psi}$	as above

(Here we put $P_{-\nu} = P^k_{k,-\nu}$)

Denote W_k^0 a subspace of W_k consisting of functions expressible as (14). Making use of formulas (chap. 3, sec 4 [17])

$$\partial_\theta p^k_{m,n}(\cos\theta)=\frac{i}{2}(\sqrt{(k+n+1)(k-n)}\,P^k_{m,n+1}(\cos\theta) + \sqrt{(k+n)(k-n+1)}P^k_{m,n-1}(\cos\theta)), \tag{18}$$

$$i(m-n\cos\theta)P^k_{m,n}(\cos\theta) = \frac{\sin\theta}{2}(\sqrt{(k+n)(k-n+1)}\,P^k_{m,n-1}(\cos\theta) - \sqrt{(k-n)(k+n+1)}P^k_{m,n+1}(\cos\theta)), \tag{19}$$

and calculating formally, we see that

$$\Delta'\Big(\sum_{\nu\geqslant-\ell}^{k} f_\nu P^k_{k,-\nu}e^{-ik\varphi+i\nu\psi}\Big) = \sum_{\nu\geqslant-\ell}^{k}\Big[-2i\nu(\partial_\tau+\operatorname{th}\tau)f_\nu - (\ell+\nu+1)\sqrt{(k+\nu+1)(k-\nu)}\frac{f_{\nu+1}}{\operatorname{ch}\tau} + (\ell-\nu+1)\sqrt{(k-\nu+1)(k+\nu)}\frac{f_{\nu-1}}{\operatorname{ch}\tau}\Big]P^k_{k,-\nu}e^{-ik\varphi+i\nu\psi}. \tag{20}$$

Similarly, applying the formulas (18) (19) and

$$\sin\theta P^k_{k,-\nu} = -2i\sqrt{\frac{(k-\nu+1)(k+\nu+1)}{(2k+1)(2k+2)}}P^{k+1}_{k+1,-\nu},$$

$$\sin^2\frac{\theta}{2}P^k_{k,-\nu+1} = -\sqrt{\frac{(k+\nu)(k+\nu+1)}{(2k+1)(2k+2)}}P^{k+1}_{k+1,-\nu},$$

$$\cos^2\frac{\theta}{2}P^k_{k,-\nu-1} = \sqrt{\frac{(k-\nu)(k-\nu+1)}{(2k+1)(2k+2)}}P^{k+1}_{k+1,-\nu}$$

we obtain

$$\begin{aligned} &F_+\Big(\sum_{\nu\geqslant-\ell}^{k} f_\nu P^k_{k,-\nu}e^{-ik\varphi+i\nu\psi}\Big) \\ &= \frac{1}{\sqrt{(2k+1)(2k+2)}}\sum_{\nu\geqslant-\ell}^{k+1}\Big[2i\sqrt{(k-\nu+1)(k+\nu+1)} \\ &\times(\partial_\tau - k\operatorname{th}\tau)f_\nu + (\ell+\nu+1)\sqrt{(k-\nu)(k-\nu+1)}\frac{f_{\nu+1}}{\operatorname{ch}\tau} \\ &+ (\ell-\nu+1)\sqrt{(k+\nu)(k+\nu+1)}\frac{f_{\nu-1}}{\operatorname{ch}\tau}\Big] \\ &\times P^{k+1}_{k+1,-\nu}e^{-i(k+1)\varphi+i\nu\psi}. \end{aligned} \tag{21}$$

Since f in W_k^0 is C^∞-function on V_{iM}, the formal calculus can be justified.

Set $c_\nu = \|e^{i\nu\psi}\|_\pi$. The isometry J_k from W_k onto $\sum_{\nu\geqslant-\ell}^{k} \oplus L^2(R)$ defined by

$$\sum_{\nu\geqslant-\ell}^{k} f_\nu P^k_{k,-\nu} e^{-ik\varphi+i\nu\varphi} \to \left(\sqrt{\frac{2}{2k+1}}\, c_\nu f_\nu(\tau)\, \mathrm{ch}\,\tau\right) \tag{22}$$

transforms $\Delta' \mid W_k^0$ to $\dot{L}_k^\pi$:

$$\dot{L}_k^\pi = -2i(\nu)\partial_\tau + \frac{1}{\mathrm{ch}\,\tau} V\,, \tag{23}$$

where $(\nu) = \begin{pmatrix} k & & & & \\ & k-1 & & & \\ & & \ddots & & \\ & & & \nu & \\ & & & & \ddots \\ & & & & & -\ell \end{pmatrix}$ and V is an hermitian matrix whose $(\nu, \nu+1)$ component is equal to $-\sqrt{(-\ell+\nu)(\ell+\nu+1)(k+1+1)}$. Since the symmetric operator $\dot{L}_k^\pi$ is essentially selfadjoint with domain $\sum_{\nu\geqslant-\ell}^{k} C_0^\infty(\boldsymbol{R})$ [7], we denote L_k^π its selfadjoint extension. Now the following proposition is selfexplanatory.

PROPOSITION 2. *For the representation* $\pi = \pi^+_{(\ell,0)}$ *the restriction* $\Delta'^{\pi,e} \mid W_k$ *is unitarily equivalent to* L_k^π *provided* $k = -\ell, -\ell+1, \cdots$.

Similarly we have

PROPOSITION 3. *For the representation* $\pi = \pi_{(\ell,0)}$ *either with* $\ell = -1/2 + i\rho$ $(\rho \geqslant 0)$ *or with* $-1 < \ell < -1/2$, *the restriction* $\Delta^{\pi,e} \mid W_0$ *is unitarily equivalent to* L_0^π *which is the selfadjoint extension of a symmetric operator* $\dot{L}_0^\pi$ *on* $L^2(R)$ *with domain* $C_0^\infty(\boldsymbol{R})$:

$$\dot{L}_0^\pi = -\partial_\tau^2 - \frac{\ell(\ell+1)}{\mathrm{ch}^2\,\tau}\,. \tag{24}$$

For a Borel set B of $\boldsymbol{R}$ and σ-finite measure σ on B, let $\int_B^\oplus \lambda d\sigma$ denote the λ-multiplication operator in $L^2(B, \sigma)$.

PROPOSITION 4. (i) *For the representation* $\pi = \pi^+_{(\ell,0)} L_k^\pi$ *is unitarily equivalent to* $[k+\ell+1]\int_R^\oplus \lambda d\lambda$. (ii) *For the representation* $\pi = \pi_{(\ell,0)}$ *either*

$\ell = -1/2 + i\rho$ ($\rho \geqslant 0$) or with $-1 < \ell < -1/2$, L_0^π is unitarily equivalent to $[2]\int_{R_+}^{\oplus} \lambda d\lambda$.

Proof. Applying the result of [7], we obtain (i). We note that L_0^π is a Schrödinger operator with a so-called short range potential. So (ii) is a direct consequence of Agmon [1] and Kato [9]. Q.E.D.

PROPOSITION 5. *For the representation $\pi = \pi_{(\ell,0)}^+$, $\Delta'^{\pi,e} | W_k \ominus F_+^{\pi,e} W_{k-1}$ is unitarily equivalent to $\int_R^{\oplus} \lambda d\lambda$ provided $k = -\ell, -\ell + 1, \cdots$.*

Proof. Lemma 4 and (i) of Proposition 4 yield the proposition. Q.E.D.

For the representation $\pi = \pi_{(\ell,0)}$ with $\ell = -1/2 + i\rho$ ($\rho \geqslant 0$) or with $-1 < \ell < -1/2$ L_k^π is unitarily equivalent to $[2k]\int_R^{\oplus} \lambda d\lambda \oplus [\aleph_0]\int_{\{0\}}^{\oplus} \lambda\delta(d\lambda)$ for any positive integer k, where δ denotes the Dirac measure. In order to show that $\Delta'^{\pi,e} | W_k \ominus F_+^{\pi,e} W_{k-1}$ is unitarily equivalent to $[2]\int_R^{\oplus} \lambda d\lambda$ we must check that $\Delta'^{\pi,e} | W_k \ominus F_+^{\pi,e} W_{k-1}$ has no eigenvectors with eigenvalue zero. This requires some calculation which we do not cite here. In this way we can manage to decompose the induced representations $\operatorname{Ind}_{SU(1,1)\uparrow SL(2,C)} \pi$ (cf. [3] [13]).

§ 5. Proof of Theorem 1 and 3

We begin with

LEMMA 10. *Let T_t and S_s be one-parameter unitary groups on $L^2(R)$:*

$$T_t f(\tau) = e^{iMt \operatorname{sh} \tau} f(\tau)\,, \quad S_s f(\tau) = f(\tau + s) \quad (M \neq 0)\,.$$

Then a closed subspace D of $L^2(R)$ which is invariant with respect to $\{T_t : t \geqslant 0\}$ and $\{S_s : s \in R\}$ is either $L^2(R)$ or the null space $\{0\}$.

Proof. Denote $\hat{f}$ the Fourier transform of f. Since D is S_s-invariant, there exists a Borel set B such that $D = \{f \in L^2(R) : \hat{f}(\lambda) = 0$ on the complement $B^c\}$. If the Lebesgue measure $|B|$ is equal to zero, we have nothing to do. Otherwise, from the fact that Laplace transform $G_\alpha = \int_{R_+} e^{-\alpha t} T_t dt$ is just the multiplication $1/(\alpha - iM \operatorname{sh} \tau)$ it follows that for non-zero element f of D Fourier transform of $G_\alpha f \in D$ is a non-zero holomorphic function on the strip $|\mathrm{Im}\,\lambda| < 1$. Thus $|B^c| = 0$. Q.E.D.

Proof of Theorem 1. First note that Theorem 2 also holds for the 2-dimensional space-time Poincaré group. Irreducible unitary representations corresponding to space-like orbits $V^{\pm iM}(2) = \{\hat{x}_0^2 - \hat{x}_3^2 = -M^2 : \hat{x}_3 \gtrless 0\}$ have the realization in $L^2(\boldsymbol{R})$:

$$U^{iM}((x_0, x_3), \omega_6(s))f(\tau) = \exp(\pm iM(x_0 \operatorname{sh}\tau + x_3 \operatorname{ch}\tau))f(\tau + s)\ .$$

Now Lemma 10 yields the theorem. Q.E.D.

Let us turn to the proof of Theorem 3. As in § 4, W_k stands for the k-th heighest weight vectors corresponding to the representation $(U^{\pi,e} \mid G, \mathfrak{H}^\pi)$ of $G = SL(2, \boldsymbol{C})$. Denote k_0 the minimum of $\{k : W_k \neq \{0\}\}$. We observe

LEMMA 11. *If there exists an invariant non-trivial closed subspace D_+ of $\mathfrak{H}^\pi$ with respect to the Poincaré subsemigroup P_+, then there exists a non-trivial closed subspace D of W_{k_0} which is invariant with respect to $\{T_t = e^{iMt\operatorname{sh}\tau} : t > 0\}$ and $\{e^{it\Delta}, e^{is\Delta'} : s \in \boldsymbol{R}\}$.*

Proof. Our reasoning depends on the results of § 3. Denoting the orthogonal complement of D_+ by $D_+^\perp$, it holds that

$$W_{k_0} = (W_{k_0} \cap D_+) \oplus (W_{k_0} \cap D_+^\perp)\ . \tag{25}$$

We know that $W_{k_0} \cap D_+$ (resp. $D_+^\perp$) is invariant with respect to T_t $(t > 0)$ resp. $t < 0$), Δ and Δ'. Thus both components on the right side of (25) have the same property. We claim none of them is a null space. We will show this for $W_{k_0} \cap D_+$. The proof for the another component is similar. If $W_{k_0} \cap D_+$ is a null space, some $k, k \geqslant k_0$ attains the maximum of $\{k' : W_{k'} \cap D_+ = \{0\}\}$. Since the decomposition (25) holds for any k, W_k is a subspace of $D_+^\perp$. Thus $F_+W_k^0$ and $F_+\bar{G}_\alpha W_k^0$ are orthogonal to $W_{k+1} \cap D_+$, where $\bar{G}_\alpha$ denotes Laplace transform $\int_{R+} e^{-\alpha t}T_{-t}dt = 1/(\alpha + iM\operatorname{sh}\tau)$. An $f \in J_{k+1}(W_{k+1} \cap D_+)$ satisfies

$$(f, J_{k+1}F_+J_k^{-1}h) = 0\ , \quad (G_\alpha f, J_{k+1}F_+J_k^{-1}h) = 0 \quad \text{for any } h \in J_kW_k^0 \tag{26}$$

(see (22) for J_k). From the second equality it follows that

$$\left(A\frac{iM\operatorname{ch}\tau}{(\alpha - iM\operatorname{sh}\tau)^2}f, \check{h}\right) + (f, J_{k+1}F_+J_k^{-1}\bar{G}_\alpha h) = 0 \qquad \text{for any } h \in J_kW_k^0\ , \tag{27}$$

where A is a constant diagonal matrix whose (ν, ν) component is equal to $2i\sqrt{(k-\nu+1)(k+\nu+1)}/\sqrt{(2k+2)(2k+3)}$ and $\check{h}$ denotes $(0, h^t)^t \in J_{k+1}W_k^0$.

Since the second term of (27) vanishes, f_ν is zero except f_{k+1}. Together with the first equality of (26) f vanishes. This completes the proof.

Q.E.D.

Proof of Theorem 3. For the representation $U^{\pi,e}$ (see (6)) with, say $\pi=\pi^+_{(\ell,0)}$, W_{k_0} coincides with $W_{-\ell}$. Since J_{k_0} transforms T_t and Δ' to T_t and $2i\ell\partial_\tau$ respectively, the theorem follows from Lemma 10 and 11.

Q.E.D.

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