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MODULAR FORMS OF DEGREE n AND REPRESENTATION BY QUADRATIC FORMS

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Let $A^{(m)}$, $B^{(n)}$ be positive definite integral matrices and suppose that B is represented by A over each p-adic integers ring $\boldsymbol{Z}_p$. Using the circle method or theory of modular forms in case of $n = 1$, B, if sufficiently large, is represented by A provided that $m \geq 5$. The approach via the theory of modular forms has been extended by [7] to Siegel modular forms to obtain a partial result in the particular case when $n = 2$, $m \geq 7$. On the other hand Kneser gave an arithmetic approach in case of $n = 1$ in his lectures [4] . Using this idea we proved that B is represented by A over $\boldsymbol{Z}$ provided that $m \geq 2n + 3$ and the minimum of B is sufficiently large [2]. Our aim here is to give an analytic proof in the case when A is an even unimodular positive definite matrix. Under the algebraic preparations of § 1 we give the Fourier expansion of Eisenstein series in the sense of Klingen and estimate coefficients from above. In § 3 we estimate Fourier coefficients of usual Eisenstein series from above and below and it is applied to our problem in § 4.

Notations. Let H_n be the space of $n \times n$ complex symmetric matrices Z with positive real Y. Let Γ_n denote the group of integral $2n \times 2n$ matrices M satisfying

$$MI{}^tM = I\,, \qquad I = \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}.$$

For $M \in \Gamma_n$ we put $M = \begin{pmatrix} A_M & B_M \\ C_M & D_M \end{pmatrix}$ where $A_M, \cdots, D_M$ are $n \times n$ matrices. For $1 \leq r \leq n-1$ we define the subgroup $\Delta_{n,r}$ of Γ_n as the group of all $M \in \Gamma_n$ whose elements in the first $n + r$ colums and last $n - r$ rows vanish. The transposed matrix of a matrix M is denoted by ${}^tM$. We don't use the usual convention $A[B] = {}^tBAB$. σ stands for the trace of matrices. $e(x)$ means $\exp(2\pi ix)$.

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§1.

Through this section we fix natural numbers r, n which satisfy $1 \leq r \leq n-1$. For a matrix $M \in M_{p,q}(C)$ $(p \geq n-r)$ we denote by $\tilde{M}$ the last $n-r$ rows of M, i.e., $M = \begin{pmatrix} * \\ \tilde{M} \end{pmatrix}$. For $M \in M_n(C)$ we decompose M as $M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$ where $M_1 \in M_r(C)$, $M_2 \in M_{r,n-r}(C)$, $M_3 \in M_{n-r,r}(C)$, $M_4 \in M_{n-r}(C)$. M_i are used in this sense if we don't refer.

LEMMA 1. *For $M, N \in \Gamma_n$, $\Delta_{n,r}M = \Delta_{n,r}N$ is equivalent to $\tilde{M} = g\tilde{N}$ for some $g \in GL(n-r, Z)$.*

Proof. Let M, N be elements of Γ_n. Suppose that $M = KN$ where $K = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{*} & \multicolumn{2}{c}{*} \\ \hline * & * & * & * \\ 0 & 0 & 0 & D_4 \end{array}\right) \in \Delta_{n,r}$. Then we have $\tilde{M} = D_4\tilde{N}$. Since K is a unimodular matrix, D_4 is unimodular. Conversely, suppose that $\tilde{M} = g\tilde{N}$ for some $g \in GL(n-r, Z)$. Put $G = \left(\begin{array}{c|cc} * & \multicolumn{2}{c}{0} \\ \hline & 1_r & 0 \\ 0 & 0 & g \end{array}\right) \in \Delta_{n,r}$; then we have $\widetilde{GN} = g\tilde{N}$. Hence we may assume $\tilde{M} = \tilde{N}$. Then $\widetilde{MN^{-1}} = (0^{(n+r,n-r)}, 1_{n-r})$ holds. Hence $MN^{-1} \in \Delta_{n,r}$. Q.E.D.

LEMMA 2. *For an element $N \in \Gamma_n$ with rank $\tilde{C}_N < n-r$, there is an element M in $\Delta_{n,n-1}$ such that $\Delta_{n,r}N \ni M\begin{pmatrix} U & 0 \\ 0 & {}^tU^{-1} \end{pmatrix}$ for some $U \in GL(n, Z)$.*

Proof. By the assumption rank $\tilde{C}_N < n-r$ there are unimodular matrices $g \in GL(n-r, Z)$, $V \in GL(n, Z)$ such that the last row of $g\tilde{C}_N V$ vanishes. Put $K = \left(\begin{array}{c|cc} * & \multicolumn{2}{c}{0} \\ \hline & 1_r & 0 \\ 0 & 0 & g \end{array}\right) N \begin{pmatrix} V & \\ & {}^tV^{-1} \end{pmatrix}$; then the last row of C_K vanishes and the elements of the last row of D_K are relatively prime. Taking a unimodular matrix $W \in GL(n, Z)$ such that $D_K W = \begin{pmatrix} & * & \\ 0 \cdots & 0 & 1 \end{pmatrix}$, we put $M = K\begin{pmatrix} {}^tW^{-1} & \\ & W \end{pmatrix}$; then we have $M \in \Delta_{n,n-1}$ since the last row of M is $(0, \cdots, 0, 1)$. We may take $V{}^tW^{-1}$ as U. Q.E.D.

LEMMA 3. *If ${}^tAC$ is symmetric for $A, C \in M_m(Z)$, then there is a symmetric coprime pair $(\mathfrak{C}^{(m)}, \mathfrak{D}^{(m)})$ such that $\mathfrak{C}A + \mathfrak{D}C = 0$.*

Proof. If $C = 0$, then we may take $\mathfrak{C} = 0$, $\mathfrak{D} = 1_m$. Suppose $C \neq 0$. First we assume $|C| \neq 0$. Then AC^{-1} is a symmetric rational matrix. Hence there is a symmetric coprime pair $(\mathfrak{C}^{(m)}, \mathfrak{D}^{(m)})$ such that $|\mathfrak{C}| \neq 0$,

$\mathfrak{C}^{-1}\mathfrak{D} = -AC^{-1}$ (p. 166 in [5]). Thus we have $\mathfrak{C}A + \mathfrak{D}C = 0$. Next we assume $|C| = 0$; then there are unimodular matrices $g_1, g_2 \in GL(m, Z)$ such that $g_1 C g_2^{-1} = c = \begin{pmatrix} c_1^{(s)} & 0 \\ 0 & 0 \end{pmatrix}$ ($|c_1| \neq 0, s \geq 1$). Putting $a = {}^t g_1^{-1} A g_2^{-1}$, we have ${}^t ac = {}^t g_2^{-1}\,{}^t A C g_2^{-1}$ and this is a symmetric matrix. Decompose a as $a = \begin{pmatrix} a_1^{(s)} & a_2 \\ a_3 & a_4 \end{pmatrix}$. ${}^t ac = {}^t ca$ implies that ${}^t a_1 c_1$ is symmetric and ${}^t a_2 c_1 = 0$ and so $a_2 = 0$. Take a symmetric coprime pair $(\mathfrak{C}_1^{(s)}, \mathfrak{D}_1^{(s)})$ such that $\mathfrak{C}_1 a_1 + \mathfrak{D}_1 c_1 = 0$, and put $\mathfrak{C} = \begin{pmatrix} \mathfrak{C}_1 & \\ & 0 \end{pmatrix}$. $\mathfrak{D} = \begin{pmatrix} \mathfrak{D}_1 & \\ & 1_{m-s} \end{pmatrix}$. Then $(\mathfrak{C}, \mathfrak{D})$ is a symmetric coprime pair and

$$\mathfrak{C}a + \mathfrak{D}c = \begin{pmatrix} \mathfrak{C}_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} \mathfrak{D}_1 & \\ & 1 \end{pmatrix}\begin{pmatrix} c_1 & \\ & 0 \end{pmatrix} = 0\,.$$

On the other hand, $\mathfrak{C}a + \mathfrak{D}c = (\mathfrak{C}\,{}^t g_1^{-1} A + \mathfrak{D} g_1 C) g_1^{-1}$ implies $\mathfrak{C}\,{}^t g_2^{-1} A + \mathfrak{D} g_1 C = 0$. Here it is easy to see that $(\mathfrak{C}\,{}^t g_1^{-1}, \mathfrak{D} g_1)$ is a symmetric coprime pair. This completes the proof. Q.E.D.

LEMMA 4. *For $M \in \Gamma_n$ there are a unimodular matrix $U \in GL(n, Z)$ and $N \in \Delta_{n,r} M \begin{pmatrix} {}^t U & \\ & U^{-1} \end{pmatrix}$ such that the first r colums of C_N vanishes.*

Proof. Take $g \in GL(n - r, Z)$, $U \in GL(n, Z)$ such that $g\tilde{C}_M {}^t U = (0, C_4)$; $C_4 \in M_{n-r}(Z)$. Put $K = \left(\begin{array}{c|c} * & \\ \hline & 1_r \\ & g \end{array}\right) M \begin{pmatrix} {}^t U & \\ & U^{-1} \end{pmatrix}$; then we have $C_K = \begin{pmatrix} C_1 & C_2 \\ 0 & C_4 \end{pmatrix}$. ${}^t AC = {}^t CA$ implies ${}^t A_1 C_1 = {}^t C_1 A_1$ where we put $A = A_K$, $C = C_K$. By Lemma 3 there is a symmetric coprime pair $(\mathfrak{C}_1^{(r)}, \mathfrak{D}_1^{(r)})$ such that $\mathfrak{C}_1 A_1 + \mathfrak{D}_1 C_1 = 0$. Take an element $G \in \Delta_{n,r}$ such that $C_G = \begin{pmatrix} \mathfrak{C}_1 & 0 \\ 0 & 0 \end{pmatrix}$, $D_G = \begin{pmatrix} \mathfrak{D}_1 & 0 \\ 0 & 1 \end{pmatrix}$; then $C_{GK} = \begin{pmatrix} \mathfrak{C}_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} + \begin{pmatrix} \mathfrak{D}_1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} C_1 & C_2 \\ & C_4 \end{pmatrix} = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}$. This completes the proof. Q.E.D.

We put $P_{n,r} = \left\{ A = \begin{pmatrix} A_1^{(r)} & A_2 \\ A_3 & A_4 \end{pmatrix} \in GL(n, Z) \,|\, A_3 = 0 \right\}$.

LEMMA 5. *Let M, N be elements of Γ_n such that $\begin{pmatrix} C_1 \\ C_3 \end{pmatrix} = \begin{pmatrix} C_1' \\ C_3' \end{pmatrix} = 0$, $|C_4||C_4'| \neq 0$ where we put $C_M = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$, $C_N = \begin{pmatrix} C_1' & C_2' \\ C_3' & C_4' \end{pmatrix}$. If $KM\begin{pmatrix} {}^t V & \\ & V^{-1} \end{pmatrix} = N\begin{pmatrix} {}^t U & \\ & U^{-1} \end{pmatrix}$ for $K \in \Delta_{n,r}$, U, $V \in GL(n, Z)$, then we have $C_K = 0$, ${}^t U \in P_{n,r} {}^t V$.*

Proof. Put $W = {}^t U {}^t V^{-1}$; then $C_{KM} = C_K A_M + D_K C_M = C_N W$. Putting $A_M = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$, $C_K = \begin{pmatrix} \mathfrak{C}_1 & 0 \\ 0 & 0 \end{pmatrix}$, $D_K = \begin{pmatrix} \mathfrak{D}_1 & \mathfrak{D}_2 \\ & \mathfrak{D}_4 \end{pmatrix}$, $W = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix}$, we have

$C_4'W_3=0$ and $\mathfrak{C}_1A_1=C_2'W_3$. The assumption $|C_4'|\neq 0$ implies $W_3=0$. Hence ${}^tU=W{}^tV\in P_{n,r}{}^tV$ and $\mathfrak{C}_1A_1=0$. We have only to prove $|A_1|\neq 0$. Since ${}^tA_MC_M=\begin{pmatrix} * & {}^tA_1C_2+{}^tA_3C_4 \\ 0 & * \end{pmatrix}$ is symmetric, we have ${}^tA_1C_2+{}^tA_3C_4=0$ and $A_3=-{}^tC_4^{-1}\,{}^tC_2A_1$. Since the rank of the first r colums of M is r, we have $r=\operatorname{rank}\begin{pmatrix} A_1 \\ A_3 \end{pmatrix}=\operatorname{rank}\begin{pmatrix} 1_r \\ -{}^tC_4^{-1t}C_2 \end{pmatrix}A_1\leq \operatorname{rank} A_1\leq r$. Thus we have $\operatorname{rank} A_1=r$, i.e., $|A_1|\neq 0$ and so $\mathfrak{C}_1=0$. Q.E.D.

The following lemma is a key in this paper.

LEMMA 6. *Let M be an element of Γ_n with* $\operatorname{rank}\tilde{C}_M=n-r$. *Then there is an element $N\in\Delta_{n,r}M$ such that $|C_N|\neq 0$, $(A_NC_N^{-1})_1\equiv 0 \bmod 1$.*

Proof. By virtue of Lemma 4 there exist $P\in\Gamma_n$, $U\in GL(n,Z)$ such that $\Delta_{n,r}M\ni P\begin{pmatrix} {}^tU & \\ & U^{-1} \end{pmatrix}$, $C_P=\begin{pmatrix} 0 & C_2 \\ 0 & C_4 \end{pmatrix}$. $\operatorname{rank}\tilde{C}_M=n-r$ yields $|C_4|\neq 0$. Put $A_P=\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$; then $|A_1|\neq 0$ holds as in the proof of Lemma 5. Take $K=\left(\begin{array}{c|c} 1_n & 0 \\ \hline \begin{matrix} 1_r & 0 \\ 0 & 0 \end{matrix} & 1_n \end{array}\right)\in\Delta_{n,r}$ and put $KP=N'$. Then $A_{N'}=A_P$, $C_{N'}=\begin{pmatrix} A_1 & A_2+C_2 \\ 0 & C_4 \end{pmatrix}$ imply $|C_{N'}|\neq 0$, $(A_{N'}C_{N'}^{-1})_1=1_r$. Taking $N'\begin{pmatrix} {}^tU & \\ & U^{-1} \end{pmatrix}$ as N, we have $|C_N|\neq 0$, $(A_NC_N^{-1})_1=1_r$ and $\Delta_{n,r}M\ni N$. Q.E.D.

LEMMA 7. *Let C_4, D_3, D_4 be elements of $M_{n-r}(Z)$, $M_{n-r,r}(Z)$, $M_{n-r}(Z)$ respectively. Suppose that $|C_4|\neq 0$, $C_4{}^tD_4$ is symmetric and (C_4, D_3, D_4) is primitive. Then there is a symmetric coprime pair $(C^{(n)}, D^{(n)})$ such that $\tilde{C}=(0, C_4)$, $\tilde{D}=(D_3\ D_4)$.*

Proof. Since $\operatorname{rank}(C_4, D_4)=n-r$, there exist matrices $U_4\in M_{n-r}(Z)$, $V\in GL(2(n-r), Z)$ such that $(C_4, D_4)=U_4(0, 1_{n-r})V$ and $|U_4|\neq 0$. Put $\mathfrak{C}_4=U_4^{-1}C_4$, $\mathfrak{D}_4=U_4^{-1}D_4$; then $(\mathfrak{C}_4, \mathfrak{D}_4)$ is primitive and $\mathfrak{C}_4{}^t\mathfrak{D}_4=U_4^{-1}C_4{}^tD_4{}^tU_4^{-1}$ is symmetric. Thus $(\mathfrak{C}_4, \mathfrak{D}_4)$ is a symmetric coprime pair. Since (C_4, D_3, D_4) is primitive, $(D_3, C_4, D_4)=(D_3, U_4(0, 1_{n-r})V)$ is also primitive. Hence (D_3, U_4) is primitive and there is a unimodular matrix $U=\begin{pmatrix} * & * \\ D_3 & U_4 \end{pmatrix}\in GL(n, Z)$. Put $C=U\begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{C}_4 \end{pmatrix}$, $D=U\begin{pmatrix} 1_r & \\ & \mathfrak{D}_4 \end{pmatrix}$; then (C, D) is a symmetric coprime pair and $\tilde{C}=(0, U_4\mathfrak{C}_4)=(0, C_4)$, $\tilde{D}=(D_3, U_4\mathfrak{D}_4)=(D_3, D_4)$. This completes the proof. Q.E.D.

Let C_4, D_3, D_4 be those as in Lemma 7 and let M be an element of Γ_n such that $\tilde{M}=(0, C_4, D_3, D_4)$. By Lemma 6 there is $K\in\Delta_{n,r}$ such that

$|C_N| \neq 0$, $(A_N C_N^{-1})_1 \equiv 0 \bmod 1$ for $N = KM$. By Lemma 1 $g\tilde{N} = \tilde{M}$ holds for some $g \in GL(n-r, Z)$. Put $P = \left(\begin{array}{c|cc} * & \multicolumn{2}{c}{0} \\ \hline 0 & 1_r & \\ & & g \end{array}\right) N$; then $\tilde{P} = g\tilde{N} = \tilde{M} = (0, C_4, D_3, D_4)$, $|C_P| \neq 0$, and $(A_P C_P^{-1})_1 \equiv 0 \bmod 1$. We denote one of such P's by $M[C_4, D_3, D_4]$. Then $\Delta_{n,r} M[C_4, D_3, D_4]$ is uniquely determined by (C_4, D_3, D_4) by Lemma 1.

Put

$$\mathfrak{S}_{n,r} = \left\{(C_4, D_3, D_4) \,\middle|\, \begin{array}{l} C_4, D_4 \in M_{n-r}(Z),\ D_3 \in M_{n-r,r}(Z),\ |C_4| \neq 0, \\ C_4\,{}^tD_4 = D_4\,{}^tC_4 \text{ and } (C_4, D_3, D_4) \text{ is primitive} \end{array}\right\}.$$

For $S, S' \in \mathfrak{S}_{n,r}$ we define $S \sim S'$ by $S' = gS$ for some $g \in GL(n-r, Z)$.

LEMMA 8. $\displaystyle\bigcup_{\substack{M \in \Gamma_n \\ \operatorname{rank} \tilde{C}_M = n-r}} \Delta_{n,r} M = \bigcup \Delta_{n,r} M[C_4, D_3, D_4]\begin{pmatrix} {}^tU & \\ & U^{-1} \end{pmatrix}$, *where the right hand is a disjoint union,* (C_4, D_3, D_4) (resp. ${}^tU$) *runs over representatives of* $\mathfrak{S}_{n,r}/\sim$ (resp. $P_{n,r}\backslash GL(n, Z)$).

Proof. Take an element M of Γ_n such that $\operatorname{rank} \tilde{C}_M = n - r$. Then there exist $g \in GL(n-r, Z)$, $U \in GL(n, Z)$ such that $g^{-1}\tilde{C}_M\,{}^tU^{-1} = (0, C_4^{(n-r)})$, $|C_4| \neq 0$. Hence $M \in \Delta_{n,r} M[C_4, D_3, D_4]\begin{pmatrix} {}^tU & \\ & U^{-1} \end{pmatrix}$. For $W = \begin{pmatrix} W_1 & W_2 \\ & W_4 \end{pmatrix} \in P_{n,r}$ and $h \in GL(n-r, Z)$, we put $N = \left(\begin{array}{c|cc} * & & \\ \hline & 1_r & \\ & & h \end{array}\right) M[C_4, D_3, D_4]\begin{pmatrix} W & \\ & {}^tW^{-1} \end{pmatrix}$. Then $C_N = \begin{pmatrix} * & * \\ 0 & hC_4W_4 \end{pmatrix}$ holds. Thus M is contained in some coset $\Delta_{n,r} M[C_4, D_3, D_4]\begin{pmatrix} {}^tU & \\ & U^{-1} \end{pmatrix}$ for any specified representatives (C_4, D_3, D_4), U. We must prove the disjointness of the right hand. Suppose $KM[C_4, D_3, D_4]\begin{pmatrix} {}^tU & \\ & U^{-1} \end{pmatrix} = M[C_4', D_3', D_4']\begin{pmatrix} {}^tU' & \\ & U'^{-1} \end{pmatrix}$ for (C_4, D_3, D_4), $(C_4', D_3', D_4') \in \mathfrak{S}_{n,r}/\sim$, ${}^tU, {}^tU' \in P_{n,r}\backslash GL(n, Z)$, $K \in \Delta_{n,r}$. There are some $G, G' \in \Delta_{n,r}$ such that the first r-colums of the C-parts of $GM[C_4, D_3, D_4]$, $G'M[C_4', D_3', D_4']$ vanish as in the proof of Lemma 4. Hence Lemma 5 implies ${}^tU \in P_{n,r}{}^tU'$. Thus we have $U = U'$ and then $KM[C_4, D_3, D_4] = M[C_4', D_3', D_4']$ implies $g'(C_4, D_3, D_4) = (C_4', D_3', D_4')$ where g' is a unimodular matrix defined by the right lower $(n-r) \times (n-r)$ submatrix of K. Hence $(C_4, D_3, D_4) = (C_4', D_3', D_4')$.

Q.E.D.

We introduce another equivalence relation $\approx$ in $\mathfrak{S}_{n,r}$. For (C_4, D_3, D_4),

$(C_4', D_3', D_4') \in \mathfrak{S}_{n,r}$ we define $(C_4, D_3, D_4) \approx (C_4', D_3', D_4')$ by $g(C_4', D_3', D_4') = (C_4, D_3 + C_4S_3, D_4 + C_4S_4)$ for some $g \in GL(n-r, Z)$, $S_3 \in M_{n-r,r}(Z)$, $S_4 = {}^tS_4 \in M_{n-r}(Z)$. It is easy to see that $(C_4', D_3', D_4') \approx (C_4\ D_3, D_4)$ if and only if $\Delta_{n,r}M[C_4', D_3', D_4'] = \Delta_{n,r}M[C_4, D_3, D_4]\left(\begin{array}{c|c} 1 & S \\ \hline 0 & 1 \end{array}\right)$ for some $S = {}^tS = \begin{pmatrix} 0 & S_2 \\ S_3 & S_4 \end{pmatrix} \in M_n(Z)$.

THEOREM.

$$\bigcup_{\substack{M \in \Gamma_n \\ \operatorname{rank} C_M = n-r}} \Delta_{n,r}M = \bigcup \Delta_{n,r}M[C_4, D_3, D_4]\left(\begin{array}{c|c} 1 & S \\ \hline 0 & 1 \end{array}\right)\begin{pmatrix} {}^tU & \\ & U^{-1} \end{pmatrix},$$

where the right hand is a disjoint union, (C_4, D_3, D_4) (resp. ${}^tU$) *runs over representatives of* $\mathfrak{S}_{n,r}/\approx$ (resp. $P_{n,r}\backslash GL(n, Z)$) *and* S *runs over* $\{S = {}^tS \in M_n(Z) \mid S_1 = 0\}$.

Proof. This is an immediate corollary of Lemma 8. Q.E.D.

We remark the following two propositions although they are not used for our aim.

PROPOSITION 1. *Take* $(C_4, D_3, D_4) \in \mathfrak{S}_{n,r}$ *and* M, $N \in \Gamma_n$ *such that* $\tilde{M} = \tilde{N} = (0, C_4, D_3, D_4)$, $|C_M||C_N| \neq 0$ $(A_MC_M^{-1})_1 \equiv (A_NC_N^{-1})_1 \equiv 0 \bmod 1$. *Then we have* $GL(r, Z)(C_M)_1 = GL(r, Z)(C_N)_1$.

Proof. Put $A_M = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$, $C_M = \begin{pmatrix} C_1 & C_2 \\ & C_4 \end{pmatrix}$; ${}^tA_MC_M = {}^tC_MA_M$ implies ${}^tA_1C_1 = {}^tC_1A_1$ and $A_1C_1^{-1} = {}^t(A_1C_1^{-1})$ is an integral matrix. Put $K = \left(\begin{array}{cc|cc} 0 & 0 & -1_r & 0 \\ 0 & 1_{n-r} & 0 & 0 \\ \hline 1_r & 0 & -A_1C_1^{-1} & 0 \\ 0 & 0 & 0 & 1_{n-r} \end{array}\right)$; then $K \in \Delta_{n,r}$ and $C_{KM} = \begin{pmatrix} 0 & * \\ 0 & C_4 \end{pmatrix}$. We define similarly $K' \in \Delta_{n,r}$ for N. Then $\Delta_{n,r}KM = \Delta_{n,r}K'N$ and Lemma 5 imply

$$K'N = \begin{pmatrix} U & US \\ & {}^tU^{-1} \end{pmatrix}KM,$$

where $\begin{pmatrix} U & US \\ & {}^tU^{-1} \end{pmatrix} \in \Delta_{n,r}$, $U = \begin{pmatrix} U_1^{(r)} & \\ U_3 & U_4 \end{pmatrix} \in GL(n, Z)$. Then $(A_{K'N})_1 = (UA_{KM} + USC_{KM})_1$ implies $-(C_N)_1 = -U_1(C_M)_1$. Thus we have $GL(r, Z)(C_M)_1 = GL(r, Z)(C_N)_1$. Q.E.D.

Let (C_4, D_3, D_4) be an element of $\mathfrak{S}_{n,r}$ and define matrices $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$, $(\mathfrak{C}_4, \mathfrak{D}_4)$ and (C, D) as in the proof of Lemma 7; then $C_4 = U_4\mathfrak{C}_4$, $D_4 = U_4\mathfrak{D}_4$

and $D_3 = U_3$. Taking, $\mathfrak{A}_4, \mathfrak{B}_4$ such that $\begin{pmatrix} \mathfrak{A}_4 & \mathfrak{B}_4 \\ \mathfrak{C}_4 & \mathfrak{D}_4 \end{pmatrix} \in \Gamma_{n-r}$ we put

$$M = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{1_n} & 0 \\ \hline 1_r & 0 & \multirow{2}{*}{1_n} \\ 0 & 0 & \end{array}\right] \left[\begin{array}{c|c} {}^tU^{-1} & 0 \\ \hline 0 & U \end{array}\right] \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & \mathfrak{A}_4 & 0 & \mathfrak{B}_4 \\ \hline 0 & 0 & 1 & 0 \\ 0 & \mathfrak{C}_4 & 0 & \mathfrak{D}_4 \end{array}\right].$$

Then $C_4 = U_4\mathfrak{C}_4$ and $|C_4| \neq 0$ imply $|U_4| \neq 0$ and $|U_1 - U_2U_4^{-1}U_3| \neq 0$ follows from

$$\begin{pmatrix} 1 & -U_2U_4^{-1} \\ 0 & 1 \end{pmatrix} U = \begin{pmatrix} U_1 - U_2U_4^{-1}U_3 & 0 \\ * & * \end{pmatrix}.$$

Hence we have

$$U^{-1} = \begin{pmatrix} (U_1 - U_2U_4^{-1}U_3)^{-1} & -(U_1 - U_2U_4^{-1}U_3)^{-1}U_2U_4^{-1} \\ -U_4^{-1}U_3(U_1 - U_2U_4^{-1}U_3)^{-1} & U_4^{-1} + U_4^{-1}U_3(U_1 - U_2U_4^{-1}U_3)^{-1}U_2U_4^{-1} \end{pmatrix}.$$

Putting ${}^tU^{-1} = V$, we have

$$A_M = \begin{pmatrix} V_1 & V_2\mathfrak{A}_4 \\ V_3 & V_4\mathfrak{A}_4 \end{pmatrix}, \qquad B_M = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix},$$
$$C_M = \begin{pmatrix} V_1 & V_2\mathfrak{A}_4 + U_2\mathfrak{C}_4 \\ 0 & C_4 \end{pmatrix}, \qquad D_M = \begin{pmatrix} U_1 & V_2\mathfrak{B}_4 + U_2\mathfrak{D}_4 \\ D_3 & D_4 \end{pmatrix},$$

For this special extension $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$ of $(0, C_4, D_3, D_4)$ we have

LEMMA 8. *$X = (B_1 - A_1C_1^{-1}D_1 - A_2C_4^{-1}D_3 + A_1C_1^{-1}C_2C_4^{-1}D_3){}^tC_1$ is integral.*

Proof.
$$\begin{aligned} X &= (-U_1 - V_2\mathfrak{A}_4C_4^{-1}D_3 + (V_2\mathfrak{A}_4 + U_2\mathfrak{C}_4)C_4^{-1}D_3){}^tV_1 \\ &= (-U_1 + U_2\mathfrak{C}_4C_4^{-1}D_3){}^tV_1 = -1_r\,. \end{aligned}$$
Q.E.D.

PROPOSITION 2. *Take $N = M[C_4, D_3, D_4]$ for $(C_4, D_3, D_4) \in \mathfrak{S}_{n,r}$ and a half-integral symmetric matrix $P^{(n)}$ such that ${}^t(C_N)_1^{-1}P_1(C_N)_1^{-1}$ is half-integral. Then $\sigma(PC_N^{-1}D_N) \bmod Z$ is uniquely determined by (C_4, D_3, D_4) and P.*

Proof. Take $M \in \Gamma_n$ such that $\tilde{M} = (0, C_4, D_3, D_4)$ $|C_M| \neq 0$, $(A_MC_M^{-1})_1 \equiv 0 \bmod 1$ and put $C_M = C$, $D_M = D$; then we have

$$C^{-1}D = \begin{pmatrix} C_1^{-1}(D_1 - C_2C_4^{-1}D_3) & {}^t(C_4^{-1}D_3) \\ C_4^{-1}D_3 & C_4^{-1}D_4 \end{pmatrix}$$

and $(C^{-1}D)_2, (C^{-1}D)_3, (C^{-1}D)_4$ are only dependent of (C_4, D_3, D_4). Take any extension $N = M[C_4, D_3, D_4]$ and define K, K', U, S as in the proof of Proposition 1; then $\tilde{C}_N = \tilde{C}_M$ implies $U_4 = 1_{n-r}$ and we have

$$C_N = \begin{pmatrix} -U_1S_1 & 0 \\ 0 & 0 \end{pmatrix} A_M + \begin{pmatrix} U_1 + U_1S_1A_1C_1^{-1} & -U_1S_2 \\ 0 & 1 \end{pmatrix} C_M ,$$

$$D_N = \begin{pmatrix} -U_1S_1 & 0 \\ 0 & 0 \end{pmatrix} B_M + \begin{pmatrix} U_1 + U_1S_1A_1C_1^{-1} & -U_1S_2 \\ 0 & 1 \end{pmatrix} D_M .$$

Hence $(C_M^{-1}D_M - C_N^{-1}D_N)_1 = C_1^{-1}S_1(B_1 - A_1C_1^{-1}D_1 - A_2C_4^{-1}D_3 + A_1C_1^{-1}C_2C_4^{-1}D_3)$. Now we suppose that M is a special extension in Lemma 8; then $C_1(C_M^{-1}D_M - C_N^{-1}D_N)^tC_1 \equiv 0 \bmod 1$. Therefore $\sigma(PC_M^{-1}D_M) - \sigma(PC_N^{-1}D_N) = \sigma(P_1(C_M^{-1}D_M - C_N^{-1}D_N)_1) = \sigma({}^tC_1^{-1}P_1C_1^{-1}(C_1(C_M^{-1}D_M - C_N^{-1}D_N)^tC_1)) \equiv 0 \bmod 1$. Q.E.D.

§ 2.

Through this section we fix natural numbers r, n ,k such that $1 \leq r \leq n-1$, $k \geq n + r + 2$, $k \equiv 0 \bmod 2$. Denote by f a cusp form of degree r and weight k which is also fixed.

For $M \in \Gamma_n$ we put

$$(f|M)(Z) = f(M\langle Z\rangle)_1)|C_MZ + D_M|^{-k} ,$$

where $M\langle Z\rangle = (A_MZ + B_M)(C_MZ + D_M)^{-1}$, $(Z \in H_n)$, and $(M\langle Z\rangle)_1$ is the upper left $r \times r$ submatrix of $M\langle Z\rangle$ as in § 1, and $(f|M)(Z) = f(M\langle Z\rangle^*)M\{Z\}^{-k}$ in the notation of Klingen [3]. It is easy to see that $(f|M)(Z) = (f|NM)(Z)$ for any $N \in \Delta_{n,r}$.

Put $E_1(Z) = \sum_M (f|M)(Z)$ (resp. $E_2(Z) = \sum_M (f|M)(Z)$) where M runs over representatives of $\Delta_{n,r}\backslash\Gamma_n$ such that rank $\tilde{C}_M = n - r$ (resp. rank $\tilde{C}_M < n - r$).

LEMMA 1. *Let N be an element of Γ_n such that* rank $\tilde{C}_N < n - r$ *and put* $\dfrac{\partial}{\partial Y} = \left(\dfrac{1}{2}(1 + \delta_{ij})\dfrac{\partial}{\partial y_{ij}}\right)$. *Then we have* $\left|\dfrac{\partial}{\partial Y}\right|(f|N)(Z) = 0$ *where* $Y = \operatorname{Im} Z$.

Proof. By Lemma 2 in § 1 there exist $M \in \Delta_{n,n-1}$, $U \in GL(n, Z)$ such that $\Delta_{n,r}N \ni M\begin{pmatrix} U & \\ & {}^tU^{-1} \end{pmatrix}$. Put $A_M = \begin{pmatrix} A_1 & 0 \\ A_3 & A_4 \end{pmatrix}$, $B_M = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$, $C_M = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}$, $D_M = \begin{pmatrix} D_1 & D_2 \\ 0 & D_4 \end{pmatrix}$ and $Z = \begin{pmatrix} Z_1 & Z_2 \\ {}^tZ_2 & Z_4 \end{pmatrix}$ where $A_1, \cdots, D_1$ and Z_1 are $(n-1) \times (n-1)$ matrices. Then we have $C_MZ + D_M = \begin{pmatrix} C_1Z_1 + D_1 & * \\ 0 & D_4 \end{pmatrix}$, $M\langle Z\rangle = \begin{pmatrix} (A_1Z_1 + B_1)(C_1Z_1 + D_1)^{-1} & * \\ * & * \end{pmatrix}$. Hence $(f|M)(Z)$ does not depend on Z_2, Z_4

and so $|\partial/\partial Y|(f|M)(Z) = 0$. Since $(f|N)(Z) = (f|M)(UZ{}^tU)$ and $|Y||\partial/\partial Y|$ is invariant under the transformation $Y \to UY{}^tU$ on $\{Y^{(n)} \mid Y > 0\}$, we have

$$\begin{aligned} |Y|\left|\frac{\partial}{\partial Y}\right|(f|N)(Z) &= |Y|\left|\frac{\partial}{\partial Y}\right|(f|M)(UZ{}^tU) \\ &= |Y|\left|\frac{\partial}{\partial Y}\right|(f|M)(UX{}^tU + iY)\Big|_{Y \to UY{}^tU} = 0\,, \end{aligned}$$

where $Z = X + iY$. Q.E.D.

PROPOSITION 1. $E_2(Z)$ *has Fourier expansion* $\sum_T a_2(T)e(\sigma(TZ))$ *such that* $a_2(T) = 0$ *for* $T > 0$.

Proof. Put $N = M\left(\begin{array}{c|c} 1 & S \\ \hline & 1 \end{array}\right)$ where $N, M \in \Gamma_n$, ${}^tS = S \in M_n(Z)$. Then rank $\tilde{C}_N =$ rank $\tilde{C}_M$. Hence $E_2(Z + S) = E_2(Z)$ for any $S = {}^tS \in M_n(Z)$. Thus $E_2(Z)$ has Fourier expansion $\sum_T a_2(T)e(\sigma(TZ))$. From Lemma 1 follows

$$\left|\frac{\partial}{\partial Y}\right| E_2(Z) = \sum_T a_2(T)\,|-2\pi T|\,e(\sigma(TZ)) = 0\,.$$

Hence $a_2(T)$ vanishes if T is positive definite. Q.E.D.

For a natural number m we put

$$\begin{aligned} \Lambda_m &= \{S \in M_m(Z) \mid S = {}^tS\}\,, \\ \Lambda_m^* &= \{S \in M_m(Q) \mid S = {}^tS\text{: half-integral}\}\,. \end{aligned}$$

Λ_m^* is the dual lattice of Λ_m via $\sigma(SS')$.

The following is well known ([1], [5], [8]).

LEMMA 2. *For a positive definite matrix* $Y^{(m)}$ *and* $\rho > m + 1$,

$$\sum_{F \in \Lambda_m} |Y + 2\pi i F|^{-\rho}$$

is absolutely convergent and

$$\Gamma_m(\rho) \sum_{F \in \Lambda_m} |Y + 2\pi i F|^{-\rho} = 2^{-m(m-1)/2} \sum_{\substack{T > 0 \\ T \in \Lambda_m^*}} |T|^{\rho - (m+1)/2}\, e^{-\sigma(TY)}\,,$$

where $\Gamma_m(\rho) = \pi^{m(m-1)/4} \prod_{\nu=0}^{m-1} \Gamma(\rho - \nu/2)$.

LEMMA 3. *For a positive number* a,

$$\begin{aligned} &\sum_{F \in \Lambda_m} |(2\pi i)^{-1}(Z + aF)|^{-\rho} \\ &\quad = 2^{-m(m-1)/2}(2\pi)^{2m\rho}\Gamma_m(\rho)^{-1}a^{-m\rho} \sum_{\substack{T > 0 \\ T \in \Lambda_m^*}} |T|^{\rho - (m+1)/2} e(a^{-1}\sigma(TZ))\,, \end{aligned}$$

where $Z \in H_m$ *and* $\rho > m+1$.

Proof. This is an immediate corollary of Lemma 2. Q.E.D.

LEMMA 4. *If a, b are complex numbers such that* $\operatorname{Re} a > 0$, *then we have*

$$\int_R \exp(-ax^2 + 2bx)dx = \sqrt{\pi/a} \exp(b^2/a) ,$$

where $\sqrt{\pi/a}$ *is real positive if* a *is real.*

Proof. This is also well known. Q.E.D.

The following is an easy generalization.

LEMMA 5. *If* A *is a symmetric matrix of* $M_m(C)$ *such that* $\operatorname{Re} A > 0$ *and* b *is an element of* C^m, *then we have*

$$\int_{R^m} \exp(-{}^t xAx + 2{}^t bx)dx = \sqrt{\det(\pi A^{-1})} \exp({}^t bA^{-1}b) ,$$

where $\sqrt{\det(\pi A^{-1})}$ *is real positive if* A *is real.*
We need the following generalization.

LEMMA 6. *If* A *is a symmetric matrix of* $M_{n-r}(C)$ *such that* $\operatorname{Re} A > 0$ *and* $W_1^{(r)} > 0$ *and* Q *is an element of* $M_{n-r,r}(C)$, *then we have*

$$\begin{aligned}&\int_{X \in M_{r,n-r}(R)} \exp(-2\pi\sigma(W_1 XA{}^t X) + 2\pi\sigma(XQ))dX \\ &\quad = |W_1|^{((r-n)/2} 2^{r(r-n)/2} \sqrt{(\det A^{-1})^r} \exp\left(\frac{\pi}{2}\sigma({}^t QA^{-1}QW_1^{-1})\right) ,\end{aligned}$$

where $\sqrt{(\det A^{-1})^r}$ *is real positive if* A *real.*

Proof. Put ${}^t X = ({}^t x_1, \cdots, {}^t x_r)$ and ${}^t x = (x_1, \cdots, x_r) \in M_{1,(n-r)r}(R)$. Then we have $\sigma(XA{}^t X) = {}^t x\begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix}x$ where rA's are on the diagonal. Denoting QW^{-1} by $(y_1, \cdots, y_r)$ where $W = \sqrt{W_1} > 0$, we have $\sigma(XQW^{-1}) = ({}^t y_1, \cdots, {}^t y_r)x$. Thus the integral of the left side is

$$\begin{aligned}&|W|^{r-n} \int_{X \in M_{r,n-r}(R)} \exp(-2\pi\sigma(XA{}^t X) + 2\pi\sigma(XQW^{-1}))dX \\ &\quad = |W_1|^{(r-n)/2} \int_{R^{(n-r)r}} \exp\left(-2\pi{}^t x\begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix}x + 2\pi({}^t y_1, \cdots, {}^t y_r)x\right)dx\end{aligned}$$

$$= |W_1|^{(r-n)/2} 2^{r(r-n)/2} \sqrt{(\det A^{-1})^r} \exp\left(\frac{\pi}{2}\sigma({}^tW^{-1}\,{}^tQA^{-1}QW^{-1})\right). \qquad \text{Q.E.D.}$$

LEMMA 7. *If N is an element of Γ_n such that $|C_N| \neq 0$, $(A_N C_N^{-1})_1 \equiv 0 \bmod 1$, then we have*

$$(f|N)(Z) = |C_N|^k |W_4|^{-k} f(W_1 - W_2 W_4^{-1}\,{}^tW_2),$$

where

$$W = \begin{pmatrix} W_1^{(r)} & W_2 \\ {}^tW_2 & W_4 \end{pmatrix} = C_N Z {}^tC_N + D_N {}^tC_N .$$

Proof. Put $N = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Since $B - AC^{-1}D = B - A^t(C^{-1}D) = (B^tC - A{}^tD){}^tC^{-1} = -{}^tC^{-1}$, we have $N\langle Z\rangle = (AZ+B)(CZ+D)^{-1} = AC^{-1} - {}^tC^{-1}(CZ+D)^{-1} = AC^{-1} - W^{-1}$. From the identity

$$W = \begin{pmatrix} 1 & W_2W_4^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} W_1 - W_2W_4^{-1}\,{}^tW_2 & 0 \\ 0 & W_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ W_4^{-1}\,{}^tW_2 & 1 \end{pmatrix}$$

follows

$$W^{-1} = \begin{pmatrix} (W_1 - W_2W_4^{-1}\,{}^tW_2)^{-1} & * \\ * & * \end{pmatrix}.$$

Hence we have

$$\begin{aligned}(f|N)(Z) &= f(-(W_1 - W_2W_4^{-1}\,{}^tW_2)^{-1})\,|W^tC^{-1}|^{-k} \\ &= f(W_1 - W_2W_4^{-1}\,{}^tW_2)\,|W_1 - W_2W_4^{-1}\,{}^tW_2|^k\,|W|^{-k}\,|C|^k \\ &= |C|^k\,|W_4|^{-k} f(W_1 - W_2W_4^{-1}\,{}^tW_2). \qquad \text{Q.E.D.}\end{aligned}$$

Let $N = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an element of Γ_n such that $|C| \neq 0$, $(AC^{-1})_1 \equiv 0 \bmod 1$, $C_3 = 0$ where we decompose $M \in M_n(C)$ as $M = \begin{pmatrix} M_1^{(r)} & M_2 \\ M_3 & M_4 \end{pmatrix}$ as in §1 and take a natural number p such that pC^{-1} is an integral matrix. We fix N, p till Lemma 13. Now we calculate

$$\sum_{S=\begin{pmatrix} 0 & S_2 \\ {}^tS_2 & S_4\end{pmatrix} \in p^2\Lambda_n} \left(f \middle| N\left(\begin{array}{c|c} 1 & C^{-1}S^tC^{-1} \\ \hline 0 & 1 \end{array}\right)\right)(Z).$$

Put $W = CZ^tC + D^tC$. For $S = \begin{pmatrix} 0 & S_2 \\ {}^tS_2 & S_4 \end{pmatrix} \in \Lambda_n$ we have

$$N\left(\begin{array}{c|c} 1 & C^{-1}S^tC^{-1} \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} * & * \\ \hline C & S^tC^{-1} + D \end{array}\right)$$

and $CZ^tC + (S^tC^{-1} + D)^tC = W + S$. If, moreover, $S \in p^2\Lambda_n$, then

$$\left(f \mid N\left(\frac{1}{0}\middle|\frac{C^{-1}S^tC^{-1}}{1}\right)\right)(Z)$$
$$= |C|^k |W_4 + S_4|^{-k} f(W_1 - (W_1 + S_2)(W_4 + S_4)^{-1}\,{}^t(W_2 + S_2)) .$$

First we calculate, for $t > 0$, $t \in \Lambda_r^*$,

$$\sum_{S_2 \in p^2 M_{r,n-r}(\boldsymbol{Z})} e(-\sigma(t(W_2 + S_2)W_4^{-1}\,{}^t(W_2 + S_2))) .$$

It equals

$$\sum_{S_2 \in M_{r,n-r}(\boldsymbol{Z})} e(-\sigma(t(W_2 + p^2S_2)W_4^{-1}\,{}^t(W_2 + p^2S_2)))$$
$$= \sum_{S_2 \in M_{r,n-r}(\boldsymbol{Z})} \int_{X \in M_{r,n-r}(\boldsymbol{R})} e(-\sigma(t(W_2 + p^2X)W_4^{-1}\,{}^t(W_2 + p^2X)))e(\sigma(S_2{}^tX))dX$$
$$= e(-\sigma(tW_2W_4^{-1}\,{}^tW_2)) \sum_{S_2 \in M_{r,n-r}(\boldsymbol{Z})} \int_{X \in M_{r,n-r}(\boldsymbol{R})} \exp(-2\pi\sigma(p^4tX(iW_4^{-1})^tX))$$
$$\times \exp(2\pi\sigma(X(-2ip^2W_4^{-1}\,{}^tW_2t + i^tS_2)))dX$$
$$= e(-\sigma(tW_2W_4^{-1}\,{}^tW_2)) \sum_{S_2 \in M_{r,n-r}(\boldsymbol{Z})} |p^4t|^{(r-n)/2}\, 2^{r(r-n)/2}\sqrt{\det(i^{-1}W_4)^r}$$
$$\times \exp((\pi/2)\sigma({}^tQ(i^{-1}W_4)Qp^{-4}t^{-1})) ,$$

where we put $Q = -2ip^2W_4^{-1}\,{}^tW_2t + i^tS_2$.

Since

$$\sigma({}^tQ(i^{-1}W_4)Qp^{-4}t^{-1})$$
$$= p^{-4}i\sigma(4p^4W_2W_4^{-1}\,{}^tW_2t - 4p^2W_2\,{}^tS_2 + S_2W_4\,{}^tS_2t^{-1}) ,$$

we have

$$\sum_{S_2 \in p^2M_{r,n-r}(\boldsymbol{Z})} e(-\sigma(t(W_2 + S_2)W_4^{-1}\,{}^t(W_2 + S_2)))$$
$$= 2^{r(r-n)/2} \cdot p^{2r(r-n)}\, |t|^{(r-n)/2}\sqrt{\det(i^{-1}W_4)^r}$$
$$\times \sum_{S_2 \in M_{r,n-r}(\boldsymbol{Z})} e((4p^4)^{-1}\sigma(-4p^2W_2{}^tS_2 + S_2W_2{}^tS_2t^{-1})) .$$

Put $f(z^{(r)}) = \sum_{t>0,\, t \in \Lambda_r^*} b(t)e(\sigma(tz))$; then $b(t) = O(|t|^{k/2})$ is known [7].

LEMMA 8.

$$\sum_{S = \left(\begin{smallmatrix} 0 & S_2 \\ {}^tS_2 & 0 \end{smallmatrix}\right) \in p^2\Lambda_n} \left(f \mid N\left(\frac{1}{0}\middle|\frac{C^{-1}S^tC^{-1}}{1}\right)\right)(Z)$$
$$= |C|^k 2^{r(r-n)/2} p^{2r(r-n)} |W_4|^{-k} \sqrt{\det(i^{-1}W_4)^r}$$
$$\times \sum_{\substack{t>0 \\ t \in \Lambda_r^* \\ S_2 \in M_{r,n-r}(\boldsymbol{Z})}} |t|^{(r-n)/2} b(t) e(\sigma(tW_1) + (4p^4)^{-1}\sigma(-4p^2W_2{}^tS_2 + S_2W_4{}^tS_2t^{-1}))$$

where the right hand is absolutely convergent.

Proof. Define a matrix $P \in M_r(\boldsymbol{R})$ by

$$P = \mathrm{Im}\,(W_1 - (W_2 + X)W_4^{-1}\,{}^t(W_2 + X)) - (X + Q)\,\mathrm{Im}\,(-W_4^{-1})\,{}^t(X + Q)\,,$$

where $Q = \mathrm{Re}\,W_2 + \mathrm{Im}\,W_2 \cdot \mathrm{Re}\,(-W_4^{-1})(\mathrm{Im}\,(-W_4^{-1}))^{-1}$, $X \in M_{r,n-r}(\boldsymbol{R})$. Then P is independent of X. Since $(W_1 - (W_2 + X)W_4^{-1}\,{}^t(W_2 + X))^{-1}$ is the upper left $r \times r$ matrix of

$$\left(W + \begin{pmatrix} 0 & X \\ {}^tX & 0 \end{pmatrix}\right)^{-1}$$

we have $W_1 - (W_2 + X)W_4^{-1}\,{}^t(W_2 + X) \in H_r$ and its imaginary part is positive definite. Hence, putting $X = -Q$, we see that P is positive definite. Now we have

$$\begin{aligned}&|e(\sigma(t(W_1 - (W_2 + S_2)W_4^{-1}\,{}^t(W_2 + S_2))))| \\ &\quad = \exp\,(-2\pi\sigma(tP + t(S_2 + Q)\,\mathrm{Im}\,(-W_4^{-1})\,{}^t(S_2 + Q))) \\ &\quad\quad < \exp\,(-2\pi\varepsilon\sigma(t + (S_2 + Q)\,{}^t(S_2 + Q)))\,,\end{aligned}$$

where $\varepsilon > 0$ is defined by

$$P > \varepsilon 1_r\,, \quad \mathrm{Im}\,(-W_4^{-1}) > \sqrt{\varepsilon}\,1_{n-r}\,, \quad t > \sqrt{\varepsilon}\,1_r \qquad \text{for } t \in \Lambda_r^*,\ t > 0\,.$$

Then it is easy to see that

$$\sum_{\substack{t>0 \\ t \in \Lambda_r^* \\ S_2 \in p^2 M_{r,n-r}(\boldsymbol{Z})}} b(t)e(\sigma(t(W_1 - (W_2 + S_2)W_4^{-1}\,{}^t(W_2 + S_2)))$$

is absolutely convergent.

To prove that the right hand is absolutely convergent, it is enough to show

$$\begin{aligned}&\sum_{S_2 \in M_{r,n-r}(\boldsymbol{Z})} |e(\sigma(tW_1) + (4p^4)^{-1}\sigma(-4p^2W_2\,{}^tS_2 + S_2W_4\,{}^tS_2t^{-1}))| \\ &\quad = O(|t|^{n-r}\exp\,(-2\pi\varepsilon\sigma(t))) \qquad \text{for some } \varepsilon > 0\,.\end{aligned}$$

$\mathrm{Im}\,(\sigma(tW_1) + (4p^4)^{-1}\sigma(-4p^2W_2\,{}^tS_2 + S_2W_4\,{}^tS_2t^{-1}))$ is equal to

$$\begin{aligned}&\sigma\left((\mathrm{Im}\,W)\begin{pmatrix} t & -(2p^2)^{-1}S_2 \\ -(2p^2)^{-1}\,{}^tS_2 & (4p^4)^{-1}\,{}^tS_2t^{-1}S_2 \end{pmatrix}\right) \\ &\quad = \sigma\left(\begin{pmatrix} 1 & -(2p^2)^{-1}t^{-1}S_2 \\ 0 & 1 \end{pmatrix}(\mathrm{Im}\,W)\begin{pmatrix} 1 & 0 \\ (-(2p^2)^{-1}\,{}^tS_2t^{-1} & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}\right)\end{aligned}$$

and, taking a positive number ε_1 such that $\operatorname{Im} W > \varepsilon_1 1_n$, we have

$$\begin{aligned}&|e(\sigma(tW_1) + (4p^4)^{-1}\sigma(-4p^2 W_2{}^t S_2 + S_2 W_4{}^t S_2 t^{-1}))| \\ &\quad \leq \exp(-2\pi\varepsilon_1\sigma(t + (4p^4)^{-1}S_2{}^t S_2 t^{-1})) .\end{aligned}$$

Hence we have only to prove that, for $\varepsilon' = 2\pi\varepsilon_1(4p^4)^{-1}$,

$$\sum_{S_2 \in M_{r,n-r}(\boldsymbol{Z})} \exp(-\varepsilon'\sigma(S_2{}^t S_2 t^{-1})) = O(|t|^{n-r}) \qquad \text{for } t > 0,\ t \in \Lambda_r^* .$$

Without loss of generality we may assume that t^{-1} is in some Siegel domain. Then there are positive constants $\varepsilon_2, \varepsilon_3$ such that

$$\varepsilon_2\delta < t^{-1} < \varepsilon_3\delta ,$$

where δ is a diagonal matrix defined by

$$\delta = \begin{bmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_r \end{bmatrix} = {}^t T^{-1} t^{-1} T^{-1} , \qquad T = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} .$$

Then $t < \varepsilon_2^{-1}\delta^{-1}$ and $t \in \Lambda_r^*$ imply that $\delta_i < \varepsilon_2^{-1}$. Therefore we have

$$\begin{aligned}
&\sum_{S_2 \in M_{r,n-r}(\boldsymbol{Z})} \exp(-\varepsilon'\sigma(S_2{}^t S_2 t^{-1})) < \sum_{S_2 \in M_{r,n-r}(\boldsymbol{Z})} \exp(-\varepsilon'\varepsilon_2\sigma(S_2{}^t S_2\delta)) \\
&= \sum \exp\Big(-\varepsilon_4 \sum_{i,j} \delta_i s_{ij}^2\Big) \qquad (\varepsilon_4 = \varepsilon'\varepsilon_2) \\
&= \prod_i \Big(\sum_{s \in \boldsymbol{Z}} \exp(-\varepsilon_4\delta_i s^2)\Big)^{n-r} \leq \prod_i \Big(1 + 2\sum_{s \geq 1} \exp(-\varepsilon_4\delta_i s)\Big)^{n-r} \\
&= \prod_i \Big(1 + 2\frac{\exp(-\varepsilon_4\delta_i)}{1 - \exp(-\varepsilon_4\delta_i)}\Big)^{n-r} \\
&= \prod_i \Big(1 + \frac{2}{\exp(\varepsilon_4\delta_i) - 1}\Big)^{n-r} < \prod_i \Big(1 + \frac{2}{\varepsilon_4\delta_i}\Big)^{n-r} \\
&= \prod_i \Big(\frac{\varepsilon_4\delta_i + 2}{\varepsilon_4\delta_i}\Big)^{n-r} < \Big(\prod_i \varepsilon_4^{-1}(\varepsilon_4\varepsilon_2^{-1} + 2)\Big)^{n-r} |t|^{n-r} .
\end{aligned}$$

Now the calculation before Lemma 8 implies the identity in Lemma 8.
Q.E.D.

LEMMA 9. *For $t > 0$, $t \in \Lambda_r^*$ we have*

$$\sum_{S_4\in p^2\Lambda_{n-r}} |W_4+S_4|^{-k}\sqrt{\det(i^{-1}(W_4+S_4))^r}\,e((4p^4)^{-1}\sigma((W_4+S_4){}^tS_2t^{-1}S_2))$$
$$= i^{(n-r)k}(2\pi)^{(n-r)(k-r/2)}2^{(r-n)(n-r-1)/2}p^{(r-n)(n-r+1)}(4p^4b_t)^{(r-n)(2k-n-1)/2}$$
$$\times \Gamma_{n-r}(k-r/2)^{-1}e((4p^4)^{-1}\sigma(W_4{}^tS_2t^{-1}S_2))$$
$$\times \sum_T |T|^{k-(n+1)/2}e((4p^4b_t)^{-1}\sigma(TW_4))\,,$$

where b_t *is any fixed natural number such that* b_tt^{-1} *is integral,* $S_2\in M_{r,n-r}(Z)$*, and* T *runs over*

$$\{T\,|\,T>0,\ T\in\Lambda^*_{n-r},\ (4p^2)^{-1}({}^tS_2t^{-1}S_2+b_t^{-1}T)\in\Lambda^*_{n-r}\}\,.$$

Proof. The left side equals

$$i^{k(n-r)}e((4p^4)^{-1}\sigma(W_4{}^tS_2t^{-1}S_2))\sum_{S_4\in\Lambda_{n-r}}|-i(W_4+p^2S_4)|^{-(k-r/2)}e((4p^2)^{-1}\sigma(S_4{}^tS_2t^{-1}S_2))$$
$$= i^{k(n-r)}e((4p^4)^{-1}\sigma(W_4{}^tS_2t^{-1}S_2))\sum_{S_4'\in\Lambda_{n-r}\bmod 4p^2b_t}e((4p^2)^{-1}\sigma(S_4'{}^tS_2t^{-1}S_2))$$
$$\times\sum_{S_4\in\Lambda_{n-r}}|-i(W_4+p^2S_4'+4p^4b_tS_4)|^{-(k-r/2)}$$
$$= i^{k(n-r)}e((4p)^{-1}\sigma(W_4{}^tS_2t^{-1}S_2))\sum_{S_4'\in\Lambda_{n-r}\bmod 4p^2b_t}e((4p^2)^{-1}\sigma(S_4'{}^tS_2t^{-1}S_2))$$
$$\times(2\pi)^{(r-n)(k-r/2)}2^{(r-n)(n-r-1)/2}(2\pi)^{2(n-r)(k-r/2)}\Gamma_{n-r}(k-r/2)^{-1}$$
$$\times(4p^4b_t)^{-(n-r)(k-r/2)}\sum_{\substack{T>0\\T\in\Lambda^*_{n-r}}}|T|^{k-(n+1)/2}e((4p^4b_t)^{-1}\sigma(T(W_4+p^2S_4')))\,.$$

From

$$\sum_{S_4'\in\Lambda_{n-r}\bmod 4p^2b_t}e((4p^2)^{-1}\sigma(S_4'{}^tS_2t^{-1}S_2)+(4p^4b_t)^{-1}\sigma(p^2TS_4'))$$
$$=\begin{cases}(4p^2b_t)^{(n-r)(n-r+1)/2} & \text{if } (4p^2)^{-1}({}^tS_2t^{-1}S_2+b_t^{-1}T)\in\Lambda^*_{n-r}\,,\\ 0 & \text{otherwise}\end{cases}$$

follows the identity in Lemma 9. Q.E.D.

Now we have

$$\sum_{S=\begin{pmatrix}0&S_2\\{}^tS_2&S_4\end{pmatrix}\in p^2\Lambda_n}\left(f\,|\,N\left(\begin{array}{c|c}1 & C^{-1}S{}^tC^{-1}\\\hline 0 & 1\end{array}\right)\right)(Z)$$
$$=\sum_{S_4\in p^2\Lambda_{n-r}}|C|^k\,2^{r(r-n)/2}p^{2r(r-n)}\sum_{\substack{t>0\\t\in\Lambda_r^*\\S_2\in M_{r,n-r}(Z)}}|t|^{(r-n)/2}b(t)$$
$$\times e(\sigma(tW_1)-p^{-2}\sigma(W_2{}^tS_2))\,|W_4+S_4|^{-k}\sqrt{\det(i^{-1}(W_4+S_4))^r}$$
$$\times e((4p^4)^{-1}\sigma((W_4+S_4){}^tS_2t^{-1}S_2))$$
$$=|C|^k\,2^{(r-n)(n-1)/2}p^{(r-n)(n+r+1)}i^{(n-r)k}(2\pi)^{(n-r)(k-r/2)}$$
$$\times\Gamma_{n-r}(k-r/2)^{-1}\sum_{t,S_2,T}|t|^{(r-n)/2}(4p^4b_t)^{(r-n)(2k-n-1)/2}$$

$$\times |T|^{k-(n+1)/2} b(t) e(\sigma(tW_1) - p^{-2}\sigma(W_2{}^tS_2) + (4p^4)^{-1}\sigma(W_4{}^tS_2t^{-1}S_2) + (4p^4b_t)^{-1}\sigma(TW_4)) \,,$$

where

$$t > 0 \,, \quad t \in \Lambda_r^* \,, \quad S_2 \in M_{r,n-r}(Z) \,, \quad T > 0 \,,$$
$$T \in \Lambda_{n-r}^* \,, \qquad (4p^2)^{-1}({}^tS_2t^{-1}S_2 + b_t^{-1}T) \in \Lambda_{n-r}^* \,.$$

Put

$$P = \begin{pmatrix} t & -(2p^2)^{-1}S_2 \\ -(2p^2)^{-1}\,{}^tS_2 & (4p^4)^{-1}(b_t^{-1}T + {}^tS_2t^{-1}S_2) \end{pmatrix} = \begin{pmatrix} P_1 & P_2 \\ {}^tP_2 & P_4 \end{pmatrix};$$

then

$$P = \begin{pmatrix} 1 & 0 \\ -(2p^2)^{-1}\,{}^tS_2t^{-1} & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & (4p^4b_t)^{-1}T \end{pmatrix} \begin{pmatrix} 1 & -(2p^2)^{-1}t^{-1}S_2 \\ 0 & 1 \end{pmatrix}$$

implies that P is positive definite and $|P| = (4p^4b_t)^{r-n}|t||T|$. Our assumptions on t, S_2, T mean that $P_1 \in \Lambda_r^*$, $2p^2P_2 \in M_{r,n-r}(Z)$, $p^2P_4 \in \Lambda_{n-r}^*$, and $b_t4p^4P_4 - b_t{}^tS_2t^{-1}S_2 \in \Lambda_{n-r}^*$. $\{t, S_2, T\}$ and P correspond bijectively and

$$\sigma(WP) = \sigma(W_1t) - p^{-2}\sigma(W_2{}^tS_2) + (4p^4)^{-1}\sigma(W_4(b_t^{-1}T + {}^tS_2t^{-1}S_2)) \,.$$

Thus we have

LEMMA 10.

$$\sum_{S=\begin{pmatrix} 0 & S_2 \\ {}^tS_2 & S_4 \end{pmatrix} \in p^2\Lambda_n} \left(f \,\middle|\, N\left(\begin{array}{c|c} 1 & C^{-1}S{}^tC^{-1} \\ \hline 0 & 1 \end{array}\right)\right)(Z)$$
$$= |C|^k 2^{(r-n)(n-1)/2} p^{(r-n)(n+r+1)} i^{(n-r)k} (2\pi)^{(n-r)(k-r/2)} \,.$$
$$\times \Gamma_{n-r}(k - r/2)^{-1} \sum_{P>0} b(P_1)|P_1|^{(r+1)/2-k}|P|^{k-(n+1)/2} e(\sigma(PW)) \,,$$

where

$$P_1 \in \Lambda_r^* \,, \quad 2p^2P_2 \in M_{r,n-r}(Z) \,, \quad p^2P_4 \in \Lambda_{n-r}^* \,.$$

LEMMA 11. *Put* $G = \{({}^tS_2{}^tC_1^{-1} - S_4{}^t(C_1^{-1}C_2C_4^{-1}),\ S_4{}^tC_4^{-1}) \mid S_2 \in p^2M_{r,n-r}(Z),\ S_4 \in p^2\Lambda_{n-r}\}$, $G' = \{(C_4{}^tS_2,\ C_4S_4) \mid S_2 \in M_{r,n-r}(Z),\ S_4 \in \Lambda_{n-r}\}$. *Then we have* $[G' : G] = p^{(n-r)(n+r+1)} abs(|C_1|^{r-n}|C_4|^{-n-1})$.

Proof. By definition $C = \begin{pmatrix} C_1 & C_2 \\ & C_4 \end{pmatrix}$ and $pC^{-1} = p\begin{pmatrix} C_1^{-1} & -C_1^{-1}C_2C_4^{-1} \\ & C_4^{-1} \end{pmatrix}$ are integral. It implies $G' \supset G$. Put $G_0 = M_{n-r,r}(Z) \times \{C_4S_4 \mid S_4 \in \Lambda_{n-r}\}$; then $[G_0 : G'] = abs|C_4|^r$. As representatives of G_0/G we can take representatives of $\{C_4S_4 \mid S_4 \in \Lambda_{n-r}\}/\{S_4{}^tC_4^{-1} \mid S_4 \in p^2\Lambda_{n-r}\}$ and then representatives of

$$M_{n-r,r}(Z)/\{{}^tS_2{}^tC_1^{-1} \mid S_2 \in p^2M_{r,r-n}(Z)\}\ .$$

Hence we have

$$\begin{aligned}[G_0\colon G] &= [\{C_4S_4 \mid S_4 \in \Lambda_{n-r}\}\colon \{S_4{}^tC_4^{-1} \mid S_4 \in p^2\Lambda_{n-r}\}] \\ &\quad\times [M_{n-r,r}(Z)\colon \{{}^tS_2{}^tC_1^{-1} \mid S_2 \in p^2M_{n-r,r}(Z)\}] \\ &= [\Lambda_{n-r}\colon p^2C_4^{-1}\Lambda_{n-r}{}^tC_4^{-1}]abs\,|p^2\,{}^tC_1^{-1}|^{n-r} \\ &= abs\,|pC_4^{-1}|^{n-r+1}\,|p^2\,{}^tC_1^{-1}|^{n-r}\ .\end{aligned}$$

Thus we have $[G'\colon G] = p^{(n-r)(n+r+1)}abs(|C_1|^{r-n}\,|C_4|^{-n-1})$. Q.E.D.

It is obvious that

$$G = \left\{\widetilde{S^tC^{-1}} \mid S = \begin{pmatrix} 0 & S_2 \\ {}^tS_2 & S_4 \end{pmatrix} \in p^2\Lambda_n\right\},$$

$$G' = \left\{\widetilde{CS} \mid S = \begin{pmatrix} 0 & S_2 \\ {}^tS_2 & S_4 \end{pmatrix} \in \Lambda_n\right\}$$

and

$$\widetilde{N\left(\begin{array}{c|c} 1 & C^{-1}S^tC^{-1} \\ \hline 0 & 1 \end{array}\right)} = (\tilde{C}, \tilde{D} + \widetilde{S^tC^{-1}})\ , \qquad \widetilde{N\left(\begin{array}{c|c} 1 & S \\ \hline 0 & 1 \end{array}\right)} = (\tilde{C}, \tilde{D} + \widetilde{CS})\ .$$

We take

$$S_i' = \begin{pmatrix} 0 & S_{i,2} \\ {}^tS_{i,2} & S_{i,4} \end{pmatrix} \in \Lambda_n$$

such that $\widetilde{CS_i'}$ is representatives of G'/G. For $M \in \Gamma_n$, $(f|M)(Z)$ is uniquely determined by $\tilde{M}$. Hence we may write $(f|\tilde{M})(Z)$ for $(f|M)(Z)$. Then we have

$$\begin{aligned}\sum_{S=\begin{pmatrix} 0 & S_2 \\ {}^tS_2 & S_4 \end{pmatrix}\in\Lambda_n} \left(f \,\middle|\, N\left(\begin{array}{c|c} 1 & S \\ \hline 0 & 1 \end{array}\right)\right)(Z) &= \sum_{g\in G'} (f|(\tilde{C}, \tilde{D} + g))(Z) \\ &= \sum_i \sum_{g\in G} (f|(\tilde{C}, \widetilde{(D + CS_i')} + g)(Z) \\ &= \sum_i \sum_{S=\begin{pmatrix} 0 & S_2 \\ {}^tS_2 & S_4 \end{pmatrix}\in p^2\Lambda_n} \left(f \,\middle|\, N\left(\begin{array}{c|c} 1 & S_i' \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} 1 & C^{-1}S^tC^{-1} \\ \hline 0 & 1 \end{array}\right)\right)(Z)\ .\end{aligned}$$

For

$$M = N\left(\begin{array}{c|c} 1 & S_i' \\ \hline 0 & 1 \end{array}\right)$$

we have $C_M = C_N = C$, $D_M = CS_i' + D$, $(A_MC_M^{-1})_1 \equiv 0 \bmod 1$ and $C_MZ^tC_M$

$+ D_M {}^tC_M = W + CS_i' {}^tC$. Applying Lemma 10, we have

$$\sum_{S=\left(\begin{smallmatrix} 0 & S_2 \\ {}^tS_2 & S_4 \end{smallmatrix}\right)\in \Lambda_n} \left(f|N\left(\frac{1}{0}\middle|\frac{S}{1}\right)\right)(Z)$$
$$= \alpha |C|^k p^{(r-n)(n+r+1)} \sum_{\substack{P>0 \\ i}} b(P_1)|P_1|^{(r+1)/2-k}|P|^{k-(n+1)/2}$$
$$\times e(\sigma(P(W + CS_i' {}^tC)) ,$$

where $\alpha = i^{(n-r)k}2^{(r-n)(n-1)/2}(2\pi)^{(n-r)(k-r/2)}\Gamma_{n-r}(k - r/2)^{-1}$, and P runs over

$$\{P > 0 | P_1 \in \Lambda_r^*,\ 2p^2P_2 \in M_{r,n-r}(Z),\ p^2P_4 \in \Lambda_{n-r}^*\} .$$

We fix P such that $P_1 \in \Lambda_r^*$, $2p^2P_2 \in M_{r,n-r}(Z)$, $p^2P_4 \in \Lambda_{n-r}^*$ and we put $\chi(g) = e(2\sigma(g_1 {}^tC_1P_2) + 2\sigma(g_2 {}^tC_2P_2) + \sigma(g_2 {}^tC_4P_4))$ for $g = (g_1^{(n-r,r)}, g_2^{(n-r,n-r)}) \in G'$. It is easy to see that $\chi(g) = 1$ for $g \in G$ and $e(\sigma(PCS{}^tC)) = \chi((C_4 {}^tS_2, C_4S_4))$ $= \chi(\widetilde{CS})$ for $S = \begin{pmatrix} 0 & S_2 \\ {}^tS_2 & S_4 \end{pmatrix} \in \Lambda_n$. Therefore the definition of S_i' implies

$$\sum_i e(\sigma(PCS_i' {}^tC)) = \sum_{g\in G'/G} \chi(g) .$$

χ is trivial if and only if $2{}^tC_1P_2C_4 \in M_{r,n-r}(Z)$ and ${}^tC_2P_2C_4 + {}^tC_4 {}^tP_2C_2 + {}^tC_4P_4C_4 \in \Lambda_{n-r}^*$. Put $T = {}^tCPC$; then we have

LEMMA 12. *$P_1 \in \Lambda_r^*$, $2p^2P_2 \in M_{r,n-r}(Z)$, $p^2P_4 \in \Lambda_{n-r}^*$, $2{}^tC_1P_2C_4 \in M_{r,n-r}(Z)$ and ${}^tC_2P_2C_4 + {}^tC_4 {}^tP_2C_2 + {}^tC_4P_4C_4 \in \Lambda_{n-r}^*$ if and only if $T \in \Lambda_n^*$ and ${}^tC_1^{-1}T_1C_1^{-1}$ $= ({}^tC^{-1}TC^{-1})_1 \in \Lambda_r^*$.*

Proof. $T = {}^tCPC$ implies $T_1 = {}^tC_1P_1C_1$, $T_2 = {}^tC_1P_1C_2 + {}^tC_1P_2C_4$, $T_4 =$ ${}^tC_2P_1C_2 + {}^tC_4 {}^tP_2C_2 + {}^tC_2P_2C_4 + {}^tC_4P_4C_4$. The assumptions on P imply ${}^tC_1^{-1}T_1C_1^{-1} = P_1 \in \Lambda_r^*$ and so $T_1 \in \Lambda_r^*$. $2T_2 = 2{}^tC_1P_1C_2 + 2{}^tC_1P_2C_4 \in M_{r,n-r}(Z)$ holds since $P_1 \in \Lambda_r^*$, $2{}^tC_1P_2C_4 \in M_{r,n-r}(Z)$. $T_4 = {}^tC_2P_1C_2 + ({}^tC_4 {}^tP_2C_2 + {}^tC_2P_2C_4$ $+ {}^tC_4P_4C_4) \in \Lambda_{n-r}^*$. Thus "only if" part has been proved. Conversely, assume $T \in \Lambda_n^*$, ${}^tC_1^{-1}T_1C_1^{-1} \in \Lambda_r^*$. Then $P_1 = {}^tC_1^{-1}T_1C_1^{-1} \in \Lambda_r^*$, ${}^tC_2P_2C_4 + {}^tC_4 {}^tP_2C_2$ $+ {}^tC_4P_4C_4 = T_4 - {}^tC_2P_1C_2 \in \Lambda_{n-r}^*$, $2{}^tC_1P_2C_4 = 2T_2 - 2{}^tC_1P_1C_2 \in M_{r,n-r}(Z)$, $2p^2P_2$ $= 2(p{}^tC_1^{-1})({}^tC_1P_2C_4)(pC_4^{-1}) \in M_{r,n-r}(Z)$ follow easily. From $P_4 - {}^tC_4^{-1}T_4C_4^{-1} +$ ${}^tC_4^{-1} {}^tC_2P_1C_2C_4^{-1} = -{}^tP_2C_2C_4^{-1} - {}^tC_4^{-1} {}^tC_2P_2 = -{}^tC_4^{-1}({}^tT_2 - {}^tC_2P_1C_1)C_1^{-1}C_2C_4^{-1} -$ ${}^t(C_1^{-1}C_2C_4^{-1})(T_2 - {}^tC_1P_1C_2)C_4^{-1}$ and $pC^{-1} = p\begin{pmatrix} C_1^{-1} & -C_1^{-1}C_2C_4^{-1} \\ 0 & C_4^{-1} \end{pmatrix} \in M_n(Z)$ follows $p^2P_4 \in \Lambda_{n-r}^*$. Q.E.D.

Summarizing we have

LEMMA 13.

$$\sum_{S=\begin{pmatrix} 0 & S_2 \\ {}^tS_2 & S_4 \end{pmatrix} \in \Lambda_n} \left(f \,\middle|\, N\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \right)(Z)$$
$$= \alpha\, |C_1|^k |C_4|^{-k} \sum_{\substack{T>0 \\ T \in \Lambda_n^*}} b(({}^tC^{-1}TC^{-1})_1)\, |T_1|^{(r+1)/2-k}$$
$$\times |T|^{k-(n+1)/2} e(\sigma(TC^{-1}D)) e(\sigma(TZ)) ,$$

where we suppose $b(({}^tC^{-1}TC^{-1})_1) = 0$ *if* $({}^tC^{-1}TC^{-1})_1 \notin \Lambda_r^*$.

This lemma, Proposition 1 and Theorem in § 1 imply the former part of the following

THEOREM. *Let* r, n, k, *be natural numbers such that* $1 \le r \le n-1$, $k \ge n+r+2$, $k \equiv 0 \bmod 2$, *and let* $f(z^{(r)}) = \sum_{t>0} b(t) e(\sigma(tz))$ *be a cusp form of degree* r *and weight* k. *If we put*

$$E_{n,r}^k(Z, f) = \sum_{M \in \Delta_{n,r} \backslash \Gamma_n} (f | M)(Z) = \sum_{\substack{T \ge 0 \\ T \in \Lambda^*}} a(T, f) e(\sigma(TZ)) ,$$

then we have

$$a(T, f) = \alpha \sum_{\substack{(C_4, D_3, D_4) \in \mathfrak{S}_{n,r}/\approx \\ {}^tU \in P_{n,r} \backslash GL(n,Z)}} |C_1|^k |C_4|^{-k} b(({}^tC^{-1}U^{-1}T^tU^{-1}C^{-1})_1)$$
$$\times |(U^{-1}T^tU^{-1})_1|^{(r+1)/2-k} |T|^{k-(n+1)/2} e(\sigma(U^{-1}T^tU^{-1}C^{-1}D))$$

for $T > 0$, *where* M_1 *stands for the upper left* $r \times r$ *matrix of* M, (C, D) *stands for any fixed symmetric coprime pair such that*

$$(\tilde{C}, \tilde{D}) = (0, C_4, D_3, D_4) , \quad C = \begin{pmatrix} C_1 & C_2 \\ 0 & C_4 \end{pmatrix} , \quad |C| \ne 0$$

and $(AC^{-1})_1 \equiv 0 \bmod 1$ for some $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$, *and*

$$\alpha = i^{(n-r)k} 2^{(r-n)(n-1)/2} (2\pi)^{(n-r)(k-r/2)}$$
$$\times \left(\pi^{(n-r)(n-r-1)/4} \prod_{\nu=0}^{n-r-1} \Gamma(k - (r+\nu)/2) \right)^{-1} .$$

Moreover we have $a(T, f) = O(|T_1|^{-(k-r-1)/2} |T|^{k-(n+1)/2})$ *if* $T > 0$ *runs over any fixed Siegel domain.*

To prove the latter part we prepare the following

LEMMA 14. *The number of* (D_3, D_4) *such that* (C_4, D_3, D_4) *is representatives of* $\mathfrak{S}_{n,r}/\approx$ *for fixed* $C_4 (|C_4| \ne 0)$ *is at most* abs $|C_4|^r \delta_1^{n-r} \cdots \delta_{n-r}$ *where* $\delta_1 | \cdots | \delta_{n-r}$ *are elementary divisors of* C_4.

Proof. By the definition of the relation $\approx$ the number of inequivalent (C_4, D_3, D_4) for fixed C_4 is at most $[M_{n-r,r}(Z): C_4 M_{n-r,r}(Z)] \times [\{D_4 \in M_{n-r}(Z) \mid C_4^{-1}D_4 = {}^t(C_4^{-1}D_4)\}: C_4 \Lambda_{n-r}]$. $[M_{n-r,r}(Z): C_4 M_{n-r,r}(Z)] = abs|C_4|^r$ is obvious. The latter part equals $[\{S = {}^tS \in M_{n-r}(Q) \mid C_4 S \in M_{n-r}(Z)\} : \Lambda_{n-r}]$. Put $C_4 = g_1 \delta g_2$ where $g_i \in GL(n-r, Z)$,

$$\delta = \begin{pmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_{n-r} \end{pmatrix}, \qquad \delta_1 | \cdots | \delta_{n-r} .$$

Then $\{S = {}^tS \in M_{n-r}(Q) \mid C_4 S \in M_{n-r}(Z)\} = \{S = {}^tS \in M_{n-r}(Q) \mid \delta S \in M_{m-r}(Z)\}$. Hence the latter part equals $\delta_1^{n-r} \cdots \delta_{n-r}$. Q.E.D.

We can take

$$\left\{ \begin{pmatrix} c_1 & & c_{ij} \\ & \ddots & \\ 0 & & c_{n-r} \end{pmatrix} \middle| c_1, \cdots, c_{n-r} > 0, 0 \le c_{ij} < c_j \right\}$$

as representatives of $GL(n-r, Z) \backslash \{C_4 \in M_{n-r}(Z) \mid |C_4| \neq 0\}$. If

$$C_4 = \begin{pmatrix} c_1 & & c_{ij} \\ & \ddots & \\ & & c_{n-r} \end{pmatrix},$$

then $\delta_1^{n-r} \cdots \delta_{n-r} = \delta_1(\delta_1\delta_2) \cdots (\delta_1 \cdots \delta_{n-r}) \le c_1(c_1c_2) \cdots (c_1 \cdots c_{n-r})$ where $\delta_1 | \cdots | \delta_{n-r}$ are elementary divisors of C_4. Hence we have

$$\begin{aligned} a(T,f) &= O\Bigl(\sum_{\substack{(C_4,D_3,D_4) \in \mathfrak{S}_{n,r}/\approx \\ {}^tU \in P_{n,r} \backslash GL(n,Z)}} |C_4|^{-k} \times |(U^{-1}T{}^tU^{-1})_1|^{(r+1-k)/2} |T|^{k-(n+1)/2} \Bigr) \\ &= \sum_{c_i=1}^{\infty} \Bigl(\prod_{i=1}^{n-r} c_i \Bigr)^{-k+r} (c_1^{n-r} \cdots c_{n-r})(c_2 \cdots c_{n-r}^{n-r-1}) \\ &\quad \times O\Bigl(\sum_{{}^tU \in P_{n,r} \backslash GL(n,Z)} |(U^{-1}T{}^tU^{-1})_1|^{(r+1-k)/2} |T|^{k-(n+1)/2} \Bigr), \end{aligned}$$

where the sum of c_i is equal to $\zeta(k-n)^{n-r}$, and the last sum is a so-called Selberg's zeta function and the order of the magnitude is $|T_1|^{(r+1-k)/2} |T|^{k-(n+1)/2}$ if T runs over any fixed Siegel domain (p. 143 and Theorem in p. 144 in [5]). This completes the proof of Theorem.

§3.

Let k, n be natural numbers such that $k \ge n+2$, $k \equiv 0 \bmod 2$ and put

$$E_n^k(Z) = \sum |CZ + D|^{-k} \qquad (Z \in H_n) ,$$

where (C, D) runs over representatives of $GL(n, Z)\backslash\{$symmetric coprime pairs$\}$. Our aim is to prove

THEOREM. *Put*

$$E_n^k(Z) = \sum_{\substack{T \geq 0 \\ T \in \Lambda_n^*}} a(T) e(\sigma(TZ)) .$$

Then there are positive numbers c_1, c_2 *such that*

$$c_1 |T|^{k-(n+1)/2} < |a(T)| < c_2 |T|^{k-(n+1)/2} \qquad \text{for } T > 0 .$$

Put $S_T = \sum_R e(-\sigma(TR))\nu(R)^{-k}$ for $T \in \Lambda_n^*$ where R runs over all $n \times n$ rational symmetric matrices modulo 1 and $\nu(R)$ is the product of denominators of elementary divisors of R. Then it is known that

$$a(T) = \text{const.} \times |T|^{k-(n+1)/2} S_T \qquad \text{for } T > 0 \text{ ([9])} .$$

On the other hand $S_T = \sum |C|^{-k} e(-\sigma(TC^{-1}D))$ where (C, D) runs over the set $\{(C, D) \mid \text{symmetric coprime pair}, |C| \neq 0\}/\sim$. Here by definition $(C', D') \sim (C, D)$ if and only if there are $U \in GL(n, Z)$, $S \in \Lambda_n$ such that $(C', D') = U(C, D + CS)$. We take as representatives of C matrices of the form

$$\begin{pmatrix} c_1 & & c_{ij} \\ & \ddots & \\ & & c_n \end{pmatrix}, \qquad 0 \leq c_{ij} < c_j ;$$

then the number of the choice of D for C is at most $c_1^n \cdots c_n$ as in the proof of Theorem in § 2. Thus we have

$$|S_T| \leq \sum_{c_i = 1}^{\infty} (\prod c_i)^{-k} (\prod c_i^{i-1}) \prod c_i^{n+1-i} = \zeta(k-n)^n .$$

To complete the proof we must show $|S_T| > \varepsilon$ where ε is a positive number independent of T.

Put $S_p(T) = \sum_R e(-\sigma(TR))\nu(R)^{-k}$ where R runs over all $n \times n$ rational symmetric matrices modulo 1 such that $\nu(R)$ is a power of p. Then $S_T = \prod_p S_p(T)$ for $T > 0$, $T \in \Lambda_n^*$. Put

$$J = \frac{1}{2}\begin{pmatrix} 0 & 1_k \\ 1_k & 0 \end{pmatrix} \quad \text{and} \quad A_q(T) = \# \{C \in M_{2k,n}(Z) \bmod q \mid q^{-1}({}^tCJC - T) \in \Lambda_n^*\} .$$

Then we have

$$S_p(T) = (p^a)^{n(n+1)/2-2kn} A_{p^a}(T)$$

for sufficiently large a.

LEMMA 1. *For $W \in \Lambda_n$ we put*

$$G(W; p^a) = \sum_{\substack{C \in M_{2k,n}(Z) \\ C \bmod p^a \\ \text{primitive mod } p}} e(p^{-a}\sigma({}^tCJCW)) .$$

Then we have

$$G(W; p^a) = \begin{cases} p^{2kn(a-1)} G(p^{-(a-1)}W; p) & \text{if } W \equiv 0 \bmod p^{a-1} , \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For a primitive element C_1 of $M_{2k,n}(Z)$ we take a unimodular matrix $U = (C_1 \ *)$. From

$$\begin{pmatrix} W \\ 0 \end{pmatrix} = \begin{pmatrix} 1_n \\ 0 \end{pmatrix} W = U^{-1} C_1 W$$

follows that $W \equiv 0 \bmod p$ if and only if $C_1 W \equiv 0 \bmod p$. Lemma is obvious for $a = 1$. We assume $a \geq 2$. Decompose $C \in M_{2k,n}(Z)$ as $C = C_1 + p^{a-1}C_2$. Then C is primitive $\bmod p$ if and only if C_1 is primitive $\bmod p$.

Hence we have

$$G(W; p^a) = \sum_{\substack{C_1 \bmod p^{a-1} \\ C_1\text{: primitive mod } p}} e(p^{-a}\sigma({}^tC_1JC_1W)) \sum_{C_2 \bmod p} e(2p^{-1}\sigma({}^tC_1JC_2W)) .$$

$2\sigma({}^tC_1JC_2W) \equiv 0 \bmod p$ for any C_2 if and only if $2W{}^tC_1J \equiv 0 \bmod p$, and it is equivalent to $W \equiv 0 \bmod p$. Thus we have

$$G(W; p^a) = \begin{cases} p^{2kn} G(p^{-1}W; p^{a-1}) & \text{if } W \equiv 0 \bmod p , \\ 0 & \text{otherwise.} \end{cases}$$

Now our lemma is inductively proved. Q.E.D.

Put

$$A'_{p^a}(T) = \sharp \left\{ C \in M_{2k,n}(Z) \bmod p^a \,\middle|\, \begin{matrix} C; \text{ primitive mod } p, \\ p^{-a}({}^tCJC - T) \in \Lambda_n^* \end{matrix} \right\} .$$

LEMMA 2.

$$(p^a)^{n(n+1)/2-2kn} A'_{p^a}(T) = p^{n(n+1)/2-2kn} A'_p(T) \qquad \text{for } a \geq 1,\ T \in \Lambda_n^* .$$

Proof.

$$\begin{aligned}(p^a)^{n(n+1)/2}A'_{p^a}(T) &= \sum_{W \in \Lambda_n \bmod p^a} \sum_{\substack{C \bmod p^a \\ \text{primitive} \bmod p}} e(p^{-a}\sigma(({}^tCJC - T)W)) \\ &= \sum_{W \in \Lambda_n \bmod p^a} G(W; p^a)e(-p^{-a}\sigma(TW)) \\ &= \sum_{W \in \Lambda_n \bmod p} p^{2kn(a-1)}G(W; p)e(-p^{-1}\sigma(TW)) \\ &= p^{2kn(a-1)}p^{n(n+1)/2}A'_p(T) .\end{aligned}$$

Q.E.D.

LEMMA 3.

$$S_p(T) \geq p^{n(n+1)/2-2kn}A'_p(T) \qquad \textit{for } T \in \Lambda_n^* .$$

Proof. This follows from $A_{p^a}(T) \geq A'_{p^a}(T)$. Q.E.D.

LEMMA 4. *There is a positive number ε such that $S_2(T) > \varepsilon$ for $T \in \Lambda_n^*$.*

Proof. $A'_2(T)$ is uniquely determined by $T \bmod 2$. Hence the values of $A'_2(T)$ is a finite set. Hence we have only to prove $A'_2(T) \neq 0$ for $T \in \Lambda_n^*$. By the theory of quadratic forms $T \in \Lambda_n^*$ is equivalent over $\boldsymbol{Z}_2$ to a direct sum of

$$2^{a-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad 2^{a-1}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad 2^a u \ (a \geq 0, u \in \boldsymbol{Z}_2^\times) .$$

Since $A'_2(T + 2T') = A'_2(T)$ for $T, T' \in \Lambda_n^*$, we may assume that T is a direct sum of

$$\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad u \in \boldsymbol{Z}_2^\times$$

and 0. Hence we may suppose

$$T = \begin{pmatrix} A_1 & & & & & \\ & \ddots & & & & \\ & & A_r & & & \\ & & & u_1 & & \\ & & & & \ddots & \\ & & & & & u_{n-2r} \end{pmatrix},$$

where

$$A_1 = \cdots = A_{r-1} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_r = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$u_3 = \cdots = u_{n-2r} = 0$ and $u_1, u_2 = 0$ or $\in Z_2^\times$ (93:18 in [6]). Denote by t the number of u_i such that $u_i = 0$. It is easy to see that $\perp_{r+t-1} \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents primitively $\perp_{i=1}^{r-1} A_i \perp_t (0)$. To prove $A_2'(T) \neq 0$ we have only to show that $\perp_{k-(r+t-1)} \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents A_r, $A_r \perp (u_1)$, $A_r \perp (u_1) \perp (u_2)$, $(u_i \in Z_2^\times)$, primitively according as $t = n - 2r, n - 2r - 1, n - 2r - 2$.

(i) The case of $t = n - 2r, n - 2r - 1$; then $2(k - r - t + 1) - 3 \geq 3$ and $\perp_{k-r-t+1} \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is Z_2-maximal. Hence $\perp_{k-r-t+1} \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents $A_r \perp (u_1)$, $(u_1 \in Z_2^\times)$. $2^{-1} \det(2A_r \perp (2u_1)) \in Z_2^\times$ implies that this representation is primitive.

(ii) The case of $t = n - 2r - 2$; then $k - r + 1 - t \geq 5$.

$$\frac{1}{2}(u, 1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ 1 \end{pmatrix} = u\,, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

yield that $\perp_4 \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents $A_r \perp (u_1) \perp (u_2)$ primitively. Thus we have proved $A_2'(T) \neq 0$ for $T \in \Lambda_n^*$. Q.E.D.

If $S_p(T) \geq (1 - p^{-2})^{4n}$ for odd prime p, then

$$S_T = \prod S_p(T) \geq S_2(T) \prod_{p \neq 2} (1 - p^{-2})^{4n} > \varepsilon' > 0$$

holds, and it completes the proof. From now we show

$$S_p(T) \geq (1 - p^{-2})^{4n} \qquad \text{for odd prime } p,\ T \in \Lambda_n^*\,.$$

We fix an odd prime p. Let L be a hyperbolic space of $\dim 2k$ over $F_p = Z/(p)$; then $A_p'(T)$ is equal to the number of isometries from the quadratic space over F_p corresponding to T to L where isometries are supposed to be injective. For quadratic spaces M, N over F_p we denote by $A(M, N)$ the number of isometries from M to N. Our aim is to prove that

$$A(T, L) \geq p^{2kn - n(n+1)/2}(1 - p^{-2})^{4n} \qquad \text{for any quadratic space}$$

T of dimension n.

Let T be a quadratic space of $\dim n$ over F_p and $T = T_0 \perp R$ where R is the radical of T. Define a quadratic space L_1 by $L \cong T_0 \perp L_1$; then $A(T, L) = A(T_0, L)A(R, L_1)$ and

$$A(T_0, L) = p^{-t(t+1)/2+2kt}\begin{cases}(1 \pm p^{-k})(1 \pm p^{-k+t/2}) \prod_{s=1}^{t/2-1} (1 - p^{2s-2k}) & \text{if } t \equiv 0 \bmod 2\,, \\ (1 \pm p^{-k}) \prod_{s=1}^{(t-1)/2} (1 - p^{2s-2k}) & \text{if } t \equiv 1 \bmod 2\,, \end{cases}$$

where $t = \dim T_0$ ([8]).

From $k \geq n + 2$ and $n \geq t$ follows

$$A(T_0, L) \geq p^{-t(t+1)/2+2kt}(1 - p^{-2})^{2n}\,.$$

Since, then, $A(T, L) \geq p^{2kn-n(n+1)/2}(1 - p^{-2})^{4n}$ follows from $A(R, L_1) \geq p^{-(n-t)(n-t+1)/2+(2k-t)(n-t)}(1 - p^{-2})^{2(n-t)}$, we have only to prove

LEMMA 5. *Let M be a regular quadratic space over F_p of* $\dim M = m$ *and let N* be *a totally isotropic quadratic space over F_p of* $\dim N = s$. *Then we have*

$$A(N, M) \geq p^{-s(s+1)/2+ms}(1 - p^{-2})^{2s} \qquad \textit{if } m \geq 2s + 3\,.$$

Proof. Put $N = F_p[v_1, \cdots, v_s]$, $M = H \perp M_1$ where H is a hyperbolic plane. For quadratic spaces we denote by Q, B associated quadratic forms and bilinear forms ($Q(x) = B(x, x)$). Take a basis $\{e, f\}$ of H such that $Q(e) = Q(f) = 0$, $B(e, f) = 1$. Let σ be an isometry from N to M such that $\sigma(v_1) = e$, and put $\sigma(v_i) = a_i e + b_i f + u_i$ ($a_i, b_i \in F_p$, $u_i \in M_1$). Then $B(v_1, v_i) = B(\sigma(v_1), \sigma(v_i)) = b_i = 0$. Since σ is injective, $u_2, \cdots, u_s$ are linearly independent and $B(u_i, u_j) = 0$ for i, j. If, conversely, $w_2, \cdots, w_s \in M_1$ are linearly independent and $B(w_i, w_j) = 0$ for i, j, then $\mu(v_1) = e$, $\mu(v_i) = a_i e + w_i$ ($a_i \in F_p$, $i \geq 2$) define an isometry from N to M. Thus we have

$$A(N_s, M) = p^{s-1}a(M)A(N_{s-1}, M_1)\,,$$

where N_i denotes a totally isotropic quadratic space of $\dim i$ and $a(M)$ is the number of (non-zero) isotropic vectors of M.

Put $M = \perp_{s-1} H \perp M_0$; then we have

$$\begin{aligned} A(N_s, M) &= p^{s-1}a\Big(\perp_{s-1} H \perp M_0\Big)\, A\Big(N_{s-1}, \perp_{s-2} H \perp M_0\Big) \\ &= \cdots \\ &= p^{s(s-1)/2} \prod_{i=0}^{s-1} a\Big(\perp_{i} H \perp M_0\Big)\,. \end{aligned}$$

If $\dim M_0 \equiv 1 \bmod 2$, then we have

$$\begin{aligned} a\Bigl(\mathop{\perp}_i H \perp M_0\Bigr) &= p^{2i+m-2(s-1)-1} - 1 \\ &\geq p^{2i+m-2(s-1)-1}(1-p^{-2}) \\ &\geq p^{2i+m-2(s-1)-1}(1-p^{-2})^2 , \end{aligned}$$

since $2i + m - 2(s-1) - 1 \geq 2$.

If $\dim M_0 \equiv 0 \bmod 2$, then we have

$$a\Bigl(\mathop{\perp}_i H \perp M_0\Bigr) = \begin{cases} p^{2r-1} + p^r - p^{r-1} - 1 & \text{if } M_0 \text{ is hyperbolic} , \\ p^{2r-1} - p^r + p^{r-1} - 1 & \text{otherwise} , \end{cases}$$

where

$$2r = 2i + m - 2(s-1) = \dim \mathop{\perp}_i H \perp M_0 .$$

Hence

$$a\Bigl(\mathop{\perp}_i H \perp M_0\Bigr) = (p^r \mp 1)(p^{r-1} \pm 1) \geq p^{2r-1}(1-p^{-2})^2$$

holds.

Thus we have

$$\begin{aligned} A(N_s, M) &\geq p^{s(s-1)/2} \prod_{i=0}^{s-1} p^{2i+m-2(s-1)-1}(1-p^{-2})^2 \\ &= p^{ms-s(s+1)/2}(1-p^{-2})^{2s} . \end{aligned} \qquad \text{Q.E.D.}$$

COROLLARY. *If n, k are natural numbers such that $k \geq 2n+2$, $k \equiv 0 \bmod 2$ and $f(Z) = \sum a(T)e(\sigma(TZ))(Z \in H_n)$ is a modular form of degree n and weight k, then we have*

$$a(T) = O(|T|^{k-(n+1)/2}) \qquad \textit{for } T > 0 .$$

Proof. It is known that there exist cusp forms f_r of degree r and weight k such that $f(Z) = \sum_{r=1}^{n-1} E^k_{n,r}(Z, f_r) + aE^k_n(Z) + f_n(Z)$, $(a \in C)$ ([3]). Since $a({}^tUTU) = a(T)$ for $U \in GL(n, Z)$, we may assume that T is in some fixed Siegel domain. Theorem in § 2 and our theorem imply the corollary. Q.E.D.

§ 4.

Let A be an even integral unimodular positive definite symmetric matrix of rank m; then $m \equiv 0 \bmod 8$. Put

$$\theta_n(Z, A) = \sum_{C \in M_{m,n}(Z)} e\left(\frac{1}{2}\sigma({}^tCACZ)\right), \qquad Z \in H_n .$$

Then $\theta_n(Z, A)$ is a modular form of degree n and weight $m/2$. Then there exist cusp forms f_r of degree r and weight $m/2$ such that

$$\theta_n(Z, A) = E_n^{m/2}(Z) + \sum_{r=1}^{n-1} E_{n,r}^{m/2}(Z, f_r) + f_n(Z) \qquad \text{for } n \leq m/4 - 1 .$$

If

$$E_n^{m/2}(Z) = \sum_{\substack{T \geq 0 \\ T \in \Lambda_n^*}} a_{m/2,n}(T) e(\sigma(TZ)) ,$$

then

$$a_{m/2,n}(T) = 2^{n(m-n+1)/2} \prod_{t=0}^{n-1} \frac{\pi^{(m-t)/2}}{\Gamma((m-t)/2)} \cdot |T|^{(m-n-1)/2} \prod_p S_p(T) \qquad \text{for } T > 0 ,$$

where $S_p(T) = \sum_R e(-\sigma(TR))\nu(R)^{-m/2}$ where R runs over all $n \times n$ rational symmetric matrices modulo 1 such that the product $\nu(R)$ of denominators of elementary divisors of R is a power of p.

Put

$$\theta_n(Z, A) = \sum_{\substack{T \geq 0 \\ T \in \Lambda_n^*}} N_n(T, A) e(\sigma(TZ)) ,$$

and

$$E_{n,r}^{m/2}(Z, f_r) = \sum_{\substack{T \geq 0 \\ T \in \Lambda_n^*}} a(T, f_r) e(\sigma(TZ)) .$$

Then we have, summarizing,

THEOREM. *If $n \leq m/4 - 1$, $T \in \Lambda_n^*$, $T > 0$, then $N_n(T, A) = a_{m/2,n}(T) + \sum_{r=1}^{n-1} a(T, f_r) + O(|T|^{m/4})$. If, moreover, T runs in any fixed Siegel domain, then*

$$a_{m/2,n}(T) \sim |T|^{(m-n-1)/2} , \qquad a(T, f_r) = O(|T_r|^{-(m/2-r-1)/2} |T|^{(m-n-1)/2}) ,$$

where T_r stands for the upper left $r \times r$ submatrix of T.

For $n \times n$ positive definite matrix S we denote by $m(S)$ the minimal value of ${}^txSx$ ($x \in Z^n - \{0\}$). It is well known that there is a constant μ_n such that $m(S) \leq \mu_n \sqrt[n]{|S|}$ for any $n \times n$ positive definite matrix S.

COROLLARY. *If $n \leq m/4 - 1$, then*

$$N_n(T, A) = a_{m/2,n}(T) + O(m(T)^{1-m/4}|T|^{(m-n-1)/2}) \qquad \textit{for } T > 0,\ T \in \Lambda_n^* .$$

Especially $N_n(T, A) > 0$ *if* $m(T)$ *is sufficiently large.*

Proof.

$$\begin{aligned}|T_r|^{-(m/2-r-1)/2} &= O(m(T_r)^{-r(m/2-r-1)/2}) \\ &= O(m(T)^{-r(m/2-r-1)/2}) .\end{aligned}$$

On the other hand

$$\begin{aligned}|T|^{m/4}(m(T)^{1-m/4}|T|^{(m-n-1)/2})^{-1} &= m(T)^{m/4-1}|T|^{(2n+2-m)/4} \\ &\le \mu_n^{m/4-1}|T|^{(m/4-1)/n+(2n+2-m)/4} .\end{aligned}$$

Since $r(m/2 - r - 1)/2 \ge m/4 - 1$ for $1 \le r \le n \le m/4 - 1$, we have $a(T, f_r) = O(m(T)^{1-m/4}|T|^{(m-n-1)/2})$, $|T|^{m/4} = O(m(T)^{1-m/4}|T|^{(m-n-1)/2})$, if $|T| \ge 1$.

There are only finitely many equivalence classes of $T \in \Lambda_n^*$ such that $T > 0$, $|T| < 1$. This completes the proof. Q.E.D.

Remark. Let $f(Z) = \sum a(T)e(\sigma(TZ))$ be a modular form of degree n, weight $k(\in \frac{1}{2}Z)$ with level such that the constant term of $f(Z)$ at any cusps vanishes. Results in [2] and here seem to suggest that $a(T) = O(m(T)^{1-k/2}|T|^{k-(n+1)/2})$ for $T > 0$ if, at least, $2k \ge 2n + 3$.

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