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ON THE GEOMETRY OF SOME SIEGEL DOMAINS

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§ 1. Introduction

In his book [2], Pyatetskii-Shapiro describes representations of classical domains as certain "fibrations" over their boundary components. The fibers are quasi-symmetric Siegel domains of the second kind [3]. Professor Kobayashi asked "how symmetric" these fibers are, or more precisely, he asked for totally geodesic directions in the fiber. The object of this paper is to determine at least a totally geodesic submanifold of the fiber, and it turns out to be complex. As the fibers over different points are analytically equivalent, we consider one particular fiber. The general calculation below holds for a reductive homogeneous submanifold through the base point of a symmetric space. Then we specify the second fundamental form of the fiber for the case of the Siegel disk (domain of type III) $\{Z \in M(p, C) | {}^tZ = Z, I_p - Z^*Z > 0\}$. For the domain of type I, $\{Z \in M(p, q, C) | I_q - Z^*Z > 0\}$, $p \geq q$, and the domain of type II, $\{Z \in M(p, C) | {}^tZ = -Z, I_p - Z^*Z > 0\}$, the calculations are similar, so we just point out some of the changes (§ 6). Since the case of a zero-dimensional boundary component is trivial, we consider only positive-dimensional boundary components. For lack of space-time, we have not yet considered the domain of type IV.

Finally, we prove that, in the above cases, the Bergman metric of the domain induces (up to a constant) the Bergman metric of the fiber. In proving that, we also have to describe the fiber as a Siegel domain of the second kind and compute Satake's mappings R and T. We include a proof that the fiber is in fact quasi-symmetric, since the proof is easy when we have the mappings R and T. (For a general proof see Ch. V, § 5 of a forthcoming book by Satake about algebraic structures on symmetric domains). The Siegel domains in the cases of domains of type I, II, III are defined over the cones of positive-definite

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matrices with entries in complex numbers, quaternions and real numbers, respectively.

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§ 2. The Siegel disk

We consider the following classical domain, where $1 \leq p \in \boldsymbol{Z}$:

$$\mathscr{D}_p := \{Z \in M(p, \boldsymbol{C}) \mid {}^tZ = Z,\ I_p - Z^*Z > 0\} ,$$

where $M(p, \boldsymbol{C})$ is the set of $p \times p$ complex matrices, ^{t} is transpose, $*$ is adjoint and I_p is the identity matrix. The automorphism group of $\mathscr{D}_p$ is

$$G = \{g \in G\ell(2p, \boldsymbol{C}) \mid {}^t g \cdot J_0 g = J_0,\ g^*H_0 g = H_0\} ,$$

where

$$J_0 = \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix} \quad \text{and} \quad H_0 = \begin{pmatrix} -I_p & 0 \\ 0 & I_p \end{pmatrix} .$$

The Lie algebra of G is

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \middle| A, B \in M(p, \boldsymbol{C}),\ A^* + A = 0,\ {}^tB = B \right\} .$$

G acts transitively on $\mathscr{D}_p$ with the action

$$g \cdot Z = (aZ + b)(cZ + d)^{-1} , \qquad \text{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } a, b, c, d \in M_p(\boldsymbol{C}) .$$

The isotropy group at $Z = 0$ is

$$K = \left\{ \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \middle| a \in U(p) \right\} .$$

So $\mathscr{D}_p = G/K$, and also the involution is $\sigma : G \ni g \mapsto H_0 g H_0^{-1} \in G$.

For realizations of $\mathscr{D}_p$ giving fibrations over different boundary components, one uses, following Pyatetskii-Shapiro [2], other choices of J_0 and H_0. The realizations take place in a Grassmannian; also the above one, where Z is represented by $\begin{pmatrix} Z \\ I_p \end{pmatrix}$ in $G_{p,p}(\boldsymbol{C})$. Put $p = r + s$, with $0 < r \in \boldsymbol{Z}$, and

$$J_s := \begin{pmatrix} 0 & 0 & 0 & I_s \\ 0 & 0 & I_r & 0 \\ 0 & -I_r & 0 & 0 \\ -I_s & 0 & 0 & 0 \end{pmatrix}, \qquad H_s := \begin{pmatrix} 0 & 0 & 0 & iI_s \\ 0 & -I_r & 0 & 0 \\ 0 & 0 & I_r & 0 \\ -iI_s & 0 & 0 & 0 \end{pmatrix}.$$

The corresponding realization is

$$\mathscr{D}_p^{(s)} = \{[U] \in G_{p,p}(\boldsymbol{C}) \mid U \in M(2p, p, \boldsymbol{C}),\ {}^tUJ_sU = 0,\ U^*H_sU > 0\},$$

where [ ] means equivalence class under the right action of $G\ell(p, \boldsymbol{C})$ on $M(2p, p, \boldsymbol{C}) = \{2p \times p$ complex matrices$\}$. For each $[U] \in \mathscr{D}_p^{(s)}$, there is a unique representation of the form

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \\ 0 & I_r \\ I_s & 0 \end{bmatrix}, \qquad \text{where } U_{11} \in M(s, \boldsymbol{C}),\ U_{12} \in M(s, r, \boldsymbol{C}),$$

$U_{21} \in M(r, s, \boldsymbol{C})$, $U_{22} \in M(r, \boldsymbol{C})$. Here ${}^tU_{11} = U_{11}$, ${}^tU_{22} = U_{22}$,

$${}^tU_{21} = U_{12} \quad \text{and} \quad W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} > 0,$$

where

$$W_{11} = \frac{1}{i}(U_{11} - U_{11}^*) - U_{21}^*U_{21}, \qquad W_{12} = W_{21}^* = \frac{1}{i}U_{12} - U_{21}^*U_{22},$$

$W_{22} = I_r - U_{22}^*U_{22}$. The positivity-condition is equivalent to $W_{22} > 0$ and

$$\frac{1}{i}(U_{11} - U_{11}^*) - U_{21}^*(I_r - U_{22}U_{22}^*)^{-1}U_{21} - U_{12}W_{22}^{-1}U_{12}^* - iU_{12}W_{22}^{-1}U_{22}^*U_{21} + iU_{21}^*U_{22}W_{22}^{-1}U_{12}^* > 0.$$

Pyatetskii-Shapiro puts this in Siegel domain form as follows: Set $t = U_{22}$, $z = 2U_{11}$, $u = U_{12}$, $v = V_{12}(\in M(s, r, \boldsymbol{C}))$, and

$$L_t(u, v) = u(I_r - t^*t)^{-1}v^* + \bar{v}(I_r - tt^*)^{-1}\,{}^tu + i\{u(I_r - t^*t)^{-1}t^{*t}v + v(I_r - t^*t)^{-1}t^*\,{}^tu\}.$$

Finally, let Ω be the cone of $s \times s$ hermitian positive definite matrices. Then $L_t(u, v)$ is $\boldsymbol{C}$-linear in u, $\boldsymbol{R}$-linear in v, and $L_t(u, v) - L_t(v, u)$ is purely imaginary, where conjugation is *. The realization $\mathscr{D}_p^{(s)}$ is then the Siegel domain of the third kind given by L_t and Ω, i.e.

$$\mathscr{D}_p^{(s)} = \left\{ \begin{bmatrix} \frac{1}{2}z & u \\ {}^t u & t \\ 0 & I_r \\ I_s & 0 \end{bmatrix} \middle| \begin{array}{r} u \in M(s, r, \boldsymbol{C}), \quad {}^t z = z \in M(s, \boldsymbol{C}), \quad {}^t t = t \in M(r, \boldsymbol{C}), \\ I_r - t^* t > 0, \quad \operatorname{Im} z - \operatorname{Re} L_t(u, u) \in \Omega \end{array} \right\} .$$

We see that we have a "fibration" of $\mathscr{D}_p^{(s)}$ over the boundary component

$$\mathscr{F}_s = \left\{ \begin{bmatrix} I_s & 0 \\ 0 & t \\ 0 & I_r \\ 0 & 0 \end{bmatrix} \middle| {}^t t = t \in M(r, \boldsymbol{C}), \; I_r - t^* t > 0 \right\} \simeq \mathscr{D}_r, \text{ by the map } (z, u, t) \mapsto t \, .$$

Let V_0 be the fiber over $t = 0$.
The automorphism group now looks like

$$G^{(s)} = \{ g \in G\ell(2p, \boldsymbol{C}) \,|\, {}^t g J_s g = J_s, \; g^* H_s g = H_s \} ,$$

with action $g[U] = [gU]$, and the Lie algebra is

$$\mathfrak{g}^{(s)} = \{ X \in M(2p, \boldsymbol{C}) \,|\, {}^t X J_s + J_s X = 0, \; X^* H_s + H_s X = 0 \} .$$

And the involution is $\sigma : g \to H_s g H_s^{-1}$. All these objects correspond to the same things in the realization $\mathscr{D}_p$, via the isomorphism $\kappa : \mathscr{D}_p \xrightarrow{\simeq} \mathscr{D}_p^{(s)}$ which takes W to MW, where $W \in M(2p, p, \boldsymbol{C})$ represents a point in $\mathscr{D}_p$, (each such point has a unique representative of the form $W = \begin{bmatrix} Z \\ I_p \end{bmatrix}$ with ${}^t Z = Z \in M(p, \boldsymbol{C})$ and $I_p - Z^* Z > 0$), and where

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} I_s & 0 & 0 & iI_s \\ 0 & \sqrt{2}\, I_r & 0 & 0 \\ 0 & 0 & \sqrt{2}\, I_r & 0 \\ iI_s & 0 & 0 & I_s \end{pmatrix} \begin{pmatrix} I_p & 0 \\ 0 & \begin{pmatrix} 0 & I_r \\ I_s & 0 \end{pmatrix} \end{pmatrix} \in U(2p) .$$

M satisfies ${}^t M J_s M = J_0$, $M^* H_s M = H_0$, and we have also the isomorphism $\kappa : G \xrightarrow{\simeq} G^{(s)}$ given by $\kappa(g) = \kappa \circ g \circ \kappa^{-1}$, which can also be written $g \mapsto MgM^*$. Then $\kappa(gW) = \kappa(g)\kappa(W)$, and κ sends $Z = 0$ in $\mathscr{D}_p$ to the point $\sigma = \begin{pmatrix} iI_s & 0 \\ 0 & 0 \\ 0 & I_r \\ I_s & 0 \end{pmatrix} \in V_0$, which we therefore take as our base point in $\mathscr{D}_p^{(s)}$.

We now look at some subgroups of $G^{(s)}$ which are relevant for the boundary fibration:

1) An element $g \in G^{(s)}$ preserves the boundary component $\mathscr{F}_s$ if and only if it has the form

$$g = \begin{array}{c} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \begin{matrix} s \\ r \\ r, \\ s \end{matrix} \\ \begin{matrix} s & \quad r & \quad r & \quad s \end{matrix} \end{array}$$ where the sizes of the blocks are as indicated .

Let $\tilde{G}^{(s)}$ be the group of these elements.

2) An element $g \in G^{(s)}$ fixes the point $\begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \in \mathscr{F}_s$

(that is the point $t = 0$) if and only if it has the form

$$g = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & 0 & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix}.$$

Let $G_0^{(s)}$ be the group of these elements.

3) An element $g \in G^{(s)}$ preserves the fiber V_0 if and only if it fixes the point $t = 0$ in $\mathscr{F}_s$. So the "group of the fiber V_0" is $G_0^{(s)}$.

4) An element $g \in G_0^{(s)}$ fixes the base point $\sigma = \begin{pmatrix} iI & 0 \\ 0 & 0 \\ 0 & I \\ I & 0 \end{pmatrix} \in V_0$ if and only

if it has the form

$$g = \begin{pmatrix} a_{11} & a_{12} & 0 & i(a_{44} - a_{11}) \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix}.$$

Let $K_0^{(s)}$ be the group of these elements.

Using the conditions satisfied by elements of $G^{(s)}$, we can then check: $G_0^{(s)}$ is the set of elements

$$g = \begin{pmatrix} {}^t a_{14}^{-1}, & i\, {}^t a_{44}^{-1} a_{24}^* \bar{a}_{33}, & {}^t a_{44}^{-1}\, {}^t a_{24} a_{33}, & a_{14} \\ 0 & \bar{a}_{33} & 0 & a_{24} \\ 0 & 0 & a_{33} & -i\bar{a}_{24} \\ 0 & 0 & 0 & a_{44} \end{pmatrix}$$

with $a_{14} \in M(s, \boldsymbol{R})$, $a_{24} \in M(r, s, \boldsymbol{C})$, $a_{33} \in U(r)$, $a_{44} \in G\ell(s, \boldsymbol{R})$ and ${}^t a_{14} a_{44} - {}^t a_{44} a_{14} = -i(a_{24}^* a_{24} - {}^t a_{24} \bar{a}_{24})$.

$$K_0^{(s)} \text{ is the set of elements } g = \begin{pmatrix} a_{44} & 0 & 0 & 0 \\ 0 & \bar{a}_{33} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{pmatrix}$$

with $a_{33} \in U(r)$, $a_{44} \in 0(s)$, i.e. $K_0^{(s)} = U(r) \times 0(s)$.
The Lie algebra of $G_0^{(s)}$ is

$$\mathfrak{g}_0^{(s)} = \left\{ \begin{pmatrix} -{}^t X_{44}, & iX_{24}^*, & {}^t X_{24}, & X_{14} \\ 0 & \bar{X}_{33} & 0 & X_{24} \\ 0 & 0 & X_{33} & -i\bar{X}_{24} \\ 0 & 0 & 0 & X_{44} \end{pmatrix} \middle| \begin{array}{r} {}^t X_{14} = X_{14},\ X_{44} \in M(s, \boldsymbol{R}), \\ X_{24} \in M(r, s, \boldsymbol{C}), \\ -X_{33}^* = X_{33} \in M(r, \boldsymbol{C}) \end{array} \right\},$$

as a subalgebra of $\mathfrak{g}\ell(2p, \boldsymbol{C})$.
Finally, one can check that
a) $\tilde{G}^{(s)}$ is transitive on $\mathscr{F}_s$
b) $G_0^{(s)}$ is transitive on V_0
c) The fibration $\mathscr{D}_p^{(s)} \ni (z, u, t) \mapsto t \in \mathscr{F}_s$ is $\tilde{G}^{(s)}$-equivariant.
d) $\tilde{G}^{(s)}/K^{(s)} \cap \tilde{G}^{(s)} \xrightarrow{\simeq} G^{(s)}/K^{(s)} = \mathscr{D}_p^{(s)}$ is an isomorphism.

The fiber $V_0 = G_0^{(s)}/K_0^{(s)}$ is a reductive homogeneous space with respect to the decomposition $\mathfrak{g}_0^{(s)} = \mathfrak{k}_0^{(s)} + \mathfrak{m}$, where $\mathfrak{k}_0^{(s)}$ is the Lie algebra of $K_0^{(s)}$ and

$$\mathfrak{m} = \left\{ \begin{pmatrix} * & * & * & X_{14} \\ 0 & 0 & 0 & X_{24} \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & X_{44} \end{pmatrix} \in \mathfrak{g}_0^{(s)} \middle| {}^t X_{44} = X_{44} \right\}.$$

The following is of course well-known, but we include it for completeness: Consider the realization $\mathscr{D}_p = G/K$. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} \middle| -A^* = A \in M(p, \boldsymbol{C}) \right\}$$

is the Lie algebra of K, and

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & B \\ \bar{B} & 0 \end{pmatrix} \middle| {}^t B = B \in M(p, \boldsymbol{C}) \right\}.$$

The tangent space at $Z = 0$ is represented by $\mathfrak{p}$, and $\mathfrak{p}$ admits the positive definite Ad K-invariant j_0-hermitian metric $B(X, Y) = \operatorname{trace}(XY)$, which is, except for a factor $2(p+1)$, the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{p}$, and where $j_0\colon \begin{pmatrix} 0 & B \\ \bar{B} & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & iB \\ -i\bar{B} & 0 \end{pmatrix}$ is the (Ad K-invariant) complex structure on $\mathfrak{p}$ corresponding to the natural complex structure on $\mathscr{D}_p$. In this way, by translation from the origin $Z = 0$, $\mathscr{D}_p$ gets its invariant Kähler metric.

§ 3. Curvature of V_0

In this section we write $G, K, \mathscr{D}, G_0, K_0$ for $G^{(s)}, K^{(s)}, \mathscr{D}_p^{(s)}, G_0^{(s)}, K_0^{(s)}$ etc. The connection on G/K can be described by ([1], Ch. 10, 11):

$$\Lambda(X) = \begin{cases} \lambda(X)\,, & X \in \mathfrak{k} \\ \Lambda_{\mathfrak{p}}(X)\,, & X \in \mathfrak{p} \end{cases}$$

where λ is the isotropy representation, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition and $\Lambda(X) \in \mathfrak{gl}(p(p+1), \boldsymbol{R})$, $(p(p+1) = \dim_{\boldsymbol{R}} \mathscr{D})$. For the riemannian connection given by the above invariant metric (Killing form), the connection is the natural torsion free and also the canonical one, i.e. $\Lambda_{\mathfrak{p}} \equiv 0$, ([1], Ch. 10, 11). By [1], p. 191, $(A_X)_0 = 0$ for $X \in \mathfrak{p}$, where $A_X\colon = L_X - \nabla_X$, (Lie derivative minus covariant derivative). If $X \in \mathfrak{g}$, then we let X also denote the vector field on G/K defined by $\exp tX$. By [1], p. 188, we have $u_0 \circ \Lambda(X) \circ u_0^{-1} = -(A_X)_0$ for $X \in \mathfrak{g}$, where u_0 is a (fixed) linear frame at 0, used to define Λ. For the isotropy representation we have the commutative diagram

$$\begin{array}{ccccc} \mathfrak{p} & \xrightarrow[\zeta]{\simeq} & T_0\mathscr{D} & \xrightarrow[u_0^{-1}]{\simeq} & \boldsymbol{R}^{p(p+1)} \\ {\scriptstyle \mathrm{ad}_X}\big\downarrow & & & & \big\downarrow{\scriptstyle \lambda(X)} \\ \mathfrak{p} & \xrightarrow[\zeta]{\simeq} & T_0\mathscr{D} & \xrightarrow[u_0^{-1}]{\simeq} & \boldsymbol{R}^{p(p+1)} \end{array}$$

where $T_0\mathscr{D}$ is the tangent space at 0, and

$$\zeta(X)\colon = \left.\frac{d}{dt}\right|_{t=0} \{(\exp tX)K\}\,.$$

So for $X \in \mathfrak{k}$, $\mathrm{ad}_X|_{\mathfrak{p}} = \zeta^{-1} \circ u_0 \circ \lambda(x) \circ u_0^{-1} \circ \zeta = \zeta^{-1} \circ u_0 \circ \Lambda(X) \circ u_0 \circ \zeta = -\zeta^{-1} \circ (A_X)_0 \circ \zeta$. We see

$$(1)\qquad -(A_X)_0 = \begin{cases} \zeta \circ \mathrm{ad}_X \circ \zeta^{-1}, & X \in \mathfrak{k} \\ 0 & X \in \mathfrak{p}\,. \end{cases}$$

To calculate the connection from this, we have ([1], p. 188).

$$(2)\qquad \nabla_Y X = -A_X Y \qquad \text{for all vector fields } X, Y \text{ on } G/K\,.$$

The similar situation for $V_0 = G_0/K_0$ is that the induced connection is G_0-invariant, and hence given by some $\Lambda_{\mathfrak{m}}: \mathfrak{m} \to \mathfrak{gl}(\dim_{\boldsymbol{R}} V_0, \boldsymbol{R})$. Here we base $\Lambda_{\mathfrak{m}}$ on a linear frame $\tilde{u}_0$ of V_0 at 0, and corresponding to the above, we have $\xrightarrow[\tilde{\zeta}]{\simeq} T_0V_0 \xrightarrow[\tilde{u}_0^{-1}]{\simeq} \boldsymbol{R}^{\dim_{\boldsymbol{R}} V_0}$. We get

$$(3)\qquad -(\tilde{A}_Y)_0 = \begin{cases} \tilde{\zeta} \circ \mathrm{ad}_Y \circ \tilde{\zeta}^{-1}, & Y \in \mathfrak{k}_0 \\ \tilde{u}_0 \circ \Lambda_{\mathfrak{m}}(Y) \circ \tilde{u}_0^{-1}, & Y \in \mathfrak{m}\,, \end{cases}$$

where $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{m}$ is the earlier decomposition, and also $\tilde{\nabla}_{W_0} Y = -(\tilde{A}_Y)_0 W$ for vector fields Y, W on V_0. We want to calculate $\Lambda_{\mathfrak{m}}$.

Let $Z \in T_0V_0$, $Y \in \mathfrak{m}$, $\alpha(Z, Y)$ be the second fundamental form of V_0 in $\mathscr{D}$, and $\tilde{\nabla}$ be the (above) induced covariant derivative on V_0. By the Gauss formula, we have

$$(4)\qquad \begin{aligned} \tilde{u}_0 \circ \Lambda_{\mathfrak{m}}(Y) \circ \tilde{u}_0^{-1} Z &= -(\tilde{A}_Y)_0 Z = \tilde{\nabla}_Z Y \\ &= \nabla_Z Y - \alpha(Z, Y) = -(A_Y)_0 Z - \alpha(Z, Y)\,. \end{aligned}$$

We must decompose Y relative to $\mathfrak{k}$ and $\mathfrak{p}$ in order to use (1), and we claim

$$(5)\qquad -(A_Y)_0 = \zeta \circ \mathrm{ad}_{(I+\sigma)Y/2} \circ \zeta^{-1}\,,$$

where σ is the involution on G.

Proof. a) The map $\mathfrak{g} \ni Y \mapsto Y \in \{\text{vector fields on } \mathscr{D}\}$ is $\boldsymbol{C}$-linear, for $Y_{gK} = \left.\dfrac{d}{dt}\right|_{t=0} \{(\exp tY)gK\} = \pi_* \circ R_{g*}(Y)$, where $\pi: G \to G/K$ is the natural map and $R_g: G \to G$ is right translation by $g \in G$.

b) Using (1), we have

$$\begin{aligned} -(A_Y)_0 X &= \nabla_{X_0}\left(\frac{I+\sigma}{2}Y + \frac{I-\sigma}{2}Y\right) = -(A_{(I+\sigma)Y/2})_0 X - (A_{(I-\sigma)Y/2})_0 X \\ &= \zeta \circ \mathrm{ad}_{(I+\sigma)Y/2} \circ \zeta^{-1} X\,, \end{aligned}$$

proving (5). Further, $\alpha(Z, Y) =$ normal component of $-(A_Y)_0 Z =$ normal component of $\zeta \circ \mathrm{ad}_{(I+\sigma)Y/2} \circ \zeta^{-1} Z$, i.e.

$$(6)\qquad \alpha(Z, Y) = \text{normal component of } \zeta\left[\frac{I+\sigma}{2}Y, \zeta^{-1}Z\right], \text{ where } Z \in T_0V_0\,, \; Y \in \mathfrak{m}\,.$$

By (4) we see that, for such Z, Y:

$$\tilde{u}_0 \circ \Lambda_{\mathfrak{m}}(Y) \circ \tilde{u}_0^{-1} Z = \text{tangential component of } \zeta\left[\frac{I+\sigma}{2} Y, \zeta^{-1} Z\right]. \tag{7}$$

We choose our (fixed) frames u_0 and $\tilde{u}_0$ as follows: Let $\tilde{u}_0 = \{e_1, \cdots, e_{p(p+1)}\}$ be an orthonormal frame at 0 of $\mathscr{D}$ such that $\tilde{u}_0 = \{e_1, \cdots, e_{\dim_{\boldsymbol{R}} V_0}\}$ is a frame of V_0. Then since the metric on $\mathfrak{p}$ is given by B (Killing form), we have

$$\Lambda_{\mathfrak{m}}(Y) = \sum_{\ell=1}^{\dim_{\boldsymbol{R}} V_0} B\left(\left[\frac{I+\sigma}{2} Y, \zeta^{-1} \tilde{u}_0(\,\cdot\,)\right], \zeta^{-1} e_\ell\right) \varepsilon_\ell\,, \tag{8}$$

as an endomorphism of $\boldsymbol{R}^{\dim_{\boldsymbol{R}} V_0}$, where the ε_ℓ's form the standard basis of the latter vector space. We want to simplify this:

The following diagram commutes, where $\theta = \dfrac{I-\sigma}{2}$ is the projection onto $\mathfrak{p}$:

$$\begin{array}{ccc} T_0 V_0 & \hookrightarrow & T_0 \mathscr{D} \\ \tilde{\zeta}\uparrow\simeq & & \simeq\uparrow\tilde{\zeta} \\ \mathfrak{m} & \xrightarrow[\theta]{} & \mathfrak{p} \end{array}\,.$$

For if $X = X' + \theta X \in \mathfrak{m}$ with $X' \in \mathfrak{k}$, and $\pi: G \to G/K$, then on the one hand

$$\tilde{\zeta}(X) = \frac{d}{dt}\bigg|_{t=0} \{(\exp tX)K_0\} = \frac{d}{dt}\bigg|_{t=0} \{(\exp tX)K\} = \pi_* X\,,$$

and on the other hand

$$(\exp t\theta X)(\exp tX') = \exp\{t(\theta X + X') + O(t^2)\} = \exp\{tX + O(t^2)\}$$

implies

$$\begin{aligned} \zeta\theta X &= \frac{d}{dt}\bigg|_{t=0} \{(\exp t\theta X)K\} = \frac{d}{dt}\bigg|_{t=0} \{(\exp t\theta X)(\exp tX')K\} \\ &= \frac{d}{dt}\bigg|_{t=0} \{\exp tX + O(t^2)\}K\} = \pi_* X\,. \end{aligned}$$

Via $\tilde{u}_0^{-1} \circ \tilde{\zeta}$ we can consider $\Lambda_{\mathfrak{m}}(Y) \in \operatorname{End}(\mathfrak{m})$, and using also (the injective map) θ, we consider $\Lambda_{\mathfrak{m}}(Y) \in \operatorname{End}(\theta\mathfrak{m})$.

PROPOSITION 1. *For* $\Lambda_{\mathfrak{m}}(Y) \in \operatorname{End}(\theta\mathfrak{m})$, *where* $Y \in \mathfrak{m}$ *and* $\theta = \dfrac{I-\sigma}{2}$ *is the projection to* $\mathfrak{p}$, *we have*

$$\Lambda_{\mathfrak{m}}(Y) = \tau \circ \operatorname{ad}_{(I+\sigma)Y/2} ,$$

where $\tau : \mathfrak{p} \to \theta\mathfrak{m}$ *is the orthogonal projection with respect to the Killing form.*

Proof. For $Y, Z \in \mathfrak{m}$, we have

$$\Lambda_{\mathfrak{m}}(Y)Z = \sum_{\ell=1}^{\dim_{\boldsymbol{R}} V_0} B\Big(\Big[\frac{I+\sigma}{2}Y, \zeta^{-1}\tilde{\zeta}Z\Big], \zeta^{-1}e_\ell\Big)\zeta^{-1}e_\ell \in \theta\mathfrak{m} .$$

So for $Y \in \mathfrak{m}$, $Z \in \theta\mathfrak{m}$, we get, since $\zeta^{-1}\tilde{\zeta} = \theta$ by (9), and considering $\Lambda_{\mathfrak{m}}(Y) \in \operatorname{End}(\theta\mathfrak{m})$:

$$\Lambda_{\mathfrak{m}}(Y)Z = \sum_{\ell=1}^{\dim_{\boldsymbol{R}} V_0} B\Big(\Big[\frac{I+\sigma}{2}Y, Z\Big], \zeta^{-1}e_\ell\Big)\zeta^{-1}e_\ell = \tau \circ \operatorname{ad}_{(I+\sigma)Y/2}Z .$$

q.e.d.

We can now calculate the curvature of V_0. We calculate at 0: Denoting the curvature transformation by $\tilde{R}(X, Y)$ where $X, Y \in \mathfrak{m}$, we have ([1], p. 192).

(10) $$\tilde{R}(X, Y) = [\Lambda_{\mathfrak{m}}(X), \Lambda_{\mathfrak{m}}(Y)] - \{\Lambda_{\mathfrak{m}}([X, Y]_{\mathfrak{m}}) + \lambda_0([X, Y]_{\mathfrak{k}_0})\} ,$$

where $[\ \ ]_{\mathfrak{m}}$ and $[\ \ ]_{\mathfrak{k}_0}$ mean $\mathfrak{m}$- and $\mathfrak{k}_0$-components, and where $\lambda_0 : \mathfrak{k}_0 \to \mathfrak{gl}(\dim_{\boldsymbol{R}} V_0, \boldsymbol{R})$ is induced by the isotropy representation $\lambda_0 : K_0 \to G\ell(\dim_{\boldsymbol{R}} V_0, \boldsymbol{R})$. As before, we have the commutative diagram ($Z \in \mathfrak{k}_0 \subset \mathfrak{k}$):

(11) $$\begin{array}{ccccc} \mathfrak{m} & \xrightarrow[\zeta]{\simeq} & T_0V_0 & \xrightarrow[\tilde{u}_0^{-1}]{\simeq} & \boldsymbol{R}^{\dim_{\boldsymbol{R}} V_0} \\ {\scriptstyle \operatorname{ad}_Z}\downarrow & & & & \downarrow{\scriptstyle \lambda_0(Z)} \\ \mathfrak{m} & \xrightarrow[\zeta]{\simeq} & T_0V_0 & \xrightarrow[\tilde{u}_0^{-1}]{\simeq} & \boldsymbol{R}^{\dim_{\boldsymbol{R}} V_0} \end{array}$$

(12) Also $\theta \circ \operatorname{ad}_Z = \operatorname{ad}_Z \circ \theta$ for $Z \in \mathfrak{k}_0 \subset \mathfrak{k}$, as one easily checks.
Now for $X, Y \in \mathfrak{m}$ write $[X, Y] = Z + W$ with $Z \in \mathfrak{k}_0$, $W \in \mathfrak{m}$. Then in $\operatorname{End}(\theta\mathfrak{m})$ we have by (11) and (12):

(13) $\lambda_0([X, Y]_{\mathfrak{k}_0}) = \lambda_0(Z) = \operatorname{ad}_Z$

Also $\Lambda_{\mathfrak{m}}([X, Y]_{\mathfrak{m}}) = \tau \circ \operatorname{ad}_{(I+\sigma)W/2}$, by Proposition 1.

Since $Z \in \mathfrak{k}_0 \subset \mathfrak{k}$, we have $\sigma Z = Z$, hence $\mathrm{ad}_Z = \mathrm{ad}_{(I+\sigma)Z/2}$. Also $\mathrm{ad}_Z(\theta\mathfrak{m}) \subset \theta\mathfrak{m}$ implies $\mathrm{ad}_Z = \tau \circ \mathrm{ad}_Z$ on $\theta\mathfrak{m}$. Therefore $\mathrm{ad}_Z = \tau \circ \mathrm{ad}_{(I+\sigma)Z/2} : \theta\mathfrak{m} \to \theta\mathfrak{m}$. We now see

$$\Lambda_{\mathfrak{m}}([X, Y]_{\mathfrak{m}}) + \lambda_0([X, Y]_{\mathfrak{k}_0}) = \tau \circ \mathrm{ad}_{(I+\sigma)W/2} + \tau \circ \mathrm{ad}_{(I+\sigma)Z/2} : \theta\mathfrak{m} \to \theta\mathfrak{m} .$$

Using Proposition 1, we now get

PROPOSITION 2. *The induced curvature on V_0 is*

$$\tilde{R}(X, Y) = [\tau \circ \mathrm{ad}_{(I+\sigma)X/2}, \tau \circ \mathrm{ad}_{(I+\sigma)Y/Z} - \tau \circ \mathrm{ad}_{(I+\sigma)[X,Y]/2} : \theta\mathfrak{m} \to \theta\mathfrak{m} ,$$

where $X, Y \in \mathfrak{m}$, and $\tau : \mathfrak{p} \to \theta\mathfrak{m}$ is the orthogonal projection with respect to the Killing form.

§ 4. The 2nd fundamental form α

We know this already; see (6): For $X, Y \in \mathfrak{m}$,

$\alpha(X, Y) =$ normal component of $\zeta\left[\frac{I+\sigma}{2}Y, \zeta^{-1}\xi X\right] =$ normal component of $\zeta\left[\frac{I+\sigma}{2}Y, \frac{I+\sigma}{2}X\right]$, using (9). So we get, (using the symmetry of α):

PROPOSITION 3. *The second fundamental form $\alpha : \mathfrak{m} \times \mathfrak{m} \to (\theta\mathfrak{m})^{\perp} \subset \mathfrak{p}$ of V_0 in $\mathscr{D}$ is*

$$\alpha(X, Y) = (I - \tau)\left[\frac{I+\sigma}{2}X, \frac{I-\sigma}{2}Y\right] ,$$

where $\tau : \mathfrak{p} \to \theta\mathfrak{m}$ and $(\theta\mathfrak{m})^{\perp}$ are orthogonal projection and complement with respect to the Killing form.

LEMMA 1. *For $X, Y \in \mathfrak{m}$, we have $\alpha(X, Y) = 0$ if and only if $[\sigma X, Y] + [\sigma Y, X] \in \theta\mathfrak{m}$.*

Proof. We have

$$\begin{aligned}\left[\frac{I+\sigma}{2}X, \frac{I-\sigma}{2}Y\right] &= \frac{1}{4}\{[X, Y] - \sigma[X, Y]\} + \frac{1}{4}\{[\sigma X, Y] - [X, \sigma Y]\} \\ &= \frac{1}{2}\theta([X, Y]) + \frac{1}{4}\{[\sigma X, Y] + [\sigma Y, X]\} ,\end{aligned}$$

and since $\theta([X, Y]) \in \theta\mathfrak{g}_0 = \theta\mathfrak{k}_0 + \theta\mathfrak{m} = \theta\mathfrak{m}$, the lemma follows. q.e.d.

We now calculate the condition for $\alpha(X, Y)$ to be zero in our concrete case $\mathscr{D} = \mathscr{D}_p^{(s)}$. The involution $\sigma(g) = H_s g H_s^{-1}$ and $\mathfrak{m}$ are described in § 2.

For $X = \begin{pmatrix} -X_{44} & iX_{24}^* & {}^tX_{24} & X_{14} \\ 0 & 0 & 0 & X_{24} \\ 0 & 0 & 0 & -i\bar{X}_{24} \\ 0 & 0 & 0 & X_{44} \end{pmatrix} \in \mathfrak{m}$ we then find

$$\sigma X = \begin{pmatrix} X_{44} & 0 & 0 & 0 \\ iX_{24} & 0 & 0 & 0 \\ -\bar{X}_{24} & 0 & 0 & 0 \\ -X_{14} & -X_{24}^* & -i{}^tX_{24} & -X_{44} \end{pmatrix}.$$

Using such expressions in Lemma 1, we find, after a matrix calculation:

LEMMA 2. *$\alpha(X, Y) = 0$ if and only if $X_{24}{}^tY_{24} + Y_{24}{}^tX_{24} = 0$.*

Then we can calculate the *null-space* $N_\alpha := \{X \mid \alpha(X, Y) = 0 \forall Y \in \mathfrak{m}\}$ of α. In Lemma 2, $X_{24}, Y_{24} \in M(r, s, \boldsymbol{C})$, and we must find those $P \in M(r, s, \boldsymbol{C})$ for which $P{}^tQ + Q{}^tP = 0 \ \forall Q \in M(r, s, \boldsymbol{C})$. Let $\{E_{\lambda\mu}\}$ be the standard basis for $M(r, s, \boldsymbol{C})$, and write $P = \sum_{\lambda\mu} P_{\lambda\mu} E_{\lambda\mu}$. Then $0 = P{}^tE_{\varepsilon\delta} + E_{\varepsilon\delta}{}^tP = \sum_{\lambda\mu} P_{\lambda\mu} E_{\lambda\mu} E_{\delta\varepsilon} + \sum_{\lambda\mu} E_{\varepsilon\delta} P_{\lambda\mu} E_{\mu\lambda} = \sum_{\lambda} P_{\lambda\delta} E_{\lambda\varepsilon} + \sum_{\lambda} P_{\lambda\delta} E_{\varepsilon\lambda} = 2P_{\varepsilon\delta} E_{\varepsilon\varepsilon} + \sum_{\lambda \neq \varepsilon} P_{\lambda\delta} E_{\lambda\varepsilon} + \sum_{\lambda \neq \varepsilon} P_{\lambda\delta} E_{\varepsilon\lambda}$. We see $P = 0$, so $X \in N_\alpha$ if and only if $X_{24} = 0$, i.e.

LEMMA 3.

$$N_\alpha = \left\{ \begin{pmatrix} -X_{44} & 0 & 0 & X_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_{44} \end{pmatrix} \middle| \ {}^tX_{14} = X_{14}, {}^tX_{44} = X_{44} \in M(s, \boldsymbol{R}) \right\}.$$

Let $\mathscr{N} := \bigcup_{x \in V_0} N_{\alpha,x}$, where $N_{\alpha,x}$ = null-space of α at x. If $g \in G_0$ and $X, Y \in T_0V_0$, then $\alpha(gX, gY) = g\alpha(X, Y)$, so $gN_\alpha = N_{\alpha, g\cdot 0}$.

PROPOSITION 4. *The distribution $\mathscr{N}$ is integrable (involutive).*

Proof. Let X, Y, Z be local vector fields on V_0 near 0, and suppose $X, Y \in \mathscr{N}$. Now $X \in \mathscr{N}$ if and only if $\nabla_X Z$ is a local vector field on V_0 for all (local vector fields on V_0)Z, by definition of $\mathscr{N}$. We have further $\nabla_{[X,Y]} Z = [\nabla_X, \nabla_Y] Z - R(X, Y) Z$. Here $\nabla_X Z, \nabla_Y Z$ are local vector fields on V_0 since $X, Y \in \mathscr{N}$, and so are, for the same reason, $\nabla_X(\nabla_Y Z)$,

$\nabla_Y(\nabla_X Z)$. So we have to prove $R(X, Y)Z$ is tangent to V_0. By invariance of V_0 and $\mathscr{N}$ under G_0, it suffices to check this at 0. Now for $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{p} \simeq T_0\mathscr{D}$, we have $R(\tilde{X}, \tilde{Y})Z = -[[\tilde{X}, \tilde{Y}], \tilde{Z}]$. So for the above $X, Y \in N_\alpha \subset \mathfrak{m}$ and $Z \in \mathfrak{m}$, one has to check that $[[\theta X, \theta Y], \theta Z] \in \theta\mathfrak{m}$, i.e. $[[\theta N_\alpha, \theta N_\alpha], \theta\mathfrak{m}] \subset \theta\mathfrak{m}$. This is straightforward, so we leave it. q.e.d.

Equally straightforward is

LEMMA 4. $[N_\alpha, N_\alpha] \subset \mathfrak{k}_0 + N_\alpha$, $[[N_\alpha, N_\alpha], N_\alpha] \subset N_\alpha$, $[[\theta N_\alpha, \theta N_\alpha], \theta N_\alpha] \subset \theta N_\alpha$.

Now let $S \subset V_0$ be a maximal connected integral submanifold for $\mathscr{N}$ through 0. By Lemma 4, $\mathfrak{g}_\alpha := [N_\alpha, N_\alpha] + N_\alpha$ is a subalgebra of $\mathfrak{g}_0$, and we let G_α be the connected subgroup of G_0 with Lie algebra $\mathfrak{g}_\alpha$. Letting $K_\alpha := K_0 \cap G_\alpha$, we have the submanifold G_α/K_α of V_0. If $g \in G_\alpha$, then $T_{g\cdot 0}(G_\alpha/K_\alpha) = gT_0(G_\alpha/K_\alpha) = gN_\alpha = \mathscr{N}_{g\cdot 0}$, since by Lemma 4 we have $T_0(G_\alpha/K_\alpha) = N_\alpha$. We see $S = G_\alpha/K_\alpha$.

By Lemma 4, we can also consider the algebra $\tilde{\mathfrak{g}}_\alpha := [\theta N_\alpha, \theta N_\alpha] + \theta N_\alpha$, which is a symmetric subalgebra of $\mathfrak{g}$ since $\theta N_\alpha \subset \mathfrak{p}$, and the corresponding groups $\tilde{G}_\alpha, \tilde{K}_\alpha := K \cap \tilde{G}_\alpha$. Then $\tilde{S} = \tilde{G}_\alpha/\tilde{K}_\alpha$ is a totally geodesic submanifold of $\mathscr{D}$. Since $T_0\tilde{S} \simeq \theta N$, we have $T_0\tilde{S} = T_0 S$.

One can calculate that

$$\mathfrak{g}_\alpha \cap \theta N_\alpha = \mathfrak{g}_0 \cap \theta\mathfrak{m} = \left\{ \begin{pmatrix} -X_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_{44} \end{pmatrix} \middle|\ {}^tX_{44} = X_{44} \in M(s, \boldsymbol{R}) \right\},$$

so for such X's we have the $\mathscr{D}$-geodesic $(\exp tX)\cdot 0 \in V_0$. However, since $(\exp tX)\cdot 0 = (\exp tX)k_t\cdot 0$ for any path $k_t \in K$, we could have that $(\exp tX)\cdot 0 \in V_0$ for all $X \in \theta N_\alpha$, i.e. that $S = \tilde{S}$. We shall see that this is in fact the case.

By Proposition 1 we have $\tilde{\nabla}_{X_0} Y = \Lambda_{\mathfrak{m}}(Y)X = \tau\left[\frac{I+\sigma}{2}Y, X\right] \in \theta\mathfrak{m}$ for $X \in \theta\mathfrak{m}$, $Y \in \mathfrak{m}$. If now $X, Y \in T_0 S$ too, then $\alpha(X, Y) = 0$, so then $\nabla_{X_0} Y = \tilde{\nabla}_{X_0} Y$. To prove that the second fundamental form of $S = G_\alpha/K_\alpha$ in $\mathscr{D}$ is zero, we therefore have to prove that $\tau\left[\frac{I+\sigma}{2}Y, X\right] \in \theta N_\alpha$ for $X \in \theta N_\alpha$, $Y \in N_\alpha$, i.e. we have to prove that $\tau\left[\frac{I+\sigma}{2}Y, \frac{I-\sigma}{2}X\right] \in \theta N_\alpha$ for X, $Y \in N_\alpha$. We have in fact:

LEMMA 5. $\left[\frac{I+\sigma}{2}Y, \frac{I-\sigma}{2}X\right] \in \theta N_\alpha$ *for* $X, Y \in N_\alpha$.

Proof. Trivial, using the matrix expressions for σ and elements of N_α. q.e.d.

We now have ([1], p. 59).

PROPOSITION 5. *The integral submanifold* $S = G_\alpha/K_\alpha$ *for* $\mathcal{N}$ *is a totally geodesic complex submanifold of* $\mathcal{D}$ *contained in* V_0, *and* $T_0S = N_\alpha$.

Proof. It only remains to prove that S is complex. In §2 we described the complex structure j_0. Transforming to our representation $\mathcal{D}_p^{(s)}$, we have that the complex structure is given by

$$j = Mj_0M^* = \begin{pmatrix} 0 & 0 & 0 & I_s \\ 0 & iI_r & 0 & 0 \\ 0 & 0 & -iI_r & 0 \\ -I_s & 0 & 0 & 0 \end{pmatrix} : \mathfrak{p} \to \mathfrak{p}\,,$$

where M is given in §2. Since

$$j\begin{pmatrix} -X_{44} & 0 & 0 & X_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ X_{14} & 0 & 0 & X_{44} \end{pmatrix} = \begin{pmatrix} X_{14} & 0 & 0 & X_{44} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ X_{44} & 0 & 0 & -X_{14} \end{pmatrix}$$

where j here acts on a typical element of θN_α, we see $j\theta N_\alpha = \theta N_\alpha$. By [1], p. 261, we see that the totally geodesic submanifold $\tilde{S}$ of $\mathcal{D}$ is a complex submanifold. Since it follows by the earlier argument that $S = \tilde{S}$, we are done. q.e.d.

§5. The Bergmann metric on V_0

Since V_0, being a Siegel domain of the second kind, is equivalent to a bounded domain, we have a Bergman metric on V_0. This metric was computed in [4] for the case of a quasi-symmetric irreducible Siegel domain, and V_0 is such a space. On the other hand, $\mathcal{D}_p$ is also a bounded domain, and has its own Bergman metric. The purpose of this section is to show

PROPOSITION 6. *The Bergman metric on* $\mathcal{D}_p$ *induces* (*up to a constant*) *the Bergman metric on* V_0, *and* V_0 *is a quasi-symmetric irre-*

ducible Siegel domain of the second kind ([2], [3], [4]).

Remark. Since the stability group of G_0 is $U(r) \times 0(s)$ (see § 2), hence not irreducible, the proposition is not immediate. That V_0 is quasi-symmetric and irreducible is of course known.

Proof. 1) First we compute the induced metric. We again write G and $G^{(s)}$ etc., just as in § 2. For the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, we have that the Killing form is

$$B\left(\begin{pmatrix} 0 & A \\ \bar{A} & 0 \end{pmatrix}, \begin{pmatrix} 0 & A' \\ \bar{A}' & 0 \end{pmatrix}\right) = \sum_{ij} \{A_{ij}\bar{A}'_{ij} + \bar{A}_{ij}A'_{ij}\} ,$$

and this is the Bergman metric on $\mathscr{D}_p$ (restricted to $T_0\mathscr{D}_p \simeq \mathfrak{p}$). The transformation between $\mathfrak{g}$ and $\mathfrak{g}^{(s)}$ is (§ 2) $\mathfrak{g}^{(s)} = \kappa(\mathfrak{g}) = M\mathfrak{g}M^*$, where

$$M = \frac{1}{\sqrt{2}}\begin{pmatrix} I_s & 0 & iI_s & 0 \\ 0 & \sqrt{2}\,I_r & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}\,I_r \\ iI_s & 0 & I_s & 0 \end{pmatrix} \in U(2p) .$$

So for $X, Y \in \mathfrak{g}^{(s)}$ we have $B_s(X, Y) = B(M^*XM, M^*YM)$ for the Killing form. For the decomposition $\mathfrak{g}_0^{(s)} = \mathfrak{k}_0^{(s)} + \mathfrak{m}$ we have

$$\theta\mathfrak{m} = \left\{ \begin{pmatrix} -X_{44} & iX_{24}^* & {}^tX_{24} & X_{14} \\ -iX_{24} & 0 & 0 & X_{24} \\ \bar{X}_{24} & 0 & 0 & -i\bar{X}_{24} \\ X_{14} & X_{24}^* & i\,{}^tX_{24} & X_{44} \end{pmatrix} \middle| \begin{array}{l} {}^tX_{14} = X_{14}, {}^tX_{44} = X_{44} \in M(s, \boldsymbol{R}), \\ X_{24} \in M(r, s, \boldsymbol{C}) \end{array} \right\} \subset \mathfrak{p}^{(s)} ,$$

where $\mathfrak{g}^{(s)} = \mathfrak{k}^{(s)} + \mathfrak{p}^{(s)}$ is the Cartan decomposition. If we write the typical element of $\theta\mathfrak{m}$ as (X_{14}, X_{44}, X_{24}), then a simple computation shows that

$$\kappa^{-1}(X_{14}, X_{44}, X_{24}) = \begin{pmatrix} 0 & B \\ \bar{B} & 0 \end{pmatrix} \qquad \text{with } B = \begin{pmatrix} X_{14} - iX_{44} & \sqrt{2}\,{}^tX_{24} \\ \sqrt{2}\,X_{24} & 0 \end{pmatrix} ,$$

and that

$$\begin{aligned} (14) \qquad & B_s(X_{14}, X_{44}, X_{24} \mid Y_{14}, Y_{44}, Y_{24}) \\ & = 2\sum_{ij} \{X_{14ij}Y_{14ij} + X_{44ij}Y_{44ij}\} + 4\sum_{\alpha\beta} \{\bar{X}_{24\alpha\beta}Y_{24\alpha\beta} + X_{24\alpha\beta}\bar{Y}_{24\alpha\beta}\} . \end{aligned}$$

2) The description of V_0 as a quasi-symmetric domain is as follows, using terminology from [3], [4]: Setting $t = 0$ in the expressions in § 2, we see

$$V_0 = \left\{ \begin{pmatrix} z/2 & u \\ {}^t u & 0 \\ 0 & I_r \\ I_s & 0 \end{pmatrix} = : (z, u) \;\middle|\; \begin{matrix} {}^t z = z \in M(s, \boldsymbol{C}),\ u \in M(s, r, \boldsymbol{C}), \\ \dfrac{z - z^*}{2i} - (uu^* + \bar{u}{}^t u) > 0 \end{matrix} \right\} .$$

We let

$$\mathscr{E} : = \{x \in M(s, \boldsymbol{R}) \,|\, {}^t x = x\} \simeq \boldsymbol{R}^{s(s+1)/2} ,$$

$\mathscr{V} : = M(s, r, \boldsymbol{C}) \simeq \boldsymbol{C}^{sr}$, and $F : \mathscr{V} \times \mathscr{V} \to \mathscr{E}_{\boldsymbol{C}}$ be the hermitian map $F(u, v) : = uv^* + \bar{v}{}^t u$. Letting Ω be the irreducible, self-adjoint (with respect to the metric below) cone $\Omega : = \{x \in M(s, \boldsymbol{R}) \,|\, x > 0\} \subset \mathscr{E}$, we have that F is Ω-positive, and we have $V_0 = \mathscr{D}(\mathscr{E}, \mathscr{V}, F, \Omega) : = \{(z, u) \in \mathscr{E}_{\boldsymbol{C}} \times \mathscr{V} \,|\, \operatorname{Im} z - F(u, u) \in \Omega\}$; the expression as a Siegel domain. As metric on $\mathscr{E}$ we take $\langle x, y\rangle : = \sum_{ij} x_{ij} y_{ij} = \operatorname{trace}(xy)$, and as base point we take $e : = 2I_s \in \Omega$. We must compute the mapping $R_x \in \operatorname{End}(\mathscr{V})$ for $x \in \mathscr{E}$, defined by $\langle x, F(u, v)\rangle = : 2\langle e, F(R_x u, v)\rangle$. We have

$$\sum_{ij} x_{ij}(uv^* + \bar{v}{}^t u)_{ij} = \langle x, uv^* + \bar{v}{}^t u\rangle = 4\langle I, R_x u \cdot v^* + \bar{v}{}^t(R_x u)\rangle .$$

Assuming (and proving) that $R_x \in M(s, \boldsymbol{R})$ and that R_x is symmetric, the above expression equals $4\langle I, R_x uv^* + \bar{v}{}^t u R_x\rangle = 4 \sum_{ij} R_{xij}(uv^* + \bar{v}{}^t u)_{ij}$.

(15) We see that $R_x = \frac{1}{4} L_x$ (left multiplication by $x/4$) .

Now we must compute the mapping $T_x \in \mathfrak{p}(\Omega)_e \subset \mathfrak{g}(\Omega) \subset \mathfrak{gl}(\mathscr{E})$ defined by $T_x e = x$, where $\mathfrak{g}(\Omega) = \mathfrak{k}(\Omega)_e + \mathfrak{p}(\Omega)_e$ is the Cartan decomposition of the Lie algebra of $G(\Omega) : = \{g \in G\ell(\mathscr{E}) \,|\, g\Omega = \Omega\}$ at e. We have first a homomorphism $\varphi : G\ell(s, \boldsymbol{R}) \to G(\Omega)$ defined by $\varphi(a)x : = ax{}^t a$ for $x \in \mathscr{E}$,

(16) and the corresponding $\varphi : \mathfrak{gl}(s, \boldsymbol{R}) \to \mathfrak{g}(\Omega)$ is $\varphi(A)x = Ax + x{}^t A$.

(17) Also $\mathfrak{p}(\Omega)_e = \{X \in \mathfrak{g}(\Omega) \,|\, {}^t X = X\}$.

Now for $x \in \mathscr{E} \subset \mathfrak{gl}(s, \boldsymbol{R})$ we have by (16) that $\langle \varphi(x)z, y\rangle = \langle xz + zx, y\rangle = \sum_{ijk} x_{ij} z_{jk} y_{ki} + \sum_{ijk} z_{ij} x_{jk} y_{ki} = \langle z, xy + yx\rangle = \langle z, \varphi(x)y\rangle$. So ${}^t\varphi(x) = \varphi(x)$ for $x \in \mathscr{E} \subset \mathfrak{gl}(s, \boldsymbol{R})$, i.e. (by (17)) $\varphi(x) \in \mathfrak{p}(\Omega)_e$ for $x \in \mathscr{E} \subset \mathfrak{gl}(s, \boldsymbol{R})$. Since $T_x 2I = x$ and $\varphi(x)2I = 2(xI + Ix) = 4x$, we see

(18) $T_x = \frac{1}{4}\varphi(x)$, where $x \in \mathscr{E} \subset \mathfrak{gl}(s, \boldsymbol{R})$.

We have to check the quasi-symmetry condition $T_x F(u, v) = F(R_x u, v)$

$+ F(u, R_x v): T_x F(u, v) = \frac{1}{4}\{x(uv^* + \bar{v}^t u) + (uv^* + \bar{v}^t u)x\} = \frac{1}{4}\{xuv^* + \bar{v}^t(xu) + u(xv)^* + \overline{xv}^t u\} = F(R_x u, v) + F(u, R_x v)$. The irreducibility of V_0 follows from the irreducibility of Ω ([3], [4]).

So V_0 is an irreducible, quasi-symmetric Siegel domain.

3) In [4] we computed the Bergman metric for such a domain. The result was, where $\partial/\partial x_{ij}$, $\partial/\partial y_{ij}$, $\partial/\partial u_{\alpha\beta}$ are vectors in $T_0 V_0$, 0 being the base point

$$(ie, 0) = (2iI_s, 0) = \begin{pmatrix} iI_s & 0 \\ 0 & 0 \\ 0 & I_r \\ I_s & 0 \end{pmatrix},$$

and where for instance $X\cdot\partial/\partial x := \sum_{i\le j} X_{ij}\partial/\partial x_{ij}$ for $X \in \mathscr{E} \subset \mathfrak{gl}(s, \boldsymbol{R})$, and $U\cdot\partial/\partial u := \sum_{\alpha\beta} U_{\alpha\beta}\partial/\partial u_{\alpha\beta}$ for $U \in \mathscr{V}$: $\langle X_1\cdot\partial/\partial x, X_2\cdot\partial/\partial x\rangle_0 = \langle X_1\cdot\partial/\partial y, X_2\cdot\partial/\partial y\rangle_0 = C\langle X_1, X_2\rangle = C\sum_{ij} X_{1ij}X_{2ij}$, $\langle X_1\cdot\partial/\partial x, X_2\cdot\partial/\partial y\rangle_0 = 0$, $\langle X\cdot\partial/\partial x, U\cdot\partial/\partial u\rangle_0 = \langle X\cdot\partial/\partial y, U\cdot\partial/\partial u\rangle_0 = 0$, $\langle U_1\cdot\partial/\partial u, U_2\cdot\partial/\partial u\rangle_0 = 2C\langle 2I_s, F(U_1, U_2)\rangle = 4C\sum_{\alpha\beta}\{U_{1\alpha\beta}\bar{U}_{2\alpha\beta} + \bar{U}_{2\alpha\beta}U_{1\alpha\beta}\} = 8C\sum_{\alpha\beta} U_{1\alpha\beta}\bar{U}_{2\alpha\beta}$, where $C > 0$ is a certain constant.

4) To compare the metric in 3) with the induced metric (14), we must translate $X = (X_{14}, X_{44}, X_{24}) \in \theta\mathfrak{m}$ to the differential expressions in 3): On the one hand we have

$$X_0 = \left.\frac{d}{dt}\right|_{t=0} \{(\exp tX)\cdot 0\}$$

$$= \begin{pmatrix} -X_{44} & iX_{24}^* & {}^tX_{24} & X_{14} \\ -iX_{24} & 0 & 0 & X_{24} \\ \bar{X}_{24} & 0 & 0 & -i\bar{X}_{24} \\ X_{14} & X_{24}^* & i\,{}^tX_{24} & X_{44} \end{pmatrix}\begin{pmatrix} iI_s & 0 \\ 0 & 0 \\ 0 & I_r \\ I_s & 0 \end{pmatrix} = \begin{pmatrix} X_{14} - iX_{44} & {}^tX_{24} \\ 2X_{24} & 0 \\ 0 & 0 \\ iX_{14} + X_{44} & i\,{}^tX_{24} \end{pmatrix}.$$

Writing $(\exp tX)\cdot 0 = (z_t, u_t)$, we have on the other hand, using the equivalence of different expressions for points in $\mathscr{D}_p^{(s)}$ (see § 2):

$$(\exp tX)\cdot 0 = \begin{pmatrix} z_t/2 & u_t \\ {}^tu_t & 0 \\ 0 & I_r \\ I_s & 0 \end{pmatrix}\begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix} \text{ with the last matrix in } G\ell(p, \boldsymbol{C}).$$

Here z_t, u_t, A_t, B_t, C_t and D_t are curves with $z_0 = iI_s$, $u_0 = 0$, $A_0 = I_s$, $B_0 = 0$, $C_0 = 0$, $D_0 = I_r$. This gives

$$X_0 = \begin{pmatrix} \frac{1}{2}\dot{z}_0 + i\dot{A}_0 & i\dot{B}_0 + \dot{u}_0 \\ {}^t\dot{u}_0 & 0 \\ \dot{C}_0 & \dot{D}_0 \\ \dot{A}_0 & \dot{B}_0 \end{pmatrix} .$$

Comparing the two expressions we see $\dot{A}_0 = iX_{14} + X_{44}$, $\dot{z}_0 = 4\{X_{14} - iX_{44}\}$, $\dot{u}_0 = 2{}^tX_{24} = 2({}^tX'_{24} + i{}^tX''_{24})$, where X'_{24}, X''_{24} are real.

(19) So $X = (X_{14}, X_{44}, X_{24}) \in \theta\mathfrak{m}$ represents

$$4X_{14}\cdot\partial/\partial x - 4X_{44}\cdot\partial/\partial y + 2{}^tX'_{24}\cdot\partial/\partial u' + 2{}^tX''_{24}\cdot\partial/\partial u'' \in T_0V_0 ,$$

where $u = u' + iu''$ with u', u'' real.

5) We now compare the two metrics. By (14)

$$B_s(X_{14}, 0, 0 \mid Y_{14}, 0, 0) = 2\sum_{ij} X_{14ij}Y_{14ij},\ B_s(X_{14}, 0, 0 \mid 0, Y_{44}, 0) = 0 ,$$
$$B_s(0, X_{44}, 0 \mid 0, Y_{44}, 0) = 2\sum_{ij} X_{44ij}Y_{44ij},\ B_s(X_{14}, X_{44}, 0 \mid 0, 0, Y_{24}) = 0 ,$$
$$B_s(0, 0, X_{24} \mid 0, 0, Y_{24}) = 4\sum_{\alpha\beta} \{\overline{X}_{24\alpha\beta}Y_{24\alpha\beta} + X_{24\alpha\beta}\overline{Y}_{24\alpha\beta}\}$$
$$= 8\sum_{\alpha\beta} \{X'_{24\alpha\beta}Y'_{24\alpha\beta} + X''_{24\alpha\beta}Y''_{24\alpha\beta}\} .$$

On the other hand we have, using (19) and 3):

$$\langle 4X_{14}\cdot\partial/\partial x, 4Y_{14}\cdot\partial/\partial y\rangle_0 = 16C\sum_{ij} X_{14ij}Y_{14ij} ,$$
$$\langle 4X_{14}\cdot\partial/\partial x, -4Y_{44}\cdot\partial/\partial y\rangle_0 = 0 ,$$
$$\langle -4X_{44}\cdot\partial/\partial x, -4Y_{44}\cdot\partial/\partial y\rangle_0 = 16C\sum_{ij} X_{44ij}Y_{44ij} ,$$
$$\langle 4X_{14}\cdot\partial/\partial x - 4X_{44}\cdot\partial/\partial y, 2{}^tY'_{24}\cdot\partial/\partial u' + 2{}^tY''_{24}\cdot\partial/\partial u''\rangle_0 = 0 .$$

The last B_s-expression above is for the real vectors indicated, while the last $\langle\ ,\ \rangle_0$-expression in 3) is for complex vectors. We see first $\langle\partial/\partial u_{\alpha\beta}, \partial/\partial\overline{u}_{\gamma\delta}\rangle_0 = 8C\delta_{\alpha\gamma}\delta_{\beta\delta}$ (Kronecker deltas), and therefore $\langle\partial/\partial u'_{\alpha\beta}, \partial/\partial u'_{\gamma\delta}\rangle_0 = \langle\partial/\partial u''_{\alpha\beta}, \partial/\partial u''_{\gamma\delta}\rangle_0 = 16C\delta_{\alpha\gamma}\delta_{\beta\delta}$ and $\langle\partial/\partial u'_{\alpha\beta}, \partial/\partial u''_{\gamma\delta}\rangle_0 = 0$. Then

$$\langle 2{}^tX'_{24}\cdot\partial/\partial u' + 2{}^tX''_{24}\cdot\partial/\partial u'', 2{}^tY'_{24}\cdot\partial/\partial u' + 2{}^tY''_{24}\cdot\partial/\partial u''\rangle_0$$
$$= 64C\sum_{\alpha\beta} \{X'_{24\alpha\beta}Y'_{24\alpha\beta} + X''_{24\alpha\beta}Y''_{24\alpha\beta}\} .$$

So we see that $\langle\ ,\ \rangle = 8CB_s$. q.e.d.

§ 6. Domains of Type I, II

The same results hold as in the case of the Siegel disk. Some of the changes are (see also [2]):

$$\text{I.}\quad V_0 = \left\{\begin{pmatrix} z/2 & U_{12} \\ U_{21} & 0 \\ 0 & I_{q_1} \\ I_r & 0 \end{pmatrix}\right. =: (z, (U_{12}, U_{21}))$$

$$=: (z, u) \left| \begin{matrix} z \in M(r, \boldsymbol{C}), U_{12} \in M(r, q_1, \boldsymbol{C}), \\ U_{21} \in M(p_1, r, \boldsymbol{C}) \;, \\ \operatorname{Im} z - \{U_{12}U_{12}^* + U_{21}^*U_{21}\} > 0 \end{matrix} \right\} ,$$

where $p_1 = p - r$, $q_1 = q - r$, and $\operatorname{Im} z = (z - z^*)/2i$. Then $V_0 = \mathscr{D}(\mathscr{E}, \mathscr{V}, F, \Omega)$, where $\mathscr{E} := \mathscr{H}(r, \boldsymbol{C}) = \{\text{hermitian matrices}\}$ (real vector space), $\mathscr{V} := M(r, q_1, \boldsymbol{C}) \oplus M(p_1, r, \boldsymbol{C})$ (complex vector space), $\Omega := \mathscr{P}(r, \boldsymbol{C}) = \{\text{positive-definite hermitian matrices}\}$ (cone) and $F: \mathscr{V} \times \mathscr{V} \to \mathscr{E}_{\boldsymbol{C}} = M(r, \boldsymbol{C})$ is the Ω-positive hermitian map $F(u', u'' \mid v', v'') := u'v'^* + v''^*u''$. The metric on $\mathscr{E}$ is $\langle x, y \rangle := \operatorname{trace}(xy)$, base point is $e = 2I_r \in \Omega$, and $R_x(u', u'') = \frac{1}{4}(xu', u''x)$ for $x \in \mathscr{E}$. Also $T_x = \frac{1}{4}\varphi(x)$ where $\varphi: \mathfrak{gl}(r, \boldsymbol{C}) \to \mathfrak{g}(\Omega)$ is $\varphi(A)y = Ay + yA^*$. Further, we can take

$$\mathfrak{m} := \left\{\begin{pmatrix} -X_{44}, & iX_{24}^*, & -iX_{34}^*, & X_{14} \\ & & & X_{24} \\ & 0 & & X_{34} \\ & & & X_{44} \end{pmatrix} \left| \begin{matrix} X_{14}^* = X_{14} \in M(r, \boldsymbol{C}),\ X_{24} \in M(p_1, r, \boldsymbol{C}) \;, \\ X_{34} \in M(q_1, r, \boldsymbol{C}),\ X_{44}^* = X_{44} \in M(r, \boldsymbol{C}) \end{matrix} \right.\right\} ,$$

$$M := \frac{1}{\sqrt{2}} \begin{pmatrix} I_r & 0 & 0 & iI_r \\ 0 & \sqrt{2}\, I_{p_1} & 0 & 0 \\ 0 & 0 & \sqrt{2}\, I_{q_1} & 0 \\ iI_r & 0 & 0 & I_r \end{pmatrix} , \text{ and we have}$$

$$j = \begin{pmatrix} 0 & 0 & 0 & I_r \\ 0 & iI_{p_1} & 0 & 0 \\ 0 & 0 & -iI_{q_1} & 0 \\ -I_r & 0 & 0 & 0 \end{pmatrix} .$$

For $X, Y \in \mathfrak{m}$, we have $\alpha(X, Y) = 0 \Leftrightarrow X_{24}Y_{34}^* + Y_{24}X_{34}^* = 0$, and

$$N_\alpha = \left\{\begin{pmatrix} -X_{44} & 0 & 0 & X_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_{44} \end{pmatrix}\right\} \subset \mathfrak{m} .$$

$$\textbf{II.}\quad V_0 = \left\{ \begin{pmatrix} z/2 & u \\ -{}^t uJ & 0 \\ 0 & I_r \\ I_{2s} & 0 \end{pmatrix} \right.$$

$$=:(z,u)\left|\begin{matrix} z \in M(2s, \boldsymbol{C}),\ {}^t zJ = Jz,\ u \in M(2s, r, \boldsymbol{C}), \\ \operatorname{Im} z - \{uu^* - J\bar{u}{}^t uJ\} \in \Omega \end{matrix}\right\},$$

where $J = \begin{pmatrix} 0 & I_s \\ -I_s & 0 \end{pmatrix}$, $p = r + 2s$, $\operatorname{Im} z = (z - z^*)/2i$, and the cone is $\Omega = \{Y \in M(2s, \boldsymbol{C}) \mid \overline{Y}J = JY,\ Y^* = Y > 0\} \simeq \mathscr{P}(s, \boldsymbol{H})$, where $\boldsymbol{H}$ denotes the quaternions. The last isomorphism is by restriction of the isomorphism

$$\mathscr{E} \ni x = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mapsto a + jb \in \mathscr{H}(s, \boldsymbol{H}) = \{\text{quaternion hermitian matrices}\},$$

where $a^* = a$, ${}^t b = -b$, j *here* denotes the **2nd** quaternionic unit, and $\mathscr{E} := \{x \in M(2s, \boldsymbol{C}) \mid x^* = x, \bar{x}J = Jx\}$ (real vector space). Letting $\mathscr{V} := M(2s, r, \boldsymbol{C})$ (complex vector space), and $F : \mathscr{V} \times \mathscr{V} \to \mathscr{E}_{\boldsymbol{C}}$ be the Ω-hermitian map $F(u, v) := uv^* - J\bar{v}{}^t uJ$, we have $V_0 = \mathscr{D}(\mathscr{E}, \mathscr{V}, F, \Omega)$. The metric on $\mathscr{E}$ is $\langle x, y\rangle = \operatorname{trace}(xy)$, base point is $e = 2I_{2s} \in \Omega$, and $R_x = \frac{1}{4}L_x$ for $x \in \mathscr{E}$, (left multiplication). Also $T_x = \frac{1}{4}\varphi(x)$ where

$$\varphi : \{A \in \mathfrak{gl}(2s, \boldsymbol{C}) \mid \overline{A}J = JA\} \to \mathfrak{g}(\Omega)$$

is $\varphi(A)y = Ay + yA^*$. Further we can take

$$\mathfrak{m} := \left\{ \begin{pmatrix} -X_{44}, & iX_{24}^*, & -J{}^t X_{24}, & \begin{matrix} X_{14} \\ X_{24} \\ -i\overline{X}_{24}J \\ X_{44} \end{matrix} \\ & 0 & & \end{pmatrix} \middle| \begin{matrix} X_{14}^* = X_{14} \in M(2s, \boldsymbol{C}),\ \overline{X}_{14}J = JX_{14}, \\ X_{24} \in M(r, 2s, \boldsymbol{C}), \\ X_{44}^* = X_{44} \in M(2s, \boldsymbol{C}),\ \overline{X}_{44}J = JX_{44}, \end{matrix} \right\},$$

$$M := \frac{1}{\sqrt{2}} \begin{pmatrix} I_{2s} & 0 & 0 & iI_{2s} \\ 0 & \sqrt{2}I_r & 0 & 0 \\ 0 & 0 & \sqrt{2}I_r & 0 \\ iI_{2s} & 0 & 0 & I_{2s} \end{pmatrix} \begin{pmatrix} I_p & 0 \\ 0 & K \end{pmatrix} \text{ with } K = \begin{pmatrix} 0 & I_r \\ -J & 0 \end{pmatrix},$$

and we have

$$j = \begin{pmatrix} 0 & 0 & 0 & I_{2s} \\ 0 & iI_r & 0 & 0 \\ 0 & 0 & -iI_r & 0 \\ -I_{2s} & 0 & 0 & 0 \end{pmatrix}$$

for the complex structure. For $X, Y \in \mathfrak{m}$, we have $\alpha(X, Y) = 0 \Leftrightarrow X_{24}J^tY_{24} + Y_{24}J^tX_{24} = 0$.
(Here we also use that the dimension of the boundary component is positive, and therefore $r > 1$.) Finally,

$$N_\alpha = \left\{\begin{pmatrix} -X_{44} & 0 & 0 & X_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_{44} \end{pmatrix}\right\} \subset \mathfrak{m} .$$

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