P. Valabrega and G. Valla
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FORM RINGS AND REGULAR SEQUENCES[1]

PAOLO VALABREGA AND GIUSEPPE VALLA

Introduction

Hironaka, in his paper [H$_1$] on desingularization of algebraic varieties over a field of characteristic 0, to deal with singular points develops the algebraic apparatus of the associated graded ring, introducing standard bases of ideals, numerical characters ν^* and τ^* etc. Such a point of view involves a deep investigation of the ideal $\boldsymbol{b}^*$ generated by the initial forms of the elements of an ideal $\boldsymbol{b}$ of a local ring, with respect to a certain ideal $\boldsymbol{a}$.

The present paper has its origin in the effort of extending to a general situation the following result (due to Hironaka: [H$_2$]):

Let $(A, \boldsymbol{m})$ be a local ring and $z \in \boldsymbol{m} - \boldsymbol{m}^2$; then the initial form z^* of z in the associated graded ring $G(\boldsymbol{m})$ is a regular element if and only if z is regular in A and $(z) \cap \boldsymbol{m}^{n+1} = (z) \cdot \boldsymbol{m}^n$, for every integer n.

Really our paper investigates in a general way the relations between an ideal $\boldsymbol{b} = (f_1, \cdots, f_r)$ and the associated graded ideal $\boldsymbol{b}^*$ generated by the initial forms of the elements of $\boldsymbol{b}$ in $G_A(\boldsymbol{a})$ = graded ring with respect to the ideal $\boldsymbol{a}$ of the ring A.

We prove the following main results:

1–a necessary and sufficient condition for $\boldsymbol{b}^*$ to be generated by the initial forms of the f_i's, valid for an arbitrary noetherian ring A;

2–a necessary and sufficient condition for $\boldsymbol{b}^*$ to be generated by a regular sequence, valid for an arbitrary noetherian ring A;

3–a condition like in **2**, valid for a local ring A, and generalizing in a natural way Hironaka's result.

The paper contains also some other properties of the associated graded ideal (concerning height and minimal generating sets) and of $G_A(\boldsymbol{a})$ (conditions to be Cohen-Macaulay: see also [S]).

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1. Form rings and ideals

If A is a noetherian ring with a unit element 1 and $\boldsymbol{a}$ any ideal in A, we denote by $G_A(\boldsymbol{a})$ the graded $A/\boldsymbol{a}$-algebra $\bigoplus_{n=0}^{\infty} \boldsymbol{a}^n/\boldsymbol{a}^{n+1}$ and call it the form ring of A relative to $\boldsymbol{a}$ (other term: associated graded ring).

Sometimes we will have to deal also with negative powers of the ideal $\boldsymbol{a}$; once for all $\boldsymbol{a}^n = A$ if $n \leq 0$.

Given an element $a \in A$, we denote by $v(a)$ the largest integer n such that $a \in \boldsymbol{a}^n$; if $a \in \bigcap_{n=1}^{\infty} \boldsymbol{a}^n$ we set: $v(a) = \infty$. When $v(a) \neq \infty$ the residue class of a in $\boldsymbol{a}^{v(a)}/\boldsymbol{a}^{v(a)+1}$ is called the initial form of a and denoted by a^*. If $v(a) = \infty$, then we set: $a^* = 0$.

The definition of multiplication in $G_A(\boldsymbol{a})$ shows that the two relations: $a^*b^* = (ab)^*$ and $a^*b^* \neq 0$ are equivalent, provided that $\boldsymbol{a}^*$ and $\boldsymbol{b}^* \neq 0$.

Let now $\boldsymbol{b}$ be any ideal of A; we shall denote by $\boldsymbol{b}^*$ the homogeneous ideal of $G_A(\boldsymbol{a})$ generated by all the initial forms of the elements in $\boldsymbol{b}$; $\boldsymbol{b}^*$ is called the form ideal of $\boldsymbol{b}$ relative to $\boldsymbol{a}$ (or the associated graded ideal). For every integer n we have: $\boldsymbol{b}_n^*$ = set of homogeneous elements in $\boldsymbol{b}^*$ of degree $n = (\boldsymbol{b} \cap \boldsymbol{a}^n + \boldsymbol{a}^{n+1})/\boldsymbol{a}^{n+1}$; furthermore $\boldsymbol{b}^*$ is the kernel of a natural epimorphism of graded rings

$$\varphi : G_A(\boldsymbol{a}) \to G_{A/\boldsymbol{b}}(\boldsymbol{b} + \boldsymbol{a}/\boldsymbol{b}) ,$$

which is homogeneous of degree 0.

The ideal $\boldsymbol{b}^*$ can be defined with respect to any other ideal $\boldsymbol{a}$; but, if $\boldsymbol{b}$ and $\boldsymbol{a}$ are comaximal (i.e. $\boldsymbol{a} + \boldsymbol{b} = A$) then $\boldsymbol{b}^* = G_A(\boldsymbol{a})$ and conversely (as one may easily check). Therefore *we assume, once for all, that* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are not comaximal.* Of course both $\boldsymbol{a}$ and $\boldsymbol{b}$ are assumed to be proper.

Since A is noetherian, then $G_A(\boldsymbol{a})$ is noetherian. In particular $\boldsymbol{b}^*$ is generated by the initial forms of a finite number of elements of $\boldsymbol{b}$. However it is not generally true that, if $\boldsymbol{b} = (f_1, \cdots, f_r)$, then $\boldsymbol{b}^* = (f_1^*, \cdots, f_r^*)$.

The following theorem is just a necessary and sufficient condition for the equality:

THEOREM 1.1. *If* $\boldsymbol{a}$ *and* $\boldsymbol{b} = (f_1, \cdots, f_r)$ *are ideals of* A, *then* $\boldsymbol{b}^* = (f_1^*, \cdots, f_r^*)$ *in* $G_A(\boldsymbol{a})$ *if and only if for all* $n \geqslant 1$ *the following equality holds:*

$$\boldsymbol{a}^n \cap \boldsymbol{b} = \sum_{i=1}^{r} \boldsymbol{a}^{n-p_i} f_i$$

where $p_i = v(f_i)$, $i = 1, \cdots, r$.

Proof. It is clear that $(f_1^*, \cdots, f_r^*)_n = \left(\sum_{i=1}^{r} \boldsymbol{a}^{n-p_i} f_i + \boldsymbol{a}^{n+1}\right) \Big/ \boldsymbol{a}^{n+1}$.

Hence, if $\boldsymbol{b} \cap \boldsymbol{a}^n = \sum_{i=1}^{r} \boldsymbol{a}^{n-p_i} f_i$ for all $n \geqslant 1$, we have: $\boldsymbol{b}^* = (f_1^*, \cdots, f_r^*)$.

Conversely, if $\boldsymbol{b}^*$ is generated by the f_i^*'s, then we have: $\boldsymbol{a}^n \cap \boldsymbol{b} \subseteq \sum_{i=1}^{r} \boldsymbol{a}^{n-p_i} f_i + \boldsymbol{a}^{n+1}$ for all $n \geqslant 1$; it follows that $\boldsymbol{a}^n \cap \boldsymbol{b} = \bigcap_{t=1}^{\infty} \left(\sum_{i=1}^{r} \boldsymbol{a}^{n-p_i} f_i + \boldsymbol{a}^{n+t} \cap \boldsymbol{b}\right)$. By the Artin-Rees lemma there exists an integer $q \geqslant 0$ such that $\boldsymbol{a}^{n+t} \cap \boldsymbol{b} = \boldsymbol{a}^{n+t-q}(\boldsymbol{a}^q \cap \boldsymbol{b})$ for all $n + t \geqslant q$. Hence, if d is an integer such that $d \geqslant n - p_i$ for $i = 1, \cdots, r$, we get the following equality:

$$\boldsymbol{a}^n \cap \boldsymbol{b} = \bigcap_{t \geqslant q-n+d} \left(\sum_{i=1}^{r} \boldsymbol{a}^{n-p_i} f_i + \boldsymbol{a}^{n+t-q}(\boldsymbol{a}^q \cap \boldsymbol{b})\right) = \sum_{i=1}^{r} \boldsymbol{a}^{n-p_i} f_i\,,$$

since, if $t \geqslant q - n + d$, then $\boldsymbol{a}^{n+t-q}(\boldsymbol{a}^q \cap \boldsymbol{b}) \subseteq \boldsymbol{a}^d \boldsymbol{b} \subseteq \sum_{i=1}^{r} \boldsymbol{a}^{n-p_i} f_i$.

Remark 1.2. Using [R, Rem. (3.7)] one can easily see (cf. [V]) that, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are ideals of A such that $\boldsymbol{a} + \boldsymbol{b} \neq A$, then $h(\boldsymbol{b}^*) \geqslant h(\boldsymbol{b})$ at least when A is local, and certainly equality holds whenever $\boldsymbol{a} \subseteq \boldsymbol{b}$; moreover there are examples with strict inequality. Therefore the following result may have some interest:

PROPOSITION 1.3. *If* $\boldsymbol{a}$ *and* $\boldsymbol{b} = (f_1, \cdots, f_r)$ *are ideals of* A *such that* $\boldsymbol{a} + \boldsymbol{b} \neq A$, *then* $h(\boldsymbol{b}^*) \leqslant r$.

Proof. Let $\boldsymbol{m}$ be a maximal ideal containing both $\boldsymbol{a}$ and $\boldsymbol{b}$; then we have: $h(\boldsymbol{m}) - r \leq h(\boldsymbol{m}/\boldsymbol{b}) = h((\boldsymbol{m}/\boldsymbol{b})^*) = h(\boldsymbol{m}^*/\boldsymbol{b}^*) \leq h(\boldsymbol{m}^*) - h(\boldsymbol{b}^*) = h(\boldsymbol{m}) - h(\boldsymbol{b}^*)$ (in the preceding chain of inequalities we use the fact that, in the isomorphism $G_A(\boldsymbol{a})/\boldsymbol{b}^* \cong G_{A/\boldsymbol{b}}(\boldsymbol{a} + \boldsymbol{b}/\boldsymbol{b})$, $\boldsymbol{m}^*/\boldsymbol{b}^*$ and $(\boldsymbol{m}/\boldsymbol{b})^*$ are corresponding ideals).

Remark 1.4. If $h(f_1^*, \cdots, f_r^*) = r$, then also $h(\boldsymbol{b}^*) = r$. However the following example shows that the condition $h(f_1^*, \cdots, f_r^*) = r$ does not imply the equality $\boldsymbol{b}^* = (f_1^*, \cdots, f_r^*)$.

EXAMPLE 1.5. Let A be the ring $k[[t^4, t^5, t^{11}]] = k[[X, Y, Z]]/(XZ - Y^3, YZ - X^4, Z^2 - X^3Y^2) = k[[x, y, z]]$, $\boldsymbol{a} = (x, y, z)$, $\boldsymbol{b} = (x)$; we have: $G_A(\boldsymbol{a}) = k[T_1, T_2, T_3]/(T_1T_3, T_2T_3, T_3^2, T_2^4) = k[t_1, t_2, t_3]$; hence $h(\boldsymbol{b}^*) = h(t_1) = 1$, but $y^3 \in \boldsymbol{a}^3 \cap \boldsymbol{b}$ and $y^3 \notin \boldsymbol{a}^2x$ and so, by Theorem 1.1, $\boldsymbol{b}^* \neq (x^*)$.

2. Regular sequences in $G_A(\boldsymbol{a})$

In this section we consider a noetherian ring A and an ideal $\boldsymbol{a}$ in A; if $f_1, \cdots, f_r \in A$ we shall state some necessary and sufficient conditions for $f_1^*, \cdots, f_r^*$ to be a regular sequence in $G_A(\boldsymbol{a})$ (always provided that $\boldsymbol{a}$ and $(f_1, \cdots, f_r)$ be not comaximal).

In the following we shall write $p_i = v(f_i)$ and $\boldsymbol{b}_i = (f_1, \cdots, f_i)$, for $i = 1, \cdots, r$ $(\boldsymbol{b}_0 = (0))$.

PROPOSITION 2.1. *If $\boldsymbol{a}$ and $\boldsymbol{b} = (f_1, \cdots, f_r)$ are ideals of A such that $f_1^*, \cdots, f_r^*$ is a $G_A(\boldsymbol{a})$-sequence, then $\boldsymbol{b}^* = (f_1^*, \cdots, f_r^*)$.*

Proof. We use induction on r. The case $r = 1$ is easy: if f^* is a non zero-divisor, then, for every $g, (fg)^* = f^*g^*$, which shows our claim. Assume now the theorem true for $r - 1$ and prove it for r.

If $a \in \boldsymbol{a}^n \cap \boldsymbol{b}$ let t be the greatest integer (if it exists, ∞ if it does not exist) such that $a \in \boldsymbol{b}_{r-1} + f_r\boldsymbol{a}^t$. So we can write $a = x + f_ry$, with $v(y) = t$, $x \in \boldsymbol{b}_{r-1}$; if $t + p_r < n$ we have: $f_ry \in (\boldsymbol{a}^n + \boldsymbol{b}_{r-1}) \cap \boldsymbol{a}^{t+p_r} \subseteq \boldsymbol{a}^{t+p_r+1} + \boldsymbol{b}_{r-1} \cap \boldsymbol{a}^{t+p_r}$.

It follows that $f_r^*y^* \in \boldsymbol{b}_{r-1}^*$; by our inductive hypothesis $\boldsymbol{b}_{r-1}^*$ is generated by the initial forms of the f_i's, $i = 1, \cdots, r - 1$, hence $y^* \in \boldsymbol{b}_{r-1}^*$. Then $y \in \boldsymbol{a}^{t+1} + \boldsymbol{b}_{r-1} \cap \boldsymbol{a}^t$, so that $a \in \boldsymbol{b}_{r-1} + f_r\boldsymbol{a}^{t+1}$, which is absurd. Therefore $t + p_r \geqslant n$, hence $a \in \boldsymbol{a}^n \cap \boldsymbol{b}_{r-1} + f_r\boldsymbol{a}^{n-p_r}$. The conclusion follows immediately from the inductive assumption together with our Theorem 1.1.

Remark 2.2. The converse of Proposition 2.1 is false even if A is local. In fact, let $A = k[[X, Y]]/(XY) = k[[x, y]]$, $\boldsymbol{b} = (x)$, $\boldsymbol{a} = (x, y)$; then we have: $\boldsymbol{a}^n \cap \boldsymbol{b} = (x^n, y^n) \cap (x) = (x^n) + (y^n) \cap (x) = (x^n) = \boldsymbol{a}^{n-1}\boldsymbol{b}$. Hence by Theorem 1.1 $\boldsymbol{b}^* = (x^*)$, but $G_A(\boldsymbol{a}) = k[T_1, T_2]/(T_1T_2) = k[t_1, t_2]$ and $x^* = t_1$ is a zero divisor in $G_A(\boldsymbol{a})$.

In the following we shall denote by $\bar{\boldsymbol{I}}$ the topological closure of an ideal $\boldsymbol{I}$ with respect to the $\boldsymbol{a}$-adic topology.

THEOREM 2.3. *Let* $\boldsymbol{a}$ *and* $\boldsymbol{b} = (f_1, \cdots, f_r)$ *be ideals of the noetherian ring* A. *Then the following facts are equivalent*:

(i) $(f_1^*, \cdots, f_r^*)$ *is a* $G_A(\boldsymbol{a})$*-sequence*;

(ii) *for each* $i = 1, \cdots, r, \boldsymbol{b}_{i-1}\colon f_i \subseteq \bar{\boldsymbol{b}}_{i-1}$ *and* $\boldsymbol{b}_i \cap \boldsymbol{a}^n = \sum_{j=1}^{i} \boldsymbol{a}^{n-p_j} f_j$, *for all* $n \geqslant 1$.

Proof. (i) $\Rightarrow$ (ii). For Proposition 2.1 it is enough to show that $\boldsymbol{b}_{i-1}\colon f_i \subseteq \bar{\boldsymbol{b}}_{i-1}$. Let $a \in \boldsymbol{b}_{i-1}\colon f_i$ with $v(a) = n$; then $a^* f_i^* \in \boldsymbol{b}_{i-1}^* = (f_1^*, \cdots, f_{i-1}^*)$, hence $a^* \in \boldsymbol{b}_{i-1}^*$. It follows that $a \in \boldsymbol{b}_{i-1} + \boldsymbol{a}^{n+1} \cap (\boldsymbol{b}_{i-1}\colon f_i)$; repeating the argument we see that $a \in \bar{\boldsymbol{b}}_{i-1}$.

(ii) $\Rightarrow$ (i). Conversely, since $\boldsymbol{a} + \boldsymbol{b} \neq A$ and $\boldsymbol{b}_{i-1}\colon f_i \subseteq \bar{\boldsymbol{b}}_{i-1}$ for each $i = 1, \cdots, r$, we have $p_i < \infty$ for each $i = 1, \cdots, r$. In fact, assume that $p_i = \infty$ for some i; then there exists $a \in \boldsymbol{a}$ such that $(1 - a) f_i = 0$. Thus $1 - a \in \bar{\boldsymbol{b}}_{i-1}$ and then $(1 - a)(1 - a') \in \boldsymbol{b}_{i-1}$, with $a' \in \boldsymbol{a}$; but since $\boldsymbol{a} + \boldsymbol{b} \neq A$, this is a contradiction.

Now let $a^* f_i^* \in (f_1^*, \cdots, f_{i-1}^*)$ with $v(a) = n$; then $a f_i \in \boldsymbol{b}_{i-1} + \boldsymbol{a}^{n+p_i+1}$, hence $b = a f_i + \sum_{j=1}^{i-1} a_j f_j \in \boldsymbol{b}_i \cap \boldsymbol{a}^{n+p_i+1}$.

We can write $b = \sum_{j=1}^{i} b_j f_j$ with $b_j \in \boldsymbol{a}^{n+p_i+1-p_j}$, whence we deduce that $a - b_i \in \boldsymbol{b}_{i-1}\colon f_i$ and from this it follows that $a \in \boldsymbol{a}^{n+1} + \bar{\boldsymbol{b}}_{i-1}$.

Finally $a \in \boldsymbol{a}^{n+1} + \boldsymbol{b}_{i-1} \cap \boldsymbol{a}^n$ and this proves that $a^* \in \boldsymbol{b}_{i-1}^* = (f_1^*, \cdots, f_{i-1}^*)$.

COROLLARY 2.4. *Let* A *be a local ring and* $\boldsymbol{I}, \boldsymbol{a}$ *ideals of* A, *such that* $\boldsymbol{I}^*$ *is generated by a* $G_A(\boldsymbol{a})$*-sequence. Then* $\boldsymbol{I}$ *is generated by an* A*-sequence.*

Proof. Let $\boldsymbol{I}^*$ be generated by $g_1, \cdots, g_r$, where the g_i's form a regular sequence. Since all the minimal generating sets of $\boldsymbol{I}^*$ have the same number of elements, we can write $\boldsymbol{I}^* = (f_1^*, \cdots, f_r^*)$ with $f_i \in \boldsymbol{I}$. Now $gr\,(\boldsymbol{I}^*) = r$, hence using the homology of Koszul complex we get $H_1(f_1^*, \cdots, f_r^*; G_A(\boldsymbol{a})) = 0$. From this it follows that $f_1^*, \cdots, f_r^*$ is a $G_A(\boldsymbol{a})$-sequence (see [A–B], Prop. 2.8); hence, by Theorem 2.3, $f_1, \cdots, f_r$ is an A-sequence. Furthermore, since $\boldsymbol{I}^* = (f_1^*, \cdots, f_r^*)$, we get: $(f_1, \cdots, f_r) \subseteq \boldsymbol{I} \subseteq \overline{(f_1, \cdots, f_r)} = (f_1, \cdots, f_r)$.

Remark 2.5. If $f_1^*, \cdots, f_r^*$ form a regular sequence, it is not necessarily true that $f_1, \cdots, f_r$ form also an A-sequence, unless $\boldsymbol{I} = \bar{\boldsymbol{I}}$ for every ideal I contained in $\boldsymbol{b}$. In fact let $A = k[x, y, z] = k[X, Y, Z]/(XZ, X - XY)$, $\boldsymbol{a} = (y)$, $f = yz$; then, since y is not a 0-divisor in A, we have: $G_A(\boldsymbol{a}) = (A/\boldsymbol{a})[T] = k[Z, T]$ which is a domain. Therefore $\bar{f} \in \boldsymbol{a}/\boldsymbol{a}^2$ is not a 0-divisor in $G_A(\boldsymbol{a})$, but $xf = 0$.

PROPOSITION 2.6. *Let $\boldsymbol{a}$ and $\boldsymbol{b} = (f_1, \cdots, f_r)$ be two ideals of A such that $f_1, \cdots, f_r$ is an A-sequence and $\boldsymbol{a}^n \cap \boldsymbol{b} = \sum_{j=1}^{r} \boldsymbol{a}^{n-p_i} f_i$ for all $n \geqslant 1$. Suppose either $\boldsymbol{b} \subseteq \boldsymbol{a}$ or A is local.*

Then $\boldsymbol{a}^n \cap \boldsymbol{b}_i = \sum_{j=1}^{i} \boldsymbol{a}^{n-p_j} f_j$, for each $i = 1, \cdots, r$ and for all $n \geqslant 1$; thus $f_1^*, \cdots, f_r^*$ is a $G_A(\boldsymbol{a})$-sequence.

Proof. It is enough to show that $\boldsymbol{a}^n \cap \boldsymbol{b}_{r-1} = \sum_{j=1}^{r-1} \boldsymbol{a}^{n-p_j} f_j$, for all $n \geqslant 1$. Let $a = \sum_{i=1}^{r-1} a_i f_i$ be an element of $\boldsymbol{a}^n$; then $a = \sum_{i=1}^{r} b_i f_i$, where $b_i \in \boldsymbol{a}^{n-p_i}$ and we get: $b_r \in \boldsymbol{a}^{n-p_r} \cap \boldsymbol{b}_{r-1}$. Thus we have: $\boldsymbol{a}^n \cap \boldsymbol{b}_{r-1} \subseteq \sum_{j=1}^{r-1} \boldsymbol{a}^{n-p_j} f_j + f_r(\boldsymbol{a}^{n-p_r} \cap \boldsymbol{b}_{r-1})$. Let m be an integer such that $m \leq p_i$ for each $i = 1, \cdots, r-1$. If $n \leq m + p_r$, we have that $\boldsymbol{b}_{r-1} \subseteq \boldsymbol{a}^{n-p_r}$ and $f_r \in \boldsymbol{a}^{n-p_i}$ for each $i = 1, \cdots, r-1$; hence $\boldsymbol{a}^n \cap \boldsymbol{b}_{r-1}$ is contained in $\sum_{j=1}^{r-1} \boldsymbol{a}^{n-p_j} f_j + f_r \boldsymbol{b}_{r-1} \subseteq \sum_{j=1}^{r-1} \boldsymbol{a}^{n-p_j} f_j$. Therefore we may assume that n is greater than $m + p_r$ and also that $\boldsymbol{a}^t \cap \boldsymbol{b}_{r-1} = \sum_{j=1}^{r-1} \boldsymbol{a}^{t-p_j} f_j$, for all $t \leq n-1$. If $\boldsymbol{b} \subseteq \boldsymbol{a}$, then $p_r > 0$; if A is local and $p_r = 0$, then we have $\boldsymbol{a}^n \cap \boldsymbol{b}_{r-1} \subseteq \sum_{j=1}^{r-1} \boldsymbol{a}^{n-p_j} f_j + f_r(\boldsymbol{a}^n \cap \boldsymbol{b}_{r-1})$, hence $\boldsymbol{a}^n \cap \boldsymbol{b}_{r-1} = \sum_{j=1}^{r-1} \boldsymbol{a}^{n-p_j} f_j$. If $p_r \geqslant 1$, since $n - p_r \leq n - 1$, we have $\boldsymbol{a}^n \cap \boldsymbol{b}_{r-1} \subseteq \sum_{j=1}^{r-1} \boldsymbol{a}^{n-p_j} f_j + f_r\left(\sum_{j=1}^{r-1} \boldsymbol{a}^{n-p_r-p_j} f_j\right) = \sum_{j=1}^{r-1} \boldsymbol{a}^{n-p_j} f_j$, and this completes the proof.

COROLLARY 2.7. *If $\boldsymbol{a}$ and $\boldsymbol{b} = (f_1, \cdots, f_r)$ are ideals of a local ring A, then $f_1^*, \cdots, f_r^*$ form a $G_A(\boldsymbol{a})$-sequence if and only if $f_1, \cdots, f_r$ form an A-sequence and moreover $\boldsymbol{a}^n \cap \boldsymbol{b} = \sum_{i=1}^{r} \boldsymbol{a}^{n-p_i} f_i$ for all $n \geqslant 1$.*

Remark 2.8. The following example justifies the hypotheses of the above proposition. Let A be the ring $k[X, Y, Z, T]/(XT - Y^2, T - Y - TZ) = k[x, y, z, t]$, $\boldsymbol{a} = (x, y)$, $f_1 = x$ and $f_2 = z$; then it is easy to see that

f_1, f_2 form a regular sequence and $p_1 = 1$, $p_2 = 0$. We have: $xt \in \boldsymbol{a}^2 \cap (x)$, but $xt \notin \boldsymbol{a}x$; on the other hand $\boldsymbol{a} \cap (x, z) = (x) + \boldsymbol{a} \cap (z) = (x) + \boldsymbol{a}z$ and, if $n \geqslant 2$, then $\boldsymbol{a}^n \cap (x, z) = \boldsymbol{a}^n = \boldsymbol{a}^{n-1}x + (y^n) = \boldsymbol{a}^{n-1}x + \boldsymbol{a}^n z$ since $y^n = xy^{n-1} + y^n z$.

Remark 2.9. The results of the present section extend to a quite general situation the theorem proved by Hironaka ([H_2], Prop. 6) for a local ring $(A, \boldsymbol{m})$, when $\boldsymbol{a} = \boldsymbol{m}$ and $\boldsymbol{b} = (f) =$ principal ideal, generated by an element in $\boldsymbol{m} - \boldsymbol{m}^2$.

3. Applications

In this section we discuss some applications of the preceding results.

PROPOSITION 3.1. *Let $\boldsymbol{a}$ and $\boldsymbol{b} = (f_1, \cdots, f_r)$ be ideals of A such that $\boldsymbol{b} \subseteq \boldsymbol{a}$, $f_1, \cdots, f_r$ is an A-sequence and $\boldsymbol{ab} = \boldsymbol{a}^2$. Then the initial forms of the f_i's form a $G_A(\boldsymbol{a})$-sequence.*

Proof. If $f_i \in \boldsymbol{a}^2 = \boldsymbol{ab}$ then we would get a relation of the form

$$a_1 f_1 + \cdots + (1 + a_i) f_i + \cdots + a_r f_r = 0 , \qquad a_j \in \boldsymbol{a}(\forall j) .$$

But since $f_1, \cdots, f_r$ is an A-sequence, in any relation $\sum x_j f_j = 0$ all the coefficients x_j must lie in the ideal $(f_1, \cdots, f_r)$. This is well known and easy to prove by induction on r. Thus $f_i \notin \boldsymbol{a}^2$ and we have $p_1 = \cdots = p_r = 1$. Therefore it is enough to prove, by Proposition 2.6, that $\boldsymbol{a}^n \cap \boldsymbol{b} = \boldsymbol{a}^{n-1}\boldsymbol{b}$. This is true for $n = 1$; if $n \geqslant 2$ we have $\boldsymbol{a}^n = \boldsymbol{a}^{n-1}\boldsymbol{b}$, hence $\boldsymbol{a}^n \cap \boldsymbol{b} = \boldsymbol{a}^{n-1}\boldsymbol{b} \cap \boldsymbol{b} = \boldsymbol{a}^{n-1}\boldsymbol{b}$.

Remark 3.2. An interesting situation in which we can apply the above proposition is the following: let $(A, \boldsymbol{m})$ be a local ring of dimension r which is Cohen-Macaulay, with embedding dimension m and multiplicity e; then one can show ([S], Theorem 1) that $m \leqslant e + r - 1$ and the equality holds if and only if there is an A-sequence $f_1, \cdots, f_r$ in $\boldsymbol{m}$ such that $\boldsymbol{m}^2 = \boldsymbol{m}\ (f_1, \cdots, f_r)$. The latter equality is exactly our condition on the ideals $\boldsymbol{a} = \boldsymbol{m}$ and $\boldsymbol{b} = (f_1, \cdots, f_r)$.

Remark 3.3. The results of the present paper can be used to give a new and simplified proof of ([V], Theorem 3.2), i.e. of the following claim:

Let A be a Cohen-Macaulay ring and let $a_1, \cdots, a_s$ be a regular *sequence,* $\boldsymbol{I} = (a_1, \cdots, a_s)$, *t an integer* $\geqslant 1$. *Then* $G_A(\boldsymbol{a})$ *is Cohen-Macaulay if* $\boldsymbol{a} = \boldsymbol{I}^t$.

Proof. As in [V] we may assume that A is a r-dimensional local ring with maximal ideal $\boldsymbol{m}$. Let $a_1, \cdots, a_s, f_{s+1}, \cdots, f_r$ be a maximal A-sequence in $\boldsymbol{m}$ and let $\boldsymbol{J} = (f_{s+1}, \cdots, f_r)$, $f_i = a_i^t$, for each $i = 1, \cdots, s$ and $\boldsymbol{b} = (f_1, \cdots, f_r)$. Since $f_1, \cdots, f_s$ is a regular sequence modulo $\boldsymbol{J}$ we have, by [V], Lemma 2.1, that $\boldsymbol{a}^n \cap \boldsymbol{b} \subseteq \boldsymbol{a}^{n-1}(f_1, \cdots, f_s) + \boldsymbol{J}$ for all $n \geqslant 1$; furthermore, since $f_{s+1}, \cdots, f_r$ is a regular sequence modulo $\boldsymbol{I}$, hence modulo $\boldsymbol{a}^n$ for all $n \geqslant 1$, by [R–V], Lemma 1.1, we get: $\boldsymbol{a}^n \cap \boldsymbol{J} = \boldsymbol{a}^n\boldsymbol{J}$. From this it follows that $\boldsymbol{a}^n \cap \boldsymbol{b} \subseteq \boldsymbol{a}^n \cap (\boldsymbol{a}^{n-1}(f_1, \cdots, f_s) + \boldsymbol{J}) = \boldsymbol{a}^{n-1}(f_1, \cdots, f_s) + \boldsymbol{a}^n \cap \boldsymbol{J} = \boldsymbol{a}^{n-1}(f_1, \cdots, f_s) + \boldsymbol{a}^n\boldsymbol{J}$; thus the f_i^*'s form a $G_A(\boldsymbol{a})$-sequence by Proposition 2.6. Since $\dim G_A(\boldsymbol{a}) = r$, by [M–R] this is enough to prove that $G_A(\boldsymbol{a})$ is Cohen-Macaulay.

We conclude the paper trying to compare the initial forms of a set of elements with respect to two different ideals $\boldsymbol{a}$ and $\boldsymbol{I}$.

First of all, it is easy to see that the initial form of the same element with respect to two different ideals may or may not be a 0-divisor; for instance, if $A = k[[x, y, z]] = k[[X, Y, Z]]/(XY - Z^2)$, $\boldsymbol{a} = (x, y, z)$, $\boldsymbol{I} = (x, z)$, then it is clear that the initial form of x relative to $\boldsymbol{a}$ is a non 0-divisor, while the initial form with respect to $\boldsymbol{I}$ is a 0-divisor. Therefore $(x)^* = (x^*)$ in $G_A(\boldsymbol{a})$ (Proposition 2.1); on the contrary $(x)^*$ in $G_A(\boldsymbol{I})$ is not generated by the initial form of x, because of Theorem 1.1.

In the following proposition we denote by f^* the initial form with respect to $\boldsymbol{a}$ and by f^0 the initial form with respect to $\boldsymbol{I}$.

PROPOSITION 3.4. *Let* $\boldsymbol{I} \subseteq \boldsymbol{a}$ *be ideals of* A *and let* $f_1, \cdots, f_r$ *be elements of* $\boldsymbol{I}$ *such that* $v_I(f_i) = v_a(f_i)$ *for each* i. *Assume that* $f_1^*, \cdots, f_r^*$ *form a* $G_A(\boldsymbol{a})$*-sequence. Then* $f_1^0, \cdots, f_r^0$ *form a minimal base of the ideal* $(f_1^0, \cdots, f_r^0)$ *of* $G_A(\boldsymbol{I})$.

Proof. By [A–B], Corollary 2.9, $f_1^*, \cdots, f_r^*$ is a $G_A(\boldsymbol{a})$-sequence in any order. Now if $f_r^0 = \sum_{i=1}^{r-1} a_i^0 f_i^0$, let $a = \sum_{i=1}^{r-1} a_i f_i$ and $p = v_a(f_r) = v_I(f_r)$; then $f_r = a + b$, where $b \in \boldsymbol{I}^{p+1}$. Hence $a \in \boldsymbol{a}^p$ and $a \notin \boldsymbol{a}^{p+1}$; it follows that $f_r^* = a^* \in (f_1, \cdots, f_{r-1})^* = (f_1^*, \cdots, f_{r-1}^*)$, which is absurd.

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Istituto matematico-Politecnico di Torino
Istituto matematico-Università di Genova