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ON THE DOI-NAGANUMA LIFTING ASSOCIATED WITH IMAGINARY QUADRATIC FIELDS

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Introduction

Similarly to the real quadratic field case by Doi and Naganuma ([3], [9]) there is a lifting from an elliptic modular form to an automorphic form on $SL_2(\boldsymbol{C})$ with respect to an arithmetic discrete subgroup relative to an imaginary quadratic field. This fact is contained in his general theory of Jacquet ([6]) as a special case. In this paper, we try to reproduce this lifting in its concrete form by using the theta function method developed first by Niwa ([10]); also Kudla ([7]) has treated the real quadratic field case on the same line. The theta function method will naturally lead to a theory of lifting to an orthogonal group of general signature (cf. Oda [11]), and the present note will give a prototype of non-holomorphic case.

Let an imaginary quadratic number field be fixed once for all throughout this paper, and let $\mathfrak{o}$ denote the ring of integers of the field. For simplicity's sake we assume that the class number is one and the discriminant $-D$ is odd prime, less than -3. For a positive even integer ν we denote by $\mathscr{S}_{\nu+1}$ the space of cusp forms of weight $\nu+1$ of Neben type χ with respect to $\Gamma_0(D)$, where χ denotes the Kronecker character $\left(\frac{-D}{*}\right)$. We shall show that each cusp form f in $\mathscr{S}_{\nu+1}$ can be lifted to a $\boldsymbol{C}^{2\nu+1}$-valued automorphic form F on $SL_2(\boldsymbol{C})$ with respect to $SL_2(\mathfrak{o})$, belonging to an irreducible representation of $SU_2(\boldsymbol{C})$ of degree $2\nu+1$, which is also an eigen-function of the Casimir operator with the eigen-value $\frac{1}{2}(\nu^2-1)$. We shall give the Fourier expansion of F explicitly. It may be remarkable that the lifted image F is cuspidal if and only if f is orthogonal to $\theta^{(\nu)}_{-D}$ in $\mathscr{S}_{\nu+1}$, where $\theta^{(\nu)}_{-D}(z)=\frac{1}{2}\sum_{r\in\mathfrak{o}} r^{\nu}\exp(2\pi i r\bar{r}z)$.

In our argument, a special polynomial of four variables, which is

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nothing but a classical Laplace's spherical harmonics in essential, plays a fundamental role, so that we shall devote the first section to summarizing its properties. One of virtues of Niwa's method is in connecting a theta function with so-called Rankin's method, and we shall follow this, but in a little more direct fashion, namely we try to avoid using Eisenstein series there. The exceptional behaviour of $\theta_{-D}^{(\nu)}$ is related to the fact that $\theta_{-D}^{(\nu)}$ is the unique primitive form in $\mathscr{S}_{\nu+1}$ whose Fourier coefficients are all real. As an application of the Doi-Naganuma lifting we shall give a proof of this in the last section.

§1. The spherical harmonic polynomial

1.1. We shall denote by $\rho_n(g)$ the n-ply symmetric tensor product of $g \in GL_2(\boldsymbol{C})$, i.e.

$$ {}^t((a, b){}^tg)_n = \rho_n(g){}^t(a, b)_n $$

with indeterminants $a, b \in \boldsymbol{C}$ and $(a, b)_n = (a^n, a^{n-1}b, \cdots, ab^{n-1}, b^n)$. Put $G = SL_2(\boldsymbol{C})$ and $K = SU_2(\boldsymbol{C})$. Each ρ_n gives an irreducible representation of K as is well known. Let us put

$$ V = \{X \in M_2(\boldsymbol{C})\,;\, X = {}^t\bar{X}\}\,, $$

which is a vector space of dimension four over $\boldsymbol{R}$, and so we regard $M_2(\boldsymbol{C})$ as $V_{\boldsymbol{C}} = V \otimes_{\boldsymbol{R}} \boldsymbol{C}$. The group G acts on V in such a way that $X^g = {}^t\bar{g}Xg$ for $X \in V$ and $g \in G$. The action restricted to K gives a representation on V equivalent to $\rho_0 \oplus \rho_2$. Let us define two symmetric bilinear forms Q and R on V by

$$ Q(X, Y) = -\mathrm{tr}\,(X\tilde{Y}),\ R(X, Y) = \mathrm{tr}\,(XY)\,; \qquad X, Y \in V\,, $$

where $\tilde{Y} = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right){}^tY\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$. $Q(X)$ and $R(X)$ denote the associated quadratic forms $Q(X, X)$ and $R(X, X)$, respectively. The form Q is of signature $(3, 1)$ and G-invariant, i.e. $Q(X^g) = Q(X)$ for every $g \in G$, while the form R is positive definite and K-invariant, i.e. $R(X^\kappa) = R(X)$ for every $\kappa \in K$. We should note the form $R(X^g)$ is a minimal majorant of $Q(X)$ for each $g \in G$. For a non-negative integer ν, let $\mathscr{H}^\nu$ denote the $\boldsymbol{C}$-linear space of polynomial functions on V spanned by $Q(X, A)^\nu$, where $A \in V_{\boldsymbol{C}}$ such that $Q(X, A) = R(X, A)$ and $Q(A) = R(A) = 0$. An element of $\mathscr{H}^\nu$ is called a spherical harmonic polynomial with respect to Q and its majorant R. The dimension of $\mathscr{H}^\nu$ is $2\nu + 1$, in fact, we can get a natural basis as follows.

Let us put $A = {}^t(a, b)(a, b)\left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right) \in V_C$, then the form $Q(X, A)^\nu$ is a homogeneous polynomial of degree ν (resp. 2ν) with respect to X (resp. a, b). Hence we may define a homogeneous polynomial $\eta_{\nu,\alpha}(X)$ of degree ν as the coefficient of $a^{\nu-\alpha}b^{\nu+\alpha}$ in $Q(X, A)^\nu$ for each $\alpha, |\alpha| \leqq \nu$. We thus put:

$$(1) \qquad \begin{aligned} &\eta_{(\nu)}(X)^t(a, b)_{2\nu} = Q(X, A)^\nu , \\ &\eta_{(\nu)}(X) = (\eta_{\nu,-\nu}(X), \cdots, \eta_{\nu,\nu}(X)) . \end{aligned}$$

Obviously we have

LEMMA 1. *The polynomials $\eta_{\nu,\alpha}(X), |\alpha| \leqq \nu$, form a basis of $\mathscr{H}^\nu$.*

Since the forms Q and R are K-invariant, $\mathscr{H}^\nu$ is regarded as a representation space of K, which is irreducible. In particular, we have

LEMMA 2. $\eta_{(\nu)}(X^\kappa) = \eta_{(\nu)}(X)\rho_{2\nu}(\kappa)$ *for every* $\kappa \in K$.

Proof. We have $\eta_{(\nu)}(X^\kappa)^t(a, b)_{2\nu} = Q(X^\kappa, A)^\nu = Q(X, A^{\kappa^{-1}})^\nu$, which is equal to $\eta_{(\nu)}(X)^t((a, b)^t\kappa)_{2\nu} = \eta_{(\nu)}(X)\rho_{2\nu}(\kappa)^t(a, b)_{2\nu}$, since $A^{\kappa^{-1}} = {}^t((a, b)^t\kappa)((a, b)^t\kappa) \cdot \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$ for $\kappa \in K$.

We should note that Lemmas 1 and 2 characterize the polynomials $\eta_{\nu,\alpha}(X)$ up to a constant multiple, in fact, by a simple observation we can show that the polynomials $\eta_{\nu,\alpha}(X)$ are essentially the same as classical Laplace's spherical harmonics (cf. [4], Chap. XI).

1.2. We write a general element X of V as follows:

$$(2) \qquad X = \begin{pmatrix} m & r \\ \bar{r} & n \end{pmatrix}; \; m, n \in \boldsymbol{R}, r \in \boldsymbol{C} ,$$

and this parametrization will be kept throughout the paper. It is, then, easy to see that $\eta_{(0)}(X) = 1$ and $\eta_{(1)}(X) = (-\bar{r}, m - n, r)$, while we can compute more by use of a recursion formula

$$(3) \qquad \eta_{\nu,\alpha}(X) = (-\bar{r})\eta_{\nu-1,\alpha+1}(X) + (m - n)\eta_{\nu-1,\alpha}(X) + (r)\eta_{\nu-1,\alpha-1}(X) ,$$

where we understand $\eta_{0,0}(X) = 1$ and $\eta_{\nu,\alpha}(X) = 0$ if $|\alpha| > \nu$. The following is an explicit formula which we need later.

LEMMA 3. *For $|\alpha| \leqq \nu$, we have*

$$(4) \qquad \eta_{\nu,\alpha}(X) = \nu! \sum_{\beta,\gamma} \frac{1}{(\alpha+\beta)!\cdot\gamma!} 2^{-\beta} r^{\alpha} L_{\beta}^{(\alpha)}(2r\bar{r}) H_{\gamma}(m-n) ,$$

where $L_{\beta}^{(\alpha)}$ and H_{γ} are Laguerre's and Hermite's polynomials respectively, and the sum is taken over all non-negative integers β, γ such that $2\beta + \gamma = \nu - \alpha$ and $\alpha + \beta \geqq 0$.

Proof. We put $q = m - n$ for abbreviation. By the definition (1) we have

$$\exp((-\bar{r}a^2 + qab + rb^2)t) = \sum_{\nu=0}^{\infty} \sum_{\alpha=-\nu}^{\nu} \eta_{\nu,\alpha}(X) a^{\nu-\alpha} b^{\nu+\alpha} \frac{t^{\nu}}{\nu!} .$$

It follows from a generating function formula of H_{γ} (e.g. [8], p. 253) that

$$\exp\left(qabt - \frac{1}{2}a^2b^2t^2\right) = \sum_{\gamma=0}^{\infty} H_{\gamma}(q) a^{\gamma} b^{\gamma} \frac{t^{\gamma}}{\gamma!} .$$

On the other hand, we have

$$\begin{aligned} \exp\left((-\bar{r}a^2 + rb^2)t + \frac{1}{2}a^2b^2t^2\right) &= \sum_{\ell=0}^{\infty} \exp(-\bar{r}a^2t)\left(1 + \frac{1}{2}r^{-1}a^2t\right)^{\ell} \frac{(rb^2t)^{\ell}}{\ell!} \\ &= \sum_{\ell=0}^{\infty} \sum_{\beta=0}^{\ell} 2^{-\beta} r^{\ell-2\beta} L_{\beta}^{(\ell-2\beta)}(2r\bar{r}) a^{2\beta} b^{2\ell-2\beta} \frac{t^{\ell}}{(\ell-\beta)!} \end{aligned}$$

by use of a generating function formula of $L_{\beta}^{(\alpha)}$ (e.g. [8], p. 242). Equating these, we obtain the proof.

For convenience' sake let us define another quadratic form on V by

$$(5) \qquad P(X) = Q(X) + R(X) ,$$

then it can be immediately seen that P is K-invariant, and $0 \leqq P(X) \leqq 2R(X)$ for $X \in V$.

LEMMA 4. $|\eta_{\nu,\alpha}(X)|^2 \leqq \binom{2\nu}{\nu-\alpha}\binom{2\nu}{\nu}^{-1} P(X)^{\nu}$ *for* $X \in V$.

Proof. It is sufficient to prove

$$(6) \qquad \sum_{\alpha=-\nu}^{\nu} \binom{2\nu}{\nu-\alpha}^{-1} |\eta_{\nu,\alpha}(X)|^2 = \binom{2\nu}{\nu}^{-1} P(X)^{\nu} .$$

Denote by J a diagonal constant matrix of degree $2\nu + 1$ satisfying $(a,b)_{2\nu} J^2\, {}^t(\bar{a},\bar{b})_{2\nu} = (a\bar{a} + b\bar{b})^{2\nu}$, then $J\rho_{2\nu}(\kappa)J^{-1}$ is a unitary matrix for every

$\kappa \in K$. Hence the left-hand side of (6), which is $\eta_{(\nu)}(X)J^{-2}\,\overline{{}^t\eta_{(\nu)}(X)}$, is K-invariant as well as the right-hand side. We may, therefore, assume $X = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}$. Then both sides are equal to $\binom{2\nu}{\nu}^{-1}$ $(m-n)^{2\nu}$. This completes the proof.

1.3. For the purpose to give another property of $\eta_{\nu,\alpha}(X)$ we recall some notions on Lie derivatives. Let S be an arbitrary element of the Lie algebra of G, i.e. $S \in M_2(C)$ with $\operatorname{tr}(S) = 0$. A differential operator S' (or S'') on $C^\infty(G)$ is defined by $S'f(g) = \left[\dfrac{\partial}{\partial t} f(g\cdot \exp(tS))\right]_{t=0}$ $\Big($or by replacing $\dfrac{\partial}{\partial t}$ with $\dfrac{\partial}{\partial \bar{t}}$, accordingly.$\Big)$ On the other hand, a function $F \in C^\infty(V)$ induces a function $F_X(g) = F(X^g) \in C^\infty(G)$ for each $X \in V$. Hence S' (or S'') operates on $C^\infty(V)$ in such way that $(S'F)_X = S'F_X$ (or $(S''F)_X = S''F_X$). It can be easily seen that S' (or S'') on $C^\infty(V)$ is equal to $m'\dfrac{\partial}{\partial m} + n'\dfrac{\partial}{\partial n} + r'\dfrac{\partial}{\partial r} + r''\dfrac{\partial}{\partial \bar{r}}$, where XS (or ${}^t\bar{S}X) = \begin{pmatrix} m' & r' \\ r'' & n' \end{pmatrix}$ with the parametrization $X = \begin{pmatrix} m & r \\ \bar{r} & n \end{pmatrix} \in V$. The elements $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ form a basis of the Lie algebra, and the Casimir operators C' and C'' are given by $C' = A'B' + B'A' + \frac{1}{2}U'U'$ and $C'' = A''B'' + B''A'' + \frac{1}{2}U''U''$, while both the operators C' and C'' coincide with each other on $C^\infty(V)$. Further, by an easy computation we can obtain a formula on $C^\infty(V)$:

$$C' = C'' = \tfrac{1}{2}L^2 + L - \tfrac{1}{2}Q(X)\Delta_Q\,, \tag{7}$$

where $L = m\dfrac{\partial}{\partial m} + n\dfrac{\partial}{\partial n} + r\dfrac{\partial}{\partial r} + \bar{r}\dfrac{\partial}{\partial \bar{r}}$ and $\Delta_Q = 2\dfrac{\partial^2}{\partial r \partial \bar{r}} - 2\dfrac{\partial^2}{\partial m \partial n}$ (cf. [13], p. 95).

LEMMA 5. *Put*

$$\eta^*_{\nu,\alpha}(X) = \eta_{\nu,\alpha}(X)P(X)^{-(\nu+\frac{1}{2})}\,, \tag{8}$$

then we have $C'\eta^*_{\nu,\alpha} = C''\eta^*_{\nu,\alpha} = \frac{1}{2}(\nu^2-1)\eta^*_{\nu,\alpha}$.

Proof. Since $\eta^*_{\nu,\alpha}(X)$ is a homogeneous function of degree $-(\nu+1)$, $L\eta^*_{\nu,\alpha} = -(\nu+1)\eta^*_{\nu,\alpha}$. On the other hand, $\Delta_Q\eta^*_{\nu,\alpha} = 0$ as well as $\Delta_Q\eta_{\nu,\alpha} = 0$. These combined with the formula (7) complete the proof.

The function $\eta^*_{\nu,\alpha}(X)$ is also called a spherical harmonics of degree $-(\nu+1)$, which coincides with $\eta_{\nu,\alpha}(X)$ on the surface: $P(X)=1$.

§ 2. Poincaré series and theta series

2.1. Let $\mathfrak{o}$ be the ring of integers of an imaginary quadratic field of odd prime discriminant $-D$. It is, as in the introduction, assumed that the class number is one and the units are 1 and -1 only. Let us define a lattice $\mathfrak{L}$ of V over $\boldsymbol{Z}$ by $\mathfrak{L}=V\cap M_2(\mathfrak{o})$, and denote by $\mathfrak{L}(\ell)$ the subset of $\mathfrak{L}$ consisting of all elements of determinant $-\ell$ for each $\ell\in\boldsymbol{Z}$. Let Λ denote the discrete subgroup $SL_2(\mathfrak{o})$ of G. Each subset $\mathfrak{L}(\ell)$ is Λ-invariant. For each positive integer ℓ we define a $\boldsymbol{C}^{2\nu+1}$-valued function on G by

$$h_{(\nu),\ell}(g)=\sum_{X\in\mathfrak{L}(\ell)}\eta^*_{(\nu)}(X^g)\,,\qquad(g\in G)\,,\tag{9}$$

where $\eta^*_{(\nu)}(X)=(\eta^*_{\nu,-\nu}(X),\cdots,\eta^*_{\nu,\nu}(X))$ and $\eta^*_{\nu,\alpha}(X)$ is given by (8). This series is, as we shall see below, absolutely convergent for $\nu>1$, hence the function $h_{(\nu),\ell}$ can be regarded as a type of Poincaré series, considering that the set $\mathfrak{L}(\ell)$ is a union of some Λ-orbits. This combined with Lemmas 2 and 5 leads to

LEMMA 6. *For each positive integer ℓ and $\nu>1$, it holds*
(i) $h_{(\nu),\ell}(\gamma g\kappa)=h_{(\nu),\ell}(g)\rho_{2\nu}(\kappa)$ *for* $\gamma\in\Lambda$ *and* $\kappa\in K$.
(ii) $C'h_{(\nu),\ell}=C''h_{(\nu),\ell}=\frac{1}{2}(\nu^2-1)h_{(\nu),\ell}$.

For the purpose to prove that the series (9) is convergent absolutely and uniformly on any compact subset of G, it is sufficient to combine Lemma 4 with next two lemmas.

LEMMA 7. *If ℓ is positive, a series $\sum_{X\in\mathfrak{L}(\ell)}P(X)^{-s}$ is absolutely convergent for* $\operatorname{Re}s>1$.

Proof. By noting $P(X)=4r\bar r+(m-n)^2$ for $X=\begin{pmatrix}m & r\\ \bar r & n\end{pmatrix}\in V$, divide the summation on $\mathfrak{L}(\ell)$ into the following three parts: one with $r=0$, one with $r\bar r=\ell$ and the others. The first, then, is a finite sum, and it can be easily seen that the second and the third are majorized by some constant multiples of $\sum_{n\in\boldsymbol{Z}}(4\ell+n^2)^{-\operatorname{Re}s}$ and $\sum_{r\in\mathfrak{o}-\{0\}}(r\bar r)^{-(\operatorname{Re}s-\varepsilon)}$ respectively, where ε is chosen so that $\operatorname{Re}s-1>\varepsilon>0$. These imply the lemma.

LEMMA 8. *For arbitrary $g \in G$ there exists a positive constant c such that $P(X^g) \geqq cP(X)$ for every $X \in V$ with non-positive determinant.*

Proof. Since $P(X)$ is K-invariant, we can assume that g is diagonal: $g = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$. Take c to be the minimum of $|\beta|^4$ and $|\beta|^{-4}$, then we have $P(X^g) = 4r\bar{r} + (m|\beta|^2 - n|\beta|^{-2})^2 \geqq P(X) + (m^2 + n^2)(c - 1)$, which is not less than $cP(X)$ because $P(X) \geqq m^2 + n^2$ and $c \leqq 1$.

Remark. $h_{(\nu),\ell}$ vanishes for every odd ν. Also $h_{(\nu),0}$ can be defined by (9) in whose summation $X = 0$ is excluded, and it is an Eisenstein series satisfying the properties in Lemma 6.

2.2. For a function $f(z)$ on the upper half complex plane $\mathfrak{H}$ and a real matrix $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of positive determinant (i.e. $\sigma \in GL_2^+(\boldsymbol{R})$), we write

$$(f\,|\,[\sigma]_k)(z) = (\det(\sigma))^{k/2}(cz + d)^{-k} f(\sigma z)$$

for $k \in \boldsymbol{Z}$, where $\sigma z = (az + b)(cz + d)^{-1}$. We define a differential operator δ_λ for $\lambda \in \boldsymbol{R}$ following Shimura ([12]) by

$$\delta_\lambda = \frac{1}{2\pi i}\left(\frac{\lambda}{2iy} + \frac{\partial}{\partial z}\right) = \frac{1}{2\pi i} y^{-\lambda} \frac{\partial}{\partial z} y^\lambda \ ,$$

where $z = x + iy$, and we also put

$$\delta_\lambda^\ell = \delta_{\lambda+2(\ell-1)} \cdots \delta_{\lambda+2}\delta_\lambda \qquad \text{for } 0 \leqq \ell \in \boldsymbol{Z} \ .$$

The "raising" operator δ_λ^ℓ acts on functions on $\mathfrak{H}$ and satisfies that $\delta_\lambda^\ell(f\,|\,[\sigma]_\lambda) = (\delta_\lambda^\ell f)\,|\,[\sigma]_{\lambda+2\ell}$ for every $\sigma \in GL_2^+(\boldsymbol{R})$. We need the following

LEMMA 9. (i) $\delta_\lambda^\ell \exp(2\pi iaz) = \ell\,!\,(-4\pi y)^{-\ell} L_\ell^{(\lambda-1)}(4\pi ay) \exp(2\pi iaz)$, $(\lambda, a \in \boldsymbol{R})$.

(ii) *Put* $t_\gamma(z\,; m, n) = \sqrt{y}(8\pi y)^{-\gamma/2} H_\gamma(\sqrt{2\pi y}(m - n)) \exp(\pi i(x(-2mn) + iy(m^2 + n^2)))$, *then* $\delta_\gamma^\ell t_\gamma = t_{\gamma+2\ell}$, $(0 \leqq \gamma \in \boldsymbol{Z},\ m, n \in \boldsymbol{R})$. *Here* $L_\ell^{(\lambda)}$ *and* H_γ *are Laguerre's and Hermite's polynomials respectively.*

Proof. We can easily prove (i) or (ii) by induction on ℓ, using a formula (cf. [8], p. 241, p. 252):

$$x\frac{d}{dx}L_\ell^{(\alpha)}(x) = (\ell + 1)L_{\ell+1}^{(\alpha)}(x) - (\ell + \alpha + 1 - x)L_\ell^{(\alpha)}(x)$$

or

$$x\frac{d}{dx}H_\ell(x) = (x^2 - (\ell + 1))H_\ell(x) - H_{\ell+2}(x) ,$$

accordingly.

2.3. Now let us define a theta series with respect to the indefinite quadratic form Q on V. Since the minimal majorant $R(X^g)$ is parametrized by $g \in G$, our function has double variables $z = x + iy \in \mathfrak{H}$ and $g \in G$. Namely we put

$$\begin{aligned} \theta_{\nu,\alpha}(z, g) &= \sqrt{y} \sum_{X \in \mathfrak{L}} \eta_{\nu,\alpha}(X^g) \exp(\pi i(xQ(X) + iyR(X^g))) , \\ \theta_{(\nu)}(z, g) &= (\theta_{\nu,-\nu}(z, g), \cdots, \theta_{\nu,\nu}(z, g)) , \end{aligned} \tag{10}$$

where $\mathfrak{L} = V \cap M_2(\mathfrak{o})$ as before. Obviously the series is absolutely convergent. We may, for non-triviality, assume that ν is even, since $\theta_{(\nu)}(z, g)$ vanishes for odd ν. For abbreviation Γ will stand for the subgroup $\Gamma_0(D)$ of $SL_2(Z)$ consisting of all $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $D \mid c$, and χ denotes the character of Γ defined by Kronecker's symbol, i.e. $\chi(\sigma) = \left(\frac{-D}{d}\right)$ for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

LEMMA 10. *For a non-negative even integer ν, it holds that*
(i) $\theta_{\nu,\alpha}(z, g) = \chi(\sigma)\theta_{\nu,\alpha}(z, g) \mid [\sigma]_{\nu+1}$ *for every* $\sigma \in \Gamma$.
(ii) $\theta_{(\nu)}(z, \gamma g \kappa) = \theta_{(\nu)}(z, g)\rho_{2\nu}(\kappa)$ *for every* $\gamma \in \Lambda$ *and* $\kappa \in K$.
(ii.a) $\theta_{(\nu)}(z, g^0) = \theta_{(\nu)}(z, g)\rho_{2\nu}\left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)$, *where* $g^0 = \left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right) g \left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)$.
(ii.b) $\theta_{(\nu)}(z, \bar{g}) = \theta_{(\nu)}(z, g)\rho_{2\nu}\left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$.

Proof. Due to Lemma 1.2 and Proposition 1.6 of Shintani [13], the theta transformation formula depends only on the form Q and the lattice $\mathfrak{L}$, hence it is sufficient to prove (i) in the case that $\alpha = \nu$ and $g = 1$ (the identity). Since $\eta_{\nu,\nu}(X) = r^\nu$ for $X = \begin{pmatrix} m & r \\ \bar{r} & n \end{pmatrix} \in \mathfrak{L}$, we can immediately see that $\theta_{\nu,\nu}(z, 1) = 2\theta^{(\nu)}_{-D}(z)\theta_0(z)$, where $\theta^{(\nu)}_{-D}(z) = \frac{1}{2}\sum_{r \in \mathfrak{o}} r^\nu \cdot \exp(2\pi i r\bar{r}z)$ and $\theta_0(z) = \sqrt{y}\sum_{m,n \in Z} \exp(\pi i(x(-2mn) + iy(m^2 + n^2)))$. As is well known, $\theta^{(\nu)}_{-D} = \chi(\sigma)\theta^{(\nu)}_{-D} \mid [\sigma]_{\nu+1}$ for $\sigma \in \Gamma$, and $\theta_0 = \theta_0 \mid [\sigma]_0$ for $\sigma \in SL_2(Z)$ since $\theta_0(z) = \sum_{c,d \in Z} \exp(-\pi y^{-1} |cz + d|^2)$ as derived by Poisson's summation formula. We have thus proved (i). For the proof of (ii) we have only to consider Λ-invariance of $\mathfrak{L}$, K-invariance of $R(X^g)$ and Lemma 2. We can derive (ii.a) in the same way as (ii), though $\left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)$ belongs to neither Λ nor K. Finally from the definition (1) of $\eta_{\nu,\alpha}$ we have $\eta_{\nu,\alpha}(\bar{X}) = (-1)^{\alpha}\eta_{\nu,-\alpha}(X)$, which imply (ii.b).

2.4. For a diagonal $g \in G$ the theta function $\theta_{\nu,\alpha}(z, g)$ splits into a convenient form. To describe this, we define two more theta series by

(11) $$\theta^{(\alpha)}_{-D,\beta}(z) = \tfrac{1}{2}\sum_{r\in\mathfrak{o}} r^{\alpha}y^{-\beta}L^{(\alpha)}_{\beta}(4\pi r\bar{r}y)\exp(2\pi i r\bar{r}z)$$

for $0 \leqq \alpha, \beta \in \boldsymbol{Z}$, and

(12) $$\theta_{\gamma}(z, v) = v^{\gamma+1}y^{-\gamma}\sum_{c,d\in\boldsymbol{Z}}(cz + d)^{\gamma}\exp(-\pi v^2 y^{-1}|cz + d|^2)$$

for $0 \leqq \gamma \in \boldsymbol{Z}$, where $z = x + iy \in \mathfrak{H}$ and $0 < v \in \boldsymbol{R}$. We abbreviate $\theta^{(\alpha)}_{-D,0}$ to $\theta^{(\alpha)}_{-D}$ simply. We should notice that $\theta^{(\alpha)}_{-D,\beta}$ (or θ_{γ}) vanishes for odd α (or odd γ). Due to Lemma 9 we have $\theta^{(\alpha)}_{-D,\beta} = (-4\pi)^{\beta}(\beta\,!)^{-1}\delta^{\beta}_{\alpha+1}\theta^{(\alpha)}_{-D}$ and $\theta_{\gamma} = (-4)^{\gamma/2}\delta_0^{\gamma/2}\theta_0$ for even γ, so that $\theta^{(\alpha)}_{-D,\beta} = \chi(\sigma)\theta^{(\alpha)}_{-D,\beta}|[\sigma]_{\alpha+2\beta+1}$ for $\sigma \in \Gamma$ and $\theta_{\gamma} = \theta_{\gamma}|[\sigma]_{\gamma}$ for $\sigma \in SL_2(\boldsymbol{Z})$. There is another expression for θ_{γ}:

(13) $$\theta_{\gamma}(z, v) = (-4)^{\gamma/2}\sum_{m,n\in\boldsymbol{Z}} t_{\gamma}(z\,; mv, nv^{-1})\ ,$$

where $t_{\gamma}(z\,; m, n)$ is the same as defined in Lemma 9. We can obtain (13) directly by Poisson's summation formula, or by applying the raising operator $\delta_0^{\gamma/2}$ to the simpler case of $\gamma = 0$.

Now let us put $g(v) = \dfrac{1}{\sqrt{v}}\begin{pmatrix} v & \\ & 1\end{pmatrix}$ for $0 < v \in \boldsymbol{R}$. Then we have $\theta_{\nu,\alpha}(z, g(v)) = (-1)^{\alpha}\theta_{\nu,-\alpha}(z, g(v))$ which, especially, vanishes for odd α.

LEMMA 11. *For a non-negative even integer α,*

$$\theta_{\nu,\alpha}(z, g(v)) = 2(\nu\,!)\sum_{\beta,\gamma}\frac{1}{(\alpha+\beta)\,!\cdot\gamma\,!}i^{\gamma}(4\pi)^{-\beta}\theta^{(\alpha)}_{-D,\beta}(z)\theta_{\gamma}(z, v)\ ,$$

where the sum is taken over $\beta \geqq 0$, $\gamma \geqq 0$ with $2\beta + \gamma = \nu - \alpha$.

Proof. Observing that

$$xQ(X) + iyR(X^g) = (2r\bar{r}z) + (x(-2mn) + iy(m^2v^2 + n^2v^{-2}))$$

for $g = g(v)$ and $X = \begin{pmatrix} m & r \\ \bar{r} & n\end{pmatrix} \in \mathfrak{L}$, and considering the explicit formula for $\eta_{\nu,\alpha}(X)$ in Lemma 3, we can easily obtain the lemma.

§ 3. The Doi-Naganuma lifting

3.1. We hereafter make the assumption that ν is even and positive. $\mathcal{S}_{\nu+1}$ will, as in the introduction, denote the space of all holomorphic cusp forms of weight $\nu+1$ of Neben type χ with respect to Γ, so that f in $\mathcal{S}_{\nu+1}$ satisfies $f=\chi(\sigma)f|[\sigma]_{\nu+1}$ for $\sigma\in\Gamma$, where $\Gamma=\Gamma_0(D)$ and $\chi(\sigma)=\left(\frac{-D}{d}\right)$ for $\sigma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma$. Let f be a cusp form in $\mathcal{S}_{\nu+1}$, and let us consider the following integral of Petersson's inner product type in view of (i) in Lemma 10:

$$F_\alpha(g)=2^{\nu-2}i^{-(\nu+1)}D^{\nu/2}\int_{\Gamma\backslash\mathfrak{H}}\theta_{\nu,\alpha}(z,g)\overline{f_T(z)}y^{\nu-1}dxdy\ , \tag{14}$$

and define a $\boldsymbol{C}^{2\nu+1}$-valued function $F(g)$ on G by

$$F(g)=(F_{-\nu}(g),\cdots,F_\nu(g))\ , \tag{15}$$

where f_T in $\mathcal{S}_{\nu+1}$ is defined by $f_T(z)=\overline{(f|[W]_{\nu+1})(-\bar{z})}$ with $W=\left(\begin{smallmatrix} & 1\\ -D & \end{smallmatrix}\right)$. In an obvious manner we can see that $\theta_{\nu,\alpha}(z,g)=O(\exp(-\varepsilon y))$ when $y\to\infty$ for some positive ε and similar estimations hold at any other cusps, hence the integral (14) is absolutely convergent. The correspondence of f with F defines a linear map, which we shall denote by I, from $\mathcal{S}_{\nu+1}$ to a space of some functions on G. This is the definition of the Doi-Naganuma lifting in our case.

THEOREM 1. *The lifted image $F=I(f)$ satisfies the followings:*
(i) $F(\gamma g\kappa)=F(g)\rho_{2\nu}(\kappa)$ *for* $\gamma\in\Lambda$ *and* $\kappa\in K$.
(i.a) $F(g^0)=F(g)\rho_{2\nu}\left(\begin{smallmatrix}1 & \\ & -1\end{smallmatrix}\right)$, *where* $g^0=\left(\begin{smallmatrix}1 & \\ & -1\end{smallmatrix}\right)g\left(\begin{smallmatrix}1 & \\ & -1\end{smallmatrix}\right)$.
(i.b) $F(\bar{g})=F(g)\rho_{2\nu}\left(\begin{smallmatrix} & -1\\ 1 & \end{smallmatrix}\right)$.
(ii) $C'F=C''F=\frac{1}{2}(\nu^2-1)F$, *where C', C'' are the Casimir operators on G.*

Proof. (i), (i.a) and (i.b) are immediate consequences of (ii), (ii.a) and (ii.b) in Lemma 10, respectively. To prove (ii), we should recall that the space $\mathcal{S}_{\nu+1}$ is generated by Poincaré series for $1\leqq\ell\in\boldsymbol{Z}$:

$$p_\ell(z)=\sum_{\sigma\in\Gamma_\infty\backslash\Gamma}\chi(\sigma)\exp(2\pi i\ell z)|[\sigma]_{\nu+1}\ ,$$

where σ runs over Γ modulo the stabilizer Γ_∞ of ∞ in Γ. We can easily obtain

$$\int_{\Gamma\backslash\mathfrak{H}} \theta_{(\nu)}(z, g)\overline{p_\ell(z)}y^{\nu-1}dxdy = \pi^{-(\nu+\frac{1}{2})}\Gamma(\nu + \tfrac{1}{2})h_{(\nu),\ell}(g) ,$$

where $h_{(\nu),\ell}(g)$ is the same as defined by (9). The computation is valid for $\nu > 1$. This combined with Lemma 6 implies (ii).

3.2. To give the Fourier expansion of the lifted image $F = I(f)$ we need some notations as follows. We should first recall that Λ has the unique non-equivalent cusp, say ∞. Let us put $g(u, v) = \frac{1}{\sqrt{v}}\begin{pmatrix} v & u \\ & 1 \end{pmatrix}$ for $u \in \boldsymbol{C}$ and $0 < v \in \boldsymbol{R}$, so that every element of G has unique expression as $g(u, v)\kappa$ with some $g(u, v)$ and $\kappa \in K$. We may abbreviate $g(0, v)$ to $g(v)$ as before. We put $S(u) = u + \bar{u}$ for $u \in \boldsymbol{C}$. $K_\alpha(v)$ denotes the modified Bessel function of order α (e.g. [8], Chap. III). For each $\alpha \in \boldsymbol{Z}$ a grössen character ξ^α is defined by $\xi^\alpha(r) = r^\alpha |r|^{-\alpha}$ for $r \in \mathfrak{o} - \{0\}$. Finally we put $\omega = \frac{1}{\sqrt{-D}}$, so that (ω) is the complementary ideal of $\mathfrak{o}$.

THEOREM 2. *Let $f \in \mathscr{S}_{\nu+1}$ and $F = (F_{-\nu}, \cdots, F_\nu)$ be the lifted image $I(f)$. Suppose we have*

$$f(z) = \sum_{n=1}^{\infty} a(n) \exp(2\pi i n z) , \qquad f_T(z) = \sum_{n=1}^{\infty} \overline{b(n)} \exp(2\pi i n z) ,$$

and put

$$C(0) = 2^{\nu-1}D^{\nu/2}\int_{\Gamma\backslash\mathfrak{H}} f(z)\overline{\theta^{(\nu)}_{-D}(z)}y^{\nu-1}dxdy ,$$

$$C(r) = C_1(r) + C_2(r) \qquad \text{for } r \in \mathfrak{o} - \{0\} ;$$

$$C_1(r) = \sum_{n|(r)} n^\nu a(n^{-2}r\bar{r}) , \qquad C_2(r) = (-i)D^{\nu/2} \sum_{n|(r\omega)} n^\nu b(n^{-2}D^{-1}r\bar{r}) .$$

Then

$$F_\alpha(g(u, v)) = \sum_{r \in \mathfrak{o}} C(r)\phi_\alpha(v, r) \exp(2\pi i S(\bar{r}\omega u)) \tag{16}$$

for $|\alpha| \leqq \nu$, where

$$\phi_\alpha(v, 0) = \delta_{\nu,|\alpha|}v , \quad \phi_\alpha(v, r) = \binom{2\nu}{\nu - \alpha}\xi^\alpha(r)K_\alpha(4\pi|r\omega| v)v^{\nu+1} \qquad \textit{for } r \neq 0 .$$

We shall prove Theorem 2 in 3.5 after some preliminary lemmas. On the other hand, Theorem 2 says that the image $F = I(f)$ is not always cuspidal even though f is a cusp form. To make clear we state this as follows, while the proof is obvious:

THEOREM 3. *Let $\mathscr{S}^1_{\nu+1}$ be the orthogonal complement of $\theta^{(\nu)}_{-D}$ with respect to the Petersson metric in $\mathscr{S}_{\nu+1}$, then the lifted image $F = I(f)$ of f in $\mathscr{S}^1_{\nu+1}$ satisfies the followings:*
(iii) *$F(g)^t\overline{F(g)}$ is bounded on G.*
(iv) *$\int_{C/\mathfrak{o}} F(g(u,1)\cdot g)du d\bar{u} = 0$ for every $g \in G$.*

3.3. The next and following two lemmas are preparations to prove Theorem 2.

LEMMA 12. *Under the same conditions and notations as in Theorem 2,*

$$F_\alpha(g(v)) = \sum_{r\in\mathfrak{o}} C(r)\phi_\alpha(v, r) . \tag{17}$$

Proof. We first notice that $F_\alpha(g(v))$ vanishes for odd α as well as $\theta_{\nu,\alpha}(z, g(v))$, and $F_\alpha(g(v)) = F_{-\alpha}(g(v))$ for even α. Hence we may assume that α is even and non-negative. We prove (17) by a direct computation of the integral (14) for $g = g(v)$. Owing to Lemma 11, it is reduced to a computation of an integral

$$\int_{\Gamma\backslash\mathfrak{H}} \theta^{(\alpha)}_{-D,\beta}(z)\theta_\gamma(z, v)\overline{f_T(z)}y^{\nu-1}dxdy \tag{18}$$

with $2\beta + \gamma = \nu - \alpha$, $\beta \geqq 0$, $\gamma \geqq 0$. By the definition (12) of θ_γ, it is plain to see that

$$\theta_\gamma(z, v) = \delta_{\gamma,0}v + \theta^1_\gamma(z, v) + D^{\gamma/2}\theta^1_\gamma(Dz, v)\,|\,[W]_\gamma ,$$

where $\theta^1_\gamma(z, v) = 2\sum_{n=1}^\infty n^\gamma \sum_{\sigma\in\Gamma_\infty\backslash\Gamma} k(z, n)\,|\,[\sigma]_\gamma$, $k(z, n) = v^{\gamma+1}y^{-\gamma}\exp(-\pi v^2n^2y^{-1})$ and $W = \left(\begin{smallmatrix} & 1 \\ -D & \end{smallmatrix}\right)$. Hence the integral (18) decomposes into three parts. For the first, because of Lemma 6 of [12] we have, when $\gamma = 0$,

$$\int_{\Gamma\backslash\mathfrak{H}} \theta^{(\alpha)}_{-D,\beta}(z)\overline{f_T(z)}y^{\nu-1}dxdy = 0 \qquad \text{unless } \beta = 0 \ (\alpha = \nu) .$$

By usual method the second part is computed as follows:

$$\int_{\Gamma\backslash\mathfrak{H}} \theta^{(\alpha)}_{-D,\beta}(z)\overline{f_T(z)}\theta^1_\gamma(z, v)y^{\nu-1}dxdy$$
$$= 2\sum_{n=1}^\infty n^\gamma \int_{\Gamma\backslash\mathfrak{H}} \theta^{(\alpha)}_{-D,\beta}(z)\overline{f_T(z)}k(z, n)y^{\nu-1}dxdy$$

$$= 2v^{\nu+1}\sum_{n=1}^{\infty} n^{\gamma}\int_0^\infty dy\, y^{\nu-\gamma-1}\exp(-\pi v^2 n^2 y^{-1})\int_0^1 \theta^{(\alpha)}_{-D,\beta}(z)\overline{f_T(z)}dx\ .$$

Here we need a formula (cf. [5], p. 175 (33), though there is a misprint.):

$$\int_0^\infty \exp\left(-at-\frac{b}{t}\right)L^{(\alpha)}_\beta(at)t^{\alpha+\beta-1}dt = (-1)^\beta\frac{2}{\beta!}a^{-\alpha/2}b^{\alpha/2+\beta}K_\alpha(2\sqrt{ab})$$

for $a, b > 0$. We thus obtain

$$\int_{\Gamma\backslash\mathfrak{H}} \theta^{(\alpha)}_{-D,\beta}(z)\overline{f_T(z)}\theta^1_\gamma(z,v)y^{\nu-1}dxdy$$
$$= (-\pi)^\beta(\beta!)^{-1}2^{1-\alpha}i^{\alpha+1}D^{-\nu/2}\sum_{r\in\mathfrak{o}-\{0\}}\xi^\alpha(r)C_2(r)K_\alpha(4\pi|r\omega|v)v^{\nu+1}\ .$$

The same computation holds for the third part. Thus by using a relation $\nu!\sum_{2\beta+\gamma=\nu-\alpha}2^\gamma(\beta!\,\gamma!\,(\alpha+\beta)!)^{-1} = \binom{2\nu}{\nu-\alpha}$, we can complete the proof of (17).

3.4. LEMMA 13. *Suppose that $F = (F_{-\nu}, \cdots, F_\nu)$ is a function on G, satisfying the properties* (i), (i.a), (i.b) *and* (ii) *in Theorem* 1, *and an additional condition that $F_\alpha(g(v)) = \delta_{\nu,|\alpha|}B(0)v + O(\exp(-\varepsilon v))$ for $v\to\infty$ with some constants $\varepsilon > 0$ and $B(0)$. Then, F_α has a Fourier expansion as follows:*

$$(19)\qquad F_\alpha(g(u,v)) = \sum_{r\in\mathfrak{o}} B(r)\phi_\alpha(v,r)\exp(2\pi iS(\bar{r}\omega u))\ ,$$

where $\phi_\alpha(v,r)$ is the same as in Theorem 2 *and the coefficient $B(r)$ $(= B(-r) = B(\bar{r}))$ does not depend on α.*

Proof. This lemma is due to Weil [14], Chap. VIII. In fact, put formally $F_\alpha(g(u,v)) = \sum_{r\in\mathfrak{o}}\psi_\alpha(v,r)\exp(2\pi iS(\bar{r}\omega u))$, then each term satisfies the Beltrami operator's equation (E_0) in [14], p. 72 for $r = 0$ or (E) in p. 74 for $r\neq 0$. So we have $\psi_\alpha(v,0) = \delta_{\nu,\alpha}B(0)v$. For $r\neq 0$ it first follows that we can put $\psi_\nu(v,r) = B(r)\phi_\nu(v,r)$, and then we obtain $\psi_\alpha(v,r) = B(r)\phi_\alpha(v,r)$ recursively by using a formula $xK'_\alpha(x) + \alpha K_\alpha(x) = -xK_{\alpha-1}(x)$ (cf. [8], p. 67) and by noting a special role of the factor $\binom{2\nu}{\nu-\alpha}\xi^\alpha(r)$. It follows from (i.a) and (i.b) that $B(r) = B(-r) = B(\bar{r})$.

LEMMA 14. *For a non-negative integer ℓ, it holds that*

$$\left[\frac{\partial^\ell}{\partial\bar{u}^\ell}\theta_{\nu,\nu}(z, g(u, v))\right]_{u=0} = 2(-2\pi i)^\ell v^{-\ell} y^\ell \theta_{-D}^{(\nu+\ell)}(z)\theta_\ell(-z, v) . \tag{20}$$

Proof. We first note that $\eta_{\nu,\nu}(X^g) = (mu + r)^\nu$ does not depend on $\bar{u}$ for $X = \begin{pmatrix} m & r \\ \bar{r} & n \end{pmatrix} \in \mathfrak{L}$ and $g = g(u, v)$. Put $Y = \pi y R(X^g)$ which is a quadratic polynomial of $\bar{u}$, $Y_1 = \dfrac{\partial}{\partial\bar{u}}Y$ and $Y_2 = \dfrac{\partial^2}{\partial\bar{u}^2}Y$, so that it holds $(-1)^\ell \exp(Y)\dfrac{\partial^\ell}{\partial\bar{u}^\ell}\exp(-Y) = Y_2^{\ell/2}H_\ell(Y_1Y_2^{-1/2})$ with Hermite's polynomial H_ℓ. Hence

$$\left[\frac{\partial^\ell}{\partial\bar{u}^\ell}\exp(-Y)\right]_{u=0} = (-rv^{-1})^\ell(2\pi y)^{\ell/2}H_\ell(\sqrt{2\pi y}(mv + nv^{-1}))$$
$$\exp(-\pi y(2r\bar{r} + m^2v^2 + n^2v^{-2})) ,$$

which leads to the proof of (20).

3.5. *Proof of Theorem* 2. Since our function $F = I(f)$ satisfies the assumption of Lemma 13, we may write $F_\alpha(g(u, v))$ in the form of (19). A simple observation of this and (17) follows that $B(0) = C(0)$. Next we must prove that $B(r) = C(r)$ for any $r \in \mathfrak{o} - \{0\}$. Observing that $\xi^\alpha(\mathfrak{a})$, $B(\mathfrak{a})$ and $C(\mathfrak{a})$ are well defined for each ideal $\mathfrak{a} = (r)$ if α is even, it is sufficient for our purpose to show

$$\sum_{\mathfrak{a}} \xi^\alpha(\mathfrak{a})B(\mathfrak{a})N\mathfrak{a}^{-s} = \sum_{\mathfrak{a}} \xi^\alpha(\mathfrak{a})C(\mathfrak{a})N\mathfrak{a}^{-s} \tag{21}$$

for all even $\alpha \in \boldsymbol{Z}$ and $s \in \boldsymbol{C}$ with sufficiently large real part, where $\mathfrak{a}$ runs over all non-zero integral ideals of $\mathfrak{o}$. For $|\alpha| \leqq \nu$ we can get (21) by the Mellin transform

$$\int_0^\infty (F_\alpha(g(v)) - \delta_{\nu,|\alpha|}C(0)v)v^{2s-(\nu+2)}dv ,$$

which, in fact, is equal to

$$\frac{1}{2}\binom{2\nu}{\nu - \alpha}D^s(2\pi)^{-2s}\Gamma\left(s + \frac{\alpha}{2}\right)\Gamma\left(s - \frac{\alpha}{2}\right)\sum_{\mathfrak{a}} \xi^\alpha(\mathfrak{a})C(\mathfrak{a})N\mathfrak{a}^{-s}$$

if we use (17), or the same in which $C(\mathfrak{a})$ is replaced by $B(\mathfrak{a})$ if we use (19). For $|\alpha| > \nu$ we may assume $\alpha = \nu + \ell$ with positive even ℓ because $\xi^{-\alpha}(\mathfrak{a}) = \xi^\alpha(\bar{\mathfrak{a}})$, $B(\mathfrak{a}) = B(\bar{\mathfrak{a}})$ and $C(\mathfrak{a}) = C(\bar{\mathfrak{a}})$. On one hand, it follows from (19) that

$$\left[\frac{\partial^{\ell}}{\partial \bar{u}^{\ell}} F_{\nu}(g(u, v))\right]_{u=0} = (-2\pi)^{\ell} D^{-\ell/2} \sum_{r \in \mathfrak{o}-\{0\}} r^{\ell} B(r) \phi_{\nu}(v, r) .$$

On the other hand, this is equal to

$$\int_{\Gamma \backslash \mathfrak{H}} \left[\frac{\partial^{\ell}}{\partial \bar{u}^{\ell}} \theta_{\nu,\nu}(z, g(u, v))\right]_{u=0} \cdot \overline{f_T(z)} y^{\nu-1} dx dy ,$$

which becomes $(-2\pi)^{\ell} D^{-\ell/2} \sum_{r \in \mathfrak{o}-\{0\}} r^{\ell} C(r) \phi_{\nu}(v, r)$ through a similar computation to Lemma 12 by using Lemma 14. We can therefore complete the proof of (21) for $\alpha = \nu + \ell$ by Mellin transform again.

3.6. We give here some supplementary remarks on Dirichlet series and their Euler products. Let $F = I(f)$ be the lifted image of a cusp form f in $\mathscr{S}_{\nu+1}$, so that F has the Fourier expansion as in Theorem 2. Let us put

$$\Phi_{\alpha}(s) = D^{s}(2\pi)^{-2s} \Gamma\left(s + \frac{\alpha}{2}\right) \Gamma\left(s - \frac{\alpha}{2}\right) \sum_{\mathfrak{a}} \xi^{\alpha}(\mathfrak{a}) C(\mathfrak{a}) N\mathfrak{a}^{-s}$$

for even $\alpha \in \boldsymbol{Z}$ and $s \in \boldsymbol{C}$ with sufficiently large real part. Because ${}^t g^{-1} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} g \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ for $g \in G$, $F({}^t g^{-1}) = F(g) \rho_{2\nu}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = F(\bar{g})$ and so $F(g(v)) = F(g(v^{-1}))$. Hence a variant expression of the Mellin transform

$$\int_1^{\infty} (F_{\alpha}(g(v)) - \delta_{\nu,|\alpha|} C(0) v)(v^{2s-(\nu+1)} + v^{(\nu+1)-2s}) \frac{dv}{v} - \delta_{\nu,|\alpha|} C(0) \left(\frac{1}{2s-\nu} - \frac{1}{\nu+2-2s}\right)$$

gives the meromorphic continuation of $\Phi_{\alpha}(s)$ and the functional equation $\Phi_{\alpha}(s) = \Phi_{\alpha}(\nu + 1 - s)$ for $|\alpha| \leqq \nu$. In particular, $\Phi_{\alpha}(s)$ is entire for $|\alpha| < \nu$, while $\Phi_{\pm\nu}(s)$ is entire if and only if $f \in \mathscr{S}^1_{\nu+1}$. For $\alpha = \nu + \ell$ with positive even ℓ(It should be noticed $\Phi_{\alpha} = \Phi_{-\alpha}$ in our case.), by Rankin's method in the convolution of f and $\theta^{(\alpha)}_{-D}$ it is also possible to get the meromorphic continuation of $\Phi_{\alpha}(s)$ and the functional equation $\Phi_{\alpha}(s) = \Phi_{\alpha}(\nu + 1 - s)$, while we can say no more about the holomorphy except the fact $\left(s - \frac{\alpha}{2}\right)\left(s - \frac{\alpha}{2} + 1\right) \cdots \left(s - \frac{\alpha}{2} + (\ell - 1)\right) \Phi_{\alpha}(s)$ is entire.

When f in $\mathscr{S}_{\nu+1}$ is a normalized primitive form (i.e. a common eigen-function of all Hecke operators with $a(1) = 1$), it is well known that $f_T = -iD^{-\nu/2}a(D)f$ and so $b(n) = iD^{-\nu/2}\overline{a(Dn)}$. Hence we can easily show

$$\sum_{\mathfrak{a}} \xi^{\alpha}(\mathfrak{a})C(\mathfrak{a})N\mathfrak{a}^{-s} = \prod_{\mathfrak{p}} (1 - \xi^{\alpha}(\mathfrak{p})C(\mathfrak{p})N\mathfrak{p}^{-s} + \xi^{2\alpha}(\mathfrak{p})N\mathfrak{p}^{\nu-2s})^{-1}$$

for every even integer α. It should be also remarked that $C(\mathfrak{p}) = a(p)$, $a(p^2) + p^{\nu}$ or $a(p) + \overline{a(p)}$ according as $\left(\frac{-D}{p}\right) = 1, -1$ or 0 for each prime $\mathfrak{p}|p$. In particular, it holds

$$\sum_{\mathfrak{a}} C(\mathfrak{a})N\mathfrak{a}^{-s} = \sum_{n=1}^{\infty} a(n)n^{-s} \cdot \sum_{n=1}^{\infty} \overline{a(n)}n^{-s} .$$

§4. A characterization of $\theta_{-D}^{(\nu)}(z)$

4.1. As an application of the Doi-Naganuma lifting we give a proof of the following

THEOREM 4. *Let f be a normalized primitive form in $\mathscr{S}_{\nu+1}$, and assume that all the eigen-values for Hecke operators are real, then $f = \theta_{-D}^{(\nu)}$.*

Remark. Our proof will be based on the fact that the lifted image $I(f)$ is cuspidal if and only if f is orthogonal to $\theta_{-D}^{(\nu)}$, i.e. $f \in \mathscr{S}_{\nu+1}^1$. In contrast with this, the lifted image of a holomorphic cusp form is always a Hilbert modular cusp form in the real quadratic field case. By using this fact we may derive an analogous result as follows: There are no such primitive forms of Neben type $\left(\Gamma_0(\Delta), \left(\frac{\Delta}{*}\right)\right)$ as all the eigen-values for Hecke operators are real, where Δ is a discriminant of a real quadratic field. In fact, we have already treated the case that the class number is one and Δ is odd (cf. [2], especially Cor. to Prop. 5).

4.2. We quote a lemma on Rankin's convolution. For the proof and some other details we can refer to [2], p. 91 and [1], Th. 3, the latter of which, however, contains an obvious mistake in its statement.

Let $f_j(z) = \sum_{n=1}^{\infty} a_j(n) \exp(2\pi i n z)$ ($j = 1$ or 2) be a normalized primitive form in $\mathscr{S}_{\nu+1}$, so that the corresponding Dirichlet series $\phi_j(s) = \sum_{n=1}^{\infty} a_j(n)n^{-s}$ has the Euler product as follows:

$$\phi_j(s) = \prod_{p} \phi_{j,p}(s) ;$$

$$\phi_{j,p}(s)^{-1} = \begin{cases} (1 - \xi_j V)(1 - \eta_j V) & \text{if } p \neq D\,, \\ (1 - a_j(p)V) & \text{if } p = D\,, \end{cases}$$

where $V = p^{-s}$, and ξ_j, η_j are two roots of the equation $x^2 - a_j(p)x + \left(\frac{-D}{p}\right)p^\nu = 0$ for each rational prime p. The convolution of these is defined by

$$\psi(s; f_1, \bar{f}_2) = \prod_p \psi_p(s)\,;$$

$$\psi_p(s)^{-1} = \begin{cases} (1 - \xi_1\bar{\xi}_2 V)(1 - \xi_1\bar{\eta}_2 V)(1 - \eta_1\bar{\xi}_2 V)(1 - \eta_1\bar{\eta}_2 V) & \text{if } p \neq D\,, \\ (1 - a_1(p)\overline{a_2(p)}V)(1 - \overline{a_1(p)}a_2(p)V) & \text{if } p = D\,. \end{cases}$$

LEMMA 15. *$\psi(s; f_1, \bar{f}_2)$ can be meromorphically continued to the whole complex s-plane and satisfies a functional equation. It is entire if $f_1 \neq f_2$ and it has a simple pole at $s = \nu + 1$ if $f_1 = f_2$.*

4.3. *Proof of Theorem* 4. Suppose that $f(z) = \sum_{n=1}^{\infty} a(n) \exp(2\pi i n z)$ is the Fourier expansion and put $f_\rho(z) = \sum_{n=1}^{\infty} \overline{a(n)} \exp(2\pi i n z)$. We have to prove that $f \neq f_\rho$ if f is a normalized primitive form in $\mathscr{S}^1_{\nu+1}$. Owing to Lemma 15, it is sufficient to prove that $\psi(s; f, \bar{f}_\rho)$ is entire. Let $F = I(f)$ denote the lifted image whose Fourier expansion is, we may assume, given by (16). We consider a Dirichlet series associated with F defined by

$$H(s) = \zeta_{-D}(2s - 2\nu) \sum_{\mathfrak{a}} C(\mathfrak{a})^2 N\mathfrak{a}^{-s}\,,$$

where ζ_{-D} is the Dedekind zeta function of the imaginary quadratic field of discriminant $-D$. By comparing their Euler products, we can obtain

$$H(s) = \psi(s; f, \bar{f})\psi(s; f, \bar{f}_\rho)\,.$$

This is the same as Proposition 6 in [2], though only the real quadratic field case is treated there, hence we omit the detail here. Therefore we have only to prove:

LEMMA 16. *If $f \in \mathscr{S}^1_{\nu+1}$, then $H(s)$ can be meromorphically continued to the whole complex s-plane and is holomorphic except a simple pole at $s = \nu + 1$.*

Proof. Observing that $F(g)J^{-2}\,{}^t\overline{F(g)}$ is left Λ- and right K-invariant where J denote the same matrix as in the proof of Lemma 4, let us consider an integral

$$\Omega(s)=\int_{\Lambda\backslash G/K}F(g)J^{-2}\,{}^t\overline{F(g)}E^*(g,s-\nu)d\dot{g}\ ,$$

where the invariant volume $d\dot{g}$ is given by $v^{-3}dud\bar{u}dv$ for $g=g(u,v)\kappa$ ($\kappa\in K$). In the above we put $E^*(g,s)=D^s(2\pi)^{-2s}\Gamma(2s)\zeta_{-D}(2s)E(g,s)$ with $E(g,s)=\sum_{\gamma\in\Lambda_\infty\backslash\Lambda}v(\gamma g)^s$ ($\mathrm{Re}\,s>1$), where $v(g)=v$ for $g=g(u,v)\kappa$ and Λ_∞ is the subgroup consisting of all $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Lambda$ with $c=0$. As is well known, $E^*(g,s)$ can be meromorphically continued to the whole complex s-plane and is holomorphic except two simple poles at $s=0,1$. We first notice that $F(g)$ decreases rapidly when $v(g)\to\infty$ because $f\in\mathscr{S}^1_{\nu+1}$. By so-called Rankin's method we obtain, on one hand, that the integral $\Omega(s)$ is absolutely convergent for all $s\in\boldsymbol{C}$ except two simple poles at $s=\nu,\nu+1$, and on the other hand, that $\Omega(s)$ is a constant multiple of $D^{2s}(2\pi)^{-4s}\Gamma(s)^2\Gamma(s-\nu)^2H(s)$ when $\mathrm{Re}\,s$ is sufficiently large. In these computations we need a formula

$$\int_0^\infty K_\alpha(v)^2v^{2s-1}dv=2^{2s-3}\Gamma(s+\alpha)\Gamma(s-\alpha)\Gamma(s)^2\Gamma(2s)^{-1}$$

for $\mathrm{Re}\,s>2|\alpha|$ (e.g. [5], p. 334 (45)), and an elementary identity

$$\sum_{\alpha=-\nu}^{\nu}\binom{2\nu}{\nu-\alpha}\Gamma(s+\alpha)\Gamma(s-\alpha)\Gamma(2s)^{-1}=\Gamma(s-\nu)^2\Gamma(2s-2\nu)^{-1}\ .$$

Thus we complete the proof of Lemma 16, and so, that of Theorem 4.

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