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REMARKS TO THE UNIQUENESS PROBLEM OF MEROMORPHIC MAPS INTO $P^N(C)$, I

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§ 1. Introduction

As generalizations of the results in [5] and [4], the author gave some uniqueness theorems of meromorphic maps into $P^N(C)$ in previous papers [2] and [3]. He studied two meromorphic maps f and g of C^n into $P^N(C)$ such that $\nu(f, H_i) = \nu(g, H_i)$ for q hyperplanes H_i located in general position in $P^N(C)$, where $\nu(f, H_i)$ and $\nu(g, H_i)$ denote the pull-backs of divisors (H_i) on $P^N(C)$ by f and g respectively. In [2], he showed that, if $q \geqq 3N + 2$ and either f or g is non-degenerate, then $f \equiv g$. And, in [3] (p. 140), he gave the following

THEOREM. *If $q \geqq 2N + 3$ and either f or g is algebraically non-degenerate, i.e., the image is not included in any proper subvariety of $P^N(C)$, then $f \equiv g$.*

Unfortunately, a gap was found in the proof of Lemma 6.5 in [3] which is essentially used to prove the above theorem.

The purposes of this paper are to give a complete proof of the above theorem and, simultaneously, to give some remarks to the uniqueness problem of meromorphic maps of C^n into $P^N(C)$. Theorem 6.9 in [3] will be improved and the results in the last section of [3] will be generalized to the higher dimensional case.

§ 2. Main results

We recall some notations and terminologies given in [3]. Let f be a meromorphic map of C^n into the N-dimensional complex projective space $P^N(C)$ and H a hyperplane in $P^N(C)$ such that $f(C^n) \not\subset H$. For an arbitrarily fixed homogeneous coordinates $w_1 : w_2 : \cdots : w_{N+1}$ on $P^N(C)$, we can take a representation $f = f_1 : f_2 : \cdots : f_{N+1}$ with holomorphic func-

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tions $f_1, f_2, \cdots, f_{N+1}$ on C^n satisfying the condition

$$\operatorname{codim}\{z \in C^n; f_1(z) = f_2(z) = \cdots = f_{N+1}(z) = 0\} \geqq 2 ,$$

which we call an admissible representation of f. Let H be given as

$$H: a^1 w_1 + a^2 w_2 + \cdots + a^{N+1} w_{N+1} = 0$$

and define a holomorphic function

$$F_f^H := a^1 f_1 + a^2 f_2 + \cdots + a^{N+1} f_{N+1} . \tag{2.1}$$

For each point z in C^n, we denote by $\nu(f, H)(z)$ the zero multiplicity of F_f^H at z. The integer-valued function $\nu(f, H)$ may be considered to be the pull-back of the divisor (H) by f.

Let us consider two meromorphic maps f, g of C^n into $P^N(C)$ and assume that there are $2N+2$ hyperplanes H_i $(1 \leqq i \leqq 2N+2)$ located in general position in $P^N(C)$ such that $f(C^n) \not\subset H_i$, $g(C^n) \not\subset H_i$ and $\nu(f, H_i) = \nu(g, H_i)$ for any i. Then,

$$h_i := F_f^{H_i} / F_g^{H_i} \qquad (1 \leqq i \leqq 2N+2) \tag{2.2}$$

are nowhere zero holomorphic functions on C^n and the ratios h_i/h_j $(1 \leqq i, j \leqq 2N+2)$ are uniquely determined independently of any choices of homogeneous coordinates and admissible representations of f and g.

In this situation, we shall prove

THEOREM I. *If either f or g is algebraically non-degenerate, then after a suitable change of indices i of H_i the functions h_i are represented as one of the following two types;*

(α) $h_1 : h_2 : \cdots h_{2N+2}$

$$= \eta_1 : \eta_1^{-1} : \eta_2 : \eta_2^{-1} : \cdots : \eta_N : \eta_N^{-1} : 1 : (-1)^N$$

(β) $N+1$ *is prime and*

$h_1 : h_2 : \cdots : h_{2N+2}$

$$= \eta_1 : \eta_2 : \cdots : \eta_N : (\eta_1 \eta_2 \cdots \eta_N)^{-1} : 1 : \zeta : \cdots : \zeta^N ,$$

where $\eta_1, \eta_2, \cdots, \eta_N$ are algebraically independent nowhere zero holomorphic functions on C^n and ζ denotes a primitive $(N+1)$-th root of unity.

This is an improvement of Proposition 6.3 in [3], which is proved without using Lemma 6.5 in it. Thus, we can prove the theorem stated in § 1 correctly by the same argument as in [3], p. 141.

We shall give also the following theorem, which is an improvement of Theorem 6.9 in [3].

THEOREM II. *If f or g is algebraically non-degenerate, then they are reduced by a suitable change of indices to one of the following two cases;*

$(\alpha)'$ *there are relations between f and g such that*

$$F_f^{H_{2i-1}} F_f^{H_{2i}} = F_g^{H_{2i-1}} F_g^{H_{2i}} \qquad 1 \leqq i \leqq N$$

$$F_f^{H_{2N+1}} = F_g^{H_{2N+1}} , \qquad F_f^{H_{2N+2}} = (-1)^N F_g^{H_{2N+2}} ,$$

$(\beta)'$ *$N+1$ is prime and f and g are related as $L \cdot g = f$ with a projective linear transformation $L: P^N(C) \to P^N(C)$ which fixes hyperplanes $H_1, H_2, \cdots, H_{N+1}$ and maps $H_{N+2}, H_{N+3}, \cdots, H_{2N+2}$ onto $H_{2N+2}, H_{N+2}, \cdots, H_{2N+1}$ respectively.*

These theorems will be proved in § 5 completely after giving some preparations in § 3 and § 4.

§ 3. Some known results

Let f, g and H_i $(1 \leqq i \leqq 2N+2)$ satisfy the conditions stated in the previous section and assume that g is algebraically non-degenerate.

As in [3], we consider the multiplicative group H^* of all nowhere zero holomorphic functions on C^n and the factor group $G := H^*/C^*$, where $C^* := C - \{0\}$. For an element $h \in H^*$, we denote by $[h]$ the class in G containing h and, for the functions $h_1, \cdots, h_{2N+2}$ defined as (2.2), by $t([h_1], \cdots, [h_{2N+2}])$ the rank of the subgroup of G generated by $[h_1], \cdots, [h_{2N+2}]$. We shall restate here Proposition 6.3 in [3] revised as follows.

PROPOSITION 3.1. *There exist elements $\beta_1, \cdots, \beta_t$ in H^*/C^* such that, after a suitable change of indices,*

$$\begin{aligned} (3.2) \quad & [h_1] : [h_2] : \cdots : [h_{2N+2}] \\ & = \beta_1 : \beta_2 : \cdots : \beta_t : (\beta_1 \cdots \beta_{a_1})^{-1} : \cdots : (\beta_{a_{k-1}+1} \cdots \beta_{a_k})^{-1} : 1 : 1 : \cdots : 1 , \end{aligned}$$

where $t = t([h_1], \cdots, [h_{2N+2}])$, 1 appears $2N - k - t + 2$ times repeatedly and $a_k - a_{k-1} \leqq t - k + 1$ (let $a_0 = 0$).

For the proof, see [3], pp. 138–140. In that place, Lemma 6.5 in

[3] whose proof contains a gap is used only to prove the assertion $a_k = t$ in Proposition 6.3 in [3] which is missed in the above Proposition 3.1.

We shall recall here another result in [3]. To state it, we choose $2s$ $(1 \leqq s \leqq N+1)$ hyperplanes among $H_1, H_2, \cdots, H_{2N+2}$ arbitrarily and change indices so that they are $H_1, \cdots, H_s, H_{N+2}, \cdots, H_{N+s+1}$. We can take homogeneous coordinates $w_1 : w_2 : \cdots : w_{N+1}$ such that

$$(3.3) \qquad \begin{aligned} &H_i : w_i = 0 && 1 \leqq i \leqq N+1 \\ &H_{N+j+1} : a_j^1 w_1 + \cdots + a_j^{N+1} w_{N+1} = 0 && 1 \leqq j \leqq N+1 , \end{aligned}$$

where (a_j^i) is a square matrix of order $N+1$ whose minors do not vanish.

PROPOSITION 3.4. *If* $s > t := t([h_1], \cdots, [h_{2N+2}])$, *then*

$$\det (a_j^i(h_i - h_{N+j+1}) ; 1 \leqq i, j \leqq s) \equiv 0 .$$

Proof. This is essentially the same as Corollary 5.4 in [3] and proved by the same argument as in its proof. In fact, if

$$\det (a_j^i(h_i - h_{N+j+1}) ; 1 \leqq i, j \leqq s) \not\equiv 0 ,$$

we have obviously

$$\det (a_j^i(H_i(u) - H_{N+j+1}(u)) ; 1 \leqq i, j \leqq s) \not\equiv 0 ,$$

where $H_i(u)$ are rational functions of $u = (u_1, \cdots, u_t)$ defined as

$$H_i(u) = c_i u_1^{\ell_{i1}} u_2^{\ell_{i2}} \cdots u_t^{\ell_{it}} u_{t+1}^{\ell_{it+1}} \qquad 1 \leqq i \leqq 2N+2$$

when h_i has representations

$$h_i = c_i \eta_1^{\ell_{i1}} \eta_2^{\ell_{i2}} \cdots \eta_t^{\ell_{it}}$$

with algebraically independent $\eta_1, \cdots, \eta_t \in H^*$ and $\ell_{it+1} = -(\ell_{i1} + \ell_{i2} + \cdots + \ell_{it})$. Let $V_{f,g}$ be the smallest algebraic set in $P^N(C) \times P^N(C)$ which includes the set $(f \times g)(C^n)$. This implies that

$$\dim V_{f,g} \leqq N - s + t < N$$

as in the proof of Theorem 5.3 in [3]. On the other hand, $V_{f,g}$ is of dimension N by (6.2) in [3]. This is a contradiction and gives Proposition 3.4.

Now, we change indices i of H_i $(1 \leqq i \leqq 2N+2)$ so that (3.2) is rewritten as

$$
\begin{aligned}
&[h_1]:[h_2]:\cdots:[h_{N+1}]:[h_{N+2}]:\cdots:[h_{2N+2}] \\
(3.5)\qquad &= (\beta_1\cdots\beta_{a_1})^{-1}:\cdots:(\beta_{a_{k-1}+1}\cdots\beta_{a_k})^{-1}: \\
&\underbrace{1:\cdots:1}_{N+1-k\ \text{times}}:\beta_1:\cdots:\beta_t:\underbrace{1:\cdots:1}_{N+1-t\ \text{times}}.
\end{aligned}
$$

And, we choose homogeneous coordinates $w_1:w_2:\cdots:w_{N+1}$ so that H_i's with this arrangement are represented as in (3.3). We put anew $\eta_i:=h_i$ for each i $(1\leqq i\leqq t)$. By a suitable choice of an admissible representation of f, we may assume $h_{N+t+2}\equiv 1$. For convenience' sake, we put $\eta_{t+1}=h_{N+t+2}$ $(\equiv 1)$. The relation (3.5) can be written as

$$
\begin{aligned}
&h_i = x_i(\eta_{a_{i-1}+1}\cdots\eta_{a_i})^{-1} \qquad 1\leqq i\leqq k \\
(3.6)\quad &h_i = x_i \qquad k+1\leqq i\leqq N+1 \quad \text{or} \quad N+t+3\leqq i\leqq 2N+2 \\
&h_{N+1+j} = \eta_j \qquad 1\leqq j\leqq t+1\,,
\end{aligned}
$$

where x_i are some constants. Then, by Proposition 3.4,

$$
(3.7)\qquad \det(a^i_j(\tilde{\eta}_i\eta_j - x_i)\,;\,1\leqq i,j\leqq t+1)\equiv 0\,,
$$

where $\tilde{\eta}_i=\eta_{a_{i-1}+1}\cdots\eta_{a_i}$ $(1\leqq i\leqq k)$ and $\tilde{\eta}_i\equiv 1$ $(k+1\leqq i\leqq t+1)$. Since $\eta_1,\cdots,\eta_t$ are algebraically independent, i.e., have no non-trivial algebraic relation by (2.9) in [3], this is regarded as an identity of polynomials with indeterminates $\eta_1,\cdots,\eta_t$.

§ 4. An algebraic lemma

For the proof of Theorems I and II, we have to investigate the relation (3.7) more precisely. We shall give the following.

LEMMA 4.1. *Let (a^i_j) be a square matrix of order $t+1$ whose minors do not vanish and* (3.7) *holds as an identity of polynomials with indeterminates $\eta_1,\cdots,\eta_t$ and η_{t+1}. Then, after a suitable change of indices, one of the following two cases occurs;*

$(\alpha)''$ $k=t,\ a_\kappa - a_{\kappa-1}=1$ *for any* κ $(1\leqq\kappa\leqq k)$ *and* $x_1=x_2=\cdots=x_t=1,\ x_{t+1}=(-1)^t$.

$(\beta)''$ $k=1,\ a_1=t$ *and* $x_1=1,\ x_2=\zeta,\ x_3=\zeta^2,\cdots,x_{t+1}=\zeta^t$,

where ζ denotes a primitive $(t+1)$-th root of unity.

Proof. Changing indices if necessary, we may assume

$$x_1 = x_2 = \cdots = x_\ell = 1\,, \qquad x_{\ell+1} \neq 1, \cdots, x_k \neq 1$$
$$x_{k+1} = x_{k+2} = \cdots = x_{k+m} = 1\,, \qquad x_{k+m+1} \neq 1, \cdots, x_{t+1} \neq 1\,,$$

where $0 \leqq \ell \leqq k$ and $0 \leqq m \leqq t - k + 1$. We divide the proof of Lemma 4.1 into several steps.

1°) $\ell \geqq m + 1$.

We note first $m \leqq t - 1$. In fact, if $m = t$, we have easily an absurd identity

$$a^1_{t+1} \det (a^i_j\,; 1 \leqq i, j \leqq t)(\tilde{\eta}_1 - x_1)(\eta_1 - 1)(\eta_2 - 1) \cdots (\eta_t - 1) \equiv 0\,.$$

Assume that $\ell \leqq m$. Then, we can choose $t - m$ η_τ's, say, $\eta_{\tau_1}, \eta_{\tau_2}, \cdots, \eta_{\tau_{t-m}}$, in the set $\{\eta_1, \cdots, \eta_t\} - \{\eta_{a_1}, \eta_{a_2}, \cdots, \eta_{a_\ell}\}$. Substitute $\eta_{\tau_1} = \eta_{\tau_2} = \cdots = \eta_{\tau_{t-m}} = 1$ in (3.7). We see $\tilde{\eta}_i\eta_j - x_i = 0$ when and only when $i = k + 1, \cdots, k + m$ and $j = \tau_1, \tau_2, \cdots, \tau_{t-m}, t + 1$. So, (3.7) is in this case reduced to

$$\det \begin{pmatrix} a^i_j\,; \begin{matrix} i \neq k+1, \cdots, k+m \\ j = \tau_1, \cdots, \tau_{t-m}, t+1 \end{matrix} \end{pmatrix} \det \begin{pmatrix} a^i_j\,; \begin{matrix} i = k+1, \cdots, k+m \\ j \neq \tau_1, \cdots, \tau_{t-m}, t+1 \end{matrix} \end{pmatrix}$$
$$\times \prod_{i \neq k+1, \cdots, k+m} (\eta^*_i - x_i) \times \prod_{j \neq \tau_1, \cdots, \tau_{t-m}, t+1} (\eta_j - 1) \equiv 0\,,$$

where η^*_i ($\not\equiv x_i$) are quantities obtained from $\tilde{\eta}_i$ by substitutions of $\eta_{\tau_1} = \eta_{\tau_2} = \cdots = \eta_{\tau_{t-m}} = 1$. This is a contradiction. We conclude $\ell \geqq m + 1$.

2°) Put $r := [(\ell - m + 1)/2]\ (\geqq 1)$, where $[a]$ denotes the largest integer not larger than a real number a. And, assume

$$\alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_\ell$$

for $\alpha_\kappa := a_\kappa - a_{\kappa-1}$ $(1 \leqq \kappa \leqq \ell)$ by a suitable change of indices, where we put $a_0 = 0$. We have then one of the followings;

(i) $a_r + m + r \leqq t$,

(ii) $\ell = t$,

(iii) $m = 0$ and $r = 1$.

To see this, we assume $a_r + m + r > t$. Then, for any chosen $i_1, i_2, \cdots, i_r$ $(1 \leqq i_1 < i_2 < \cdots < i_r \leqq \ell)$,

$$\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_r} \geqq t - m - r + 1\,.$$

Therefore,

$$\sum_{1\leq i_1<\cdots<i_r\leq \ell}(\alpha_{i_1}+\cdots+\alpha_{i_r})=(\alpha_1+\cdots+\alpha_\ell)\frac{(\ell-1)!}{(r-1)!\,(\ell-r)!}$$
$$\geqq (t-m-r+1)\frac{\ell!}{r!\,(\ell-r)!}$$

and so

$$rt\geqq ra_\ell = r(\alpha_1+\cdots+\alpha_\ell)\geqq \ell(t-m-r+1)\ .$$

Since $\ell-m+1\geqq 2r$ in any case, we have

$$\ell(m+r-1)\geqq t(\ell-r)\geqq t(m+r-1)\ .$$

If $m+r-1>0$, then $t\leqq \ell$ and so the case (ii) occurs. If $m+r-1=0$, then we have the case (iii).

3°) The case (i) of 2°) is impossible.

In fact, if it occurs, we can choose distinct indices $\sigma_1,\cdots,\sigma_{t-(m+r)}$ such that $\{1,2,\cdots,a_r\}\subset\{\sigma_1,\sigma_2,\cdots,\sigma_{t-(m+r)}\}$ and $\{\sigma_1,\sigma_2,\cdots,\sigma_{t-(m+r)}\}\cap\{a_{r+1},a_{r+2},\cdots,a_\ell\}=\varnothing$, because

$$a_r\leqq t-m-r\leqq t-(\ell-r)\ .$$

Substitute $\eta_{\sigma_1}=\cdots=\eta_{\sigma_{t-(m+r)}}=1$ in (3.7). Then, $\tilde{\eta}_i\eta_j-x_i=0$ when and only when $i=1,2,\cdots,r,k+1,\cdots,k+m$ and $j=\sigma_1,\sigma_2,\cdots,\sigma_{t-(m+r)}$, $t+1$. And, as is easily seen, the relation (3.7) contradicts the assumption that any minor of (a_j^i) does not vanish. Therefore the case (i) of 2°) does not occur.

4°) The case (ii) of 2°) is reduced to the case $(\alpha)''$ of Lemma 4.1.

Let $\ell=t$. We see easily $k=t$ and $a_\kappa-a_{\kappa-1}=1\ (1\leqq\kappa\leqq t)$. The identity (3.7) can be rewritten as

$$\det(a_i^1(\eta_i\eta_1-1),\cdots,a_i^t(\eta_i\eta_t-1),a_i^{t+1}(\eta_i-x_{t+1});1\leqq i\leqq t+1)\equiv 0\ ,$$

where $\eta_{t+1}=1$. Put $s=[(t+1)/2]$. And, substitute $\eta_1=\eta_2=\cdots=\eta_s=(-1)^t$. We can conclude easily $x_{t+1}=(-1)^t$. This gives the case $(\alpha)''$.

5°) The case (iii) of 2°) is reduced to the case $(\beta)''$.

Assume that $m=0$ and $r=1$. If $a_1\leqq t-1$, we substitute $\eta_1=\eta_2=\cdots=\eta_{t-\ell}=1$ in (3.7), where $\ell=1$ or $=2$. This leads to a contradiction. Let $a_1=t$. We have then $\ell=1$ and

$$\det(a_i^1(\eta_1 \cdots \eta_t)\eta_i - 1), a_i^2(\eta_i - x_2), \cdots, a_i^{t+1}(\eta_i - x_{t+1}); 1 \leqq i \leqq t+1) \equiv 0 .$$

For each $u(1 \leqq u \leqq t)$, substitute $\eta_1 = \eta_2 = \cdots = \eta_t = \zeta^u$, where ζ denotes a primitive $(t+1)$-th root of unity. Since $\zeta^{st} \neq 1$ for any $s\,(1 \leqq s \leqq t)$, some $x_i\,(2 \leqq i \leqq t+1)$ is equal to ζ^u. By a suitable change of indices, we have

$$x_2 = \zeta\ , \quad x_3 = \zeta^2, \cdots, x_{t+1} = \zeta^t\ ,$$

because $\zeta, \zeta^2, \cdots, \zeta^t$ are mutually distinct. This is the case $(\beta)''$ of Lemma 4.1. Lemma 4.1 is completely proved.

§ 5. Proofs of Theorems I and II

We shall prove first Theorem I. By the results in § 3 and Lemma 4.1, we may put

$$(5.1) \qquad \begin{aligned}&(h_1, h_2, \cdots, h_{2N+2}) \\ &\quad = (\eta_1, \cdots, \eta_t, \eta_1^{-1}, \cdots, \eta_t^{-1}, 1, (-1)^t, c_{2t+3}, \cdots, c_{2N+2})\end{aligned}$$

or

$$(5.2) \qquad \begin{aligned}&(h_1, h_2, \cdots, h_{2N+2}) \\ &\quad = (\eta_1, \cdots, \eta_t, (\eta_1 \cdots \eta_t)^{-1}, 1, \zeta, \cdots, \zeta^t, c_{2t+3}, \cdots, c_{2N+2})\end{aligned}$$

after a suitable change of indices, where $t = t([h_1], \cdots, [h_{2N+2}])$ and c_i are some constants. In this place, we shall show $t = N$. Since Proposition 3.4 remains valid even if the indices of H_i's are changed in any given order, it is easily seen that any chosen $2t+2$ elements $h_{i_1}, h_{i_2}, \cdots, h_{i_{2t+2}}$ among $h_1, h_2, \cdots, h_{2N+2}$ ought to be of the type similar to $h_1, h_2, \cdots, h_{2t+2}$ in (5.1) or (5.2) up to changes of the order and multiplication of a common factor. If $t < N$, for example, $h_2, h_3, \cdots, h_{2t+3}$ cannot be of such types, because there exist three distinct indices i, j, k among $2, 3, \cdots, 2t+3$ (let $i = 2t+1, j = 2t+2, k = 2t+3$) such that h_i/h_j and h_i/h_k are both constants, but not for $h_1, \cdots, h_{2t+2}$ in (5.1) and (5.2). This concludes $t = N$.

To complete the proof of Theorem I, we have only to prove that $N+1$ is prime for the case $t = N$ of $(\beta)''$ of Lemma 4.1. For convenience' sake, we change again indices of H_i so that

$$(h_1, h_2, \cdots, h_{2N+2}) = (\zeta, \zeta^2, \cdots, \zeta^N, 1, \eta_1, \cdots, \eta_N, (\eta_1 \cdots \eta_N)^{-1})$$

and let H_i's with these labels be given as (3.3), where ζ denotes a primitive $(N+1)$-th root of unity. For admissible representations $f = f_1 : f_2 : \cdots : f_{N+1}$ and $g = g_1 : g_2 : \cdots : g_{N+1}$, we have

$$f_i = \zeta^i g_i \qquad 1 \leqq i \leqq N+1 \tag{5.3}$$

and

$$\textstyle\sum_{i=1}^{N+1} a_j^i f_i = \eta_j(\sum_{i=1}^{N+1} a_j^i g_i) \qquad 1 \leqq j \leqq N \tag{5.4}$$

$$\textstyle\sum_{i=1}^{N+1} a_{N+1}^i f_i = (\eta_1 \eta_2 \cdots \eta_N)^{-1}(\sum_{i=1}^{N+1} a_{N+1}^i g_i) .$$

Substitute (5.3) into (5.4) and multiply all relations in (5.4) together. We get a relation

$$\textstyle\prod_{j=1}^{N+1} (\sum_{i=1}^{N+1} a_j^i \zeta^i g_i) = \prod_{j=1}^{N+1} (\sum_{i=1}^{N+1} a_j^i g_i) . \tag{5.5}$$

Since g is algebraically non-degenerate by the assumption, this is regarded as an identity of polynomials with indeterminates $g_1, g_2, \cdots, g_{N+1}$. By the unique factorization theorem for polynomials, each factor in one side of (5.5) coincides with a factor in the other up to a constant multiplication. We may assume here $a_j^i = 1$ if $i = N+1$ or $j = N+1$. Under this condition, we can conclude easily $a_j^i = \zeta^{ij}$ $(1 \leqq i, j \leqq N+1)$ by a suitable change of indices. If $N+1$ is not prime and so $N+1 = k\ell$ for some k, ℓ $(1 \leqq k \leqq \ell \leqq N)$, then

$$\begin{vmatrix} a_\ell^k & a_\ell^{N+1} \\ a_{N+1}^k & a_{N+1}^{N+1} \end{vmatrix} = 0 ,$$

which contradicts the assumption that any minor of (a_j^i) does not vanish. Therefore, $N+1$ is prime.

We shall prove next Theorem II. We know that the case (α) or (β) of Theorem I occurs. It is obvious that the case (α) implies the case $(\alpha)'$ of Theorem II. Assume that the case (β) occurs. We choose homogeneous coordinates satisfying the above conditions. Meromorphic maps f and g are related as (5.3) and (5.4). The relation (5.3) is rewritten as $L \cdot g = f$ if we take a projective linear transformation

$$L : w_i' = \zeta^i w_i \qquad 1 \leqq i \leqq N+1 .$$

We have shown in the above that $a_j^i = \zeta^{ij}$. It follows that L fixes $H_1, \cdots, H_{N+1}$ and maps $H_{N+2}, H_{N+3}, \cdots, H_{2N+2}$ onto $H_{2N+2}, H_{N+2}, \cdots, H_{2N+1}$ respectively. Thus, Theorem II is completely proved.

§ 6. An additional remark

In the previous paper [3], pp. 141–142, the author gave an example of mutually distinct algebraically non-degenerate meromorphic maps f and g of C^n into $P^N(C)$ such that $\nu(f, H_i) = \nu(g, H_i)$ for $2N + 2$ hyperplanes H_i in general position. This is a special case of $(\alpha)'$ of Theorem II and the case that f and g are related as $L \cdot g = f$ with a projective linear transformation $L: P^N(C) \to P^N(C)$ which maps $H_1, H_2, \cdots, N_N$ onto $H_{N+2}, H_{N+3}, \cdots, H_{2N+1}$ respectively and fixes H_{N+1} and H_{2N+2} after a suitable change of indices. As is shown in [3], we have always a relation of this type between f and g for the case $N = 1$ or $= 2$ of $(\alpha)'$ of Theorem II, but not for the case $N \geqq 3$. We shall remark here the following fact, which implies that the case $(\beta)'$ occurs actually.

PROPOSITION 6.1. *Let $A = (\zeta^{ij}; 1 \leqq i, j \leqq N + 1)$, where ζ denotes a primitive $(N + 1)$-th root of unity. If $N + 1$ is prime, then any minor of A does not vanish.*

For the proof, we give

LEMMA 6.2. *Let $F(x)$ be a polynomial with integral coefficients. If $F(\zeta) = 0$, then $F(1) \equiv 0 \pmod{N + 1}$.*

Proof. We can find easily a polynomial $g(x)$ with integral coefficients such that

$$F(x) = (1 + x + x^2 + \cdots + x^N)g(x) .$$

Therefore,

$$F(1) = (N + 1)g(1) \equiv 0 \qquad (\operatorname{mod} N + 1) .$$

LEMMA 6.3. *Let $f_1(x), \cdots, f_r(x)$ be polynomials and define a polynomial $\Psi(\zeta_1, \cdots, \zeta_r)$ with indeterminates $\zeta_1, \cdots, \zeta_r$ so that it satisfies the condition*

$$\det(f_j(\zeta_i); 1 \leqq i, j \leqq r) = \Psi(\zeta_1, \cdots, \zeta_r) \prod_{i>j} (\zeta_i - \zeta_j) .$$

Then,

$$\Psi(1, 1, \cdots, 1) = \det\left(\frac{f_i^{(j-1)}(1)}{(j-1)!}; 1 \leqq i, j \leqq r\right), \tag{6.4}$$

where $f_i^{(j-1)}$ denotes the $(j - 1)$-th derivative of f_i.

Proof. We expand each $f_i(x)$ as

$$f_i(x) = \sum_{\nu} \alpha_\nu^i (x-1)^\nu$$

with constants α_ν^i and put

$$g_{j,i}(x) = \sum_{\nu \geqq j-1} \alpha_\nu^i (x-1)^{\nu-j+1}$$

Then,

$$g_{j,i}(x) - g_{j,i}(1) = (x-1) g_{j+1,i}(x) .$$

As is easily seen by the induction on k, it holds that

$$\begin{aligned} &\Psi(1, \cdots, 1, \zeta_{k+1}, \cdots, \zeta_r) \prod_{k<j<i\leqq r} (\zeta_i - \zeta_j) \\ &\quad = \det (g_{1,i}(1), \cdots, g_{k,i}(1), g_{k+1,i}(\zeta_{k+1}), \cdots, g_{k+1,i}(\zeta_r) ; 1 \leqq i \leqq r) . \end{aligned}$$

For the case $k = r$, we get (6.4) because

$$g_{j,i}(1) = f_i^{(j-1)}(1)/(j-1)! .$$

Proof of Proposition 6.1. Obviously, a minor of A of degree $N+1$ does not equal to zero. Take a minor

$$\Delta = \det (\zeta^{k_i \ell_j} : 1 \leqq i, j \leqq r)$$

of A arbitrarily, where $1 \leqq k_1 < \cdots < k_r \leqq N+1$ and $1 \leqq \ell_1 < \ell_2 \cdots < \ell_r \leqq N+1$ $(1 \leqq r \leqq N)$. Apply Lemma 6.3 to the polynomials $f_1(x) = x^{\ell_1}, \cdots, f_r(x) = x^{\ell_r}$. For the polynomial $\Psi(\zeta_1, \cdots, \zeta_r)$ as in Lemma 6.3, putting $\zeta_1 = \zeta^{k_1}, \cdots, \zeta_r = \zeta^{k_r}$, we see

$$\Delta = \prod_{i>j} (\zeta^{k_i} - \zeta^{k_j}) \Psi(\zeta^{k_1}, \zeta^{k_2}, \cdots, \zeta^{k_r}) .$$

Let $g(x) = \Psi(x^{k_1}, x^{k_2}, \cdots, x^{k_r})$. This is a polynomial with integral coefficients. If $\Delta = 0$, then $g(\zeta) = 0$. By Lemma 6.2,

$$g(1) \equiv 0 \qquad (\operatorname{mod} N+1) .$$

Therefore, according to Lemma 6.3, we can conclude an absurd identity

$$\det \left(\frac{\ell_i(\ell_i - 1) \cdots (\ell_i - j + 1)}{(j-1)!} ; 1 \leqq i, j \leqq r \right)$$

$$
\begin{aligned}
&= \frac{1}{1!\,2!\cdots(r-1)!}\begin{vmatrix} 1 & 1 & \cdots 1 \\ \ell_1 & \ell_2 & \cdots \ell_r \\ & \cdots & \\ \ell_1^{r-1} & \ell_2^{r-1} & \cdots \ell_r^{r-1} \end{vmatrix} \\
&\equiv 0 \pmod{N+1}\,.
\end{aligned}
$$

Thus, $\Delta \neq 0$. Proposition 6.1 is completely proved.

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