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THE GROWTH OF THE BERGMAN KERNEL ON PSEUDOCONVEX DOMAINS OF HOMOGENEOUS FINITE DIAGONAL TYPE

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Introduction

In this article we continue the investigations on invariant metrics on a certain class of weakly pseudoconvex domains which we began in [H 1]. While in that paper the differential metrics of Caratheodory and Kobayashi were estimated precisely, the present paper contains a sharp estimate of the singularity of the Bergman kernel and metric on domains belonging to that class.

The boundary behavior of the Bergman kernel K_D and the Bergman metric B_D^2 of a smooth bounded pseudoconvex domain $D \subset \mathbf{C}^n$ is completely understood near the strictly pseudoconvex boundary points ([Di 1], [Di 2], [F], and [B-S]). Contrary to the strictly pseudoconvex case, much less is known about the growth of the Bergman kernel and metric for weakly pseudoconvex domains. Results for the Bergman kernel were obtained by Bonami-Lohué in [B-L] on the "ellipsoids"

$$\mathscr{E}_{\alpha_1,\dots,\alpha_n} = \{z \in C^n \mid \sum_{i=1}^{n} |z_i|^{2\alpha_i} < 1\},\ \alpha_i > 1, \text{ for } 1 \leq i \leq n,$$

and by the author in [H 2] on certain generalizations of these ellipsoids, as well as by Ohsawa in [Oh] for smooth bounded pseudoconvex domains, and by Diederich-Herbort-Ohsawa in [D-H-O] on uniformly extendable pseudoconvex domains. The Bergman metric was estimated in [D-F-H] and recently in [N 2] on pseudoconvex domains with a subelliptic $\bar\partial$-Neumann operator. In [Ca] Catlin treated completely the case of two-dimensional pseudoconvex domains of finite type. McNeal extended and generalized Catlin's result to the situation of boundary points of pseudoconvex domains of finite type in C^n, in case that the defining function is decoupled, [N 1].

We will in this article, using the method of interior domains of comparison, study the boundary behavior of the Bergman kernel within the class of pseudocon-

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vex domains of homogeneous finite diagonal type (this notion will be explained at the beginning of Section 1 below). This class contains also domains which do not admit holomorphic supporting functions, in particular, it contains the example of Kohn-Nirenberg, [K-N]. We will also discuss the order of growth for the Bergman kernel of domains which can be mapped by polynomial mappings of the type

$$z \longrightarrow (z_1, z'^{\alpha^{(2)}}, \cdots, z'^{\alpha^{(d)}}),$$

for $z = (z_1, z')$, and $z' = (z_2, \cdots, z_n)$, and nonzero multiindices $\alpha^{(i)} \in \mathbf{N}_0{}^{n-1}$, onto domains of homogeneous finite diagonal type; the result is given in Theorem 1 and Corollary 3 in Section 1 below. In particular Theorem (5.3.1) in [H 2] will be generalized.

The plan of this paper is as follows: In Section 1 we state all the results, in Section 2 we recall some important geometric tools developed in [H 1] and construct the relevant interior domains of comparison for the Bergman kernel. Sections 3 and 4 contain the proof of Lemma 2 of Section 1 and Theorem 1. Finally, in Section 5 we will prove a lemma from convex geometry. Since the author is not an expert in linear optimization he cannot exclude that this lemma can already be found in the literature; in any case we give the proof for reader's convenience.

Notational conventions

By $\Delta_k(a, r)$ we always mean the polydisc in $\mathbf{C}^k$ around $a \in \mathbf{C}^k$ with radius r. Further, $d\lambda_k$ is to denote the Lebesgue measure in $\mathbf{C}^k$. For a domain $D \subset \mathbf{C}^n$ we let $H^2(D) = \{f \mid f$ holomorphic and square-integrable with respect to $d\lambda_n$ on all of $D\}$. If for any point $z \in D$ there exists a function $f \in H^2(D)$, such that $f(z) \neq 0$, we are given the Bergman kernel function

$$K_D(z, \bar{z}) = \max\{|f(z)|^2 \mid f \in H^2(D), \|f\|^2_{L^2(D)} = 1\}.$$

It is smooth and positive, and $\log K_D(z, \bar{z})$ is plurisubharmonic. If for any $z \in D$ and $X \in C^n$ there is an $f \in H^2(D)$ such that $f(z) = 0$ and $(\partial f(z), X) \neq 0$ then it is even strictly plurisubharmonic and hence the potential of a Kählerian metric, the Bergman metric B^2_D of D. If we further donote by $b^2_D(z, X)$ the functional

$$b^2_D(z, X) = \max\{|(\partial f(z), X)|^2 \mid f \in H^2(D), f(z) = 0,$$

$$\|f\|_{L^2(D)} = 1\}, \ z \in D, X \in \mathrm{C}^n,$$

then we have (see [Be, pp. 198/199])

$$B^2_D(z, X) = \frac{b^2_D(z, X)}{K_D(z, \bar{z})}$$

on $D \times C^n$. We also note the following monotonicity properties of K_D and b_D^2: If $D' \subset D$, then on $D' \times C^n$ we have

$$\mathrm{K}_D(z, \bar{z}) \leq K_{D'}(z, \bar{z})$$

and

$$b_D^2(z, X) \leq b_{D'}^2(z, X).$$

§ 1. Notations and results

Let $d \geq 2$ be an integer and $P = P(v_2, \cdots, v_d)$ be a real-valued plurisubharmonic polynomial in C^{d-1} without pluriharmonic terms. Let

$$\rho(v) := \mathrm{Re} v_1 + P(v_2, \cdots, v_d).$$

We will say that the pseudoconvex domain

$$G := \{v \in \mathbf{C}^d \mid \rho(v) < 0\}$$

is of *homogeneous finite diagonal type* if the following hypotheses on P are satisfied;

(1.1) For positive integers $m_2, \cdots, m_d$ and all $\lambda > 0$ we have $P(\lambda^{\frac{1}{2m_2}} v_2, \cdots, \lambda^{\frac{1}{2m_2}} v_d) = \lambda P(v_2, \cdots, v_d)$

(1.2) For a small positive number s also the function $P(v_2, \cdots, v_d) - 2s \sum_{j=2}^d |v_j|^{2m_j}$ is plurisubharmonic on C^{d-1}.

We are interested in a sharp estimation of the growth of the Bergman kernel on domains G of homogeneous finite diagonal type and certain "covering" domains Ω of G in case that P has the special form

$$P(v_2, \cdots, v_d) = \sum_{j=2}^{d} P_j(v_j) + \sum_{j<k} P_{jk}(v_j, v_k). \tag{1.3}$$

In order to be able to describe our estimates we introduce the functions

$$A_{lj}(v') = \max\{|\frac{\partial^{\nu+\mu}\mathrm{P}}{\partial v_j^{\nu} \partial \bar{v}_j^{\mu}}(v')\| \nu, \mu \geq 1, \nu + \mu = l\} \tag{1.4}$$

for $v' = (v_2, \cdots, v_d) \in \mathbf{C}^{d-1}$, $2 \leq \ell \leq 2m_j$, $2 \leq j \leq d$, and for $t > 0$:

$$\mathscr{C}_j(v,t) = \sum_{\ell=2}^{2m_j} (\frac{A_{lj}(v')}{t})^{\frac{1}{l}}. \tag{1.5}$$

We first estimate the Bergman kernel $K_G(z, \bar{z})$ of G from above.

LEMMA 1. *Assume, for the polynomial P the conditions* (1. 1), (1. 2) *and* (1. 3) *are fulfilled. Let as abbreviate for* $(v, X) \in G \times C^d$

$$F_G(v, \bar{v}) = |\rho(v)|^{-2} \prod_{j=2}^{d} \mathscr{C}_j(v, |\rho(v)|)^2$$

and

$$M_G^2(v, X) = \frac{|(\partial\rho(v), X)|^2}{|\rho(v)|^2} + \sum_{j=2}^{d} \mathscr{C}_j(v, |\rho(v)|)^2 |X_j|^2$$

Then, with a universal positive constant c_1 we have for any $v \in G \cap \Delta_d(0, 1)$

$$K_G(v, \bar{v}) \leq c_1 F_G(v, \bar{v}) \tag{1.6a}$$

and for $X \in C^d$:

$$b_G^2(v, X) \leq F_G(v, \bar{v}) M_G^2(v, X). \tag{1.6b}$$

We will also study the situation where G is covered by means of monomial holomorphic mappings. To make this precise we let $A \subset \mathbf{N}_0^{n-1} \setminus \{0\}$ for $n \geq 2$ be a set of $d-1$ multiindices $\alpha^{(j)} = (\alpha_2^{(j)}, \cdots, \alpha_n^{(j)})$, $2 \leq j \leq d$. Further we denote by $e^{(i)} \in \mathbf{N}_0^{n-1}$ that unit multiindex with 1 at the $(i-1)$.th position and 0 elsewhere, and let $\bar{\mathbf{A}} = A \cup \{e^{(i)} | 2 \leq i \leq n, e^{(i)} \notin A\}$. Then let us consider the holomorphic mappings

$$f_A : \mathbf{C}^{n-1} \longrightarrow \mathbf{C}^{d-1},\ z' = (z_2, \cdots, z_n) \longrightarrow (z'^{\alpha^{(2)}}, \cdots, z'^{\alpha^{(d)}})$$

and

$$F_A(z) = (z_1, f_A(z')),\ for\ z \in C^n.$$

We want to estimate the singular boundary behavior of the Bergman kernel of the domain

$$\Omega = F_A^{-1}(G) \cap [\mathbf{C} \times \Delta_{n-1}(0,1)].$$

For the domain G we can, given a point $v \in G$ near $0 \in \partial G$, find an optimal interior domain of comparison for the Bergman kernel, on which $|\rho|$ does not change by more than $O(|\rho(v)|)$. This comparison domain induces on Ω also a good comparison domain at any $z \in \Omega$ near $O \in \partial\Omega$. This is the content of

Lemma 2. *Let $r = \rho \circ F_A$ and $\Omega = \{z \in C^n \mid r(z) < 0\} \cap [C \times \Delta_{n-1}(0,1)]$. Then, with universal positive constants c_2, c_3, and c_4, we have for all $z \in \Omega$ sufficiently close to $0 \in \partial\Omega$, that*

$$c_2 K_{E'c_3(z')}(z', \bar{z}') \leq |r(z)|^2 K_\Omega(z, \bar{z}) \leq \frac{1}{c_2} K_{E'c_4(z')}(z', \bar{z}'), \tag{1.7}$$

where

$$E'_c(z') = \{w' \in \Delta_{n-1}(O, 1) | \sum_{j=2}^{d} \sum_{l_j=2}^{2m_j} A_{l_j j}(f_A(z')) | w'^{\alpha^{(j)}} - z'^{\alpha^{(j)}} |^{l_j} < c | r(z)|\}.$$

From this we will obtain an estimate for $K_G(z, \bar{z})$ by estimating the Bergman kernel of $E'_{c_4}(z')$. In order to be able to state the estimate we need some preparations:

On $\mathbf{R}^{n-1}$ we define the aim functional $L(x') = x_2 + \cdots + x_n$ for $x' = (x_2, \cdots, x_n) \in \mathbf{R}^{n-1}$. Let $\mathcal{M} \in GL(n-1,\mathbf{Q})$ be a matrix, the rows of which belong to A and $I_{\mathcal{M}}$ be the set $I_{\mathcal{M}} = \{2 \leq i \leq n | e^{(i)}\mathcal{M} \in A\}$. For $i \in I_{\mathcal{M}}$ there is a unique $j(\mathcal{M}, i) \in \{2, \cdots, d\}$ with $e^{(i)}\boldsymbol{M} = \alpha^{(j(M,i))}$. We call a matrix $\mathcal{M} \in GL(n-1, \mathbf{Q})$ a minimum matrix for A, if one has

(M1) The rows of $\mathcal{M}$ belong to $\bar{A}$

(M2) The simplex

$$\Sigma_{\mathcal{M}} := \{x' \in \mathbf{R}^{n-1} | x_i \geq 0 \text{ for } i \notin I_{\mathcal{M}} \text{ and } e^{(i)}\mathcal{M}x'^T \geq 1/2m_{j(\mathcal{M},i)} \text{ for } i \in I_{\mathcal{M}}\}$$

has a minimun corner $x'_{\mathcal{M}}$ with respect to L (i.e. a corner where L attains its minimum on $\Sigma_{\mathcal{M}}$).

For reasons of dimensions the corner $x'^T_{\mathcal{M}}$ is uniquely determined, and

$$\mathcal{M}x'^T_{\mathcal{M}} = \frac{1}{2} \sum_{i \in I_{\mathcal{M}}} \frac{1}{m_{j(\mathcal{M},i)}} (e^{(i)})^T$$

(for a row vector y we denote by y^T the corresponding column vector). Finally, let for $2 \leq i \leq n : \delta_i(\mathcal{M}) = L(\mathcal{M}^{-1}(e^{(i)T})$. Then, for all such i we have $\delta_i(\mathcal{M}) \geq 0$, because $x_{(i)} = x'_{\mathcal{M}} + \mathcal{M}^{-1}(e^{(i)})^T$ belongs to $\Sigma_{\mathcal{M}}$, and thus $\delta i(\mathcal{M}) = L(x_{(i)}) - L(x'_{\mathcal{M}}) \geq 0$.

The corner $x'_{\mathcal{M}}$ is the common intersection point of $n-1$ edges (of dimension one) of $\Sigma_{\mathcal{M}}$ namely, the $E_i = \mathcal{M}^{-1}(e^{(i)})^T$; the numbers $\delta_i(\mathcal{M})$ are, intuitively speaking, a measure for the angle included by the $x'_{\mathcal{M}} + E_i$, and the affine plane $\{L = L(x'_{\mathcal{M}})\}$ at $x'_{\mathcal{M}}$. Because of

$$(\delta_2(\mathcal{M}), \cdots, \delta_n(\mathcal{M}))\mathcal{M} = (1, \cdots, 1)$$

the number $d(\mathcal{M})$ of all numbers $i \in \{2, \cdots, n)$ for which $\delta_i(\mathcal{M}) = 0$ must be less than n-1. With all these notations we can state our result as follows :

THEOREM 1. *Let Ω be as in Lemma 2. Then, with a suitable positive constant c_3 we have for any $z \in \Omega$ sufficiently close to* 0:

$$K_\Omega(z, \bar{z}) \geq c_3 |r(z)|^{-2} \sum_{\mathcal{M} \in M(A)} \frac{\Pi_{i \in I_{\mathcal{M}}} \mathscr{C}(\mathcal{M}, i; z)^{2\delta_i(\mathcal{M})}}{(\log \sum_{i \in I_{\mathcal{M}}} \mathscr{C}(\mathcal{M}, i; z))^{d(\mathcal{M})}},$$

where $M(A)$ is the set of all minimum matrices for A and

$$\mathscr{C}(\mathcal{M}, i; z) = \mathscr{C}_{j(M,i)}(F_A(z), |r(z)|).$$

For suitable choices of the set A we obtain from this:

COROLLARY 1. *Suppose* $G=\{\rho<0\}$ *is as in Lemma* 1. *Then, with a suitable positive constant* c_4 *we have, near* $0 \in G$:

$$K_G(v,\bar{v}) \geq c_4 \mid \rho(v) \mid^{-2} \prod_{j=2}^{d} \mathscr{C}_j(v, \mid \rho(v) \mid)^2.$$

Proof. We let $A=\{e^{(i)} \mid 2 \leq i \leq n\}$, so that $F_A=\mathrm{id}_G$. The only contribution to the right side of (1.8) will then come from the unity matrix $\mathscr{E}_{n-1}$. Obviously $\delta_i(\mathscr{E}_{n-1})=1$, for all i and $d(\mathscr{E}_{n-1})=0$. Here, $n=d$.

We therefore gain an exact description of the singular behavior of the Bergman kernel K_G of G. It has been announced in [H 1].

COROLLARY 2. *On* G *the Bergman metric* B_G^2 *of* G *can be estimated near* $0 \in \partial G$ *as follows:*

$$\frac{1}{c_5} \leq \frac{B_G^2(v, X)}{M_G^2(v, X)} \leq c_5$$

for $X \in C^d$. *Here the constant* $c_5>0$ *is again universal, and* M_G *is defined as in Lemma* 1.

Proof. Let Cara_G denote the pseudodifferential metric of Caratheodory on G. In [H 1, Theorem 1], it was shown that $\mathrm{Cara}_G^2 \geq$ a constant times M_G^2. This, combined with the well-known inequality $B_G^2 \geq \mathrm{Cara}_G^2$, (see [Ha] implies the lower estimate. The upper estimate is obtained by using (1.6b) and Corollary 1 together with $B_G^2(v, \mathrm{X})=b_G^2(v, X)/K_G^2(v,\bar{v})$.

COROLLARY 3. *There exist positive constants* c_6, R_0 *and a number* $d_1 \in \{0,\cdots, n-2\}$, *such that for any* $z \in \Omega \cap \Delta_n(0, R_0)$ *one has*

$$K_\Omega(z,\bar{z}) \geq c_6 \mid \log \mid r(z) \| ^{-d_1} \mid (z) \mid^{-2-2\mu_A}. \tag{1.9}$$

Here μ_A *denotes the minimum of* L *on the simplex*

$$\Sigma_A=\{x' \in \mathbf{R}^{n-1} \mid \sum_{i=2}^{n} \alpha_i^{(j)} x_i \geq \frac{1}{2m_j},\ 2 \leq j \leq d, \text{ and } x_2,\cdots, x_n \geq 0\}.$$

In the appendix we will show

LEMMA 3. *If* x_0 *is a minimum corner for* L *on* Σ_A, *then there exists a minimum matrix* $\mathscr{M}$ *for* A *with* $x'_{\mathscr{M}}=x_0$.

Proof of Corollary 3. For any $j \in\{2, \cdots, d\}$ and $v \in G$ we have $\mathscr{C}_j(v, \mid \rho(v) \mid) \geq c_7 \mid \rho(v) \mid^{-\frac{1}{2m_j}}$, $c_7>0$ independent of v. If M is chosen according to Lemma 3 for x_0, we obtain from (1.8) (with some constant $c_3'>0$):

$$K_\Omega(z,\bar{z}) \geq c_3' \mid r(z) \mid^{-2} \mid \log \mid r(z) \| ^{-d(\mathscr{M})} \prod_{i \in I_{\mathscr{M}}} \mathscr{C}_j(\mathscr{M}, i)(F_A(z), \mid r(z) \mid)^{2\delta_i(\mathscr{M})}$$

$$\geq c_8 c_3 |\log|r(z)||^{-d(\mathcal{M})}|r(z)|^{-2-p'},$$

where $p' = \sum_{i\in I_{\mathcal{M}}} \frac{1}{m_{j(\mathcal{M},i)}} \delta_i(\mathcal{M})$. To see that $p' = 2\mu_A$ we let $\mathscr{C}$ denote the diagonal matrix $\mathscr{C} = \operatorname{diag}(c_2, \cdots, c_n)$, where $c_i = 2m_{j(\mathcal{M},i)}$, for $i \in I_{\mathcal{M}}$, and $c_i = 1$ otherwise, and let $\hat{e} = (\hat{e}_2, \cdots, \hat{e}_n)$ be the vector with $\hat{e}_i = 1$, if $i \in I_{\mathcal{M}}$, and $\hat{e}_i = 0$ otherwise. Then

$$p' = 2(\delta_2(\mathcal{M}), \cdots, \delta_n(\mathcal{M}))\mathscr{C}^{-1}\hat{e}^T = 2L(\mathcal{M}^{-1}\mathscr{C}^{-1}\hat{e}^T).$$

On the other hand

$$\mathcal{M}x'_{\mathcal{M}} = \frac{1}{2}\sum_{i\in I_{\mathcal{M}}} \frac{1}{m_{j(\mathcal{M},i)}} (e^{(i)})^T = \mathscr{C}^{-1}\hat{e}^T,$$

and thus $x'_{\mathcal{M}} = \mathcal{M}^{-1}\mathscr{C}^{-1}\hat{e}^T$. This yields $p' = 2L(x'_{\mathcal{M}}) = 2L(x_0) = 2\mu_A$.

Remarks. a) This corollary generalizes Theorem (5.3.1) of [H 2]. In that paper the estimate (1.9) was obtained for $P = |v_2|^2 + \cdots + |v_d|^2$ under the assumption that $A = \bar{A}$ and Lemma 3 is true.

b) The growth exponent μ_A from (1.9) cannot be improved (cf. [H 2, Satz (5.2.12)]). The estimate (1.9) is sharp under nontangential approach of z towards $0 \in \partial\Omega$:

$$K_\Omega(z, \bar{z}) \leq c'_6 |r(z)|^{-2-2\mu_A},$$

when $z \to 0 \in \partial\Omega$ nontangentially; $c'_6 > 0$ is universal). In general, one also cannot get rid of the log-term on the right side of (1.9). This was observed in [H 3].

c) The domains of Theorem 1 need neither be of finite type in the sense of d'Angelo, [A], nor be uniformly extendable in a pseudoconvex way. If, for instance, P_0 is a subharmonic homogeneous polynomial in the plane without harmonic terms, of degree $2k$, then the class of domains covered by Theorem 1 contains the domain

$$\Omega = \{z \in \mathbf{C} \times \Delta_{n-1}(0,1) \,|\, \operatorname{Re} z_1 + P_0(z_2 \cdot \cdots \cdot z_n) < 0\}.$$

Here K_Ω grows at least like $\operatorname{dist}(\cdot, \partial\Omega)^{-2-\frac{1}{k}} |\log|r(z)||^{2-n}$ near 0. The method applied in [D-H-O] would lead to a growth order of only $2 + \frac{1}{k(n-1)}$.

§ 2. The upper estimate for $K_G(v, \bar{v})$

We recall some observations made in [H 1, Sec. 1]. Here we always assume that (1.1), (1.2), and (1.3) are fulfilled for P. Let $Q' \in \mathbf{C}^{d-1}$ be such that $Q = (-P(Q'), Q')$ is a boundary point of G within $\Delta_d(0, 1)$. By Taylor expansion of

P around Q' we see that, with a holomorphic polynomial $h(Q', \zeta')$ of the form

$$h(Q', \zeta') = \sum_{j=2}^{d} \frac{\partial \rho}{\partial \zeta_j}(Q')\zeta_j + \mathcal{O}(|\zeta|^2) \tag{2.1}$$

our P can be written as

$$\begin{aligned} P(v') = {} & P(Q') + \operatorname{Re} h(Q', v' - Q') \\ & + \sum_{j=2}^{d} \widehat{P}_j(Q', v_i - Q_j) + \sum_{j<k} \widehat{P}_{jk}(Q_j, Q_k; v_j - Q_j, v_k - Q_k). \end{aligned} \tag{2.2}$$

Here the $\widehat{P}_j$ and $\widehat{P}_{jk}$ are real-valued polynomials without pluriharmonic terms, and

$$\widehat{P}_{jk}(Q_j, Q_k; \zeta_j, \zeta_k) = \mathcal{O}(|\zeta_j\|\zeta_k|),$$

for $j < k$. Therefore, the mapping $F(Q', \cdot): \mathbf{C}^d \longrightarrow \mathbf{C}^d$, given by

$$F(Q', \zeta) = (\zeta_1 + P(Q') + h(Q', \zeta' - Q'), \zeta' - Q')$$

defines a biholomorphism $F(Q', \cdot)$ of G onto the domain

$$\widetilde{G}_{Q'} = \{\xi \in \mathbf{C}^d \mid \tilde{\rho}_{Q'}(\xi) := \operatorname{Re}\xi_1 + \sum_{j=2}^{d} \widehat{P}_j(Q', \xi_j) + \sum_{j<k} \widehat{P}_{jk}(Q_j, Q_k; \xi_j, \xi_k) < 0\}.$$

We next represent $\widehat{P}_j(Q', \cdot)$as $\widehat{P}_j = \sum_{l=2}^{2m_j} \widehat{P}_{j,l}(Q', \xi_j)$, where $\widehat{P}_{j,l}(Q', \cdot)$is a homogeneous polynomial of degree l. If we agree upon the further notations
$\|\widehat{P}_{j,l}(Q', \cdot)\| =$ maximum of the absolute values of the coefficients appearing in $\widehat{P}_{j,l}(Q', \cdot)$, and

$$\mathcal{B}_j(Q', \xi_j) := \sum_{l=2}^{2m_j} \|\widehat{P}_{j,l}(Q', \cdot)\| \, |\xi_j|^l$$

for $2 \leq j \leq d$, we can state the following bumping lemma, which is shown in Lemma 3 of [H 1].

LEMMA 2.1. *There exists a radius $r_0 > 0$, a constant $A > 0$, and for any $Q' \in \Delta_{d-1}(0', 2)$ a continuous function $\phi = \phi_{Q'}$ on $\mathbf{C}^d$, which is plurisubharmonic on the tube $\mathrm{T}_{r_0} = \mathbf{C} \times \Delta_{d-1}(0' r_0)$ satisfying*

$$-\frac{1}{A} \sum_{j=2}^{d} \mathcal{B}_j(Q', \xi_j) \leq \phi(\xi) - \tilde{\rho}_{Q'}(\xi) \leq -\frac{1}{2} A \sum_{j=2}^{d} \mathcal{B}_j(Q', \xi_j) \tag{2.3}$$

on T_{r_0}.

The bumping lemma will be crucial for the proof of Lemma 2 of this paper. In [H 1, Lemma 2] we showed how the coupling terms $\widehat{P}_{jk}(Q_j, Q_k; \xi_j, \xi_k)$ appearing in (2.1) can be estimated in absolute value by the $\mathcal{B}_j(Q', \xi_j)$, $2 \leq j \leq d$. This gives us also

LEMMA 2.2. *With a certain positive constant A_1 we have for any $Q' \in \Delta_{d-1}(0', 2)$*

and $\xi \in \mathbf{C}^d$:

$$|\tilde{\rho}_{Q'}(\xi) - \mathrm{Re}\xi_1| \leq A_1 \sum_{j=2}^{d} \mathscr{B}_j(Q', \xi_j).$$

We are now ready for the

Proof of Lemma 1. Let $v \in G$ be a point in $\Delta_d(0,1)$, such that $t = -\rho(v) \in (0,1)$ (Otherwise there is nothing to be shown). Also we can assume $\mathrm{Im}v_1 = 0$. With $Q' := v'$ we then have $v = Q - te_1$, where $Q = (-P(Q'), Q')$ as before, and $e_1 = (1, 0, \cdots, 0) \in \mathbf{C}^d$. Since $F(Q', \cdot)$ takes v into $-te_1$ we see from the transformation rule for the Bergman kernel that $K_G(v, \bar{v}) = K_{\tilde{G}_{Q'}}(-te_1, -te_1)$. Because of Lemma 2.2 the Reinhardt domain

$$\tilde{D}_{Q',t} = \Delta_1(-t, \frac{t}{2}) \times \{\xi' \in \mathbf{C}^{d-1} | A_1 \sum_{j=2}^{d} \mathscr{B}_j(Q', \xi_j) < \frac{t}{2}\}$$

is contained in $\tilde{G}_{Q'}$. This implies

$$K_{\tilde{G}_{Q'}}(-te_1, -te_1) \leq K_{\tilde{D}_{Q',t}}(-te_1, -te_1) = \frac{1}{\mathrm{Vol}(\tilde{D}_{Q',t})}. \tag{2.4}$$

It is easy to see that, with a universal positive constant A_2 the estimates

$$\frac{1}{A_2} \leq \frac{\| \hat{P}_{j,l}(Q', \cdot)\|}{A_{lj}(v')} \leq A_2 \tag{2.5}$$

and

$$\mathscr{B}_j(Q', \xi_j) \leq 2m_j A_2\, t \max_{2 \leq l \leq 2m_i} [\mathscr{C}_j(v, t) | \xi_j |]^l \tag{2.6}$$

hold. If we choose $\gamma_j = \frac{1}{2\sqrt{1+m_j, A_1 A_2 d}}$, $2 \leq j \leq d$, then the polydisc

$$D_{Q',t} = \Delta_1(-t, \frac{t}{2}) \times \Delta_1(0, \frac{\gamma_2}{\mathscr{C}_2(v, t)}) \times \cdots \times \Delta_1(0, \frac{\gamma_d}{\mathscr{C}_d(v, t)})$$

is a subset of $\tilde{D}_{Q',t}$ and thus

$$\mathrm{Vol}(\tilde{D}_{Q',t}) \geq \frac{1}{4}\pi^d \gamma_2^2 \cdot \ldots \cdot \gamma_d^2 \prod_{j=2}^{d} \mathscr{C}_j(v, t)^{-2}.$$

This, combined with (2.4), implies the estimate (1.6a).

For the proof of (1.6b) we introduce for any $X \in \mathbf{C}^d$ the vector $Y = F'(Q',v) X^T$. Since

$$F(Q', \zeta) = (\zeta_1 + P(Q') + h(Q', \zeta' - Q'), \zeta' - Q'),$$

and h is of the form (2.1), we must have

$$Y = ((\partial\rho(v), X), X_2, \cdots, X_d).$$

The transformation rule for the functional b_G^2 now yields:

$$b_G^2(v, X) = b_{\tilde{G}_{Q'}}^2(-te_1, Y) \leq b_{D_{Q',t}}(-te_1, Y)$$

$$= 8(\gamma_2 \cdots \gamma_d)^{-2} t^{-2} \prod_{j=2}^{d} \mathscr{C}_j(v, t)^{-2} \Big(\frac{|\partial\rho(v), X)|^2}{t^2} + \sum_{j=2}^{d} \mathscr{C}_j(v, t)^2 |X_j|^2 \Big),$$

since $D_{Q',t}$ is a polydisc; so we obtain (1.6b).

§ 3. The comparison lemma for the Bergman kernel

In this section we want to prove Lemma 2 of Section 1. The idea of the proof is similar to that in the proof of Theorem (6.1) of [Ca]. It is based on the $\bar{\partial}$-technique for the construction of holomorphic L^2 functions. We begin with the computation of a Levi form. Let us fix a point $Q' \in \Delta_{d-1}(0, 1)$.

LEMMA 3.1. *Let G be a domain of homogeneous finite diagonal type as described in Section* 1. *Suppose S is a real-valued C^2-function on $\mathbf{C}^{d-1}$ satisfying* $\operatorname{Re} v_1 \leq S(v_2, \cdots, v_d)$ *for any $v = (v_1, \cdots, v_d) \in \tilde{G}_{Q'}$. For $t > 0$ we define on $\tilde{G}_{Q'}$ the function*

$$V_t(v) = \Big| \frac{t}{v_1 - t - S(v')} \Big|^2.$$

Then

a) *The Levi form of V_t, on $\tilde{G}_{Q'}$ is given by*

$$\partial\bar{\partial}V_t(X, \bar{X}) = \tag{3.1}$$

$$\frac{V_t^2}{t^2}(|X_1|^2 - 2\operatorname{Re}\Big(3 + \frac{4i\operatorname{Im} v_1}{|v_1 - t - S(v')|^2}(v_1 - t - S(v')\Big) \cdot (\partial' S, X')\bar{X}_1$$

$$+ 2[(-1 + \frac{4(\operatorname{Re} v_1 - t - S(v'))^2}{t^2} V_t)|(\partial' S, X')|^2$$

$$+ \operatorname{Re}(v_1 - t - S(v'))\partial'\bar{\partial}' S(X, X')]).$$

b) *If $S \geq 0$, both S and $\log S$ are plurisubharmonic, and* $\operatorname{Re} v_1 \leq \frac{1}{2} S(v')$ *on $\tilde{G}_{Q'}$, then*

$$2\partial\bar{\partial}V_t(X, \bar{X}) \geq \frac{t^2}{|v_1 - t - S(v')|^4}|X_1|^2 - 500\frac{\partial'\bar{\partial}' S(X', \bar{X}')}{t} \tag{3.2}$$

Here ∂' (resp. $\bar{\partial}'$) is the operator ∂ (resp. $\bar{\partial}$) in $\mathbf{C}^{d-1}_{(v')}$, $X \in \mathbf{C}^d$, $X' = (X_2, \cdots, X_d)$. so that $X = (X_1, X')$.

Proof. Part a) follows from a direct computation. For the proof of part b) we estimate

$$|\,2\mathrm{Re}(3+\frac{4i\mathrm{Im}\,v_1}{|\,v_1-t-S(v')|^2}(v_1-t-S(v')))(\partial' S,\,X')\bar{X}_1\,|$$

$$\leq 10\,|(\partial' S,\,X')\|\,X_1\,|\leq 50\,|(\partial' S,\,X')|^2+\frac{1}{2}\,|\,X_1\,|^2$$

Furthermore

$$|-1+\frac{4(\mathrm{Re}\,v_1-t-S(v'))^2}{t^2}V_t\,|\leq 5,$$

and

$$|\,\mathrm{Re}\,v_1-t-S(v')|\leq\frac{t}{\sqrt{V_t}}.$$

This implies for $X\in\mathbf{C}^d$:

$$\partial\bar{\partial}V_t(X,\bar{X})\geq\frac{V_t^2}{2t^2}\Big[|\,X_1\,|^2-120\,|(\partial' S,\,X')|^2-4\frac{t}{\sqrt{V_t}}\partial'\bar{\partial}' S(X',\,\bar{X}')\Big].$$

From the hypotheses $\mathrm{Re}\,v_1\leq\frac{1}{2}S(v')$ for $v\in\tilde{G}_{Q'}$, we get

$$\sqrt{V_t}=t/|\,v_1-t-S(v')|\leq t/t+\frac{1}{2}S(v'),$$

and thus

$$S(v')\leq\frac{2\mathrm{t}}{\sqrt{V_t}}. \tag{3.3}$$

Next we use the plurisubharmonicity of log S, and can estimate

$$120\,|(\partial' S,\,X')|^2\leq 240\frac{t}{V_t}\partial'\bar{\partial}' S(X',\,\bar{X}').$$

So we obtain

$$2\partial\bar{\partial}V_t(X,\bar{X})\geq\frac{V_t}{t^2}|\,X_1\,|^2-250\,V_t^{3/2}\frac{\partial'\bar{\partial}' S(X',\,X')}{t}.$$

Now the claim follows, since $V_t=\dfrac{t^2}{|\,v_1-t-S(v')|^2}\leq 1.$

LEMMA 3.2. *For $Q'\in\Delta_{d-1}(0,1)$, $t>0$ let*

$$\tilde{\phi}_t(v)=\frac{|\,v_1+t\,|^2}{t^2}+\frac{1}{\mathrm{t}}\sum_{j=2}^{d}\mathcal{B}_j(Q',\,v_j).$$

Then, given an $\varepsilon>0$ there exists a positive constant β which depends only on ε (and not on t) and a continuous plurisubharmonic function Ψ_t on $\tilde{G}_{Q'}\cap T_{r_0}$ satisfying $\Psi_t\leq 1$ on $\bar{G}_{Q'}\cap T_{r_0}$ and

1) *On* $E_{\varepsilon,t} := \{\tilde{\phi}_t < \varepsilon\}$ *we have* $-\frac{1}{\beta} \leq \Psi_t - \beta\log\tilde{\phi}_t \leq \frac{1}{\beta}$

2) *The function* $\Psi_t - \beta\,\tilde{\phi}_t$ *is plurisubharmonic on* $E_{\varepsilon,t}$.

Proof. Let A (resp. A_1) denote the constants from Lemma 2.1 (resp. Lemma 2.2). We may assume that $A < A_1$. We know from Lemma 2.2 that

$$\mathrm{Re}\xi_1 \leq \tilde{\rho}_{Q'}(\xi) + A_1 \sum_{j=2}^{d} \mathscr{B}_j(Q',\xi_j) \text{ on } \tilde{G}_{Q'}.$$

Consequently, Lemma 3.1 applies to the function $S(\xi') = 2A_1 \sum_{j=2}^{d} \mathscr{B}_j(Q',\xi_j)$. The function V_t defined by this choice of S now satisfies

$$0 \leq V_t \leq 1 \tag{3.4a}$$

$$2\partial\bar{\partial}V_t(X,\bar{X}) \geq \frac{t^2}{|v_1 - t - S(v')|^4} |X_1|^2 - 500\frac{\partial'\bar{\partial}'S(X',\bar{X}')}{t}, \tag{3.4b}$$

for any $X \in \mathbf{C}^d$. Let h be a smooth monotone function on $\mathbf{R}$ such that $h(x) = x$, for $x < \varepsilon/2$, $h(x) = 1$, for $x \geq 3\varepsilon/4$ and $|h'|, |h''| \leq 16/\varepsilon^2$. Furthermore, let $\phi_{Q'}$ be the function from Lemma 2.1. Then we choose

$$\Phi_t(v) = \frac{1}{t}\Big(\phi_{Q'}(v) + \frac{1}{4}A\sum_{j=2}^{d} \mathscr{B}_j(Q',v_j)\Big) + \frac{A}{8000A_1} V_t(v)$$

and, with a positive constant β which we will choose later we define

$$\Psi_t = \Phi_t + \beta\log h \circ \tilde{\phi}_t.$$

Then Φ_t is plurisubharmonic on $\tilde{G}_{Q'} \cap T_{r_0}$, and on $E_{\varepsilon,t}$ also $\Phi_t - \gamma\tilde{\phi}_t$ is plurisubharmonic, if $0 < \gamma \leq 1/(2(1 + A_1)\varepsilon + 1/A)^4$. This follows easily from (3.4b). Further we have on $E_{\varepsilon,t}$

$$-(1 + A_1\varepsilon + \varepsilon/A) \leq \Phi_t \leq 2 + 2(A + A_1)\varepsilon. \tag{3.5}$$

Now, on $E_{\varepsilon,t} \cup [\mathbf{C}^d \setminus E_{3\varepsilon/4,t}]$ the function $\log h \circ \tilde{\phi}_t$ is plurisubharmonic, and on $E_{3\varepsilon/4,t} \setminus E_{\varepsilon/2,t}$ its Levi form is

$$\partial\bar{\partial}\log h \circ \tilde{\phi}_t \geq -\frac{1000}{\varepsilon^{12}}\partial\tilde{\phi}_t \wedge \bar{\partial}\tilde{\phi}_t \geq -\frac{1000}{\varepsilon^{11}}\partial\bar{\partial}\tilde{\phi}_t.$$

If we choose $0 < \beta \ll \varepsilon^{11}/4000$, then Ψ_t will satisfy all the requirements.

We can now give the

Proof of Lemma 2. Choose r_1 so small that for any $q' \in \Delta_{n-1}(0, 2r_1)$ we have (with Tr_0 as in Lemma 2.1)

$$\mathrm{C} \times \Delta_{n-1}(0, 2r_1) \subset (F(Q',\cdot) \circ F_A)^{-1}(T_{r_0}),$$

where $Q' = f_A(q')$. Now $\Omega = \{r < 0\} \cap [\mathbf{C} \times \Delta_{n-1}(0,1)]$ is biholomorphic to a bounded domain. By the localization lemma of [Oh, p. 898] we have therefore

$$K_\Omega(z, \bar{z}) \geq c'_2 K_{\Omega'}(z, \bar{z})$$

for $z \in \Omega \cap [\mathbf{C} \times \Delta_{n-1}(0, r_1)]$. Here $c'_2 > 0$ is a universal constant and $\Omega' = \Omega \cap [\mathbf{C} \times \Delta_{n-1}(0, 2r_1)]$. Let us from now on fix a point $z \in \Omega \cap [\mathbf{C} \times \Delta_{n-1}(0, r_1)]$. Since Ω is invariant under translation in $\mathrm{Im}\, z_1$-direction we may assume that $\mathrm{Im}\, z_1 = 0$. We let $t = -r(z), q' = (z_2, \cdots, z_n)$, and $Q' = f_A(q')$. Then we have $Q = (-P(Q'), Q') \in \partial G$, and $z = q - te_1$, where $q = (-P(Q'), q')$. For $\varepsilon > 0$ we denote by $E_{\varepsilon,t}$ the ellipsoid from Lemma 3.2 and define further the "pre-image" $P_{\varepsilon,t} = (F(Q', \cdot) \circ F_A)^{-1}(E_{\varepsilon,t}) \cap [\mathbf{C} \times \Delta_{n-1}(0,1)]$. Our goal is to compare the Bergman kernel functions of Ω' and $P_{\varepsilon,t}$ at $(z, \bar{z})$.

To this end let $g \in H^2(P_{\varepsilon,t})$, $g \neq 0$, be arbitrarily chosen. If χ is a smooth cut-off function with $\chi(x) = 1$, if $x \leq 1/2$, and $\chi(x) = 0$, if $x \geq 1$ and $|\chi'(x)| \leq 4$, for all $x \in \mathbf{R}$, then the (0,1) form

$$v = \bar{\partial}\left(X\left(\frac{\phi_t}{\varepsilon}\right)\right) \cdot g$$

is well-defined on Ω', with smooth coefficients. Here $\phi_t := \tilde{\phi}_t \circ F(Q', \cdot) \circ F_A$, where $\tilde{\phi}_t$ is defined as in Lemma 3.2. Further we have $\mathrm{supp}(v) \subset P_{\varepsilon,t} \backslash P_{\frac{\varepsilon}{2}t}$. Let us define for a positive integer N the following function U on Ω':

$$U(w) = N \cdot \Psi_t \circ F(Q', \cdot) \circ F_A(w) + U_0, \tag{3.6}$$

where U_0 is a strictly plurisubharmonic function on Ω' with $0 \leq U_0 \leq 1$, (note that such a function exists, since Ω' is biholomorphic to a bounded domain!) and Ψ_t is as in Lemma 3.2. By our choice of r_1 the function U is continuous and plurisubharmonic on Ω'. Also the function $U - \hat{\phi}_t$, where

$$\hat{\phi}_t(w) = \beta N \cdot \phi_t(w) + U_0, \tag{3.7}$$

for $w \in \Omega'$, is plurisubharmonic on $P_{\varepsilon,t}$, and

$$U \geq -\frac{N}{\beta} + \beta N \cdot \log\frac{\varepsilon}{2} \tag{3.8}$$

on $\mathrm{supp}(v)$. This follows from Lemma 3.2, part (1). The length $|v|_{\partial\bar{\partial}\hat{\phi}_t}$ of v with respect to the Kähler metric with potential $\hat{\phi}_t$ on Ω' is bounded above by

$$|v|_{\partial\bar{\partial}\hat{\phi}_t} \leq \left|\chi'\left(\frac{\phi_t}{\varepsilon}\right)\right| \frac{1}{\varepsilon} |g| \|\bar{\partial}\phi_t|_{\partial\bar{\partial}\hat{\phi}_t} \leq \frac{4}{\varepsilon}(\beta N\varepsilon + 1)|g|,$$

since $\partial\bar{\partial}\hat{\phi}_t \geq \partial\phi_t \wedge \bar{\partial}\hat{\phi}_t / \hat{\phi}_t + U_0$, and, on $\mathrm{supp}(v)$ one has the estimate $\hat{\phi}_t \leq 1 + \varepsilon\beta N$. We then can estimate the L^2-integral

$$I(v) = \int_{\Omega'} |v|^2_{\partial\bar\partial\hat\varphi_t} e^{-U} d\lambda_{2n}$$

as follows:

$$I(v) \leq \frac{16}{\varepsilon^2}(\beta N\varepsilon + 1)^2 e^{\frac{N}{\beta}}(\frac{2}{\varepsilon})^{\beta N} \| g \|^2_{L^2(P_{\varepsilon,t})}.$$

Let us abbreviate $c_2''(\epsilon) = \frac{16}{\varepsilon}(\beta N\epsilon + 1)^2 e^{\frac{N}{\beta}}(\frac{2}{\varepsilon})^{\beta N}$.

In this situation we can imitate the proof of Lemma (4.4.1) [Hör, p.92] (any problems coming from lack of differentiability of the function U can be overcome by a standard smoothing argument). We obtain a smooth solution u of the equation $\bar\partial u = v$ on Ω' satisfying

$$\int_{\Omega'} |u|^2 e^{-U} d\lambda_{2n} \leq 2\, c_2''(\varepsilon) \| g \|^2_{L^2(P_{\varepsilon,t})}. \tag{3.9}$$

Now by Lemma 3.2 the weight function e^{-U} is not locally integrable at z, if N is large enough. So (3.9) implies that $u(z) = 0$. Because $\Psi_t \leq 1$, our function U is bounded above by N, and therefore, by (3.9) we see that $u \in L^2_{\Omega'}$ and

$$\| u \|^2_{L^2(\Omega')} \leq 2c_2''(\varepsilon) e^N \| g \|^2_{L^2(P_{\varepsilon,t})}$$

and the function $\hat g = \chi(\psi_t/\varepsilon)\cdot g - u$ belongs to $H^2(\Omega')$. Further we have $\hat g(z) = g(z)$, and

$$\| \hat g \|^2_{L^2(\Omega')} \leq 4(1 + c_2''(\epsilon) e^N \| g \|^2_{L^2(P_{\varepsilon,t})}.$$

As g was chosen arbitrarily, we obtain (setting $c_2''' = 1/4(1 + c_2''(\varepsilon)e^N)$

$$K_{\Omega'}(z, \bar z) \geq c_2''' K_{P_{\varepsilon,t}}(z, \bar z). \tag{3.10}$$

On the other hand, for small enough ε (independent of z) we can arrange that $E_{\varepsilon,\, t} \subset \bar G_{Q'}$, and thus $P_{\varepsilon,\, t} \subset \Omega$. This gives

$$c_2'' c_2''' K_{P_{\varepsilon,t}}(z, \bar z) \leq K_\Omega(z,\bar z) \leq K_{P_{\varepsilon,t}}(z,\bar z).$$

Let us now estimate $K_{P_{\varepsilon,t}}(z,\bar z)$. We define the following biholomorphism of $\mathbf{C}^n$ into itself:

$$H(w) = (w_1 + P(Q') + h(Q', f_A(w') - Q'), w').$$

Then we have $P_{\varepsilon,t} = [\mathbf{C} \times \Delta_{n-1}(0, 1)] \cap H^{-1}(\{\zeta \mid \tilde\psi_t(\zeta_1, f_A(\zeta') - Q') < \varepsilon\})$. Recall that for $c > 0$ we defined the sets

$$E_c'(z') = \{w' \in \Delta_{n-1}(O, 1) | \sum_{j=2}^{d} \sum_{l_j=2}^{2m_j} A_{l_j j}(f_A(z')) | w'^{a(j)} - z'^{a(j)}|^{l_j} < c \mid r(z)|\}.$$

With our definitions of Q' and t we obtain by means of (2.5):

$$(3.12)\qquad H^{-1}(\Delta_1(-t, \frac{\sqrt{\varepsilon}}{2}t) \times E'_{\varepsilon/4A_2}(z')) \subset P_{\varepsilon,t} \subset H^{-1}(\Delta_1(-t, \sqrt{\varepsilon}t) \times E'_{\varepsilon A_2}(z')).$$

Since $\det H' \equiv 1$ the transformation rule for Bergman kernels gives

$$(\frac{\pi\varepsilon}{4}t^2)^{-1}K_{E'_{\varepsilon/4A_2}(z')}(z', \bar{z}') \leq K_{P_{\varepsilon,t}}(z, \bar{z}) \leq (\pi\varepsilon t^2)^{-1}K_{E'_{\varepsilon A_2}(z')}(z', \bar{z}').$$

Combining this with (3.11) we obtain the desired estimate (1.7), if the constants c_2, c_3, and c_4 are suitably chosen.

§ 4. Proof of Theorem 1

Let $z \in \Omega$ be a point sufficiently close to $0 \in \partial\Omega$, such that Lemma 2 applies. With universal positive constants c_2, c_3 we have

$$(4.1)\qquad K_\Omega(z, \bar{z}) \geq c_2 |r(z)|^{-2}K_{E'_{c3}(z')}(z', \bar{z}').$$

Let us use the abbreviations $t = |r(z)|$ and $Q' = f_A(z')$ again. If we write $m = m_2 + \cdots + m_d$, we can choose for any $j \in \{2,\ldots, d\}$ a number $l'_j \in \{2,\ldots, 2m_j\}$ such that

$$(4.2)\qquad 2m\left(\frac{A_{l'_j j}(Q')}{t}\right)^{\frac{1}{l'_j}} \geq \mathscr{C}_j(Q, t).$$

For any $w' \in E'_{c_3}(z')$ we have the estimate

$$\mathscr{C}_j(Q,t)|w'^{\alpha(j)} - Q'_j| \leq (2mc_3)^{1/l'_j} \leq (2m)^{2m}\sqrt{c_3}$$

for $2 \leq j \leq d$ and therefore

$$(4.3)\qquad \sum_{j=2}^{d}\mathscr{C}_j(Q,t)^2|w'^{\alpha(j)} - Q'_j|^2 < (2m)^{4m}dc_3.$$

So let us consider the domain

$$U_1 = \{w \in \mathbf{C} \times \Delta_{n-1}(0, 1) | \operatorname{Re} w_1 + \sum_{j=2}^{d}\mathscr{C}_j(Q,t)^2|w'^{\alpha(j)} - Q'_j|^2 < 0\}.$$

With the notation $m' = (2m)^{4m}dc_3$, we see that $D = \Delta_1(-m' - 1, 1) \times E'_{c3}(z') \subset U$, and consequently

$$(4.4)\qquad K_{E'_{c3}(z')}(z', \bar{z}') = \pi K_D((-m', z'), (-m', \bar{z}')) \geq K_{U_1}((-m', z'), (-m', \bar{z}'))$$

In order to estimate $K_{U_1}((-m', z'), (-m', \bar{z}'))$ from below fix a minimum matrix $\mathscr{M}$ for A. Recall that $\mathscr{M} \in GL(n-1, \mathbf{Q})$, the rows of $\mathscr{M}$ belong to $\bar{A} = A \cup \{e^{(2)},\ldots, e(n)$, and on the simplex

$$\Sigma_{\mathcal{M}} = \{x' = (x_2,\dots,x_n) \in \mathbf{R}^{n-1} \mid x_1 \geq 0 \text{ for } i \notin I_{\mathcal{M}}, \times \text{ and } e^{(i)}\mathcal{M}x'^T \geq \frac{1}{2m_{j(\mathcal{M},\, i)}}, \text{ for } i \in I_{\mathcal{M}}\}$$

there exists a corner $x'_{\mathcal{M}}$, where the aim functional $L(x') = x_2 + \cdots + x_n$, attains its minimum value ($I_{\mathcal{M}}$ and $j(\mathcal{M},i)$ are defined as in Section 1). We write $\delta_i = \delta_i(\mathcal{M})$, for $i = 2,\dots, n$, and arrange (by permuting the rows of $\mathcal{M}$, if necessary) that $\{i \mid \delta_i = 0\} = \{2,\dots, d'\}$, where $d' = d(\mathcal{M}) + 1$, and with suitable integers d'', k, satisfying $2 \leq d'' \leq \max\{d', 2\}$, and $d' \leq k \leq n$, the set $I_{\mathcal{M}}$ equals$\{d'',\dots, k\}$ (the case $I_{\mathcal{M}} = \emptyset$ corresponds to $\mathcal{M}$ = unity matrix $\mathcal{E}_{n-1}$, $k = d' = 1$, $d'' = 2$. Let us further abbreviate $j_i = j(\mathcal{M}, i)$. Then we can represent the matrix $\mathcal{M}$ as follows

$$\mathcal{M} = \begin{bmatrix} e^{(i_2)} \\ \vdots \\ e^{(i_{d''-1})} \\ \alpha^{(j_{d''})} \\ \vdots \\ \alpha^{(j_k)} \\ e^{(i_{k+1})} \\ \vdots \\ e^{(i_n)} \end{bmatrix}$$

where $\{i_2,\dots, i_{d''-1}, i_{k+1},\dots, i_n\} \subset \{2,\dots, n\}$. Our goal is to estimate the L^2-norm of the function $g = (w_1 - 1)^{-2n}$ over U_1 following an idea in the proof of [H 2, Theorem (5.3.1)]. In our situation a simplified version of [H 2, Lemma (5.3.3)] will suffice. We have $g(-m', z') = (m' + 1)^{-2n}$. So we must show that

$$\|g\|^2_{L^2(U_1)} \leq \text{constant}\left(\prod_{i=d'+1}^{k} \mathscr{C}_{j_i}(Q,\, t)^{-2\delta_i}\right)\left(\log \sum_{i=d'+1}^{k} \mathscr{C}_{j_i}(Q,\, t)\right)^{d'-1}. \tag{4.5}$$

In order to do so we want to work with a coordinate tranformation Φ in the w'-space of the form

$$\Phi(w') = (w'^{\beta(2)},\dots, w'^{\beta(n)})$$

where the $\beta^{(i)}$, $2 \leq i \leq n$, are (n $-$ 1)-multiindices with nonnegative integers as entries, which are related to $\mathcal{M}$. In order to be able to define Φ and to estimate reasonably the norm of g over U_1, we need to modify $\mathcal{M}$ in the first $d'-1$ rows. By the Steinitz exchange process we find $\{l_2,\dots, l_{d'}\} \subset \{2,\dots, n\} \setminus \{i_{k+1},\dots, i_n\}$, such that the matrix

$$\mathcal{M}' = \begin{bmatrix} e^{(l_2)} \\ \vdots \\ e^{(l_{d'})} \\ \alpha^{(j_{d''})} \\ \vdots \\ \alpha^{(j_k)} \\ e^{(i_{k+1})} \\ \vdots \\ e^{(i_n)} \end{bmatrix}$$

is invertible. We also have $(\delta_2, \cdots, \delta_n)\mathcal{M}' = (0, \cdots, 0, \delta_{d'+1}, \cdots, \delta_n)\mathcal{M}' = (1, 1, \cdots, 1)$. So, at the same time it follows that $d' < k$, otherwise one would have $(1, 1, \cdots, 1)\,\mathcal{M}' = (1, 1, \cdots, 1)$, since all the alpha's would have been eliminated from $\mathcal{M}$. We now define the mappings

$$\Phi(w') = (w'^{\beta(2)}, \cdots, w'^{\beta(n)}),$$

where we choose $\beta^{(i)} = e^{(i)}\mathcal{M}'$, $2 \leq i \leq n$, and

$$\Psi(w) = (w_1, \Phi(w')).$$

Then Ψ is a holomorphic mapping of $\mathbf{C}^n$ into itself, and outside the analytic set $B = \{w \in \mathbf{C}^n \mid w_1 \cdot \cdots \cdot w_n = 0\}$ the Jacobian determinant $\det \Psi'$ of Ψ has no zero. Indeed, Ψ is a proper mapping from $U_1 \backslash \Psi^{-1}(B)$ onto $U_2 = \Psi(U_1 \backslash \Psi^{-1}(B))$, with a finite number n_Ψ of sheets. On U_2 the function $g_1 : U_2 \longrightarrow \mathbf{R}^+$,

$$g_1(v) = \frac{1}{|\det \mathcal{M}'|} |v_1 - 1|^{-2n} \prod_{j=2}^{n} |v_i|^{\delta_i - 1}$$

is continuous, and for $w \in U_1 \backslash \bar{\psi}^{-1}(B)$, we have

$$|g(w)| = g_1 \circ \Psi |\det \Psi'|(w);$$

this implies

$$\text{(4.6)} \qquad \begin{aligned} \|g\|^2_{L^2(U_1)} &= \|g\|^2_{L^2(U_1 \backslash \Psi^{-1}(B))} \\ &= n_\Psi \|g_1\|^2_{L^2(U2)} \\ &= n_\Psi \|g_1\|^2_{L^2(\Psi(U_2))}. \end{aligned}$$

Now we compute $\|g_1\|^2_{L^2(\Psi(U_2))}$ by integrating at first with respect to v_1 and then with respect to $v' = (v_2, \cdots, v_n)$. We obtain, using the notation

$$p(v') = \sum_{j=d'+1}^{k} \mathscr{C}_{ji}(Q, t)^2 |v_i - Q'_{ji}|^2,$$

that

$$(4.7)\quad \|g_1\|^2_{L^2(\Psi(U_1))} = \int_{\Phi(\Delta_{n-1}(0,1))} |g_1(0, v')|^2 \left[\int_{\{\mathrm{Re}\, v_1 < -p(v')\}} \frac{d\lambda_1(v_1)}{|v_1 - 1|^{4n}}\right] d\lambda_{n-1}(v').$$

Here we used that $U_2 \subset U_3 := \{(v = (v_1, v') \in \mathbf{C}^n \mid \mathrm{Re}\; v_1 + p(v') < 0\}$. By explicit computation we find that

$$(4.8)\quad \int_{\{\mathrm{Re}\, v_1 < -p(v')\}} \frac{d\lambda_1(v_1)}{|v_1 - 1|^{4n}} = c_{10}(1 + p(v'))^{-(4n-2)},$$

where $c_{10} = \int_{\mathbf{R}}(1 + s^2)^{-2n}ds$. Next we estimate

$$(4.9)\quad (1 + p(v'))^{4n} \geq \prod_{i=d'+1}^{k} (1 + \mathscr{C}_{j_i}(Q, t)^2 |v_i - Q_{j_i}|^2)^4,$$

and substitute this into (4.6). So we get

$$(4.10)\quad \|g_1\|^2_{L^2(\Psi(U_1))} \leq c_{10}\int_{(v_{d'+1},\ldots,v_n) \in \Delta_{n-d'}(0,\,1)} \left[\prod_{i=d'+1}^{n} h_i(v_i)\right] J(v_{d'+1},\cdots, v_n)\, d\lambda_{n-d'},$$

where

$$h_i(v_i) = \begin{cases} |v_i|^{2\delta_i-2}(1 + \mathscr{C}_{j_i}(Q, t)^2 |v_2 - Q_{j_i}|^2)^{-4}, & \text{for } d' + 1 \leq i \leq k \\ |v_i|^{2\delta_i-2}, & \text{for } k + 1 \leq i \leq n, \end{cases}$$

and

$$J(v_{d'+1},\ldots, v_n) = \int_{S(v_{d'+1},\ldots,v_n)} |v_2|^{-2} \cdot \cdots \cdot |v_{d'}|^{-2} d\lambda_{d'-1}(v_2,\ldots,v_{d'}).$$

The integral is extended over the domain

$$S(v_{d'+1},\ldots, v_n) := \{(v_2,\ldots, v_{d'}) \in \Delta_{d'-1}(0, 1) : \\ |(v_2,\ldots, v_{d'}, v_{d'+1},\ldots, v_n) \in \Phi(\Delta_{n-1}(0, 1))\}.$$

In order to estimate $J(v_{d'+1},\ldots, v_n)$ we observe that we have for each $v \in \Psi(U_1)$:

(4.11) For any $2 \leq l \leq d'$ there exists an index $\lambda(l) \in \{d' + 1,\ldots, k\}$ independent of v such that $|v_l| \geq |v_{\lambda(l)}|$. To see this, we note that there exists a number $\lambda(l) \in \{d' + 1,\ldots, k\}$ with $\alpha_l = e^{(\lambda(l))}\mathscr{M}'(e^{(i_l)})^T > 0$. Otherwise one would have

$$\begin{aligned} 1 = (1,\ldots, 1)(e^{(i_l)})^T &= (0, 0,\ldots, 0, \delta_{d'+1},\ldots, \delta_n)\mathscr{M}'((e^{(i_l)})^T \\ &= \sum_{\nu=d'+1}^{n} \delta_\nu e^{(\nu)}\mathscr{M}'(e^{(i_l)})^T \\ &= \sum_{\nu=d'+1}^{k} \delta_\nu e^{(\nu)}\mathscr{M}'(e^{(i_l)})^T \\ &= 0. \end{aligned}$$

Here the second last equality follows from the fact that $e^{(\nu)}\mathscr{M}'(e^{(i_l)})^T = 0$ for $k+1$

$\leq \nu \leq n$. We now choose a pre-image $w' \in \Delta_{n-1}(0,1)$ for v' under Φ. Then

$$|v_{\lambda(l)}| = |w'^{\beta(\lambda(l))} \prod_{p=2}^{n} |w_p|^{e(\lambda(l))\mathcal{M}'(e(p))^T} \leq |w_{i_l}|^{a_l} \leq |w_{i_l}| = |v_l|.$$

Using this and integrating $|v_2|^{-2} \cdot \cdots \cdot |v_{d'}|^{-2}$ over $S(v_{d'+1},\ldots, v_n)$ we obtain

$$J(v_{d'+1},\ldots, v_n) \leq (2\pi)^{n-d'} \prod_{d'+1}^{k} \left(\log \frac{1}{|v_\lambda|}\right)^{\nu_\lambda}.$$

Here the ν_λ are certain integers in $\{0,\ldots, d'-1\}$ for which $\nu_{d'+1} + \cdots + \nu_k = d'-1$. We substitute this into (4.9). Also the integration over the $v_{k+1},\ldots, v_n$-variables can be carried out without any difficulties, since the δ_i are positive for $k+1 \leq i \leq n$. We then are left with an integral to which Fubini's theorem applies and gives us

$$\| g_1 \|^2_{L^2(\Psi(U_1))} \leq c_{11} \prod_{i=d'+1}^{k} \mathcal{T}_i, \tag{4.13}$$

where

$$\mathcal{T}_i = \int_{|v_i|<1} h_i(v_i) \left(\log \frac{1}{|v_i|}\right)^{\nu_i} d\lambda_1(v_i).$$

Now let us set $\tilde{Q}_i = Q_{ji}\mathcal{C}_{ji}(Q,t)$ and introduce the new variable $u_i = \mathcal{C}_{ji}(Q,t)v_i$ Then

$$\mathcal{T}_i = \mathcal{C}_{ji}(Q,t)^{-2\delta_i} \int_{|u_i| \leq \mathcal{C}_{ji}(Q,t)} |u_i|^{2\delta_i - 2} \frac{[\log(\mathcal{C}_{ji}(Q,t)/|u_i|)]^{\nu_i}}{(1+|u_i - \tilde{Q}_i|^2)^4} \times d\lambda_1(u_i). \tag{4.14}$$

If t is small enough, we have $\mathcal{C}_2(Q,t) \cdot \cdots \cdot \mathcal{C}_d(Q,t) > 3$. Then we can estimate

$$0 < \log \frac{\mathcal{C}_{ji}}{|u_i|} \leq (\log \mathcal{C}_{ji})(1 + \max\{\log \frac{1}{|u_i|}, 0\}). \tag{4.15}$$

Substituting this into (4.13) we obtain

$$\mathcal{T}_i \leq \mathcal{C}_{ji}(Q,t)^{-2\delta_i} \{\log \mathcal{C}_{ji})^{\nu_i} \mathcal{T}'_i, \tag{4.16}$$

where

$$\mathcal{T}'_i = \int_{\mathbf{C}} |u_i|^{2\delta_i - 2} \frac{(1 + \max\{\log \frac{1}{|u_i|}, 0\})^{\nu_i}}{(1+|u_i - \tilde{Q}_i|^2)^4} d\lambda_1(u_i).$$

Obviously we have for all $d'+1 \leq i \leq k$:

$$\mathcal{T}'_i \leq \int_{|u_i| \leq 1} |u_i|^{2\delta_i - 2}(1 + \log \frac{1}{|u_i|})^{\nu_i} d\lambda_1(u_i) + \int_{|u_i|>1} (1+|u_i - \tilde{Q}_i|^2)^{-4} d\lambda_1(u_i)$$

$$\leq 2\pi \int_0^1 x^{2\delta_i - 1}(\log(e/x))^{\nu_i} dx + \int_{\mathbf{C}} (1+|u_i|^2)^{-4} d\lambda_1(u_i).$$

So by virtue of (4.13), (4.14) we get the estimate

$$\| g_1 \|^2_{L^2(\Psi(U_1))} \leq c_{12} \Big(\prod_{i=d'+1}^{k} \mathscr{C}_{ji}(Q,t)^{-2\delta_i} \Big) \Big(\log \sum_{i=d'+1}^{k} \mathscr{C}_{ji}(Q,t) \Big)^{d'-1}$$

(since $\nu_{d'+1} + \cdots + \nu_k = d' - 1$). This, combined with (4.6) implies (4.5). The theorem now follows since $\mathscr{M}$ was chosen arbitrarily in $GL(n-1, \mathbf{Q})$ such that (M 1) and (M 2) are satisfied.

§ 5. Appendix: Proof of Lemma 3

We will argue in a more general setting. Let m, N be positive integers, $N \geq m$, and $\alpha^{(1)}, \ldots, \alpha^{(N)} \in \mathbf{R}^m \setminus \{0\}$ be vectors with nonnegative entries, and $b_1, \ldots, b_N \in \mathbf{R}^+$. We consider the simplex

$$\Sigma = \{x \in \mathbf{R}^m \,|\, (\alpha^{(i)}, x) \geq b_i,\ 1 \leq i \leq N\},$$

where $(\cdot, \cdot)$ denotes the euclidean scalar product in R^m, and fix a linear functional $\tilde{L}: \mathbf{R}^m \longrightarrow \mathbf{R}$ such that $\tilde{L}_0 = \min\{\tilde{L}(x) \,|\, x \in \Sigma\} > -\infty$ exists. Further we define the set

$$\mathscr{A}_0 := \{M \in GL(\mathrm{m}, \mathbf{R}) \,|\, e_i M \in \{a^{(1)}, \ldots, a^{(N)}\},\ 1 \leq i \leq m\},$$

where for $1 \leq i \leq m$ we denote by e_i the i th unit vector in $\mathbf{R}^m$. For each $M \in \mathscr{A}_0$ we denote by $\nu(M,i)$ the unique number in $\{1, \ldots, N\}$ such that $e_i M = \alpha^{(\nu(M,i))}$, then we will derive Lemma 3 from the following lemma:

Lemma 4. *Assume $x^0 \in \Sigma$ is a corner and it is the only one where $\tilde{L}$ attains its minimum value $\tilde{L}_0$ on Σ. Then there exists an invertible matrix M satisfying*

(5.1) $M \in \mathscr{A}_0$

(5.2) $M(x^0)^T = \sum_{i=1}^m b_{\nu(M,i)} e_i^T$

(5.3) $\tilde{L}(M^{-1} e_i^T) \geq 0$ *for* $1 \leq i \leq m$.

Here y^T is the column vector associated to $y \in \mathbf{R}^m$.

Proof. After renumbering the $\alpha^{(i)}$'s we can assume that with a number $S \in \{1, \ldots, N\}$ we have $(\alpha^{(j)}, x^0) = b_j$ for $1 \leq j \leq S$, and $(\alpha^{(j)}, x^0) > b_j$ if $j \geq S+1$. We set

$$\mathscr{A} = \{M \in \mathscr{A}_0 \,|\, \text{The rows of } M \text{ belong to } \{\alpha^{(i)} \,|\, 1 \leq i \leq S\}\}.$$

Since x^0 is a corner of Σ, the set $\mathscr{A}$ is not empty, and $S \geq m$. The following number ρ is positive:

$$\rho = \min\{| \widetilde{M}(x^0)^T - \sum_{j=i}^{m} b_{\nu(M,j)} e_j^T \| \, \widetilde{M} \in \mathscr{A}_o \setminus \mathscr{A}\}.$$

We can then choose a positive number $\delta < \rho / 2\sum_{l=1}^{N} | \alpha^{(l)}|$ so small that x^0 is the only corner of $\sum$ within the ball $B_m(x^0, 2\delta)$. Also let γ be a positive number for which $\tilde{L}(x) \geq \tilde{L}_0 + \gamma$ for $x \in \sum$, $B^m(x^0, 2\delta)$. For positive numbers $\varepsilon_1, \dots, \varepsilon_N$, which will be chosen later appropriately, we define the simplex

$$\textstyle\sum' = \{x \in \mathbf{R}^m \mid (\alpha^{(j)}, x) \geq \frac{b_j}{1 + \varepsilon_j}, 1 \leq j \leq N\}$$

and let $\varepsilon_0 = \max\{\varepsilon_1, \cdots, \varepsilon_N\}$. Since $b_j > 0$ for all $1 \leq j \leq N$ we have $(1 + \varepsilon_0)x \in \sum$ whenever $x \in \sum'$, and hence $\tilde{L}_0' := \min\{\tilde{L}(x) | x \in \sum'\} \geq \tilde{L}_0/(1 + \varepsilon_0)$. Since $\sum \subset \sum'$, also $\tilde{L}_0' \leq \tilde{L}_0$. Now we define the $\varepsilon_1, \dots, \varepsilon_N$. For each pair $(i, \widetilde{M}) \in \mathscr{T} = \{\{1, \cdots, S\} \times \mathscr{A}$ with $i \notin \{\nu(\widetilde{M}, 1), \cdots, \nu(\widetilde{M}, m)\}$ let us consider the real linear form $\lambda_{i,\widetilde{M}}$ on $\mathbf{R}^S$:

$$\lambda_{i,\widetilde{M}}(t_1, \cdots, t_S) = b_i t_i - \sum_{j=2}^{m} b_{\nu(\widetilde{M},j)} \left(a^{(i)} \widetilde{M}^{-1}, e_j\right) t_{\nu(\widetilde{M},j)}$$

(Note that for $\widetilde{M} \varepsilon \mathscr{A}$ always $\{\upsilon(\widetilde{M}, l) \mid 1 \leq l \leq m\} \subset \{1, \cdots, S\}$!}. For any $0 < \varepsilon' < 1$ we can find $0 < \varepsilon_1, \cdots, \varepsilon_S < \varepsilon'$ for which

$$\lambda_{t,\widetilde{M}}\left(\frac{\varepsilon_1}{1 + \varepsilon_1}, \cdots, \frac{\varepsilon_S}{1 + \varepsilon_S}\right) \neq 0 \tag{5.4}$$

for any pair $(i, \overline{M} \in \mathscr{T}$. If $S=N$, let

$$\varepsilon' < \eta := \min\{\frac{\rho}{4mb}, \frac{\gamma}{1 + |\tilde{L}_0|}, \frac{\delta}{2 + 2\delta + \| x^0 \|_{\mathbf{R}^m}}\}.$$

(Here we set $b = b_1 + \cdots + b_N$). If $S < N$ we choose $0 < \varepsilon' < \eta$, such that

$$\varepsilon' \sum_{j=s+1}^{N} \sum_{j=1}^{S} b_j \| \alpha^{(i)} \widetilde{M}^{-1} \|_{\mathbf{R}^m} < \frac{1}{2} b_l \tag{5.5}$$

for any $\widetilde{M} \in \mathscr{A}$, $l \in \{S + 1, \cdots, N\}$.

Let $x^{(1)}$ be a corner of $\sum'$ with $\tilde{L}(x^{(1)}) = \tilde{L}_0'$. Then there is a matrix $M \in \mathscr{A}_0$ such that

$$M(x^{(1)})^T = \sum_{j=1}^{m} \frac{1}{1 + \varepsilon_{\nu(M,j)}} b_{\nu(M,j)} e_j^T, \tag{5.6}$$

We claim that $M \in \mathscr{A}$. If not, we would have $x^{(1)} \notin B_m(x^0, \frac{3}{2}\delta)$. Otherwise

$$\begin{aligned} \frac{3}{4}\rho > \frac{3}{2}\delta \sum_{l=1}^{N} \| \alpha^{(l)} \|_{\mathbf{R}^m} &\geq \| M(x^{(1)} - x^0) \|_{\mathbf{R}^m} \\ &\geq | \sum_{j=1}^{m} b_{\nu(M,j)} e_j^T - Mx^0 | - \sum_{j=1}^{m} \frac{\varepsilon_{\nu(M,j)}}{1 + \varepsilon_{\nu(M,j)}} b_{\nu(M,j)} \\ &\geq \rho - m \epsilon_0 > \frac{3}{4}\rho, \end{aligned}$$

a contradiction. Since $\varepsilon_0 < \delta/2 + 2\delta + \|x^0\|_{\mathbf{R}^m}$, we have $(1+\varepsilon_0)x^{(1)} \in \Sigma \cap \mathbf{R}^m \setminus B_m(x_0,\delta)$. So this implies $\tilde{L}_0 \geq \tilde{L}_0' = \frac{1}{1+\varepsilon_0}\tilde{L}((1+\varepsilon_0)x^{(1)}) \geq \frac{1}{1+\varepsilon_0}(\tilde{L}_0 + \gamma) > \tilde{L}_0$, since $\varepsilon_0 < \gamma/1 + |\tilde{L}_0|$; so we can conclude that $M \varepsilon \mathscr{A}$, and, in particular, (5.1) is satisfied by M. As M even belongs to $\mathscr{A}$ it satisfies also (5.2), and we see from (5.6) that we can write $x^{(1)}$ as

$$x^{(1)} = x^0 - \sum_{j=1}^{m} \frac{\varepsilon_{\nu(M,j)}}{1+\varepsilon_{\nu(M,j)}} b_{\nu(M,j)} M^{-1} e_j^T.$$

In order to check property (5.3) we observe at first that for all $i \notin \{\upsilon(M,j) \mid j = 1,\dots, m\}$ the inequality $(a^{(i)}, x^{(1)}) > b_i/1 + \epsilon_i$ holds. To see this we distinguish two cases : Let $i \in \{1,\dots, S\}$. Then, as $x^{(1)} \in \Sigma'$:

$$0 \leq (\alpha^{(i)}, x^{(1)}) - \frac{b_i}{1+\varepsilon_i} = \lambda_{i,M}\left(\frac{\varepsilon_1}{1+\varepsilon_i}, \cdots, \frac{\varepsilon_S}{1+\varepsilon_S}\right).$$

But since the right side is nonzero, it must be positive.

Let $i \in \{S+1,\dots, N\}$. In this case the assertion follows from the choice of the $\varepsilon_{S+1}, \cdots, \varepsilon_N$. So for a small enough positive number τ the point $x^{(1)} + \tau M^{-1}e_i^T$ belongs to Σ' for $1 \leq i \leq m$, and therefore $\tilde{L}(M^{-1}e_i^T) = \frac{1}{\tau}(\tilde{L}(x^{(1)} + \tau M^{-1}e_i^T) - \tilde{L}(x^{(1)}) \geq 0$. The proof of Lemma 4 is now complete.

We can now easily prove Lemma 3.

Let us keep up the notations from Section 1. In the situation of Lemma 3 we have $m = n-1$, and $N =$(number of elements of $\bar{A}$) $+1$. Let us suppose at first that $\tilde{L}$ is a real linear form on $\mathbf{R}^{n-1}$, such that x^0 is the only minimum corner for $\tilde{L}$ on Σ_A. Then we write $A = \{\alpha^{(2)},\dots, \alpha^{(d)}\}$ and $\bar{A} \setminus A = \{a^{(d+1)},\dots, \alpha^{(N)}\}$. Since $\alpha^{(i)} \in \mathbf{N}_0^{n-1}\setminus\{0\}$, the numbers $c_i = \eta(\alpha^{(i)}, e)$, where $e = (1, 1,\dots, 1)$, are all positive, if $\eta > 0$. Now let us set

$$b_i = \begin{cases} \dfrac{1}{2m_i} + c_i, & \text{for } 2 \leq i \leq d \\ c_i, & \text{for } d+1 \leq i \leq N. \end{cases}$$

Therefore $x^0 + \eta e$ is the only minimum corner for $\tilde{L}$ on $\Sigma = \{x \in \mathbf{R}^{n-1} \mid (\alpha^{(i)},x) \geq b_i, \text{ for } 2 \leq i \leq N\}$. By Lemma 4 there exists a matrix $\mathcal{M} \in GL(n-1, \mathbf{Q})$, the rows of which belong to $\{\alpha^{(2)},\dots, \alpha^{(N)}\}$, such that

$$\mathcal{M}(x^0 + \eta e)^T = \sum_{l=2}^{n} b_{\nu(\mathcal{M},l)}(e^{(l)})^T,$$

and

$$\tilde{L}(\mathcal{M}^{-1}(e^{(l)})^T) \geq 0,$$

for $l = 2,\dots, n$. Here $\nu(\mathcal{M}, l)$ is supposed to have the same meaning as in the proof of Lemma 4). Then x^0 is a minimum corner for $\tilde{L}$ on the simplex

$$\sum_{\mathcal{M}} = \{x \in \mathbf{R}^{n-1} \,|\, (\alpha^{(i)}, x) \geq b_{\nu(\mathcal{M},i)} - c_{\nu(\mathcal{M},i)}, \text{ for } 2 \leq i \leq \mathrm{n}\}.$$

Namely, we have

$$\begin{aligned} x - x^0 &= \mathcal{M}^{-1}(\mathcal{M}(x+\eta e)^T - \mathcal{M}(x^0+\eta e)^T) \\ &= \sum_{j=2}^{n}\Big[(\alpha^{(\nu\mathcal{M},i))}, x+\eta e) - b_{\nu(\mathcal{M},i)}\Big]\mathcal{M}^{-1}(e^{(i)})^T \\ &= \sum_{j=2}^{n}\Big[(\alpha^{(\nu\mathcal{M},i))}, x) - (b_{\nu(\mathcal{M},i)} - c_{\nu(\mathcal{M},i)})\Big]\mathcal{M}^{-1}(e^{(i)})^T, \end{aligned}$$

and hence

$$\widetilde{L}(x) - \widetilde{L}(x^0) = \sum_{j=2}^{n}\Big[(\alpha^{(\nu\mathcal{M},i))}, x) - (b_{\nu(\mathcal{M},i)} - c_{\nu(\mathcal{M},i)})\Big]\,\widetilde{L}(\mathcal{M}^{-1}(e^{(i)})^T) \geq 0.$$

Finally we approximate the aim functional $L(x) = x_2 + \cdots + x_n$ in the operator norm by linear functionals $\widetilde{L}_j$, such that x^0 is the only mimimum corner for $\widetilde{L}_j$ on $\sum_A$. After selecting a convergent subsequence from the $\widetilde{L}_j$'s we will find a matrix $\mathcal{M} \in GL(n-1, \mathbf{Q})$ satisfying (M 1) and (M 2), and for all $j \geq 1$ and $x \in \sum_{\mathcal{M}}$ one has $\widetilde{L}_j(x) \geq \widetilde{L}(x^0)$. By letting $j \to \infty$ we find that $\mathcal{M}$ is the desired matrix.

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