THE ARC-LENGTH VARIATION OF ANALYTIC CAPACITY
AND A CONFORMAL GEOMETRY

TAKAFUMI MURAI

§1. Introduction

For a domain Ω in the extended complex plane $\mathbf{C} \cup \{\infty\}$, $H^\infty(\Omega)$ denotes the Banach space of bounded analytic functions in Ω with supremum norm $\| \cdot \|_{H^\infty}$. For $\zeta \in \Omega$, we put

$$c(\zeta; \Omega) = \sup\{|f'(\zeta)|; f \in H^\infty(\Omega), \|f\|_{H^\infty} \le 1\},$$

where $f'(\infty) = \lim_{z \to \infty} z\{f(\infty) - f(z)\}$ if $\zeta = \infty$. The analytic capacity of a compact set E in $\mathbf{C}$ is defined by

$$\gamma(E) = c(\infty; \Omega_E),$$

where Ω_E is the component of $E^c = \mathbf{C} \cup \{\infty\} - E$ containing ∞. Ahlfors [1] shows that, for a domain Ω and a compact set $E \subset \Omega$, the equality $\gamma(E) = 0$ holds if and only if any function in $H^\infty(\Omega \cap \Omega_E)$ has an analytic extension to Ω. Thus a compact set E satisfying Riemann's theorem of removable singularities is characterized by $\gamma(E) = 0$. Garabedian [8] studies analytic capacity from the point of view of the dual extremum problem. The quantitative properties of analytic capacity are important in the study of conformal mappings, the 2-dimensional fluid dynamics and singular integrals ([16], [17], [23]). Vitushkin [34], Gamelin [7], Garnett [11], Zalcman [35] show that $\gamma(\cdot)$ is applicable to study approximation problems, and Ahlfors-Beurling [2], Pommerenke [22], Suita [30], [31], [32] study $\gamma(\cdot)$ from the point of view of a conformal invariant. The author studied $\gamma(\cdot)$ in terms of integral geometry [17], [18] and fluid dynamics [19].

In this paper, we are concerned with the variation of $\gamma(E)$ for a small change of E. The approach in this direction is given by Havinson [13], Garabedian-Schiffer [10], Schiffer-Hawley [27], Schiffer-Spencer [28], Smith [29]. We shall discuss, in this paper, a variation of $\gamma(E)$ for a

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finite union E of arcs. Here are two motivations to deal with such a set.
Given a compact set $F \subset \mathbf{C}$ with a smooth boundary, we can study the
Hadamard variation and the Schiffer variation [25]. Then we obtain a
finite union E of arcs such that $\gamma(F)$ is expressed as a perturbation from
$\gamma(E)$ and the number of components of E is less than or equal to that
of F. Thus, in order to get global properties, it is necessary to study a
variation of $\gamma(E)$. Another motivation is as follows. Given a compact
set F, we can find a finite union G of closed disks so that $|\gamma(F) - \gamma(G)|$
is arbitrarily small. Removing some arcs from G, we obtain a finite union
E of arcs such that $|\gamma(G) - \gamma(F)|$ is arbitrarily small and the number of
components of E is less than or equal to that of F. Thus a variation of
$\gamma(E)$ is necessary. As is seen in the second motivation, for our variation,
we may restrict our attention to compact sets E such that E consists of a
finite number of mutually disjoint arcs $\{C_j\}_{j=1}^n$ and each arc C_j is expressed
as a finite union of closed analytic arcs, where a closed arc is analytic
if it is contained in an open analytic arc. Let $\mathscr{A}$ denote the totality of
such compact sets and let $\mathscr{F}$ denote the totality of domains Ω such that
$\Omega^c \in \mathscr{A}$. For $E \in \mathscr{A}$, it is natural to study $\gamma(E)$ in terms of the arc-length
$|dz|$. From this point of view, we shall focus on the variation of $\gamma(E)$
with respect to $|dz|$, and, as application of our results, we shall study the
structure of $\gamma(A \cup B)$ for two bounded continua A and B. For $f \in H^\infty(\Omega_E)$,
$-f'(\infty)$ is the $1/z$-coefficient of f at ∞ and, in the case where E is con-
nected, the Ahlfors function (i.e., the function attaining $\gamma(E)$) is nothing
but Riemann's mapping from Ω_E onto the unit disk $\mathbf{D}$. Thus our principle
is related to Löwner's differential equation for multiply-connected domains
(cf. [15, p. 116]). In fact, we shall deduce, in § 9, Löwner's D. E. for simply-
connected domains [33, p. 387] from our formula in § 3. For $\Omega \in \mathscr{F}$, $\partial\Omega$
denotes the boundary of Ω having two sides; to each $z \in F$ ($F = \Omega^c -$
{endpoints}), there correspond two points $z_\pm \in \partial\Omega$. Let $H^2(\Omega)$ denote the
H^2-space of analytic functions in Ω such that

$$(1) \quad \lim_{\delta \downarrow 0} \int_{S(\zeta,\delta)} |f(z)|^2 |dz| = 0 \ (\zeta \in \{\text{endpoints of } F\})\,,$$

$$(2) \quad \|f\|_{H^2} = \left(\int_F \{|f(z_+)|^2 + |f(z_-)|^2\} |dz| \right)^{1/2} < \infty\,,$$

where $f(z_\varepsilon)$ is the non-tangential limit of f at $z_\varepsilon \in \partial\Omega$ ($\varepsilon = \pm$) and $S(\zeta, \delta)$
is the circle of center ζ and radius δ. For $\Omega \in \mathscr{F}$, there uniquely exists
a pair $(g(\cdot\,; \Omega), \phi(\cdot\,; \Omega))$ of functions in $H^2(\Omega)$ such that $g(\infty; \Omega) = 0$,

$\phi(\infty; \Omega) = 1$ and

(3) $\quad \dfrac{1}{i}\phi(z; \Omega)\ dz = \overline{g(z; \Omega)}|dz|$ almost everywhere (a.e.) on $\partial\Omega$,

where the orientation of dz is chosen so that Ω lies to the left [8].   For any $\zeta \in \Omega - \{\infty\}$, there exists uniquely a pair $(K(\cdot, \zeta; \Omega), L(\cdot, \zeta; \Omega))$ of functions such that $K(\cdot, \zeta; \Omega)$, $(\cdot - \zeta)L(\cdot, \zeta; \Omega) \in H^2(\Omega)$, $K(\infty, \zeta; \Omega) = L(\infty, \zeta; \Omega) = 0$, $\lim_{z \to \zeta}(z - \zeta)L(z, \zeta; \Omega) = 1$ and

(4) $\quad \dfrac{1}{i}L(z, \zeta; \Omega)dz = \overline{K(z, \zeta; \Omega)}|dz|$ a.e. on $\partial\Omega$ [4] .

The kernel $K(z, \zeta; \Omega)$ is called the Szegö kernel and $L(z, \zeta; \Omega)$ is called the L-kernel [4]. (The Szegö kernel is written by $\hat{K}$ in [4]. In this note, we use the notation K.) For $z, \zeta \in \Omega - \{\infty\}$, $z \neq \zeta$, we define

$$Dc(z, \infty; \Omega) = |\phi(z; \Omega)|^2 - |g(z; \Omega)|^2 ,$$
$$D^2c(z, \zeta, \infty; \Omega) = 2\,\mathrm{Re}\,[\{L(z, \zeta; \Omega)\overline{g(z; \Omega)} - \overline{K(z, \zeta; \Omega)}\phi(z; \Omega)\}\overline{\phi(\zeta; \Omega)}$$
$$- \{L(z, \zeta; \Omega)\overline{\phi(z; \Omega)} - \overline{K(z, \zeta; \Omega)}g(z; \Omega)\}\overline{g(\zeta; \Omega)}] ,$$

where Re denotes the real part. For a closed analytic arc Γ, a continuous function $w_t \in \Gamma$ on an interval $[0, |\Gamma|]$ ($|\Gamma|$ is the length of Γ) is called the arc-length representation of Γ, if w_0, $w_{|\Gamma|}$ are endpoints of Γ and $|\Gamma_t| = t$ $(0 \leq t \leq |\Gamma|)$, where $\Gamma_t = \{w_s; 0 \leq s \leq t\}$. For $\Omega \in \mathscr{F}$ such that $\Omega - \Gamma \in \mathscr{F}$ and $\Omega^c \cap \Gamma$ is at most a singleton, we write $\Omega_t = \Omega - \Gamma_t$ $(0 \leq t \leq |\Gamma|)$; we assume that $\Omega^c \cap \Gamma = \{w_0\}$ if $\Omega^c \cap \Gamma \neq \emptyset$. We shall show

THEOREM 1.
  (5) *For any $0 < t \leq |\Gamma|$, the derivative $\partial c(\infty; \Omega_t)/\partial t$, the limit*
    $\lim_{u \downarrow t} Dc(w_u, \infty; \Omega_t)$ $(= Dc(w_t, \infty; \Omega_t)$, say) *exist and $\partial c(\infty; \Omega_t)/\partial t$*
    $= Dc(w_t, \infty; \Omega_t)/4$. *The right-derivative $\partial c(\infty; \Omega_0)/\partial t$ at $t = 0$ exists*
    and $\partial c(\infty; \Omega_t)/\partial t$ is continuous on $[0, |\Gamma|]$.
  (6) *For any $0 < t \leq |\Gamma|$ and any $z \in \Omega - (\Gamma \cup \{\infty\})$, the derivative*
    $\partial Dc(z, \infty; \Omega_t)/\partial t$, *the limit* $\lim_{u \downarrow t} D^2c(w_u, z, \infty; \Omega_t)$ $(= D^2c(w_t, z, \infty; \Omega_t)$,
    say) exist and $\partial Dc(z, \infty; \Omega_t)/\partial t = D^2c(w_t, z, \infty; \Omega_t)/4$. For any $z \in \Omega$
    $- (\Gamma \cup \{\infty\})$, *the right-derivative $\partial D(z, \infty; \Omega_0)/\partial t$ at $t = 0$ exists and*
    $\partial Dc(z, \infty; \Omega_t)/\partial t$ *is continuous on $[0, |\Gamma|]$.*

In this theorem, the derivative and the limit at $t = |\Gamma|$ are defined by the arc-length representation of an open analytic arc containing Γ. The right-derivative $\partial c(\infty; \Omega_0)/\partial t$ is not, in general, equal to $Dc(w_0, \infty; \Omega_0)/4$.

(See Remark 19.) To study $Dc(z, \infty; \Omega)$ and $D^2c(z, \zeta, \infty; \Omega)$, we introduce a wider class of domains. Let $\mathscr{G}$ denote the totality of domains Ω with the following property: Ω is expressed as $\Omega = \Omega^* - E$ with $E \in \mathscr{A}$ and a domain Ω^*, $\Omega^* \supset E$ bounded by a finite number of Jordan curves $\{C_j\}_{j=1}^n$ such that each C_j is a finite union of closed analytic arcs. For $\Omega \in \mathscr{G}$, the Szegö kernel and the L-kernel exist; the condition at ∞ is removed if $\Omega \ni \infty$. For three distinct numbers $w, z, \zeta \in \Omega$, we define

$$Dc(z, \zeta; \Omega) = |L(z, \zeta; \Omega)|^2 - |K(z, \zeta; \Omega)|^2,$$
$$D^2c(w, z, \zeta; \Omega) = 2\,\mathrm{Re}\,\{DL(w, z, \zeta; \Omega)\overline{L(z, \zeta; \Omega)}$$
$$- DK(w, z, \zeta; \Omega)\overline{K(z, \bar{\zeta}; \Omega)}\},$$

where

$$DK(w, z, \zeta; \Omega) = L(w, z; \Omega)\overline{L(w, \zeta; \Omega)} - \overline{K(w, \bar{z}; \Omega)}K(w, \zeta; \Omega),$$
$$DL(w, z, \zeta; \Omega) = L(w, z; \Omega)\overline{K(w, \bar{\zeta}; \Omega)} - \overline{K(w, \bar{z}; \Omega)}L(w, \zeta; \Omega).$$

In the above definition, we replace $K(\cdot, \overline{\infty}; \Omega) = \overline{K(\infty, \cdot\,; \Omega)}$ by $-g(\cdot\,; \Omega)$, and replace $L(\cdot, \infty; \Omega) = -L(\infty, \cdot\,; \Omega)$ by $-\phi(\cdot\,; \Omega)$ if $\Omega \ni \infty$ and one of w, z, ζ is ∞. The following proposition plays an important role in our conformal geometry.

PROPOSITION 2. *The differential forms $Dc(z, \zeta; \Omega)|dz||d\zeta|$ and $D^2c(w, z, \zeta; \Omega)|dw||dz||d\zeta|$ are conformally invariant, $Dc(z, \zeta; \Omega) = Dc(\zeta, z; \Omega)$ and $D^2c(w, z, \zeta; \Omega)$ is invariant for any permutation of a triple (w, z, ζ).*

In this proposition, $Dc(z, \zeta; \Omega)|dz||d\zeta|$ is conformally invariant in the following sense: $Dc(z, \zeta; \Omega) = Dc(h(z), h(\zeta); h(\Omega))|h'(z)h'(\zeta)|$, if h is conformal, where

$$h'(z) = \begin{cases} \lim_{\zeta \to \infty} \zeta\{h(\zeta) - h(\infty)\} & \text{if } z = \infty,\ h(\infty) \neq \infty \\ \lim_{\zeta \to \infty} h(\zeta)/\zeta & \text{if } z = \infty,\ h(\infty) = \infty \\ \lim_{\zeta \to z} (z - \zeta)h(\zeta) & \text{if } z \neq \infty,\ h(z) = \infty. \end{cases}$$

The meaning of the conformal invariance of $D^2c(w, z, \zeta; \Omega)|dw||dz||d\zeta|$ is analogous. Our results are applied as follows. Given $E \in \mathscr{A}$, we can write $E = C_1 \cup \cdots \cup C_n$ with mutually disjoint arcs $\{C_j\}_{j=1}^n$. Using the arc-length representations of C_j $(1 \leq j \leq n)$, we define a right-continuous arc-length representation W_t $(0 \leq t \leq |E|)$ of E. Let $\Omega_t = \{W_s; 0 \leq s \leq t\}^c$ $(0 \leq t \leq |E|)$. Then $c(\infty; \Omega_t)$ is continuous on $[0, |E|]$. Theorem 1 shows that

$$(7) \quad \gamma(E) = c(\infty\,;\,E^c) = \frac{1}{4}\int_0^{|E|} Dc(W_t,\,\infty\,;\,\Omega_t)dt\,.$$

The quantity $Dc(W_t,\,\infty\,;\,\Omega_t)$ is defined by the limit $\lim_{u\downarrow t} Dc(W_u,\,\infty\,;\,\Omega_t)$. Thus, to study $Dc(W_t,\,\infty\,;\,\Omega_t)$, it is sufficient to investigate $Dc(W_u,\,\infty\,;\,\Omega_t)$ for $u > t$ sufficiently near to t. Given $0 < t < |E|$ and $z \in \Omega_t - \{\infty\}$, we take a right-continuous arc-length representation W_s^* $(0 \le s \le t)$ of Ω_t^c; W_s^* may not be equal to W_s, in general. Since $Dc(z,\,\infty\,;\,\Omega_s^*)$ $(\Omega_s^* = \{W_u^*\,; 0 \le u \le s\}^c)$ is continuous on $[0, t]$, we have

$$(8) \quad Dc(z,\,\infty\,;\,\Omega_t) = Dc(z,\,\infty\,;\,\Omega_s^*) + \frac{1}{4}\int_s^t D^2c(W_u^*,\,z,\,\infty\,;\,\Omega_u^*)du$$

$$(0 \le s \le t)\,.$$

Using Proposition 2, we can study $D^2c(w,\,z,\,\infty\,;\,\Omega)$ by taking conformal mappings from Ω onto canonical domains. We shall show

Theorem 3. *If Ω is simply or doubly connected, then $D^2c(w,\,z,\,\zeta\,;\,\Omega) \le 0$.*

As application of this theorem, we here deduce Suita's subadditivity [32]: $\gamma(A \cup B) \le \gamma(A) + \gamma(B)$, if A and B are disjoint two continua. For the sake of simplicity, we work only with $A, B \in \mathscr{A}$. Using (7) with $E = A \cup B$ and a right-continuous arc-length representation W_t $(0 \le t \le |A \cup B|)$ such that $W_0 \in A$, we have

$$\gamma(A \cup B) = \gamma(A) + \frac{1}{4}\int_{|A|}^{|A \cup B|} Dc(W_t,\,\infty\,;\,\Omega_t)dt\,.$$

For $|A| < t \le |A \cup B|$, we take an arc-length representation W_s^* $(0 \le s \le t)$ of $\{W_s\,; 0 \le s \le t\}$ so that $W_0^* = W_{|A|}(\in B)$. Then $W_{t-|A|}^* = W_t$. Using (8), we have, with $\Omega_s^* = \{W_u^*\,; 0 \le u \le s\}^c$,

$$Dc(z,\,\infty\,;\,\Omega_t) = Dc(z,\,\infty\,;\,\Omega_{t-|A|}^*) + \frac{1}{4}\int_{t-|A|}^t D^2c(W_s^*,\,z,\,\infty\,;\,\Omega_s^*)ds$$
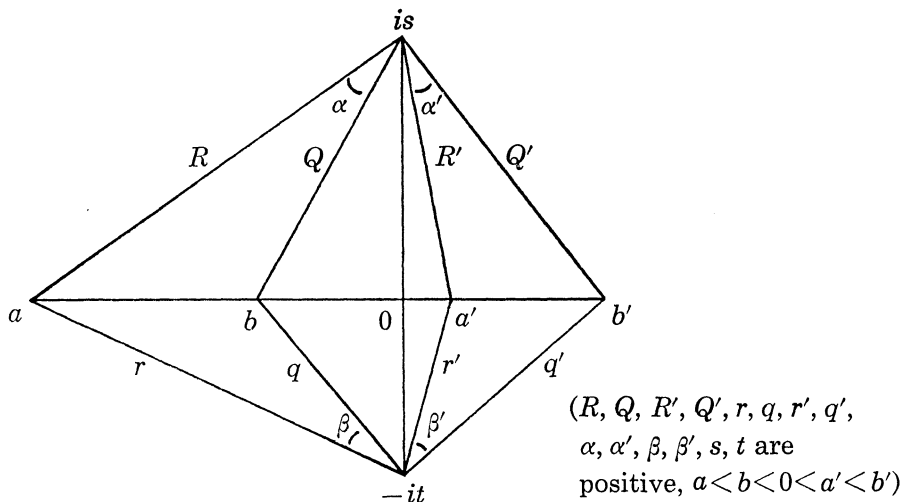
$$\le Dc(z,\,\infty\,;\,\Omega_{t-|A|}^*) \quad (z \in \Omega_t - \{\infty\})\,.$$

Letting z tend to $W_{t-|A|}^* = W_t$ along B, we obtain $Dc(W_t,\,\infty\,;\,\Omega_t) \le Dc(W_{t-|A|}^*,\,\infty\,;\,\Omega_{t-|A|}^*)$. Since

$$\frac{1}{4}\int_{|A|}^{|A \cup B|} Dc(W_t,\,\infty\,;\,\Omega_t)dt \le \frac{1}{4}\int_{|A|}^{|A \cup B|} Dc(W_{t-|A|}^*,\,\infty\,;\,\Omega_{t-|A|}^*)dt = \gamma(B)\,,$$

we obtain $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$. Suita's method is based on Rengel's inequality [33, p. 393] which is an application of Bieberbach's area method. On the other hand, our method is based on a differential equation for arcs and yields a quantitative estimate of $\gamma(A) + \gamma(B) - \gamma(A \cup B)$. The inequality $D^2c \leq 0$ for simply-connected domains is closely related to the triangle inequality with respect to the hyperbolic distance. The study of D^2c for doubly-connected domains is related to the theory of elliptic functions [12] and the proof of $D^2c \leq 0$ consequently reduces to an inequality of Möbius type in elementary geometry:

$$\frac{QQ'rr'}{RR'qq'} + \frac{RR'qq'}{QQ'rr'} \leq 2\cos(\alpha + \alpha')\cos(\beta + \beta')$$

$$+ \frac{s^2 + t^2}{st}\sin(\alpha + \alpha')\sin(\beta + \beta').$$

$(R, Q, R', Q', r, q, r', q', \alpha, \alpha', \beta, \beta', s, t$ are positive, $a < b < 0 < a' < b')$

In §§ 8–9, we shall study the structure of D^2c for simply and doubly connected domains. For general multiply-connected domains, the behaviour of D^2c is very complicated, and we do not refer to the general case in this paper. In § 2, we shall prepare some elementary properties which will be used later. In § 3, we shall show a formula for Szegö kernels and shall deduce Theorem 1 from this formula. The proof of the formula will be given in §§ 4–7. The proof of Theorem 3 will be given in §§ 8–9.

§2. Elementary properties

In this section, we show some elementary properties which are used later. For $p \geq 1$ and $\Omega \in \mathscr{G}$, $H^p(\Omega)$ denotes the H^p-space of functions in Ω defined by (1) and (2) with the power 2 replaced by p; $\partial\Omega$ has two sides on the components of Ω^c corresponding to arcs, $\partial\Omega$ is single on the components of Ω^c corresponding to Jordan curves, and the integral is taken over all $\partial\Omega$ with respect to $|dz|$. We put

$$c^*(\zeta; \Omega) = \inf\left\{\frac{1}{2\pi}\int_{\partial\Omega}\left|\frac{\psi(w)}{w - \zeta}\right|^2|dw|; \psi \in H^2(\Omega), \psi(\zeta) = 1\right\} \quad (\zeta \neq \infty),$$

$$c^*(\infty; \Omega) = \inf\left\{\frac{1}{2\pi}\|\phi\|_{H^2}^2; \phi \in H^2(\Omega), \phi(\infty) = 1\right\} \quad (\Omega \ni \infty).$$

Then $c^*(\zeta; \Omega) = c(\zeta; \Omega)$ $(\zeta \in \Omega)$ [8]. The pair $(K(z, \bar{\zeta}; \Omega), L(z, \zeta; \Omega))$ of the Szegö kernel and the L-kernel is unique and satisfies

$$K(z, \bar{\zeta}; \Omega) = \overline{K(\zeta, \bar{z}; \Omega)}, \quad L(z, \zeta; \Omega) = -L(\zeta, z; \Omega).$$

The Szegö kernel $K(z, \bar{\zeta}; \Omega)$ is a reproducing kernel in the following sense:

$$f(\zeta) = \frac{1}{2\pi}\int_{\partial\Omega}\overline{K(z, \bar{\zeta}; \Omega)}f(z)|dz| \quad (f \in H^2(\Omega), f(\infty) = 0) \text{ [4]}.$$

Making a double of Ω, we can treat the pair as a kernel on the Riemann surface. When no confusion can arise, we write simply $K(z, \bar{\zeta})$ and $L(z, \zeta)$. Let $D(z, r)$ denote the open disk of center z and radius r, and let $\mathbf{D} = D(0, 1)$. We remark

LEMMA 4. *Let $\Omega \ni \infty$. Then $\phi(z; \Omega)$ is the function attaining $c^*(\infty; \Omega)$ and*

$$g(z; \Omega) = \frac{1}{2\pi}\int_{\partial\Omega}\frac{1}{w - z}\overline{\phi(w; \Omega)}|dw|.$$

Let $\zeta \neq \infty$. Then

$$(9) \quad K(z, \bar{\zeta}) = \frac{1}{2\pi}\int_{\partial\Omega}\frac{1}{(w - z)\overline{(w - \zeta)}}\overline{\psi_\zeta(w)}|dw|,$$

$$L(z, \zeta) = \psi_\zeta(z)/(z - \zeta),$$

where ψ_ζ is the function attaining $c^*(\zeta; \Omega)$.

This lemma is essentially known in [4], [8]. Only (9) is not evident

in the case of $\Omega \ni \infty$. Let $\zeta \in \Omega - \{\infty\}$. Since ψ_ζ is the solution of the minimum problem $c^*(\zeta; \Omega)$, a measure $|w - \zeta|^{-2} \overline{\psi_\zeta(w)}|dw|$ on $\partial\Omega$ annihilates all $g \in H^2(\Omega)$, $g(\zeta) = 0$. Thus

$$\frac{1}{|w - \zeta|^2}\overline{\psi_\zeta(w)}|dw| - \frac{c}{2\pi i(w - \zeta)}dw \quad \left(c = \int_{\partial\Omega}\frac{1}{|w - \zeta|^2}\overline{\psi_\zeta(w)}|dw|\right)$$

annihilates all $g \in H^2(\Omega)$, $g(\infty) = 0$. By Riesz's theorem, we can write

$$\frac{1}{|w - \zeta|^2}\overline{\psi_\zeta(w)}|dw| = \left\{\psi_\zeta^*(w) + \frac{c}{2\pi i(w - \zeta)}\right\}dw$$

on $\partial\Omega$ with some $\psi_\zeta^* \in H^1(\Omega)$, $\psi_\zeta^*(\infty) = 0$ [11, p. 15]. Note that $c = \int_{\partial\Omega}\psi_\zeta^*(w)\,dw$. Let $K^*(z, \bar\zeta)$ denote the kernel in the right-hand side of (9). Then, for almost all z on $\partial\Omega$,

$$K^*(z, \bar\zeta) = \frac{1}{2\pi}\int_{\partial\Omega}\frac{w - \zeta}{w - z}\frac{1}{|w - \zeta|^2}\overline{\psi_\zeta(w)}|dw|$$

$$= \frac{1}{2\pi}\int_{\partial\Omega}\frac{w - \zeta}{w - z}\left\{\psi_\zeta^*(w) + \frac{c}{2\pi i(w - \zeta)}\right\}dw$$

$$= \frac{1}{2\pi}\int_{\partial\Omega}\left\{\frac{z - \zeta}{w - z} + 1\right\}\psi_\zeta^*(w)\,dw$$

$$= i(z - \zeta)\left\{\psi_\zeta^*(z) + \frac{c}{2\pi i(z - \zeta)}\right\},$$

which shows that

$$K^*(z, \bar\zeta)dz = i\overline{\psi_\zeta(z)/(z - \zeta)}|dz| \quad \text{a.e. on } \partial\Omega, \text{ i.e.,}$$

$$\frac{1}{i}\psi_\zeta(z)/(z - \zeta)dz = \overline{K^*(z, \bar\zeta)}|dz| \quad \text{a.e. on } \partial\Omega.$$

Thus the pair $(K^*(z, \bar\zeta), \psi_\zeta(z)/(z - \zeta))$ satisfies (4). From the form of the integrand in (9), we see that $K^*(z, \bar\zeta)$ has no inner function, and hence this pair satisfies all the conditions of the Szegö kernel and the L-kernel. Since (K, L) is unique, we have $K(z, \bar\zeta) = K^*(z, \bar\zeta)$ and $L(z, \zeta) = \psi_\zeta(z)/(z - \zeta)$.

LEMMA 5. *Let* $\Omega \ni \infty$ *and* $z \in \Omega - \{\infty\}$. *Then*

$$(10) \quad g(z; \Omega) = \frac{1}{2\pi}\int_{\partial\Omega}L(\zeta, z)|d\zeta| = \overline{\frac{1}{2\pi i}\int_{\partial\Omega}K(\zeta, \bar z)\,d\zeta},$$

$$(11) \quad \phi(z; \Omega) = 1 - \frac{1}{2\pi}\int_{\partial\Omega}\overline{K(\zeta, \bar z)}|d\zeta| = 1 - \frac{1}{2\pi i}\int_{\partial\Omega}L(\zeta, z)\,d\zeta.$$

In fact, (4) shows that

$$g(z;\Omega) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{1}{\zeta-z}g(\zeta;\Omega)d\zeta = \frac{1}{2\pi i}\int_{\partial\Omega}L(\zeta,z)g(\zeta;\Omega)d\zeta$$

$$= \frac{1}{2\pi}\int_{\partial\Omega}\overline{K(\zeta,\bar{z})}g(\zeta;\Omega)|d\zeta| = \overline{\frac{1}{2\pi i}\int_{\partial\Omega}K(\zeta,\bar{z})\phi(\zeta;\Omega)d\zeta}$$

$$= \overline{\frac{1}{2\pi i}\int_{\partial\Omega}K(\zeta,\bar{z})d\zeta} = \frac{1}{2\pi}\int_{\partial\Omega}L(\zeta,z)|d\zeta|.$$

Since

$$\frac{1}{2\pi i}\int_{\partial\Omega}L(\zeta,z)\phi(\zeta;\Omega)d\zeta = \frac{1}{2\pi}\int_{\partial\Omega}\overline{K(\zeta,\bar{z})}\phi(\zeta;\Omega)|d\zeta|$$

$$= \overline{\frac{1}{2\pi i}\int_{\partial\Omega}K(\zeta,\bar{z})g(\zeta;\Omega)d\zeta} = 0,$$

we have

$$\phi(z;\Omega) = 1 + \frac{1}{2\pi i}\int_{\partial\Omega}\frac{1}{\zeta-z}\{\phi(\zeta;\Omega)-1\}d\zeta$$

$$= 1 + \frac{1}{2\pi i}\int_{\partial\Omega}L(\zeta,z)\{\phi(\zeta;\Omega)-1\}d\zeta$$

$$= 1 - \frac{1}{2\pi i}\int_{\partial\Omega}L(\zeta,z)d\zeta = 1 - \frac{1}{2\pi}\int_{\partial\Omega}\overline{K(\zeta,\bar{z})}|d\zeta|.$$

This lemma plays an important role to study $\phi(z;\Omega)$; $\phi(z;\Omega)^2$ is called the Garabedian function of Ω^c [11, p. 19]. The Ahlfors function $f(z;\Omega)$ of Ω^c (i.e., $f(\cdot;\Omega)\in H^\infty(\Omega)$, $\|f(\cdot;\Omega)\|_{H^\infty}=1$, $f'(\infty;\Omega)=\gamma(\Omega^c)$) is expressed as $g(z;\Omega)/\phi(z;\Omega)$ [11, pp. 18–24]. Thus (10) gives that

$$\phi(z;\Omega) = f(z;\Omega)^{-1}\overline{\frac{1}{2\pi i}\int_{\partial D(0,r)^c}K(\zeta,\bar{z})d\zeta}$$

for $r>0$ satisfying $D(0,r)^c\subset\Omega$. Using this expression, we can define the Garabedian function of any domain (cf. [29], [30]).

LEMMA 6. *Let $\Omega\ni 0$, ∞ and $z\in\Omega-\{0,\infty\}$. Then*

(12) $g(z;\Omega) = -K(\tilde{z},\bar{0};\tilde{\Omega})\tilde{z}$, $\phi(z;\Omega) = L(\tilde{z},0;\tilde{\Omega})\tilde{z}$,

(13) $g(0;\tilde{\Omega}) = \overline{g(0;\Omega)}$, $\phi(0;\tilde{\Omega}) = \phi(0;\Omega)$,

where $\tilde{\Omega} = \{\tilde{\zeta}=1/\zeta; \zeta\in\Omega\}$.

To see (12), we put

$$\tilde{g}(\tilde{z}) = -g(1/\tilde{z};\Omega)/\tilde{z}, \qquad \tilde{\phi}(\tilde{z}) = \phi(1/\tilde{z};\Omega)/\tilde{z} \ (\tilde{z}\in\tilde{\Omega}).$$

Then $\tilde{g}, \check{\phi}\tilde{z} \in H^2(\tilde{\Omega})$, $\tilde{g}(\infty) = 0$, $\lim_{\tilde{z}\to 0} \check{\phi}(\tilde{z})\tilde{z} = 1$ and

$$\frac{1}{i}\check{\phi}(\tilde{z})d\tilde{z} = -\frac{1}{i}\phi(z;\Omega)z\,dz/z^2$$

$$= -\overline{g(z;\Omega)}|dz|/z = \overline{\tilde{g}(\tilde{z})}|d\tilde{z}| \text{ a.e. on } \partial\Omega.$$

Thus the unicity yields that $\tilde{g}(\tilde{z}) = K(\tilde{z}, \bar{0}; \tilde{\Omega})$ and $\check{\phi}(\tilde{z}) = L(\tilde{z}, 0; \tilde{\Omega})$, i.e., (12). Combined with (10) and (11), it follows that

$$g(0; \tilde{\Omega}) = -\lim_{\zeta\to\infty}\overline{K(\tilde{\zeta}, \bar{0}; \tilde{\Omega})\tilde{\zeta}} = \overline{g(0;\Omega)},$$

$$\phi(0; \tilde{\Omega}) = \lim_{\zeta\to\infty}L(\tilde{\zeta}, 0; \tilde{\Omega})\tilde{\zeta} = \phi(0;\Omega).$$

Thus (13) holds.

As application of (13), we have $Dc(0, \infty; \tilde{\Omega}) = Dc(0, \infty; \Omega)$. If $\Omega = \tilde{\Omega}$, then $g(0; \Omega)$ is real.

LEMMA 7. *Let $\Omega \ni 0, \infty$. Then*

$$\gamma(\Omega^c) = c(\infty; \Omega) = K(0, \bar{0}; \tilde{\Omega})$$

$$= \lim_{r\to\infty}\frac{1}{(2\pi)^2}\int_{\partial D(0,r)^c}\int_{\partial D(0,r)^c}K(z, \zeta; \Omega)\,dz\,\overline{d\zeta}.$$

In fact, (10) shows that, for $r > 0$ satisfying $D(0, r)^c \subset \Omega$,

$$c(\infty; \Omega) = c^*(\infty; \Omega) = \frac{1}{2\pi}\int_{\partial\Omega}|\phi(w;\Omega)|^2|dw|$$

$$= \frac{1}{2\pi}\int_{\partial\Omega}|g(w;\Omega)|^2|dw| = \frac{1}{2\pi}\int_{\partial\Omega}\left|\frac{1}{2\pi}\int_{\partial D(0,r)^c}K(z, \overline{w})\,dz\right|^2|dw|$$

$$= \frac{1}{(2\pi)^2}\int_{\partial D(0,r)^c}\int_{\partial D(0,r)^c}\left\{\frac{1}{2\pi}\int_{\partial\Omega}K(z, \overline{w})\overline{K(\zeta, \overline{w})}|dw|\right\}dz\,\overline{d\zeta}$$

$$= \frac{1}{(2\pi)^2}\int_{\partial D(0,r)^c}\int_{\partial D(0,r)^c}K(z, \bar{\zeta})\,dz\,\overline{d\zeta}.$$

It is known that $g'(\infty; \Omega) = c(\infty; \Omega)$ [11, pp. 18–24]. Thus the first equality in (12) shows that $c(\infty; \Omega) = K(0, \bar{0}; \tilde{\Omega})$.

Here is a version of the Hadamard variation to Szegö kernels. For a domain $\Omega \ni \infty$ whose boundary consists of a finite number of analytic Jordan curves, we take a non-negative smooth function ρ on $\partial\Omega$. For a sufficiently small number $\varepsilon > 0$, let Ω_ε denote a domain bounded by $\xi_\varepsilon = \xi + i\varepsilon\rho(\xi)d\xi/|d\xi|$, $\xi \in \partial\Omega$ containing ∞. The variations of the Szegö kernel and the L-kernel with respect to ρ are defined by

$$D_\rho J(z, \cdot\,; \Omega) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \{ J(z, \cdot\,; \Omega_\varepsilon) - J(z, \cdot\,; \Omega) \} \quad (J = K, L) \,.$$

Then

$$(14) \quad D_\rho J(z, \cdot\,; \Omega) = \frac{1}{4\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_\rho} DJ(\xi, z, \cdot\,; \Omega) \rho(\xi) |d\xi| \quad (J = K, L) \,,$$

where $\partial/\partial n_\rho$ is the inner normal derivative at $\xi \in \partial\Omega$.

This equality plays an important role in the study of the capacity $K(z, \bar{z}; \Omega)$. Since

$$DK(\xi, z, z; \Omega) = Dc(\xi, z; \Omega) > 0 \quad (\xi \in \Omega)$$

and $= 0$ $(\xi \in \partial\Omega)$, we have $\partial DK(\xi, z, z; \Omega)/\partial n_\rho \geq 0$ $(\xi \in \partial\Omega)$, and hence $D_\rho K(z, \bar{z}; \Omega) \geq 0$, by (14). Thus $K(z, \bar{z}; \Omega)$ is monotone with respect to Ω. This is a typical example how to use (14). In the following section, we shall show a version of (14) to the arc-length variation. The outline of the proof of (14) is as follows. Functions $J(z, \cdot) = J(z, \cdot\,; \Omega)$, $J_\varepsilon(z, \cdot) = J(z, \cdot\,; \Omega_\varepsilon)$ $(J = K, L; \varepsilon > 0)$ are analytic in a neighborhood of $\partial\Omega$ with respect to z. The reproducing property yields that

$$K_\varepsilon(z, \bar{\zeta}) = \frac{1}{2\pi} \int_{\partial\Omega} K_\varepsilon(\xi, \bar{\zeta}) \overline{K(\xi, \bar{z})} |d\xi| \,,$$

$$K(z, \bar{\zeta}) = \frac{1}{2\pi} \int_{\partial\Omega_\varepsilon} K_\varepsilon(\xi, \bar{\zeta}) \overline{K(\xi, \bar{z})} |d\xi| \,.$$

Studying the difference, we obtain

$$(15) \quad D_\rho K(z, \bar{\zeta}) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\partial K}{\partial \xi}(\xi, \bar{\zeta}) \overline{K(\xi, \bar{z})} \rho(\xi) d\xi$$

$$- \frac{1}{2\pi i} \int_{\partial\Omega} K(\xi, \bar{\zeta}) \overline{\frac{\partial K}{\partial \xi}(\xi, \bar{z})} \rho(\xi) \overline{d\xi}$$

$$+ \frac{1}{2\pi} \int_{\partial\Omega} K(\xi, \bar{\zeta}) \overline{K(\xi, \bar{z})} \rho(\xi) d\arg(d\xi) \,.$$

Using (4), we have

$$K(z, \bar{\zeta}) = \frac{1}{2\pi} \int_{\partial\Omega} \overline{L_\varepsilon(\xi, \zeta)} L(\xi, z) |d\xi| \,,$$

$$K_\varepsilon(z, \bar{\zeta}) = \frac{1}{2\pi} \int_{\partial\Omega_\varepsilon} \overline{L_\varepsilon(\xi, \zeta)} L(\xi, z) |d\xi| \,.$$

Studying the difference, we obtain

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$$(16) \quad D_\rho K(z, \bar\zeta) = - \frac{1}{2\pi i} \int_{\partial\Omega} \overline{L(\xi, \zeta)} \frac{\partial L}{\partial \xi}(\xi, z)\rho(\xi)d\xi$$

$$+ \frac{1}{2\pi i} \int_{\partial\Omega} \overline{\frac{\partial L}{\partial \xi}(\xi, \zeta)} L(\xi, z)\rho(\xi)\overline{d\xi}$$

$$- \frac{1}{2\pi} \int_{\partial\Omega} \overline{L(\xi, \zeta)} L(\xi, z)\rho(\xi)d\arg(d\xi).$$

Note that $\overline{L(\xi, \zeta)}L(\xi, z) = K(\xi, \bar\zeta)\overline{K(\xi, \bar z)}$ ($\xi \in \partial\Omega$). Thus, adding (15), (16), and dividing by 2, we obtain (14) ($J = K$). Using

$$L_\varepsilon(z, \zeta) - L(z, \zeta) = \frac{1}{2\pi} \int_{\partial\Omega} L_\varepsilon(\xi, \zeta)\overline{K(\xi, \bar z)}|d\xi|$$

$$= \frac{1}{2\pi} \int_{\partial\Omega_\varepsilon} \overline{K_\varepsilon(\xi, \zeta)}L(\xi, z)|d\xi|,$$

we obtain (14) ($J = L$).

Finally we remark that, for $z \in \Omega - \{\infty\}$,

$$\frac{1}{i} \frac{\partial L}{\partial \xi}(\xi, z)(d\xi)^2 = \overline{\frac{\partial K}{\partial \xi}(\xi, \bar z)}d\xi|d\xi| - i\overline{K(\xi, \bar z)}|d\xi|d\arg(d\xi) \text{ on } \xi \in \partial\Omega.$$

In fact, let $\xi = \xi(t)$ ($0 \le t \le t_0$) be an arc-length representation of $\partial\Omega$ in a neighborhood of $\xi(0) \in \partial\Omega$, and let $\xi'(t) = e^{i\theta(t)}$. Then

$$\frac{1}{i} \frac{\partial L}{\partial \xi}(\xi(t), z)\xi'(t) = \frac{1}{i} \frac{\partial L}{\partial t}(\xi(t), z) = \frac{\partial}{\partial t}\{\overline{K(\xi(t), \bar z)}e^{-i\theta(t)}\}$$

$$= \overline{\frac{\partial K}{\partial \xi}(\xi(t), \bar z)}\xi'(t)e^{-i\theta(t)} - i\overline{K(\xi(t), \bar z)}e^{-i\theta(t)}\theta'(t)$$

$$= \overline{\frac{\partial K}{\partial \xi}(\xi(t), \bar z)}\xi'(t)^2 - i\overline{K(\xi(t), \bar z)}\xi'(t)\frac{\partial}{\partial t}\arg(\xi'(t)).$$

Multiplying this quantity by $\xi'(t)$, we obtain the required equality. As application, we have, for $z, \zeta \in \Omega - \{\infty\}$, $z \ne \zeta$,

$$\frac{1}{i} \frac{\partial L}{\partial \xi}(\xi, \zeta)\overline{L(\xi, z)}d\xi = \frac{1}{i} \overline{\frac{\partial K}{\partial \xi}(\xi, \bar\zeta)}K(\xi, \bar z)\overline{d\xi}$$

$$- \overline{K(\xi, \bar\zeta)}K(\xi, \bar z)d\arg(d\xi) \text{ on } \xi \in \partial\Omega,$$

$$\frac{1}{i} \frac{\partial L}{\partial \xi}(\xi, \zeta)\overline{K(\xi, \bar z)}d\xi = \frac{1}{i} \overline{\frac{\partial K}{\partial \xi}(\xi, \bar\zeta)}L(\xi, z)\overline{d\xi}$$

$$- \overline{K(\xi, \bar\zeta)}L(\xi, z)d\arg(d\xi) \text{ on } \xi \in \partial\Omega.$$

Using these identities, we can rewrite (14) ($J = K, L$) according as our purpose.

§ 3.   A formula and deduction of Theorem 1

To prove Theorem 1, it is necessary to study the Szegö kernels and the L-kernels. For $\Omega \in \mathscr{G}$ and a closed analytic arc Γ such that $\Omega - \Gamma \in \mathscr{G}$ and $\Omega^c \cap \Gamma$ is at most a singleton, we take the arc-length representation w_t $(0 \leq t \leq |\Gamma|)$ of Γ; we assume that $\Omega^c \cap \Gamma = \{w_0\}$ if $\Omega^c \cap \Gamma \neq \emptyset$. We write $\Gamma_t = \{w_s; 0 \leq s \leq t\}$, $\Omega_t = \Omega - \Gamma_t$ $(0 \leq t \leq |\Gamma|)$ and $X = \{(z, \zeta);$ $z, \zeta \in \Omega - (\Gamma \cup \{\infty\})\}$. Recall the definition of DJ $(J = K, L)$. We see that $DJ(w_u, z, \zeta; \Omega_t)$ $(J = K, L)$ are defined for $0 < t < u \leq |\Gamma|$, $(z, \zeta) \in X$. We shall show

THEOREM 8.   *For any $0 < t \leq |\Gamma|$ and any $(z, \zeta) \in X$, the derivative $\partial K(z, \bar{\zeta}; \Omega_t)/\partial t$, the limit $\lim_{u \downarrow t} DK(w_u, z, \zeta; \Omega_t)$ $(= DK(w_t, z, \zeta; \Omega_t)$, say) exist and*

(17) $$\frac{\partial K}{\partial t}(z, \bar{\zeta}; \Omega_t) = \frac{1}{4} DK(w_t, z, \zeta; \Omega_t) .$$

For any $(z, \zeta) \in X$, the right-derivative $\partial K(z, \bar{\zeta}; \Omega_0)/\partial t$ at $t = 0$ exists and $\partial K(z, \bar{\zeta}; \Omega_t)/\partial t$ is continuous on $[0, |\Gamma|]$. These properties hold with K replaced by L.

In this theorem, we put $\partial L(z, z; \Omega_t)/\partial t = 0$. From the definition of DL, we have $DL(w_t, z, z; \Omega_t) = 0$, and hence, in the case of $z = \zeta$, (17) holds with K replaced by L. Recall that w_t is an endpoint of Γ_t $(0 < t \leq |\Gamma|)$. Thus this theorem is near to Löwner's D. E. than Schiffer's variational formula. In § 8, we shall deduce Löwner's D. E. for simply-connected domains from this theorem.

Once this theorem is established, Theorem 1 is deduced as follows. Note that, for $R \in \mathscr{G}$, $R \ni 0$, ∞ and $w, z \in R - \{0, \infty\}$, $w \neq z$,

(18) $Dc(w, \infty; R) = DK(\tilde{w}, 0, 0; \tilde{R})|\tilde{w}|^2$,

(19) $D^2 c(w, z, \infty; R) = D^2 c(\tilde{w}, \tilde{z}, 0; \tilde{R})|\tilde{w}\tilde{z}|^2$,

where $\tilde{\xi} = 1/\xi$ $(\xi = w, z)$ and $\tilde{R} = \{\tilde{\xi} = 1/\xi; \xi \in R\}$. In fact, Lemma 7 and (12) yield (18). The unicity of (K, L) shows that $K(w, \bar{z}; R) = K(\tilde{w}, \bar{\tilde{z}}; \tilde{R})\tilde{w}\bar{\tilde{z}}$ and $L(w, z; R) = - L(\tilde{w}, \tilde{z}; \tilde{R})\tilde{w}\tilde{z}$, and hence (12) gives that

$$
\begin{aligned}
D^2 c(w, z, \infty; R) &= 2\,\mathrm{Re}\,[\{L(w, z; R)\overline{g(w; R)} - \overline{K(w, \bar{z}; R)}\phi(w; R)\}\overline{\phi(z; R)} \\
&\quad - \{L(w, z; R)\overline{\phi(w; R)} - \overline{K(w, \bar{z}; R)}g(w, R)\}\overline{g(z; R)}] \\
&= 2\,\mathrm{Re}\,[\{L(\tilde{w}, \tilde{z}; \tilde{R})\overline{K(\tilde{w}, \bar{0}; \tilde{R})} - \overline{K(\tilde{w}, \bar{\tilde{z}}; \tilde{R})}L(\tilde{w}, 0; \tilde{R})\}\overline{L(\tilde{z}, 0; \tilde{R})} \\
&\quad - \{L(\tilde{w}, \tilde{z}; \tilde{R})\overline{L(\tilde{w}, 0; \tilde{R})} - \overline{K(\tilde{w}, \bar{\tilde{z}}; \tilde{R})}K(\tilde{w}, \bar{0}; \tilde{R})\}\overline{K(\tilde{z}, \bar{0}; \tilde{R})}]|\tilde{w}\tilde{z}|^2 \\
&= D^2 c(\tilde{w}, \tilde{z}, 0; \tilde{R})|\tilde{w}\tilde{z}|^2 ,
\end{aligned}
$$

which implies (19).

Without loss of generality, we may assume that $\Omega - \Gamma \ni 0$. Let w_s^* ($0 \leq s \leq |\tilde{\Gamma}|$) be the arc-length representation of $\tilde{\Gamma} = \{\tilde{\xi}; \xi \in \Gamma\}$ such that $w_0^* = \tilde{w}_0$, and let $s = s(t)$ be the function on $[0, |\Gamma|]$ defined by $w_s^* = \tilde{w}_t$. Then $ds/dt = |w_s^*|^2$. Using (18) with $R = \Omega_t$, $w = w_u$ ($u > t$), we see, by Theorem 8, that $\lim_{u \downarrow t} Dc(w_u, \infty; \Omega_t)$ $(= Dc(w_t, \infty; \Omega_t))$ exists and

$$(20) \quad Dc(w_t, \infty; \Omega_t) = DK(w_s^*, 0, 0; \Omega_s^*)|w_s^*|^2 \quad (\Omega_s^* = \tilde{\Omega}_t).$$

Since $s = s(t)$ is continuous on $[0, |\Gamma|]$, $Dc(w_t, \infty; \Omega_t)$ is also continuous on $[0, |\Gamma|]$. By Lemma 7, Theorem 8 and (20), we have

$$\frac{\partial c}{\partial t}(\infty; \Omega_t) = \frac{\partial K}{\partial s}(0, \bar{0}; \Omega_s^*)\frac{ds}{dt}$$

$$= \frac{1}{4}DK(w_s^*, 0, 0; \Omega_s^*)|w_s^*|^2 = \frac{1}{4}Dc(w_t, \infty; \Omega_t).$$

Thus (5) holds. Using (19) with $R = \Omega_t$, $w = w_u$ ($u > t$), we have

$$D^2 c(w_t, z, \infty; \Omega_t) = D^2 c(w_s^*, \tilde{z}, 0; \Omega_s^*)|w_s^*\tilde{z}|^2.$$

By Theorem 8 and (19), we obtain

$$\frac{\partial c}{\partial t}(z, \infty; \Omega_t) = \frac{\partial}{\partial s}DK(\tilde{z}, 0, 0; \Omega_s^*)\frac{ds}{dt}|\tilde{z}|^2$$

$$= \left(\frac{\partial}{\partial s}\{|L(\tilde{z}, 0; \Omega_s^*)|^2 - |K(\tilde{z}, \bar{0}; \Omega_s^*)|^2\}\right)|w_s^*\tilde{z}|^2$$

$$= \frac{1}{4}D^2 c(w_s^*, \tilde{z}, 0; \Omega_s^*)|w_s^*\tilde{z}|^2 = \frac{1}{4}D^2 c(w_t, z, \infty; \Omega_t).$$

Thus (6) holds. This completes the deduction of Theorem 1.

Our method of the proof of Theorem 8 requires us to distinguish the following four cases:

Case I. Ω is bounded by a finite number of analytic Jordan curves, $\Gamma = [0, 1]$, $\Omega \supset \Gamma$ and $t = 1$.

Case II. $0 < t \leq |\Gamma|$.

Case III. $\Omega \supset \Gamma$ and $t = 0$.

Case IV. $w_0 \in \partial\Omega$ and $t = 0$.

The proof will be given in §§ 4–7. Finally we give the proof of Proposition 2. A simple calculation shows that $Dc(z, \zeta; \Omega) = Dc(\zeta, z; \Omega)$ and $D^2 c(w, z, \zeta; \Omega)$ is invariant for any permutation of a triple (w, z, ζ) of distinct numbers in Ω. Let h be a conformal mapping from Ω onto Ω^*.

If $\Omega \ni \infty$, $\Omega^* \ni \infty$ and $h(\infty) = \infty$, then the unicity of (K, L) shows that

$$K(z, \zeta; \Omega) = K(h(z), \overline{h(\zeta)}; \Omega^*)\sqrt{h'(z)}\,\overline{\sqrt{h'(\zeta)}}\,,$$

$$L(z, \zeta; \Omega) = L(h(z), h(\zeta); \Omega^*)\sqrt{h'(z)}\,\sqrt{h'(\zeta)} \quad (z, \zeta \in \Omega - \{\infty\}, z \neq \zeta)\,.$$

Note that h' is single-valued and that the values $\sqrt{h'(z)}\,\overline{\sqrt{h'(\zeta)}}$, $\sqrt{h'(z)}\,\sqrt{h'(\zeta)}$ are determined independent of the choice of the branch of $\sqrt{}$. Thus a simple calculation shows that, for three distinct numbers $w, z, \zeta \in \Omega - \{\infty\}$,

$$(21) \quad D^2c(w, z, \zeta; \Omega) = D^2c(h(w), h(z), h(\zeta); \Omega^*)|h'(w)h'(z)h'(\zeta)|\,.$$

This equality holds in the case where $\zeta = h(\zeta) = \infty$, and in the case where $\Omega \not\ni \infty$, $\Omega^* \not\ni \infty$, also. Equality (19) is equivalent to (21) in the case where $\zeta = \infty$ and $h(\xi) = 1/\xi$. Since any conformal mapping h is expressed in the form $h(\xi) = h^*((1/\xi) + a)$ with $a \in \mathbf{C}$ and a conformal mapping h^* such that $h^*(\infty) = \infty$, we have (21) for any $\zeta \in \Omega$. Since $Dc(w, z, \zeta; \Omega)$ is invariant for any permutation of (w, z, ζ), we consequently see that (21) holds for any triple (w, z, ζ) of distinct numbers in Ω. This completes the proof of Proposition 2.

§4. Proof of Case I

In this section, we prove Theorem 8 in the case I. Since $\Gamma = [0, 1]$, the function $w_t = t$ is the arc-length representation of Γ. We write $\Omega_t = \Omega - [0, t]$ $(t \geq 0)$. For a domain $R \ni \infty$, let $q(z; R)$ denote the conformal mapping from R onto $\overline{\mathbf{D}}^c$ such that $q(\infty; R) = \infty$, $q'(\infty; R) = \lim_{z \to \infty} q(z; R)/z > 0$. We divide the proof into three steps.

First step. We show that $\partial K(z, \zeta; \Omega_t)/\partial t$ exists at $t = 1$.

Let $d_0 = \min\{1/2, \text{(the distance between } \partial\Omega \text{ and } [0, 1])\}$. Given $1/2 \leq u \leq 1 + d_0$, we take $\iota_u(w) = w/u$. Then the boundary of $\iota_u(\Omega_u)$ consists of $\partial\iota_u(\Omega)$ and $\partial[0, 1]^c$. We write by $J_u(\cdot, \cdot) = J(\cdot, \cdot; \Omega_u)$, $J_u^*(\cdot, \cdot) = J(\cdot, \cdot; \iota_u(\Omega_u))$ $(J = K, L)$. We write also $J = J_1$ $(J = K, L)$. Equality (4) and the reflection principle show that there exists $0 < \varepsilon_0 < d_0$ such that $J_u(z, \cdot)$, $J_u^*(z, \cdot)$, $(J = K, L; |u - 1| < \varepsilon_0)$ are analytic in a neighborhood of $\partial\Omega$ as a function of z. We have

$$(22) \quad K_u(z, \zeta) = K_u^*(\iota_u(z), \overline{\iota_u(\zeta)})\sqrt{\iota_u'(z)}\,\overline{\sqrt{\iota_u'(\zeta)}} = K_u^*(z/u, \overline{\zeta}/u)/u$$

$$= K_u^*(z, \zeta) - (u - 1)\left\{K_u^*(z, \zeta) + z\frac{\partial K_u^*}{\partial z}(z, \zeta) + \zeta\frac{\overline{\partial K_u^*}}{\partial \zeta}(\zeta, \overline{z})\right\}$$

$$+ o(|u - 1|)$$

$$= K_u^*(z, \zeta) - (u - 1)\left\{K(z, \zeta) + z\frac{\partial K}{\partial z}(z, \zeta) + \bar{\zeta}\overline{\frac{\partial K}{\partial \zeta}(\zeta, \bar{z})}\right\}$$
$$+ o(|u - 1|).$$

The reproducing property yields that

$$K_u^*(z, \zeta) = \frac{1}{2\pi}\int_{\partial\Omega_1} K_u^*(w, \zeta)\overline{K(w, \bar{z})}|dw| = \frac{1}{2\pi}\int_{\partial\Omega \cup \partial[0,1]^c}$$

$$= \frac{u}{2\pi}\int_{\partial\iota_u(\Omega)} K_u^*(uw, \zeta)\overline{K(uw, \bar{z})}|dw| + \frac{1}{2\pi}\int_{\partial[0,1]^c} K_u^*(w, \zeta)\overline{K(w, \bar{z})}|dw|$$

$$= \frac{1}{2\pi}\int_{\partial\iota_u(\Omega_u)} K_u^*(w, \zeta)\overline{K(w, \bar{z})}|dw| + \frac{u - 1}{2\pi}\int_{\partial\iota_u(\Omega)}\left\{K_u^*(uw, \zeta)\overline{K(uw, \bar{z})}\right.$$

$$\left. + w\frac{\partial K_u^*}{\partial w}(w, \zeta)\overline{K(w, \bar{z})} + K_u^*(w, \zeta)\bar{w}\overline{\frac{\partial K}{\partial w}(w, \bar{z})}\right\}|dw| + o(|u - 1|)$$

$$= K(z, \zeta) + \frac{u - 1}{2\pi}\int_{\partial\Omega}\left\{K(w, \zeta)\overline{K(w, \bar{z})} + w\frac{\partial K}{\partial w}(w, \zeta)\overline{K(w, \bar{z})}\right.$$

$$\left. + K(w, \zeta)\bar{w}\overline{\frac{\partial K}{\partial w}(w, \bar{z})}\right\}|dw| + o(|u - 1|).$$

Let $\lambda_\delta = \partial\{[\delta, 1 - \delta] \cup \overline{D(0, \delta)} \cup \overline{D(1, \delta)}\}^c$ $(0 < \delta < d_0)$. Then, combined with (4), this yields that

$$\lim_{u \to 1}\frac{1}{u - 1}\{K_u^*(z, \bar{\zeta}) - K(z, \zeta)\} = \frac{1}{2\pi}\int_{\partial\Omega}\left\{K(w, \zeta)\overline{K(w, \bar{z})}\right.$$

$$\left. + w\frac{\partial K}{\partial w}(w, \zeta)\overline{K(w, \bar{z})} + K(w, \zeta)\bar{w}\overline{\frac{\partial K}{\partial w}(w, \bar{z})}\right\}|dw|$$

$$= \frac{1}{2\pi i}\int_{\partial\Omega}\left\{K(w, \zeta)L(w, z) + w\frac{\partial K}{\partial w}(w, \zeta)L(w, z)\right\}dw$$

$$+ \overline{\frac{1}{2\pi i}\int_{\partial\Omega} L(w, \zeta)w\frac{\partial K}{\partial w}(w, \bar{z})dw} = K(z, \zeta) + z\frac{\partial K}{\partial z}(z, \zeta)$$

$$+ \bar{\zeta}\overline{\frac{\partial K}{\partial \zeta}(\zeta, \bar{z})} - \frac{1}{2\pi i}\int_{\lambda_\delta}\left\{K(w, \zeta)L(w, z) + w\frac{\partial K}{\partial w}(w, \zeta)L(w, z)\right\}dw$$

$$- \overline{\frac{1}{2\pi i}\int_{\lambda_\delta} L(w, \zeta)w\frac{\partial K}{\partial w}(w, \bar{z})dw} \quad (0 < \delta < d_0),$$

where, for the integrals on λ_δ, the orientation of dw is chosen so that $\{[\delta, 1 - \delta] \cup \overline{D(0, \delta)} \cup \overline{D(1, \delta)}\}^c$ lies to the left. By (22), we have

$$\frac{\partial K}{\partial t}(z, \zeta; \Omega_1) = \frac{\partial K}{\partial t}(z, \zeta; \Omega_t)\Big|_{t=1} = T(\delta) \quad (0 < \delta < d_0)$$

with

$$T(\delta) = -\frac{1}{2\pi i}\int_{\lambda_\delta}\left\{K(w,\zeta)L(w,z)+w\frac{\partial K}{\partial w}(w,\zeta)L(w,z)\right\}dw$$

$$-\overline{\frac{1}{2\pi i}\int_{\lambda_\delta}L(w,\zeta)w\frac{\partial K}{\partial w}(w,\bar z)dw}\,.$$

Thus $\partial K(z,\zeta;\Omega_1)/\partial t$ exists.

Second step. *We show that* $DK(w_1,z,\zeta;\Omega_1)$ *exists and* (17) *holds at* $t=1$.

Since $\partial K(z,\zeta;\Omega_1)/\partial t$ is independent of δ, we have

$$\frac{\partial K}{\partial t}(z,\zeta;\Omega_1)=\lim_{\delta\downarrow 0}T(\delta)\,.$$

Express $T(\delta)$ as a summation of $T_k(\delta)$ $(k=1,\cdots,5)$ which are defined by

$$T_1(\delta) = -\frac{1}{2\pi i}\int_\delta^{1-\delta}\left\{K(x+i0,\zeta)L(x+i0,z)+x\frac{\partial K}{\partial w}(x+i0,\zeta)L(x+i0,z)\right\}dx$$

$$-\overline{\frac{1}{2\pi i}\int_\delta^{1-\delta}L(x+i0,\zeta)x\frac{\partial K}{\partial w}(x+i0,\bar z)dx}\,,$$

$$T_2(\delta) = \frac{1}{2\pi i}\int_\delta^{1-\delta}\left\{K(x-i0,\zeta)L(x-i0,z)+x\frac{\partial K}{\partial w}(x-i0,\zeta)L(x-i0,z)\right\}dx$$

$$+\overline{\frac{1}{2\pi i}\int_\delta^{1-\delta}L(x-i0,\zeta)x\frac{\partial K}{\partial w}(x-i0,\bar z)dx}\,,$$

$$T_3(\delta) = -\frac{1}{2\pi i}\int_{S(1,\delta)}w\frac{\partial K}{\partial w}(w,\zeta)L(w,z)dw\,,$$

$$T_4(\delta) = -\overline{\frac{1}{2\pi i}\int_{S(1,\delta)}L(w,\zeta)w\frac{\partial K}{\partial w}(w,\bar z)dw}\,,$$

$$T_5(\delta) = -\frac{1}{2\pi i}\int_{S(0,\delta)\cup S(1,\delta)}K(w,\zeta)L(w,z)dw$$

$$-\frac{1}{2\pi i}\int_{S(0,\delta)}w\frac{\partial K}{\partial w}(w,\zeta)L(w,z)dw$$

$$-\overline{\frac{1}{2\pi i}\int_{S(0,\delta)}L(w,\zeta)w\frac{\partial K}{\partial w}(w,\bar z)dw}\,.$$

The behaviour of $K(z,\zeta)$, $L(z,\zeta)$ near $z=1$ is as follows. Taking

$$q(z)=q(z;[0,1]^c)=2\left\{\left(z-\frac{1}{2}\right)+\sqrt{\left(z-\frac{1}{2}\right)^2-\frac{1}{4}}\right\},$$

we put $R = q(\Omega_1)$. Note that $q(1) = 1$ and $K(\xi, \overline{q(\zeta)}; R)$, $L(\xi, q(\zeta); R)$ are analytic across $S(0, 1)$ as a function of ξ. Since

$$q(z) = 1 + 2\sqrt{z - 1} + O(|z - 1|),$$
$$\sqrt{q'(z)} = (z - 1)^{-1/4} + (z - 1)^{1/4} + O(|z - 1|^{3/4}) \quad (z \to 1),$$

we have

$$K(z, \zeta) = K(q(z), \overline{q(\zeta)}; R) \sqrt{q'(z)} \overline{\sqrt{q'(\zeta)}}$$
$$= \left\{ K(1, \overline{q(\zeta)}; R) + 2\sqrt{z - 1} \frac{\partial K}{\partial \xi}(1, \overline{q(\zeta)}; R) + O(|z - 1|) \right\}$$
$$\times \{(z - 1)^{-1/4} + (z - 1)^{1/4} + O(|z - 1|^{3/4})\} \overline{\sqrt{q'(\zeta)}}$$
$$= \{(z - 1)^{-1/4} K(1, \overline{q(\zeta)}; R) + (z - 1)^{1/4} A_\zeta + O(|z - 1|^{3/4})\} \overline{\sqrt{q'(\zeta)}} \quad (z \to 1)$$

and

$$\frac{\partial K}{\partial z}(z, \zeta) = \left\{ -\frac{1}{4}(z - 1)^{-5/4} K(1, \overline{q(\zeta)}; R) \right.$$
$$\left. + \frac{1}{4}(z - 1)^{-3/4} A_\zeta + O(|z - 1|^{-1/4}) \right\} \overline{\sqrt{q'(\zeta)}} \quad (z \to 1),$$

where, in general,

$$A_\sigma = \overline{K(1, \overline{q(\sigma)}; R) + 2 \frac{\partial K}{\partial \xi}(1, \overline{q(\sigma)}; R)},$$

$$B_\sigma = L(1, q(\sigma); R) + 2 \frac{\partial L}{\partial \xi}(1, q(\sigma); R).$$

In the same manner,

$$L(z, \zeta) = \{(z - 1)^{-1/4} L(1, q(\zeta); R) + (z - 1)^{1/4} B_\zeta + O(|z - 1|^{3/4})\} \overline{\sqrt{q'(\zeta)}}$$
$$(z \to 1)$$

and

$$\frac{\partial L}{\partial z}(z, \zeta) = \left\{ -\frac{1}{4}(z - 1)^{-5/4} L(1, q(\zeta); R) \right.$$
$$\left. + \frac{1}{4}(z - 1)^{-3/4} B_\zeta + O(|z - 1|^{-1/4}) \right\} \overline{\sqrt{q'(\zeta)}} \quad (z \to 1).$$

The behaviour of $K(z, \zeta)$, $L(z, \zeta)$ near $z = 0$ is analogous. We have

$$J(z, \cdot) = O(|z|^{-1/4}), \frac{\partial J}{\partial z}(z, \cdot) = O(|z|^{-5/4}) \quad (z \to 0, \ J = K, L).$$

We now return to the study of $T_k(\delta)$ $(k = 1, \cdots, 5)$. We have

$$T_1(\delta) = -\frac{1}{2\pi i}\int_0^{1-\delta} + o(1)$$

$$= -\frac{1}{2\pi}\int_0^{1-\delta}\left\{K(x+i0,\zeta)\overline{K(x+i0,\bar{z})}+x\frac{\partial K}{\partial x}(x+i0,\zeta)\overline{K(x+i0,\bar{z})}\right.$$

$$\left.+ K(x+i0,\zeta)x\overline{\frac{\partial K}{\partial x}(x+i0,\bar{z})}\right\}dx + o(1)$$

$$= -\frac{1}{2\pi}\int_0^{1-\delta}\frac{\partial}{\partial x}\{xK(x+i0,\zeta)\overline{K(x+i0,\bar{z})}\}dx + o(1)$$

$$= -\frac{1}{2\pi}K(1-\delta+i0,\zeta)\overline{K(1-\delta+i0,\bar{z})} + o(1)$$

$$= -\frac{1}{2\pi}\{\delta^{-1/4}e^{-i\pi/4}K(1,\overline{q(\zeta)};R) + \delta^{1/4}e^{i\pi/4}A_\zeta + O(\delta^{3/4})\}$$

$$\times \{\delta^{-1/4}e^{i\pi/4}\overline{K(1,\overline{q(z)};R)} + \delta^{1/4}e^{-i\pi/4}\overline{A}_z + O(\delta^{3/4})\}\sqrt{q'(z)}\sqrt{q'(\zeta)} + o(1)$$

$$= \left[-\frac{1}{2\pi}\delta^{-1/2}K(1,\overline{q(\zeta)};R)\overline{K(1,\overline{q(z)};R)}\right.$$

$$\left.+ \frac{i}{2\pi}\{K(1,\overline{q(\zeta)};R)\overline{A}_z - \overline{K(1,\overline{q(z)};R)}A_\zeta\}\right]\sqrt{q'(z)}\sqrt{q'(\zeta)} + o(1).$$

In the same manner,

$$T_2(\delta) = \left[-\frac{1}{2\pi}\delta^{-1/2}K(1,\overline{q(\zeta)};R)\overline{K(1,q(z);R)}\right.$$

$$\left.- \frac{i}{2\pi}\{K(1,\overline{q(\zeta)};R)\overline{A}_z - \overline{K(1,q(z);R)}A_\zeta\}\right]\sqrt{q'(z)}\sqrt{q'(\zeta)} + o(1).$$

We have

$$T_3(\delta) = -\frac{1}{2\pi i}\int_{S(1,\delta)}\frac{\partial K}{\partial w}(w,\zeta)L(w,z)dw + o(1)$$

$$= -\frac{1}{2\pi i}\int_{S(0,\delta)}\left\{-\frac{1}{4}w^{-5/4}K(1,\overline{q(\zeta)};R) + \frac{1}{4}w^{-3/4}A_\zeta + O(|w|^{-1/4})\right\}$$

$$\times \{w^{-1/4}L(1,q(z);R) + w^{1/4}B_z + O(|w|^{3/4})\}dw\sqrt{q'(z)}\sqrt{q'(\zeta)} + o(1)$$

$$= -\frac{1}{2\pi i}\int_{S(0,\delta)}\left[-\frac{1}{4}w^{-3/2}K(1,\overline{q(\zeta)};R)L(1,q(z);R)\right.$$

$$\left.- \frac{1}{4w}\{K(1,\overline{q(\zeta)};R)B_z - L(1,q(z);R)A_\zeta\}\right]dw\sqrt{q'(z)}\sqrt{q'(\zeta)} + o(1)$$

$$= \left[-\frac{1}{2\pi}\delta^{-1/2}K(1,\overline{q(\zeta)};R)L(1,q(z);R)\right.$$

$$\left.- \frac{1}{4}\{K(1,\overline{q(\zeta)};R)B_z - L(1,q(z);R)A_\zeta\}\right]\sqrt{q'(z)}\sqrt{q'(\zeta)} + o(1).$$

In the same manner,

$$T_4'(\delta) = \left[- \frac{1}{2\pi} \delta^{-1/2} \overline{L(1, q(\zeta); R)} K(1, \overline{q(z)}; R) \right.$$

$$\left. - \frac{1}{4} \{ \overline{K(1, \overline{q(z)}; R)B_\zeta} - \overline{L(1, q(\zeta); R)A_z} \} \right] \sqrt{q'(z)} \sqrt{\overline{q'(\zeta)}} + o(1) .$$

Evidently, $T_5'(\delta) = o(1)$.   Using $L(1, q(\sigma); R) = - \overline{K(1, \overline{q(\sigma)}; R)}$ $(\sigma = z, \zeta)$,
we obtain

$$T(\delta) = \sum_{k=1}^{5} T_k(\delta) = \left[- \frac{1}{2\pi} \delta^{-1/2} K(1, \overline{q(\zeta)}; R) \overline{K(1, \overline{q(z)}; R)} \right.$$

$$+ \frac{i}{2\pi} \{ K(1, \overline{q(\zeta)}; R)\overline{A}_z - \overline{K(1, \overline{q(z)}; R)}A_\zeta \}$$

$$- \frac{1}{2\pi} \delta^{-1/2} K(1, \overline{q(\zeta)}; R) \overline{K(1, \overline{q(z)}; R)}$$

$$- \frac{i}{2\pi} \{ K(1, \overline{q(\zeta)}; R)\overline{A}_z - \overline{K(1, \overline{q(z)}; R)}A_\zeta \}$$

$$- \frac{1}{2\pi} \delta^{-1/2} K(1, \overline{q(\zeta)}; R)L(1, q(z); R)$$

$$- \frac{1}{4} \{ K(1, \overline{q(\zeta)}; R)B_z - L(1, q(z); R)A_\zeta \}$$

$$- \frac{1}{2\pi} \delta^{-1/2} \overline{L(1, q(\zeta); R)} K(1, \overline{q(z)}; R)$$

$$\left. - \frac{1}{4} \{ \overline{K(1, \overline{q(z)}; R)B_\zeta} - \overline{L(1, q(\zeta); R)A_z} \} \right] \sqrt{q'(z)} \sqrt{\overline{q'(\zeta)}} + o(1)$$

$$= \frac{1}{4} [- K(1, \overline{q(\zeta)}; R)B_z + L(1, q(z); R)A_\zeta$$

$$- \overline{K(1, \overline{q(z)}; R)B_\zeta} + \overline{L(1, q(\zeta); R)A_z}] \sqrt{q'(z)} \sqrt{\overline{q'(\zeta)}} + o(1)$$

$$= \frac{1}{4} [L(1, q(z); R)\overline{B}_\zeta + \overline{L(1, q(\zeta); R)}B_z$$

$$- \overline{K(1, \overline{q(z)}; R)}A_\zeta - K(1, \overline{q(\zeta)}; R)\overline{A}_z] \sqrt{q'(z)} \sqrt{\overline{q'(\zeta)}} + o(1)$$

$$(= T_0/4 + o(1), \text{ say})$$

which shows that

$$\frac{\partial K}{\partial t}(z, \bar{\zeta}; \Omega_1) = \lim_{\delta \downarrow 0} T(\delta) = T_0/4 .$$

By (4), we have

$$L(1, q(z); R)\overline{L(1, q(\zeta); R)} = \overline{K(1, \overline{q(z)}; R)}K(1, \overline{q(\zeta)}; R),$$

and hence

$$DK(1, z, \zeta; \Omega_1) = \lim_{u \downarrow 1}\{L(u, z)\overline{L(u, \zeta)} - \overline{K(u, \bar{z})}K(u, \zeta)\}$$

$$= \lim_{u \downarrow 1}[\{(u - 1)^{-1/4}L(1, q(z); R) + (u - 1)^{1/4}B_z + O((u - 1)^{3/4})\}$$

$$\times \{(u - 1)^{-1/4}\overline{L(1, q(\zeta); R)} + (u - 1)^{1/4}\bar{B}_\zeta + O((u - 1)^{3/4})\}$$

$$- \{(u - 1)^{-1/4}\overline{K(1, \overline{q(z)}; R)} + (u - 1)^{1/4}\bar{A}_z + O((u - 1)^{3/4})\}$$

$$\times \{(u - 1)^{-1/4}K(1, \overline{q(\zeta)}; R) + (u - 1)^{1/4}A_\zeta + O((u - 1)^{3/4})\}$$

$$\times \sqrt{q'(z)}\sqrt{\overline{q'(\zeta)}}]$$

$$= \{L(1, q(z); R)\bar{B}_\zeta + \overline{L(1, q(\zeta); R)}B_z$$

$$- \overline{K(1, \overline{q(z)}; R)}A_\zeta - K(1, \overline{q(\zeta)}; R)\bar{A}_z\}\sqrt{q'(z)}\sqrt{\overline{q'(\zeta)}} = T_0.$$

Thus $DK(1, z, \zeta; \Omega_1)$ exists and (17) holds.

Final step. *We show that* $\partial L(z, \zeta; \Omega_1)/\partial t$, $DL(w_1, z, \zeta; \Omega_1)$ *exist and* (17) *holds with K replaced by L.*

We may assume that $z \neq \zeta$. For $1 < u < 1 + d_0$, we have

$$L_u(z, \zeta) - L(z, \zeta) = \frac{1}{2\pi i}\int_{\partial\Omega_u} L_u(w, \zeta)L(w, z)\,dw$$

$$= -\frac{1}{2\pi i}\overline{\int_{\partial\Omega_1} K_u(w, \bar{\zeta})K(w, \bar{z})\,dw} + \frac{1}{2\pi i}\int_{\partial[1, u]^c} L_u(w, \zeta)L(w, z)\,dw$$

$$= T_1^* + T_2^*, \text{ say.}$$

Taking

$$q_u(z) = q(z; [0, u]^c) = \frac{2}{u}\left\{z - \frac{u}{2} + \sqrt{\left(z - \frac{u}{2}\right)^2 - \left(\frac{u}{2}\right)^2}\right\},$$

we put $R_u = q_u(\Omega_u)$. We have

$$q_u(z) = 1 + 2\sqrt{z - u} + O(u - 1),$$

$$\sqrt{q_u'(z)} = u^{-1/4}(z - u)^{-1/4} + (z - u)^{1/4} + O((u - 1)^{3/4}),$$

where $O(\cdot)$ means the uniform estimate with respect to $z \in \partial[1, u]^c$ when $u \downarrow 1$. Thus

$$K_u(z, \zeta) = K(q_u(z), \overline{q_u(\zeta)}; R_u)\sqrt{q_u'(z)}\sqrt{\overline{q_u'(\zeta)}}$$

$$= \left\{K(1, \overline{q_u(\zeta)}; R_u) + 2\sqrt{z - u}\frac{\partial K}{\partial \xi}(1, \overline{q_u(\zeta)}; R_u) + O(u - 1)\right\}$$

$$\times \{u^{-1/4}(z - u)^{-1/4} + (z - u)^{1/4} + O((u - 1)^{3/4})\}\sqrt{\overline{q_u'(\zeta)}}$$

$$= \{u^{-1/4}(z-u)^{-1/4}K(1, \overline{q_u(\zeta)}; R_u) + (z-u)^{1/4}A_{u,\zeta}$$
$$+ O((u-1)^{3/4})\}\overline{\sqrt{q_u'(\zeta)}} \quad (z \in \partial[1, u]^c, u \downarrow 1),$$

where $\dfrac{\partial K}{\partial \xi}(1, \overline{q_u(\zeta)}; R_u) = \dfrac{\partial K}{\partial \xi}(\xi, \overline{q_u(\zeta)}; R_u)\Big|_{\xi=1}$, and, in general,

$$A_{u,\sigma} = K(1, \overline{q_u(\sigma)}; R_u) + 2u^{-1/4}\frac{\partial K}{\partial \xi}(1, \overline{q_u(\sigma)}; R_u),$$

$$B_{u,\sigma} = L(1, q_u(\sigma); R_u) + 2u^{-1/4}\frac{\partial L}{\partial \xi}(1, q_u(\sigma); R_u).$$

In the same manner,

$$L_u(z, \zeta) = \{u^{-1/4}(z-u)^{-1/4}L(1, q_u(z); R_u) + (z-u)^{1/4}B_{u,\zeta}$$
$$+ O((u-1)^{3/4})\}\sqrt{q_u'(\zeta)} \quad (z \in \partial[1, u]^c, \ u \downarrow 1).$$

Note that

$$\frac{1}{2\pi i}\overline{\int_{\partial[1,u]^c}(w-u)^{-1/4}(w-1)^{-1/4}dw} = \frac{1}{2\pi i}\{e^{-i\pi/4} - e^{i\pi/4}\}$$
$$\times \int_0^{u-1}(u-1-x)^{-1/4}x^{-1/4}dx = \frac{\sqrt{u-1}}{\sqrt{2}\,\pi}B\left(\frac{3}{4}, \frac{3}{4}\right),$$

$$\frac{1}{2\pi i}\overline{\int_{\partial[1,u]^c}(w-u)^{-1/4}(w-1)^{1/4}dw} = \frac{1}{2\pi i}\{e^{-i\pi/4} - e^{i\pi/4}\}$$
$$\times \int_0^{u-1}(u-1-x)^{-1/4}x^{1/4}dx = \frac{u-1}{\sqrt{2}\,\pi}B\left(\frac{3}{4}, \frac{5}{4}\right) = \frac{u-1}{4},$$

$$\frac{1}{2\pi i}\overline{\int_{\partial[1,u]^c}(w-u)^{1/4}(w-1)^{-1/4}dw} = \frac{u-1}{2\pi i}\{e^{i\pi/4} - e^{-i\pi/4}\}B\left(\frac{5}{4}, \frac{3}{4}\right)$$
$$= -\frac{u-1}{4},$$

where the orientation of dw is chosen so that $[0, u]^c$ lies to the left.
Thus

$$T_1^* = \frac{1}{2\pi i}\overline{\int_{\partial[1,u]^c}K_u(w, \zeta)}K(w, \bar{z})\,dw$$

$$= \left[\frac{1}{2\pi i}\overline{\int_{\partial[1,u]^c}\{u^{-1/4}(w-u)^{-1/4}K(1, \overline{q_u(\zeta)}; R_u) + (w-u)^{1/4}A_{u,\zeta}}\right.$$
$$\overline{+ O((u-1)^{3/4})\}}$$
$$\overline{\times \{(w-1)^{-1/4}K(1, \overline{q(z)}; R) + (w-1)^{1/4}A_z}$$
$$\left.\overline{+ O(|w-1|^{3/4})\}dw}\right]\sqrt{q'(z)}\overline{\sqrt{q_u'(\zeta)}}$$

$$
\begin{aligned}
&= \left[u^{-1/4} \overline{K(1, \overline{q_u(\zeta)}; R_u)} \overline{K(1, \overline{q(z)}; R)} \frac{1}{2\pi i} \overline{\int_{\partial[1,u]^c} (w-u)^{-1/4}(w-1)^{-1/4}\, dw} \right. \\
&\quad + u^{-1/4} \overline{K(1, \overline{q_u(\zeta)}; R_u)A_z} \frac{1}{2\pi i} \overline{\int_{\partial[1,u]^c} (w-u)^{-1/4}(w-1)^{1/4}\, dw} \\
&\quad \left. + \overline{K(1, \overline{q(z)}; R)A_{u,\zeta}} \frac{1}{2\pi i} \overline{\int_{\partial[1,u]^c} (w-u)^{1/4}(w-1)^{-1/4}\, dw} \right] \sqrt{\overline{q'(z)}}\sqrt{\overline{q_u'(\zeta)}} \\
&\quad + o(u-1) \\
&= \left[\frac{1}{\sqrt{2}\,\pi}(u-1)^{1/2}u^{-1/4}B\!\left(\frac{3}{4}, \frac{3}{4}\right)\overline{K(1, \overline{q_u(\zeta)}; R_u)}\overline{K(1, \overline{q(z)}; R)} \right. \\
&\quad \left. + \frac{u-1}{4}\{u^{-1/4}\overline{K(1, \overline{q_u(\zeta)}; R_u)A_z} - \overline{K(1, \overline{q(z)}; R)A_{u,\zeta}}\} \right]\sqrt{\overline{q'(z)}}\sqrt{\overline{q_u'(\zeta)}} \\
&\quad + o(u-1).
\end{aligned}
$$

In the same manner,

$$
\begin{aligned}
T_2^* &= \left[-\frac{1}{\sqrt{2}\,\pi}(u-1)^{1/2}u^{-1/4}B\!\left(\frac{3}{4}, \frac{3}{4}\right)L(1, q_u(\zeta); R_u)L(1, q(z); R) \right. \\
&\quad \left. - \frac{u-1}{4}\{u^{-1/4}L(1, q_u(\zeta); R_u)B_z - L(1, q(z); R)B_{u,\zeta}\} \right]\sqrt{\overline{q'(z)}}\sqrt{\overline{q_u'(\zeta)}} \\
&\quad + o(u-1).
\end{aligned}
$$

Since

$$
L(1, q_u(\zeta); R_u)L(1, q(z); R) = \overline{K(1, \overline{q_u(\zeta)}; R_u)K(1, \overline{q(z)}; R)}\,,
$$

we have

$$
\begin{aligned}
&\lim_{u \downarrow 1} \frac{1}{u-1}\{L_u(z, \zeta) - L(z, \zeta)\} \\
&= \lim_{u \downarrow 1} \frac{1}{4}[u^{-1/4}\overline{K(1, \overline{q_u(\zeta)}; R_u)A_z} - \overline{K(1, \overline{q(z)}; R)A_{u,\zeta}} \\
&\quad - u^{-1/4}L(1, q_u(\zeta); R_u)B_z + L(1, q(z); R)B_{u,\zeta}]\sqrt{\overline{q'(z)}}\sqrt{\overline{q_u'(\zeta)}} \\
&= \frac{1}{4}[\overline{K(1, \overline{q(\zeta)}; R)A_z} - \overline{K(1, \overline{q(z)}; R)A_\zeta} \\
&\quad - L(1, q(\zeta); R)B_z + L(1, q(z); R)B_\zeta]\sqrt{\overline{q'(z)}}\sqrt{\overline{q'(\zeta)}} = T_0^*/4,\ \text{say}.
\end{aligned}
$$

Thus the right-derivative $\partial^+ L(z, \zeta; \Omega_1)/\partial t$ exists and equals $T_0^*/4$. In the same manner, we see that the same property holds for the left-derivative. Consequently, $\partial L(z, \zeta; \Omega_1)/\partial t$ exists and equals $T_0^*/4$. Since

$$DL(1, z, \zeta; \Omega_1) = \lim_{u \downarrow 1} \{L(u, z)\overline{K(u, \overline{\zeta})} - \overline{K(u, \overline{z})}L(u, \zeta)\}$$

$$= \lim_{u \downarrow 1} [\{(u - 1)^{-1/4}L(1, q(z); R) + (u - 1)^{1/4}B_z + O((u - 1)^{3/4})\}$$

$$\times \{(u - 1)^{-1/4}\overline{K(1, \overline{q(\zeta)}; R)} + (u - 1)^{1/4}\overline{A}_\zeta + O((u - 1)^{3/4})\}$$

$$- \{(u - 1)^{-1/4}\overline{K(1, \overline{q(z)}; R)} + (u - 1)^{1/4}\overline{A}_z + O((u - 1)^{3/4})\}$$

$$\times \{(u - 1)^{-1/4}L(1, q(\zeta); R) + (u - 1)^{1/4}B_\zeta + O((u - 1)^{3/4})\}]$$

$$\times \sqrt{q'(z)}\sqrt{q'(\zeta)}$$

$$= [- L(1, q(\zeta); R)\overline{A}_z + L(1, q(z); R)\overline{A}_\zeta$$

$$+ \overline{K(1, \overline{q(\zeta)}; R)}B_z - \overline{K(1, \overline{q(z)}; R)}B_\zeta]\sqrt{q'(z)}\sqrt{q'(\zeta)}$$

$$= [\overline{K(1, \overline{q(\zeta)}; R)}A_z - \overline{K(1, \overline{q(z)}; R)}A_\zeta$$

$$- L(1, q(\zeta); R)B_z + L(1, q(z); R)B_\zeta]\sqrt{q'(z)}\sqrt{q'(\zeta)} = T_0^* \,.$$

Thus $DL(1, z, \zeta; \Omega_1)$ exists and $\partial L(z, \zeta; \Omega_1)/\partial t = DL(1, z, \zeta; \Omega_1)/4$. This completes the proof of Theorem 8 in the case I.

§5. Deduction of Case II from Case I

In this section, we deduce the case II from the case I. We work only with K; the argument for L is analogous. The arc-length representation of a closed analytic arc Γ is denoted by w_t $(0 \leq t \leq |\Gamma|)$. For $-\pi < \theta \leq \pi$, let $\mathcal{L}_\theta$ denote the totality of closed arcs Γ such that $\Gamma - \{0\}$ is analytic and Γ is expressed as $e^{i\theta}x + x^{3/2}P(\sqrt{x})$, $0 \leq x < r$ in a neighborhood of 0 with an analytic function P in $D(0, r^2)$. For $\Gamma \in \mathcal{L}_\theta$, let w_t $(0 \leq t \leq |\Gamma|)$ denote the arc-length representation of Γ such that $w_0 = 0$. We put $\Lambda_t = [- 1, 0] \cup \Gamma_t$ $(0 \leq t \leq |\Gamma|)$. Here are some lemmas necessary for the proof of the cases II–IV.

LEMMA 9. *Let* Γ *be a closed analytic arc and let* Λ *be an arc in* $\mathcal{A}$ *with an endpoint* w_0 *such that* $\Lambda \cap \Gamma = \{w_0\}$. *Let* h *be the conformal mapping from* Λ^c *onto* $[- 1, 0]^c$ *such that* $h(\infty) = \infty$ *and* $h(0) = 0$. *Then* $h'(0)$ *exists,* $h'(w_t)$ *is continuous on* $t \in [0, |\Gamma|]$ *and* $h(\Gamma) \in \mathcal{L}_\theta$ *for some* $-\pi < \theta \leq \pi$.

To see this lemma, we may assume that $w_0 = 0$ and Λ belongs to $\mathcal{L}_\pi$ in a neighborhood of 0. Then $\Gamma \in \mathcal{L}_\theta$ for some $-\pi < \theta \leq \pi$. Take an analytic arc γ passing through 0 such that $\gamma^2 \subset \Lambda$. Then ζ^2 is a conformal mapping from a domain U bounded by γ and a small circle $S(0, r)$ onto $D(0, r^2) - \Lambda$. Form an analytic function

$$h^*(\zeta) = q(h(\zeta^2); [- 1, 0]^c) = 2\left\{h(\zeta^2) + \frac{1}{2} + \sqrt{\left\{h(\zeta^2) + \frac{1}{2}\right\}^2 - \frac{1}{4}}\right\}$$

in U. Since $h^*(\gamma)$ is an arc on $S(0, 1)$, the reflection principle shows that h^* is analytic in a small disk $D(0, r')$ $(0 < r' < r)$. Since $\Lambda \in \mathscr{L}_0$, $h^*(0) = 1$ and h^* is one-to-one, we can write $h^*(\zeta) = 1 + a\zeta \hat{h}(\zeta)$ with $a \in \mathbf{R}$, $a \neq 0$ and an analytic function $\hat{h}$ in $D(0, r')$ such that $\hat{h}(0) = 1$. This shows that

$$h(\zeta^2) = \frac{1}{4}\left\{h^*(\zeta) + \frac{1}{h^*(\zeta)} - 2\right\}$$

$$= \frac{1}{4}\{1 + a\hat{h}(\zeta) + (1 + a\hat{h}(\zeta))^{-1} - 2\}$$

$$= 4^{-1}a^2\zeta^2\hat{h}(\zeta)^2\{1 + a\zeta\hat{h}(\zeta)\}^{-1},$$

and hence we can write

$$h(z) = 4^{-1}a^2 z\hat{h}(\sqrt{z})^2\{1 + a\sqrt{z}\,\hat{h}(\sqrt{z})\}^{-1} \quad (z \in D(0, r'))$$

with a suitable branch of $\sqrt{}$. Thus $h'(0)$ exists and equals $a^2/4$. Since $h'(z) = a^2/4 + o(1)$ $(z \to 0)$, $h'(w_t)$ is continuous at $t = 0$, which yields that $h'(w_t)$ is continuous on $[0, |\Gamma|]$. Since $\Gamma \in \mathscr{L}_\theta$ and Γ is analytic, there exists an analytic function g in a small disk $D(0, r'')$ $(0 < r'' < r')$ such that $g(0) = 0$, $g'(0) = e^{i\theta}$ and $g([0, r'']) \subset \Gamma$ in a neighborhood of 0. Then $h(\Gamma)$ is expressed as

$$4^{-1}a^2 e^{i\theta} x\left[e^{-i\theta}\frac{g(x)}{x}\hat{h}(\sqrt{g(x)})^2\{1 + a\sqrt{g(x)}\hat{h}(\sqrt{g(x)})\}^{-1}\right], \quad 0 \le x < r''$$

in a neighborhood of 0. Substituting the variable x by $4^{-1}a^2 x$, we obtain $h(\Gamma) \in \mathscr{L}_\theta$.

LEMMA 10. *Let Γ be a closed analytic arc and let k_t be the conformal mapping from Γ_t^c onto $\overline{D(0, r_t)}^c$ such that $k_t(\infty) = \infty$, $|k_t'(\infty)| = 1$ and $k_t(w_t) = r_t$ $(0 < t \le |\Gamma|)$. Then the limit $\lim_{u \downarrow t}\{k_t(w_u) - r_t\}|k_t'(w_u)|$ $(= a_t,$ say) exists, a_t is continuous in $(0, |\Gamma|]$, $a_t > 0$ and $\lim_{t \downarrow 0} a_t/r_t = 1/2$.*

In the same manner as above, we see that a_t exists, a_t is continuous in $(0, |\Gamma|]$ and $a_t > 0$. There remains to prove $\lim_{t \downarrow 0} a_t/r_t = 1/2$. We can write

$$k_t(w_u) = k_t(w_t) + b_t\sqrt{w_u - w_t} + O(u - t),$$

$$k_t'(w_u) = \frac{1}{2}b_t(w_u - w_t)^{-1/2} + O(1) \quad (u \downarrow t)$$

with $b_t\sqrt{dw_t/dt} > 0$. Then $a_t = |b_t|^2/2$. Thus it is sufficient to show that

$\lim_{t \downarrow 0} |b_t|/\sqrt{r_t} = 1$. To do this, we may assume that $w_0 = 0$. Then $\Gamma \in \mathscr{L}_\theta$ for some $-\pi < \theta \leq \pi$. Let g be the conformal mapping in the proof of Lemma 9 and, for a small number $t > 0$, let $s = s_0(t)$ be the number defined by $g(s) = w_t$. Form a conformal mapping

$$k_t^*(z) = k_t \circ g\left(z + \frac{s^2}{16z} + \frac{s}{2}\right)$$

in a small ring $D(0, r^*) - \overline{D(0, s/4)}$ such that $k_t^*(s/4) = r_t$ and $k_t^*(S(0, s/4)) = S(0, r_t)$; r^* is a small number independent of t. The reflection principle shows that k_t^* is analytic in a ring $D(0, r^*) - \overline{D(0, s^2/(16r^*))}$ with the identity $k_t^*(z) = r_t^2 \overline{(k_t^*(s^2/(16\bar{z})))}^{-1}$ in $D(0, s/4) - \overline{D(0, s^2/(16r^*))}$. Since $k_t(\zeta)$ uniformly converges to ζ on $S(0, r^*)$, we have

$$\begin{aligned}
k_t^{*\prime}(s/4) &= \frac{1}{2\pi i} \int_{\partial\{D(0,r^*) - D(0,s^2/(16r^*))\}} \frac{1}{(z - (s/4))^2} k_t^*(z)\, dz \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\{r^* e^{i\psi} - (s/4)\}^2} k_t^*(r^* e^{i\psi}) r^* e^{i\psi}\, d\psi \\
&\quad - \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\{(s^2/(16r^*))e^{i\psi} - (s/4)\}^2} \overline{k_t^*(r^* e^{i\psi})}^{-1} \frac{r_t^2 s^2}{16r^*} e^{i\psi}\, d\psi \\
&= \frac{1}{2\pi r^*} \int_0^{2\pi} g(r^* e^{i\psi}) e^{-i\psi}\, d\psi + o(1) = 1 + o(1) \quad (t \downarrow 0).
\end{aligned}$$

Since

$$|k_t^{*\prime}(s/4)| = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} |k_t \circ g\{s + 4\varepsilon^2/s + o(\varepsilon^2)\} - k_t \circ g(s)|$$

$$= |b_t| \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} |g\{s + 4\varepsilon^2/s + o(\varepsilon^2)\} - g(s)|^{1/2} = 2|b_t g'(s)|/\sqrt{s},$$

we have

$$2|b_t g'(s)|/\sqrt{s} = 1 + o(1) \quad (t \downarrow 0).$$

Since $|g'(0)| = 1$, $s_0'(0) = 1$ and $\lim_{t \downarrow 0} t/r_t = 4$, this shows that $\lim_{t \downarrow 0} |b_t|/\sqrt{r_t} = 1$. Consequently, $\lim_{t \downarrow 0} a_t/r_t = 1/2$.

LEMMA 11. *Let* $\Gamma \in \mathscr{L}_0$. *Then* $\gamma(\Lambda_t) = (1 + t)/4 + o(t) \quad (t \downarrow 0)$.

Since $\Gamma \in \mathscr{L}_0$, the diameter of Λ_t is larger than or equal to $1 + t + o(t)$, and hence $\gamma(\Lambda_t) \geq \{1 + t + o(t)\}/4$ [11, p. 9]. There exists a constant C such that Λ_t is contained in a rectangle $\{x + iy; -1 \leq x \leq t, |y| \leq Ct^{3/2}\}$ $(0 \leq t \leq |\Gamma|)$. A simple calculation yields that

$$\gamma(\{x + iy; -1 \le x \le t, |y| \le Ct^{3/2}\}) = \frac{1+t}{4} + o(t),$$

which gives that $\gamma(\Lambda_t) \le (1+t)/4 + o(t)$. Thus the required equality holds.

LEMMA 12.   Let $\Gamma \in \mathscr{L}_\theta$, $-\pi < \theta \le \pi$. For $0 \le s \le 4\gamma([-1,0] \cup \Gamma) - 1$, let $t = t^*(s)$ be the number defined by $\gamma(\Lambda_t) = (1+s)/4$. Then, for any compact set F in $([-1,0] \cup \Gamma)^c$,

(23) $\displaystyle\sup_{z \in F} |q(z; \Lambda^c_{t^*(s)}) - q(z; [-1,s]^c)| = o(s)$ $(s \downarrow 0)$.

To see this, we write simply $q^*_s(z) = q(z; \Lambda^c_{t^*(s)})$, $q^0_s(z) = q(z; [-1,s]^c)$. Note that

$$q^0_s(z) = \frac{2}{s+1}\left\{z - \frac{s-1}{2} + \sqrt{\left(z - \frac{s-1}{2}\right)^2 - \left(\frac{s+1}{2}\right)^2}\right\},$$

$$\frac{\partial q^0_s}{\partial s}(z) = \frac{1}{s+1}q^0_s(z)\frac{1 + q^0_s(z)}{1 - q^0_s(z)}.$$

Substituting the parameter of Löwner's D. E. by a suitable one, we obtain

$$\frac{\partial q^*_s}{\partial s}(z) = \frac{1}{s+1}q^*_s(z)\frac{1 + \kappa_s q^*_s(z)}{1 - \kappa_s q^*_s(z)}   (\kappa_s = \overline{q^*_s(w_{t^*(s)})}).$$

(See (56).) There exists $0 < \varepsilon_0 < 1$ such that $|q^0_s(z)| \ge 1 + \varepsilon_0$, $|q^*_s(z)| \ge 1 + \varepsilon_0$ $(z \in F,\ 0 \le s \le 4\gamma([-1,0] \cup \Gamma) - 1)$. Then

$$\left|\frac{\partial}{\partial s}|q^*_s(z) - q^0_s(z)|\right| \le \left|\frac{\partial q^*_s}{\partial s}(z) - \frac{\partial q^0_s}{\partial s}(z)\right|$$

$$= \frac{1}{s+1}\left|q^*_s(z)\frac{1 + \kappa_s q^*_s(z)}{1 - \kappa_s q^*_s(z)} - q^0_s(z)\frac{1 + q^0_s(z)}{1 - q^0_s(z)}\right|$$

$$= \frac{1}{s+1}\left|\{q^*_s(z) - q^0_s(z)\}\frac{1 + \kappa_s q^*_s(z)}{1 - \kappa_s q^*_s(z)}\right.$$

$$\left. + q^0_s(z)\left(\frac{1 + \kappa_s q^*_s(z)}{1 - \kappa_s q^*_s(z)} - \frac{1 + q^0_s(z)}{1 - q^0_s(z)}\right)\right|$$

$$\le 3\varepsilon_0^{-1}|q^*_s(z) - q^0_s(z)| + 4\varepsilon_0^{-1}\frac{|\kappa_s q^*_s(z) - q^0_s(z)|}{|q^*_s(z)| - 1}$$

$$\le 11\varepsilon_0^{-2}\{|q^*_s(z) - q^0_s(z)| + |\kappa_s - 1|\}.$$

Since $q^*_0 = q^0_0$, we have

$$|q^*_p(z) - q^0_s(z)| \le 11\varepsilon_0^{-2}\int_0^s \{|q^*_x(z) - q^0_x(z)| + |\kappa_x - 1|\}dx$$

$$\le 11\varepsilon_0^{-2}\int_0^s \{\rho(x) + |\kappa_x - 1|\}dx,$$

where $\rho(x) = \sup_{z \in F} |q_x^*(z) - q_x^0(z)|$. Thus

$$(24) \quad \rho(s) \leq 11 \varepsilon_0^{-2} \int_0^s \{\rho(x) + |\kappa_x - 1|\} dx .$$

Let $C = \max\{\rho(x) + |\kappa_x - 1|; 0 \leq x \leq 4\gamma([-1,0] \cup \Gamma) - 1\}$. Then (24) shows that $\rho(s) \leq 11 C \varepsilon_0^{-2} s$. Using (24) again, we have

$$\rho(s) \leq 121 C \varepsilon_0^{-4} s^2 + 11 \varepsilon_0^{-2} \int_0^s |\kappa_x - 1| dx .$$

Since $\lim_{s \downarrow 0} \kappa_s = 1$, this inequality gives (23).

Now we deduce the case II from the case I. Let $\Omega \in \mathscr{G}$ and let Γ be a closed analytic arc such that $\Omega^c \cap \Gamma$ is at most a singleton. If $\Omega^c \cap \Gamma = \{w_0\}$, we denote by Λ the component of Ω^c such that $\Lambda \cap \Gamma = \{w_0\}$. The condition $\Omega - \Gamma \in \mathscr{G}$ shows that $\Lambda \cup \Gamma \in \mathscr{A}$. If $\Omega^c \cap \Gamma = \emptyset$, we put $\Lambda = \emptyset$. We divide the proof into several steps.

First step. There exists a conformal mapping h from $\Omega \cup \Lambda$ onto a domain bounded by a finite number of analytic Jordan curves. Then $h(\Lambda \cup \Gamma) \in \mathscr{A}$ and $h(\Gamma)$ is a closed analytic arc. Let w_s^* $(0 \leq s \leq |h(\Gamma)|)$ be the arc-length representation of $h(\Gamma)$ such that $w_0^* = w_0$, and let $s = \hat{s}(t)$ be the function defined by $w_s^* = h(w_t)$. Then $ds/dt = |h'(w_t)|$ and ds/dt is continuous in $(0, |\Gamma|]$. If the required properties are established for a domain $h(\Omega)$, an arc $h(\Gamma)$ and $0 < s \leq |h(\Gamma)|$, then we have, with $\Omega_s^* = h(\Omega) - \{w_x^*; 0 \leq x \leq s\}$,

$$\frac{\partial K}{\partial t}(z, \bar{\zeta}; \Omega_t) = \frac{\partial K}{\partial s}(h(z), \overline{h(\zeta)}; \Omega_s^*) \frac{ds}{dt} \sqrt{h'(z)} \overline{\sqrt{h'(\zeta)}} ,$$

$$DK(w_t, z, \zeta; \Omega_t) = DK(w_s^*, h(z), h(\zeta); \Omega_s^*)|h'(w_t)| \sqrt{h'(z)} \overline{\sqrt{h'(\zeta)}}$$
$$(0 < t \leq |\Gamma|) ,$$

which yields the required properties for Ω, Γ and $0 < t \leq |\Gamma|$. Thus, from the beginning, we may assume that $\Omega - \Lambda$ is bounded by a finite number of analytic Jordan curves.

Second step. We show that the right-derivative $\partial^+ K(z, \bar{\zeta}; \Omega_t)/\partial t$ at $t = 0$ exists if $\Lambda = [-1, 0]$ and $\Gamma \in \mathscr{L}_0$.

Here is an equality necessary for the proof:

$$(25) \quad K(z, \bar{\zeta}; \Omega_t) = K(z, \bar{\zeta}; \Omega - [0, t]) + o(t) \quad (t \downarrow 0) .$$

To see this, we form a conformal mapping $\iota_s^*(z) = (q_s^0)^{-1} \circ q_s^*(z)$ from

$\Omega_{\iota^*(s)}$ onto a domain Ω_s^* whose boundary consists of $[-1, s]$ and $\partial \iota_s^*(\Omega \cup \Lambda)$, where $q_s^*(z) = q(z; \Lambda_{\iota^*(s)}^c)$, $q_s^0(z) = q(z; [-1, s]^c)$ and $t^*(s)$ is the number defined in Lemma 12 for $\Gamma \in \mathscr{L}_0$. Then

$$
\begin{aligned}
\iota_s^*(z) &= \frac{s+1}{4}\left\{q_s^*(z) + \frac{1}{q_s^*(z)}\right\} + \frac{s-1}{2} \\
&= \frac{s+1}{4}\left\{q_s^0(z) + \frac{1}{q_s^0(z)}\right\} + \frac{s-1}{2} \\
&\quad + \frac{s+1}{4}\left\{(q_s^*(z) - q_s^0(z)) + \left(\frac{1}{q_s^*(z)} - \frac{1}{q_s^0(z)}\right)\right\} \\
&= z + \frac{s+1}{4}(q_s^*(z) - q_s^0(z))\left(1 - \frac{1}{q_s^*(z)q_s^0(z)}\right),
\end{aligned}
$$

and hence (23) shows that

$$
\iota_s^*(z) = z + o(s), \qquad \iota_s^{*\prime}(z) = 1 + o(s) \quad (s \downarrow 0)
$$

locally uniformly with respect to z. Thus

$$
\begin{aligned}
K(z, \zeta; \Omega_{\iota^*(s)}) &= K(\iota_s^*(z), \overline{\iota_s^*(\zeta)}; \Omega_s^*)\sqrt{\iota_s^{*\prime}(z)}\,\overline{\sqrt{\iota_s^{*\prime}(\zeta)}} \\
&= K(z, \zeta; \Omega_s^*) + o(s) \quad (s \downarrow 0)
\end{aligned}
$$

and

$$
\begin{aligned}
K(z, \zeta; \Omega - [0, s]) &= \frac{1}{2\pi}\int_{\partial \Omega_s^*} K(w, \zeta; \Omega - [0, s])\overline{K(w, \bar{z}; \Omega_s^*)}|dw| \\
&= \frac{1}{2\pi}\int_{\partial(\Omega \cup \Lambda)} K(\iota_s^*(\xi), \zeta; \Omega - [0, s])\overline{K(\iota_s^*(\xi), \bar{z}; \Omega_s^*)}|\iota_s^{*\prime}(\xi)||d\xi| \\
&\quad + \frac{1}{2\pi}\int_{\partial[-1,s]^c} K(w, \zeta; \Omega - [0, s])\overline{K(w, \bar{z}; \Omega_s^*)}|dw| \\
&= \frac{1}{2\pi}\int_{\partial(\Omega - [0,s])} K(w, \zeta; \Omega - [0, s])\overline{K(w, \bar{z}; \Omega_s^*)}|dw| + o(s) \\
&= K(z, \zeta; \Omega_s^*) + o(s) \quad (s \downarrow 0),
\end{aligned}
$$

which shows that

$$
K(z, \zeta; \Omega_{\iota^*(s)}) = K(z, \zeta; \Omega - [0, s]) + o(s) \quad (s \downarrow 0).
$$

Let $s = s^*(t)$ denote the inverse function of $t^*(s)$. Then this equality is rewritten as

$$
(26) \quad K(z, \zeta; \Omega_t) = K(z, \zeta; \Omega - [0, s^*(t)]) + o(s^*(t)) \quad (t \downarrow 0).
$$

By Lemma 11, we have

$$\frac{1 + s^*(t)}{4} = \gamma(\Lambda_t) = \frac{1 + t}{4} + o(t) \quad (t \downarrow 0), \quad \text{i.e.,}$$

$$s^*(t) = t + o(t) \quad (t \downarrow 0).$$

Using this equality, we have, in the same manner as above,

$$K(z, \bar{\zeta}; \Omega - [0, s^*(t)]) = K(z, \bar{\zeta}; \Omega - [0, t]) + o(t) \quad (t \downarrow 0).$$

Combined with (26), this equality yields (25). Consequently, the case I and (25) show that the right-derivative $\partial^+ K(z, \bar{\zeta}; \Omega_0)/\partial t$ exists and equals $\dfrac{\partial K}{\partial t}(z, \bar{\zeta}; \Omega - [0, t])\Big|_{t=0}$.

Third step. We show that the right-derivative $\partial^+ K(z, \bar{\zeta}; \Omega_t)/\partial t$ exists in $(0, |\Gamma|]$.

Take the conformal mapping h_t from Λ_t^c onto $[-1, 0]^c$ such that $h_t(w_t) = 0$ $(0 < t \le |\Gamma|)$. Then Lemma 9 shows that $h_t'(w_t)$ exists and $h_t(\Gamma - \Gamma_t) \in \mathscr{L}_0$. Since h_t is conformal, we have $h_t'(w_t)(dw_t/dt) > 0$. Since

$$K(z, \bar{\zeta}; \Omega_u) = K(h_t(z), \overline{h_t(\zeta)}; h_t(\Omega_u)) \sqrt{h_t'(z)} \overline{\sqrt{h_t'(\zeta)}} \quad (0 < t \le u),$$

the second step shows that $\partial^+ K(z, \bar{\zeta}; \Omega_t)/\partial t$ exists and equals

$$(27) \quad \frac{\partial K}{\partial s}(h_t(z), \overline{h_t(\zeta)}; h_t(\Omega_t) - [0, s])\Big|_{s=0} |h_t'(w_t)| \sqrt{h_t'(z)} \overline{\sqrt{h_t'(\zeta)}}.$$

Fourth step. We show that $DK(w_t, z, \zeta; \Omega_t)$ exists and is continuous in $(0, |\Gamma|]$.

Let k_t be the conformal mapping in Lemma 10. Then we have, with $\hat{\Omega}_t = k_t(\Omega_t)$,

$$DK(w_u, z, \zeta; \Omega_t) = DK(k_t(w_u), k_t(z), k_t(\zeta); \hat{\Omega}_t) |k_t'(w_u) k_t'(z) k_t'(\zeta)|$$
$$(0 < t < u).$$

Since $J(\xi, \cdot; \hat{\Omega}_t)$ $(J = K, L)$ are analytic in a neighborhood of r_t with respect to ξ, the right-hand side is rewritten as

$$\left[\left\{ L(r_t, k_t(z); \hat{\Omega}_t) + (k_t(w_u) - r_t)\frac{\partial L}{\partial \xi}(r_t, k_t(z); \hat{\Omega}_t) + o(\sqrt{u - t}) \right\} \right.$$

$$\times \overline{\left\{ L(r_t, k_t(\zeta); \hat{\Omega}_t) + (k_t(w_u) - r_t)\frac{\partial L}{\partial \xi}(r_t, k_t(\zeta); \hat{\Omega}_t) + o(\sqrt{u - t}) \right\}}$$

$$\left. - \overline{\left\{ K(r_t, \overline{k_t(z)}; \hat{\Omega}_t) + (k_t(w_u) - r_t)\frac{\partial K}{\partial \xi}(r_t, \overline{k_t(z)}; \hat{\Omega}_t) + o(\sqrt{u - t}) \right\}} \right.$$

$$\times \left\{ K(r_\iota, \overline{k_\iota(\zeta)}; \hat{\Omega}_\iota) + (k_\iota(w_u) - r_\iota) \frac{\partial K}{\partial \xi}(r_\iota, \overline{k_\iota(\zeta)}; \hat{\Omega}_\iota) + o(\sqrt{u - t}) \right\}$$

$$\times |k'_\iota(w_u)k'_\iota(z)k'_\iota(\zeta)| \quad (0 < t < u) ,$$

where $\dfrac{\partial J}{\partial \xi}(r_\iota, \cdot ; \hat{\Omega}_\iota) = \dfrac{\partial J}{\partial \xi}(\xi, \cdot ; \hat{\Omega}_\iota)\Big|_{\xi = r_\iota}$ $(J = K, L)$. Recall the quantity a_ι in Lemma 10. Since

$$L(r_\iota, k_\iota(z); \hat{\Omega}_\iota)\overline{L(r_\iota, k_\iota(\zeta); \hat{\Omega}_\iota)} = \overline{K(r_\iota, \overline{k_\iota(z)}; \hat{\Omega}_\iota)}K(r_\iota, \overline{k_\iota(\zeta)}; \hat{\Omega}_\iota) ,$$

Lemma 10 shows that $DK(w_\iota, z, \zeta; \Omega_\iota)$ exists and

$$(28) \quad DK(w_\iota, z, \zeta; \Omega_\iota) = \bigg[L(r_\iota, k_\iota(z); \hat{\Omega}_\iota)\overline{\frac{\partial L}{\partial \xi}(r_\iota, k_\iota(\zeta); \hat{\Omega}_\iota)}$$

$$+ \frac{\partial L}{\partial \xi}(r_\iota, k_\iota(z); \hat{\Omega}_\iota)\overline{L(r_\iota, k_\iota(\zeta); \hat{\Omega}_\iota)}$$

$$- \overline{K(r_\iota, \overline{k_\iota(z)}; \hat{\Omega}_\iota)} \frac{\partial K}{\partial \xi}(r_\iota, \overline{k_\iota(\zeta)}; \hat{\Omega}_\iota)$$

$$- \overline{\frac{\partial K}{\partial \xi}(r_\iota, \overline{k_\iota(z)}; \hat{\Omega}_\iota)}K(r_\iota, \overline{k_\iota(\zeta)}; \hat{\Omega}_\iota) \bigg] a_\iota |k'_\iota(z)k'_\iota(\zeta)| .$$

Since a_ι is continuous in $(0, |\Gamma|]$, $DK(w_\iota, z, \zeta; \Omega_\iota)$ is continuous in $(0, |\Gamma|]$.

Fifth step. We show that $\partial^+ K(z, \zeta; \Omega_\iota)/\partial t = DK(w_\iota, z, \zeta; \Omega_\iota)/4$ $(0 < t \leq |\Gamma|)$.

Let h_ι be the same as in the third step. Then we have

$$(29) \quad DK(w_u, z, \zeta; \Omega_\iota)$$
$$= DK(h_\iota(w_u), h_\iota(z), h_\iota(\zeta); h_\iota(\Omega_\iota))|h'_\iota(w_u)|\sqrt{h'_\iota(z)}\overline{\sqrt{h'_\iota(\zeta)}} \quad (0 < t < u) .$$

In the same manner as in the fourth step,

$$\lim_{u \downarrow t} DK(h_\iota(w_u), h_\iota(z), h_\iota(\zeta); h_\iota(\Omega_\iota)) = \lim_{s \downarrow 0} DK(s, h_\iota(z), h_\iota(\zeta); h_\iota(\Omega_\iota)) .$$

The case I shows that this quantity is equal to

$$4 \frac{\partial K}{\partial s}(h_\iota(z), \overline{h_\iota(\zeta)}; h_\iota(\Omega_\iota) - [0, s])\Big|_{s=0} .$$

Lemma 9 shows that $\lim_{u \downarrow t} h'_\iota(w_u) = h'_\iota(w_\iota)$. Thus, letting u tend to t in (29), we see that $DK(w_\iota, z, \zeta; \Omega_\iota)/4$ exists and equals (27), which yields the required equality.

Final step. There remains to prove that the left-derivative

$\partial^- K(z, \zeta; \Omega_t)/\partial t$ exists in $(0, |\Gamma|]$ and equals $\partial^+ K(z, \zeta; \Omega_t)/\partial t$.

Fixing $0 < t_0 < |\Gamma|$, we form a function

$$\rho^*(t) = K(z, \zeta; \Omega_t) - K(z, \zeta; \Omega_{t_0}) - \int_{t_0}^t \frac{\partial^+}{\partial s} K(z, \zeta; \Omega_s) ds$$

on $[t_0, |\Gamma|]$. Then $\rho^*(t_0) = 0$ and the right-derivative is identically 0, which yields that $\rho^*(t) \equiv 0$. Thus $\partial^- K(z, \zeta; \Omega_t)/\partial t$ exists and equals $\partial^+ K(z, \zeta; \Omega_t)/\partial t$ in $(t_0, |\Gamma|]$, and hence the derivative exists in $(t_0, |\Gamma|]$. Since $0 < t_0 < |\Gamma|$ is arbitrary, $\partial K(z, \zeta; \Omega_t)/\partial t$ exists in $(0, |\Gamma|]$. This completes the proof of Theorem 8 in the case II.

§ 6. Proof of Case III by the operator $\mathscr{H}$

In this section, we prove Theorem 8 in the case III. This case corresponds to the variation by cutting a hole [28] and the proof is easier than that of the other cases. There are three methods of the proof. The first method is as follows: For a small number $t > 0$, we define a competing function of form $K(z, \zeta; \Omega) + t\sigma(z, \zeta, t)DK(w_0, z, \zeta; \Omega)$ so that the difference from $K(z, \zeta; \Omega_t)$ is negligible. The second method is as follows: Taking a conformal mapping from Γ_t^c outside a small disk, we use the reflection principle on the boundary of this disk in order to study $K(z, \bar{\zeta}; \Omega_t) - K(z, \bar{\zeta}; \Omega)$. In this note, we use the third method, which is an application of the singular integral operator $\mathscr{H}$ and completely different from the methods by conformal mappings. This method is not short, however, this is a powerful tool to study $\gamma(\cdot)$. Evidently, $DJ(w_0, z, \zeta; \Omega)$ $(J = K, L)$ exist. We shall see that (17) is still valid in this case. For $E \in \mathscr{A}$, $L^2(E)$ denotes the L^2-space of functions μ on E with norm $\|\mu\|_E = \left\{ \int_E |\mu|^2 |dz| \right\}^{1/2}$. The operator $\mathscr{H}_E$ from $L^2(E)$ to itself is defined by

$$\mathscr{H}_E \mu(z) = \frac{1}{\pi} p.v. \int_E \frac{\mu}{\zeta - z} |d\zeta|$$

$$= \frac{1}{\pi} \lim_{\epsilon \downarrow 0} \int_{E, |\zeta - z| > \epsilon} \frac{1}{\zeta - z} \mu(\zeta) |d\zeta| \quad (z \in E).$$

We define $\bar{\mathscr{H}}_E$ by $\bar{\mathscr{H}}_E \mu = \overline{\mathscr{H}_E \bar{\mu}}$. The operator $\mathscr{H}_E$ is anti-symmetric, i.e., $(\mathscr{H}_E \mu, \nu)_E = -(\mu, \bar{\mathscr{H}}_E \nu)_E$, where $(\cdot, \cdot)_E$ is the inner product. The operator $\text{Id} - \mathscr{H}_E \bar{\mathscr{H}}_E$ is invertible and its inverse operator is denoted by $\mathscr{T}_E$, where Id is the identity operator. Using $\mathscr{T}_E$, we can express $K(z, \bar{\zeta}; E^c)$, $L(z, \zeta; E^c)$ in the following forms.

PROPOSITION 13.

$$(30) \quad K(z, \bar{\zeta}; E^c) = \frac{1}{\pi} \int_E \frac{1}{w - z} \bar{\mathscr{T}}_E\left(\frac{1}{\cdot - \zeta}\right)(w)|dw|,$$

$$(31) \quad L(z, \zeta; E^c) = \frac{1}{z - \zeta} + \frac{1}{\pi} \int_E \frac{1}{w - z} \bar{\mathscr{H}}_E \mathscr{T}_E\left(\frac{1}{\cdot - \zeta}\right)(w)|dw|.$$

To see this proposition, we begin by showing

$$(32) \quad c(z; E^c) = \frac{1}{\pi} \int_E \frac{1}{w - z} \bar{\mathscr{T}}_E\left(\frac{1}{\cdot - z}\right)(w)|dw| \quad (z \in E^c - \{\infty\}).$$

The dual extremum problem yields that

$$c(z; E^c) = \inf\left\{\frac{1}{2\pi} \int_{\partial E^c} \left|\frac{1}{w - z} + \psi(w)\right|^2 |dw|; \psi \in H^2(E^c), \psi(\infty) = 0\right\}$$

$$= \frac{1}{2\pi} \int_{\partial E^c} |L(w, z; E^c)|^2 |dw|$$

$$= \frac{1}{2\pi} K(z, \bar{z}; E^c) \quad [4, \text{Chap. VII}].$$

For $\psi \in H^2(E^c)$, $\psi(\infty) = 0$, there exists $\mu \in L^2(E)$ such that $\psi(w) = \frac{1}{\pi} \int_E \frac{\mu}{\zeta - w} |d\zeta|$ $(w \in E^c)$. Then

$$\psi(w) = \mathscr{H}_E \mu(w) + i\mu(w)\frac{|dw|}{dw} \quad \text{a.e. on } \partial E^c.$$

To each $w \in E - \{\text{endpoints of } C_j \ (j = 1, \cdots, n)\}$, there correspond two points $w^{\pm} \in \partial E^c$ so that $dw^+ = -dw^-$, where $\{C_j\}_{j=1}^n$ are closed analytic arcs such that $E = \bigcup_{j=1}^n C_j$. Since

$$\frac{1}{2\pi} \int_{\partial E^c} \left|\frac{1}{w - z} + \psi(w)\right|^2 |dw|$$

$$= \frac{1}{2\pi} \int_E \left(\left|\frac{1}{w - z} + \mathscr{H}_E \mu(w) + i\mu(w)\frac{|dw|}{dw^+}\right|^2 \right.$$

$$\left. + \left|\frac{1}{w - z} + \mathscr{H}_E \mu(w) + i\mu(w)\frac{|dw|}{dw^-}\right|^2\right)|dw|$$

$$= \frac{1}{\pi}\left\{\left\|\frac{1}{\cdot - z} + \mathscr{H}_E \mu\right\|_E^2 + \|\mu\|_E^2\right\}$$

we have

$$c(z; E^c) = \inf\left\{\frac{1}{\pi}\left(\left\|\frac{1}{\cdot - z} + \mathscr{H}_E\mu\right\|_E^2 + \|\mu\|_E^2\right); \mu \in L^2(E)\right\}.$$

There exists $\mu_0 \in L^2(E)$ attaining $c(z; E^c)$. Then a variational method shows that

$$\left(\frac{1}{\cdot - z} + \mathscr{H}_E \mu_0, \mathscr{H}_E \mu\right)_E + (\mu_0, \mu)_E = 0$$

for all $\mu \in L^2(E)$, and hence

$$(\mathrm{Id} - \bar{\mathscr{H}}_E \mathscr{H}_E)\mu_0 = \bar{\mathscr{H}}_E\left\{\frac{1}{\cdot - z}\right\}, \quad \text{i.e.,} \quad \mu_0 = \bar{\mathscr{T}}_E \bar{\mathscr{H}}_E\left\{\frac{1}{\cdot - z}\right\}.$$

Since $\mathrm{Id} + \mathscr{H}_E \mathscr{T}_E \bar{\mathscr{H}}_E = \mathscr{T}_E$, we have

$$c(z; E^c) = \frac{1}{\pi}\left\{\left\|\frac{1}{\cdot - z} + \mathscr{H}_E \mu_0\right\|_E^2 + \|\mu_0\|_E^2\right\}$$

$$= \frac{1}{\pi}\left(\frac{1}{\cdot - z} + \mathscr{H}_E \mu_0, \frac{1}{\cdot - z}\right)_E = \frac{1}{\pi}\int_E \frac{1}{w - z}\mathscr{T}_E\left\{\frac{1}{\cdot - z}\right\}(w)|dw|$$

$$= \frac{1}{\pi}\int_E \frac{1}{w - z}\mathscr{T}_E\left\{\frac{1}{\cdot - z}\right\}(w)|dw|.$$

Thus (32) holds.

We now deduce (30) from (32). Let $K^*(z, \zeta)/\pi$ be the kernel in the right-hand side of (30). For each $\zeta \in E^c$, $K^*(\cdot, \zeta)$ is analytic in E^c. Evidently, $K^*(\infty, \zeta) = 0$ and $K^*(z, \zeta) = \overline{K^*(\bar\zeta, \bar z)}$. Equality (32) shows that

$$(33) \qquad \frac{1}{\pi}K(z, \bar z; E^c) = c(z; E^c) = \frac{1}{\pi}K^*(z, \bar z) \quad (z \in E^c).$$

Let

$$K(z, \zeta; E^c) = \sum_{n=1}^\infty \sum_{m=1}^\infty a_{n,m} z^{-n}\overline{\zeta^{-m}}, \qquad K^*(z, \zeta) = \sum_{n=1}^\infty \sum_{m=1}^\infty b_{n,m} z^{-n}\overline{\zeta^{-m}}$$

be the expansions at ∞. Then (33) yields that $a_{n,m} = b_{n,m}$ $(m, n = 1, 2, \cdots)$ (cf. [30]). Thus $K(z, \zeta; E^c) = K^*(z, \zeta)$. We next prove (31). We have

$$(34) \quad L(z, \zeta; E^c) = \frac{1}{z - \zeta} + \frac{1}{2\pi i}\int_{\partial E^c} \frac{1}{w - z}\left\{L(w, \zeta; E^c) - \frac{1}{w - \zeta}\right\}dw$$

$$= \frac{1}{z - \zeta} + \frac{1}{2\pi}\int_{\partial E^c} \frac{1}{w - z}\overline{K(w, \zeta; E^c)}|dw|.$$

Equality (30) shows that

$$K(w, \zeta; E^c) = \mathscr{H}_E \mathscr{T}_E\left(\frac{1}{\cdot - \zeta}\right)(w) + i\mathscr{T}_E\left(\frac{1}{\cdot - \zeta}\right)(w)\frac{|dw|}{dw} \quad \text{a.e. on } \partial E^c.$$

Replace $K(w, \zeta; E^c)$ in (34) by the right-hand side of this equality. Since $dw^+ = -dw^-$ a.e. on E, we obtain (31). This completes the proof of Propositon 13.

Using this proposition, we prove Theorem 8 in the case III. We divide the proof into three steps.

First step. *We show that* $\partial K(z, \zeta; \Omega_0)/\partial t$ *exists and equals* $DK(w_0, z, \zeta; \Omega)/4$. *($\partial K(z, \zeta; \Omega_0)/\partial t$ means the right-derivative at* $t = 0$.)

Let $\Omega \in \mathscr{G}$ and Γ be a closed analytic arc in Ω. Considering a suitable conformal mapping if necessary, we may assume that $\Omega \in \mathscr{F}$ and $w_0 = 0$. Let $E = \Omega^c$, $E_t = E \cup \Gamma_t$ $(0 < t \leq |\Gamma|)$. Given a small number $t > 0$, we denote by M the multiplier operator $\mu \to \chi_E \mu$ from $L^2(E_t)$ to itself, and denote by N the multiplier operator $\mu \to \chi_{\Gamma_t}\mu$, where χ_F is the characteristic function of F. We write simply

$$\mathscr{H} = \mathscr{H}_{E_t}, \quad \mathscr{T} = \mathscr{T}_{E_t}, \quad S = M\mathscr{T}_E M, \quad R = N\mathscr{T}_{\Gamma_t} N.$$

Then (30) shows that

$$K(z, \zeta; \Omega_t) = \frac{1}{\pi}(\mathscr{T}\mu_z, \mu_\zeta), \qquad K(z, \zeta; \Omega) = \frac{1}{\pi}(S\mu_z, \mu_\zeta),$$

where $\mu_\xi(w) = 1/(w - \xi)$ $(\xi = z, \zeta)$ and $(\cdot, \cdot) = (\cdot, \cdot)_{E_t}$. To study $\mathscr{T} - S$, we rewrite the identity $(\mathrm{Id} - \mathscr{H}\overline{\mathscr{H}})\mathscr{T} = \mathrm{Id}$ as

$$\{\mathrm{Id} - \mathscr{H}M\overline{\mathscr{H}}M - \mathscr{H}N\overline{\mathscr{H}}N - \mathscr{H}M\overline{\mathscr{H}}N - \mathscr{H}N\overline{\mathscr{H}}M\}\mathscr{T} = \mathrm{Id}.$$

Composing S from the left-hand side, we have

$$(35) \qquad M\mathscr{T} = S + S\mathscr{H}N\overline{\mathscr{H}}N\mathscr{T} + S\mathscr{H}M\overline{\mathscr{H}}N\mathscr{T} + S\mathscr{H}N\overline{\mathscr{H}}M\mathscr{T}.$$

Composing R from the left-hand side, we have

$$(36) \qquad N\mathscr{T} = R + R\mathscr{H}M\overline{\mathscr{H}}M\mathscr{T} + R\mathscr{H}M\overline{\mathscr{H}}N\mathscr{T} + R\mathscr{H}N\overline{\mathscr{H}}M\mathscr{T}.$$

Replace $N\mathscr{T}$ in (35) by the right-hand side of (36). Adding the resulting quantity and (36), we obtain

$$\begin{aligned}
\mathscr{T} = \ &S + S\mathscr{H}N\overline{\mathscr{H}}R + S\mathscr{H}N\overline{\mathscr{H}}R\mathscr{H}M\overline{\mathscr{H}}M\mathscr{T} + S\mathscr{H}N\overline{\mathscr{H}}R\mathscr{H}M\overline{\mathscr{H}}N\mathscr{T} \\
&+ S\mathscr{H}N\overline{\mathscr{H}}R\mathscr{H}N\overline{\mathscr{H}}M\mathscr{T} + S\mathscr{H}M\overline{\mathscr{H}}R + S\mathscr{H}M\overline{\mathscr{H}}R\mathscr{H}M\overline{\mathscr{H}}M\mathscr{T} \\
&+ S\mathscr{H}M\overline{\mathscr{H}}R\mathscr{H}M\overline{\mathscr{H}}N\mathscr{T} + S\mathscr{H}M\overline{\mathscr{H}}R\mathscr{H}N\overline{\mathscr{H}}M\mathscr{T} + S\mathscr{H}N\overline{\mathscr{H}}M\mathscr{T} \\
&+ R + R\mathscr{H}M\overline{\mathscr{H}}M\mathscr{T} + R\mathscr{H}M\overline{\mathscr{H}}N\mathscr{T} + R\mathscr{H}N\overline{\mathscr{H}}M\mathscr{T} \\
= \ &S + K_1 + \cdots + K_{13}, \quad \text{say.}
\end{aligned}$$

In the estimates of K_j $(j = 1, \cdots, 13)$, the following inequality plays an important role:

$$(37) \qquad |(\mathscr{H}M\mu, N\nu) - \mathscr{H}M\mu(0)(1, N\nu)|$$

$$\leq \frac{C}{\pi}|\Gamma_t|\int_E |\mu||dw|\int_{\Gamma_t}|\nu||dw| \quad (\mu, \nu \in L^2(E_t)),$$

where $C = $ (the distance between E and $\Gamma)^{-2}$. In fact,

$$(\mathscr{H}M\mu, N\nu) - \mathscr{H}M\mu(0)(1, N\nu) = \frac{1}{\pi}\int_{\Gamma_t}\overline{\nu(z)}\Big\{\int_E \frac{z}{(w-z)w}\mu(w)|dw|\Big\}|dz|,$$

which gives (37). Since $N\mathscr{H}R$ is anti-symmetric, we have $(1, N\mathscr{H}R1) = 0$, and hence

$$(K_{13}\mu_z, \mu_\zeta) = -(\mathscr{H}M\mathscr{T}\mu_z, N\mathscr{H}R\mu_\zeta) = -\mathscr{H}M\mathscr{T}\mu_z(0)(1, N\mathscr{H}R\mu_\zeta) + o(t)$$

$$= -\mathscr{H}M\mathscr{T}\mu_z(0)\overline{\mu_\zeta(0)}(1, N\mathscr{H}R1) + o(t) = o(t).$$

Since

$$(1, R1) = \pi\gamma(\Gamma_t) = \frac{\pi}{4}t + o(t) \quad [18], \quad \mathscr{H}M\mathscr{H}N\mathscr{T}\mu_z(0) = o(1),$$

we have

$$(K_{12}\mu_z, \mu_\zeta) = (\mathscr{H}M\mathscr{H}N\mathscr{T}\mu_z, R\mu_\zeta) = \mathscr{H}M\mathscr{H}N\mathscr{T}\mu_z(0)(1, R\mu_\zeta) + o(t)$$

$$= \mathscr{H}M\mathscr{H}N\mathscr{T}\mu_z(0)\overline{\mu_\zeta(0)}(1, R1) + o(t) = o(t).$$

Since $\mathscr{H}M\mathscr{H}M\mathscr{T}\mu_z(0) = \mathscr{H}M\mathscr{H}S\mu_z(0) + o(1)$, we have

$$(K_{11}\mu_z, \mu_\zeta) = (\mathscr{H}M\mathscr{H}M\mathscr{T}\mu_z, R\mu_\zeta) = \mathscr{H}M\mathscr{H}M\mathscr{T}\mu_z(0)\overline{\mu_\zeta(0)}(1, R1) + o(t)$$

$$= \frac{\pi}{4}t\mathscr{H}M\mathscr{H}S\mu_z(0)\overline{\mu_\zeta(0)} + o(t).$$

In the same manner, we have

$$(K_{10}\mu_z, \mu_\zeta) = \mu_z(0)\overline{\mu_\zeta(0)}(R1, 1) + o(t) = \frac{\pi}{4}t\mu_z(0)\overline{\mu_\zeta(0)} + o(t),$$

$$(K_9\mu_z, \mu_\zeta) = -(\mathscr{H}M\mathscr{T}\mu_z, N\mathscr{H}S\mu_\zeta) = -\mathscr{H}M\mathscr{T}\mu_z(0)\mathscr{H}\overline{S\mu_\zeta}(0)(1, N1) + o(t)$$

$$= -t\mathscr{H}S\mu_z(0)\mathscr{H}\overline{S\mu_\zeta}(0) + o(t),$$

$$(K_6\mu_z, \mu_\zeta) = \frac{\pi}{4}t\mathscr{H}M\mathscr{H}S\mu_z(0)\mathscr{H}M\mathscr{H}\overline{S\mu_\zeta}(0) + o(t),$$

$$(K_5\mu_z, \mu_\zeta) = \frac{\pi}{4}t\mu_z(0)\mathscr{H}M\mathscr{H}\overline{S\mu_\zeta}(0) + o(t),$$

$$(K_4\mu_z, \mu_\zeta) = -\mathscr{H}M\mathscr{T}\mu_z(0)\mathscr{H}\overline{S\mu_\zeta}(0)(N\mathscr{H}R\mathscr{H}N1, 1) + o(t)$$

$$= -\left(\frac{\pi}{4} - 1\right)t\mathscr{H}S\mu_z(0)\mathscr{H}\overline{S\mu_\zeta}(0) + o(t),$$

$$(K_j\mu_z, \mu_\zeta) = o(t) \quad (j = 1, 2, 3, 7, 8).$$

Proposition 13 shows that

$$(38) \quad \mathscr{H}S\mu_\xi(0) = \overline{K(0, \bar{\xi}; \Omega)}, \quad \mu_\xi(0) + \mathscr{H}M\mathscr{H}S\mu_\xi(0) = L(0, \xi; \Omega) \quad (\xi = z, \zeta).$$

Thus

$$\pi\{K(z, \zeta; \Omega_t) - K(z, \zeta; \Omega)\}$$

$$= (\mathscr{T}\mu_z, \mu_\zeta) - (S\mu_z, \mu_\zeta) = \sum_{j=1}^{13}(K_j\mu_z, \mu_\zeta)$$

$$= \frac{\pi}{4}t\{\mu_z(0)\overline{\mu_\zeta(0)} + \mathscr{H}M\mathscr{H}S\mu_z(0)\overline{\mu_\zeta(0)} + \mu_z(0)\overline{\mathscr{H}M\mathscr{H}S\mu_\zeta}(0)$$

$$+ \mathscr{H}M\mathscr{H}S\mu_z(0)\overline{\mathscr{H}M\mathscr{H}S\mu_\zeta}(0) - \mathscr{H}S\mu_z(0)\mathscr{H}\overline{S\mu_\zeta}(0)\} + o(t)$$

$$= \frac{\pi}{4}t\{(L(0, z; \Omega)\overline{L(0, \zeta; \Omega)} - \overline{K(0, \bar{z}; \Omega)}K(0, \zeta; \Omega) + o(t)$$

$$= \frac{\pi}{4}tDK(0, z, \zeta; \Omega) + o(t),$$

which shows that $\partial K(z, \zeta; \Omega_0)/\partial t$ exists and equals $DK(0, z, \zeta; \Omega)/4$.

Second step. *We show that* $\partial L(z, \zeta; \Omega_0)/\partial t$ *exists and equals* $DL(0, z, \zeta; \Omega)/4$.

We may assume that $z \neq \zeta$. Equality (31) shows that

$$L(z, \zeta; \Omega_t) = \frac{1}{z - \zeta} + \frac{1}{\pi}(\mathscr{H}\mathscr{T}\mu_\zeta, \bar{\mu}_z),$$

$$L(z, \zeta; \Omega) = \frac{1}{z - \zeta} + \frac{1}{\pi}(M\mathscr{H}S\mu_\zeta, \bar{\mu}_z).$$

To study $\mathscr{H}\mathscr{T} - M\mathscr{H}S$, we rewrite the identity $(\mathrm{Id} - \mathscr{H}\mathscr{H})\mathscr{H}\mathscr{T} = \mathscr{H}$ as

$$\{\mathrm{Id} - \mathscr{H}M\mathscr{H}M - \mathscr{H}N\mathscr{H}N - \mathscr{H}M\mathscr{H}N - \mathscr{H}N\mathscr{H}M\}\mathscr{H}\mathscr{T} = \mathscr{H}.$$

Composing $\overline{S}$ from the left-hand side, we have

$$M\mathscr{H}\mathscr{T} = \overline{S\mathscr{H}} + \overline{S\mathscr{H}}N\mathscr{H}N\mathscr{H}\mathscr{T} + \overline{S\mathscr{H}}M\mathscr{H}N\mathscr{H}\mathscr{T} + \overline{S\mathscr{H}}N\mathscr{H}M\mathscr{H}\mathscr{T}.$$

Composing $\overline{R}$ from the left-hand side, we have

$$N\mathscr{H}\mathscr{T} = \overline{R\mathscr{H}} + \overline{R\mathscr{H}}M\mathscr{H}M\mathscr{H}\mathscr{T} + \overline{R\mathscr{H}}M\mathscr{H}N\mathscr{H}\mathscr{T} + \overline{R\mathscr{H}}N\mathscr{H}M\mathscr{H}\mathscr{T}.$$

Using these identities, we obtain, in the same manner as above,

$$
\begin{aligned}
\mathcal{R}\mathcal{T} = \ & \overline{S\mathcal{H}} + \overline{S\mathcal{H}N\mathcal{H}R\mathcal{H}} + \overline{S\mathcal{H}N\mathcal{H}R\mathcal{H}M\mathcal{H}M\mathcal{R}\mathcal{T}} \\
& + \overline{S\mathcal{H}N\mathcal{H}R\mathcal{H}M\mathcal{H}N\mathcal{R}\mathcal{T}} + \overline{S\mathcal{H}N\mathcal{H}R\mathcal{H}N\mathcal{H}M\mathcal{R}\mathcal{T}} \\
& + \overline{S\mathcal{H}M\mathcal{H}R\mathcal{H}} + \overline{S\mathcal{H}M\mathcal{H}R\mathcal{H}M\mathcal{H}M\mathcal{R}\mathcal{T}} \\
& + \overline{S\mathcal{H}M\mathcal{H}R\mathcal{H}M\mathcal{H}N\mathcal{R}\mathcal{T}} + \overline{S\mathcal{H}M\mathcal{H}R\mathcal{H}N\mathcal{H}M\mathcal{R}\mathcal{T}} \\
& + \overline{S\mathcal{H}N\mathcal{H}M\mathcal{R}\mathcal{T}} + \overline{R\mathcal{H}} + \overline{R\mathcal{H}M\mathcal{H}M\mathcal{R}\mathcal{T}} + \overline{R\mathcal{H}M\mathcal{H}N\mathcal{R}\mathcal{T}} \\
& + \overline{R\mathcal{H}N\mathcal{H}M\mathcal{R}\mathcal{T}} = \overline{S\mathcal{H}} + L_1 + \cdots + L_{13}, \quad \text{say.}
\end{aligned}
$$

In the same manner as above,

$$
\begin{aligned}
(\overline{S\mathcal{H}}\mu_\zeta, \bar\mu_z) &= -(\mu_\zeta, M\mathcal{H}\overline{S}\mu_z) - t\mu_\zeta(0)\mathcal{H}S\mu_z(0) + o(t) \\
&= (M\overline{\mathcal{H}}S\mu_\zeta, \bar\mu_z) - t\mu_\zeta(0)\mathcal{H}S\mu_z(0) + o(t),
\end{aligned}
$$

$$
(L_1\mu_\zeta, \bar\mu_z) = -\left(\frac{\pi}{4} - 1\right)t\mu_\zeta(0)\mathcal{H}S\mu_z(0) + o(t),
$$

$$
(L_4\mu_\zeta, \bar\mu_z) = -\left(\frac{\pi}{4} - 1\right)t\mathcal{H}M\overline{\mathcal{H}}S\mu_\zeta(0)\mathcal{H}S\mu_z(0) + o(t),
$$

$$
(L_5\mu_\zeta, \bar\mu_z) = \frac{\pi}{4}t\overline{\mathcal{H}}M\mu_\zeta(0)\mathcal{H}M\overline{\mathcal{H}}S\mu_z(0) + o(t),
$$

$$
(L_6\mu_\zeta, \bar\mu_z) = \frac{\pi}{4}t\overline{\mathcal{H}}M\mathcal{H}M\overline{\mathcal{H}}S\mu_\zeta(0)\mathcal{H}M\overline{\mathcal{H}}S\mu_z(0) + o(t),
$$

$$
(L_9\mu_\zeta, \bar\mu_z) = -t\mathcal{H}M\overline{\mathcal{H}}S\mu_\zeta(0)\overline{\mathcal{H}}S\mu_z(0) + o(t),
$$

$$
(L_{10}\mu_\zeta, \bar\mu_z) = \frac{\pi}{4}t\overline{\mathcal{H}}M\mu_\zeta(0)\mu_z(0) + o(t),
$$

$$
(L_{11}\mu_\zeta, \bar\mu_z) = \frac{\pi}{4}t\overline{\mathcal{H}}M\mathcal{H}M\overline{\mathcal{H}}S\mu_\zeta(0)\mu_z(0) + o(t),
$$

$$
(L_j\mu_\zeta, \bar\mu_z) = o(t) \quad (j = 2, 3, 7, 8, 12, 13).
$$

Thus, by (38), we obtain

$$
\begin{aligned}
\pi\{&L(z, \zeta; \Omega_t) - L(z, \zeta; \Omega)\} \\
&= (\mathcal{R}\mathcal{T}\mu_\zeta, \bar\mu_z) - (M\overline{\mathcal{H}}S\mu_\zeta, \bar\mu_z) = -t\mu_\zeta(0)\mathcal{H}S\mu_z(0) + \sum_{j=1}^{13}(L_j\mu_\zeta, \bar\mu_z) \\
&= \frac{\pi}{4}t[\{\mu_z(0) + \mathcal{H}M\overline{\mathcal{H}}S\mu_z(0)\}\{\overline{\mathcal{H}}M\mu_\zeta(0) + \overline{\mathcal{H}}M\mathcal{H}M\overline{\mathcal{H}}S\mu_\zeta(0)\} \\
&\quad - \mathcal{H}S\mu_z(0)\{\mu_\zeta(0) + \mathcal{H}M\overline{\mathcal{H}}S\mu_\zeta(0)\}] + o(t) \\
&= \frac{\pi}{4}t\{L(0, z; \Omega)\overline{K(0, \zeta; \Omega)} - \overline{K(0, \bar{z}; \Omega)}L(0, \zeta; \Omega)\} + o(t) \\
&= \frac{\pi}{4}tDL(0, z, \zeta; \Omega) + o(t),
\end{aligned}
$$

which shows that $\partial L(z, \zeta; \Omega_0)/\partial t$ exists and equals $DL(0, z, \zeta; \Omega)/4$.

Final step. We show that $DJ(w_t, z, \zeta; \Omega_t)/\partial t$ $(J = K, L)$ are right-continuous at $t = 0$.

We may assume that $w_0 = 0$. Here is a lemma necessary for the proof.

LEMMA 14. *Let k_t, r_t $(0 < t \leq |\Gamma|)$ be the same as in Lemma 10. Then we have, with $\hat{\Omega}_t = k_t(\Omega_t)$,*

$$(39) \qquad \lim_{t \downarrow 0} K(r_t, \overline{k_t(z)}; \hat{\Omega}_t) = K(0, \bar{z}; \Omega) - \overline{L(0, z; \Omega)},$$

$$(40) \qquad \lim_{t \downarrow 0} L(r_t, k_t(z); \hat{\Omega}_t) = L(0, z; \Omega) - \overline{K(0, \bar{z}; \Omega)},$$

$$(41) \qquad \lim_{t \downarrow 0} r_t \frac{\partial K}{\partial \xi}(r_t, \overline{k_t(z)}; \hat{\Omega}_t) = \overline{L(0, z; \Omega)},$$

$$(42) \qquad \lim_{t \downarrow 0} r_t \frac{\partial L}{\partial \xi}(r_t, k_t(z); \hat{\Omega}_t) = \overline{K(0, \bar{z}; \Omega)},$$

where $\quad \dfrac{\partial J}{\partial \xi}(r_t, \cdot\,; \hat{\Omega}_t) = \dfrac{\partial J}{\partial \xi}(\xi, \cdot\,; \hat{\Omega}_t)\Big|_{\xi = r_t} \quad (J = K, L).$

Choose $\varepsilon_0 > 0$ so that $\overline{D(0, \varepsilon_0)} \subset \Omega$ and $k_t(z) \in \Omega - \overline{D(0, \varepsilon_0)}$ $(0 < t \leq |\Gamma|)$. Then $K(\xi, \overline{k_t(z)}; \hat{\Omega}_t)$, $L(\xi, k_t(z); \hat{\Omega}_t)$ uniformly converge to $K(\xi, \bar{z}; \Omega)$, $L(\xi, z; \Omega)$, respectively, on $\xi \in S(0, \varepsilon_0)$, when $t \downarrow 0$. Form

$$\mathscr{K}_t(\xi) = \begin{cases} K(\xi, \overline{k_t(z)}; \hat{\Omega}_t) & (\xi \in D(0, \varepsilon_0) - D(0, r_t)) \\ -\overline{L(r_t^2/\bar{\xi}, k_t(z); \hat{\Omega}_t)}r_t/\xi & (\xi \in D(0, r_t) - \overline{D(0, r_t^2/\varepsilon_0)}). \end{cases}$$

Since

$$K(\xi, \overline{k_t(z)}; \hat{\Omega}_t)\xi = -\overline{L(\xi, k_t(z); \hat{\Omega}_t)}r_t \quad (\xi \in S(0, r_t)),$$

$\mathscr{K}_t(\xi)$ is analytic in $D(0, \varepsilon_0) - \overline{D(0, r_t^2/\varepsilon_0)}$. This shows that

$$K(r_t, \overline{k_t(z)}; \hat{\Omega}_t) = \frac{1}{2\pi i} \int_{\partial\{D(0,\varepsilon_0) - D(0, r_t^2/\varepsilon_0)\}} \frac{1}{\xi - r_t} \mathscr{K}_t(\xi) d\xi$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\varepsilon_0 e^{i\psi} - r_t} K(\varepsilon_0 e^{i\psi}, \overline{k_t(z)}; \hat{\Omega}_t) \varepsilon_0 e^{i\psi} d\psi$$

$$+ \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{(r_t^2/\varepsilon_0)e^{i\psi} - r_t} \overline{L(\varepsilon_0 e^{i\psi}, k_t(z); \hat{\Omega}_t)} r_t d\psi$$

$$= \frac{1}{2\pi} \int_0^{2\pi} K(\varepsilon_0 e^{i\psi}, \bar{z}; \Omega) d\psi - \frac{1}{2\pi} \int_0^{2\pi} \overline{L(\varepsilon_0 e^{i\psi}, z; \Omega)} d\psi + o(1)$$

$$= K(0, \bar{z}; \Omega) - \overline{L(0, z; \Omega)} + o(1) \quad (t \downarrow 0),$$

which gives (39). Since

$$
r_\iota \frac{\partial K}{\partial \xi}(r_\iota, \overline{k_\iota(z)}; \hat{\Omega}_\iota) = \frac{1}{2\pi i} \int_{\partial\{D(0,\varepsilon_0)-D(0,r_\iota^2/\varepsilon_0)\}} \frac{r_\iota}{(\xi - r_\iota)^2} \mathscr{K}_\iota(\xi) d\xi
$$

$$
= \frac{1}{2\pi} \int_0^{2\pi} \frac{r_\iota}{(\varepsilon_0 e^{i\psi} - r_\iota)^2} K(\varepsilon_0 e^{i\psi}, \overline{k_\iota(z)}; \hat{\Omega}_\iota) \varepsilon_0 e^{i\psi} d\psi
$$

$$
+ \frac{1}{2\pi} \int_0^{2\pi} \frac{r_\iota}{\{(r_\iota^2/\varepsilon_0) e^{i\psi} - r_\iota\}^2} \overline{L(\varepsilon_0 e^{i\psi}, k_\iota(z); \hat{\Omega}_\iota)} r_\iota d\psi
$$

$$
= \overline{\frac{1}{2\pi} \int_0^{2\pi} L(\varepsilon_0 e^{i\psi}, z; \Omega) d\psi} + o(1) = \overline{L(0, z; \Omega)} + o(1),
$$

which gives (41). In the same manner, we obtain (40) and (42). This completes the proof of Lemma 14.

Recall (28) and $\lim_{t\downarrow 0} a_\iota/r_\iota = 1/2$ in Lemma 10. Lemma 14 shows that

$$
\lim_{\iota\downarrow 0} DK(w_\iota, z, \zeta; \Omega_\iota) = \frac{1}{2}[\{L(0, z; \Omega) - \overline{K(0, \bar{z}; \Omega)}\}K(0, \zeta; \Omega)
$$

$$
+ \overline{K(0, \bar{z}; \Omega)}\{\overline{L(0, \zeta; \Omega)} - K(0, \zeta; \Omega)\} - \{\overline{K(0, \bar{z}; \Omega)} - L(0, z; \Omega)\}\overline{L(0, \zeta; \Omega)}
$$

$$
- L(0, z; \Omega)\{K(0, \zeta; \Omega) - \overline{L(0, \zeta; \Omega)}\}]
$$

$$
= L(0, z; \Omega)\overline{L(0, \zeta; \Omega)} - \overline{K(0, \bar{z}; \Omega)}K(0, \zeta; \Omega) = DK(0, z, \zeta; \Omega).
$$

Thus $DK(w_\iota, z, \zeta; \Omega_\iota)$ is continuous at $t = 0$, i.e., $\partial K(z, \zeta; \Omega_\iota)/\partial t$ is continuous at $t = 0$. In the same manner as in the proof of (28), we have

$$
DL(w_\iota, z, \zeta; \Omega_\iota) = \left[L(r_\iota, k_\iota(z); \hat{\Omega}_\iota) \overline{\frac{\partial K}{\partial \xi}(r_\iota, \overline{k_\iota(\zeta)}; \hat{\Omega}_\iota)} \right.
$$

$$
+ \frac{\partial L}{\partial \xi}(r_\iota, k_\iota(z); \hat{\Omega}_\iota) \overline{K(r_\iota, \overline{k_\iota(\zeta)}; \hat{\Omega}_\iota)} - \overline{K(r_\iota, \overline{k_\iota(z)}; \hat{\Omega}_\iota)} \frac{\partial L}{\partial \xi}(r_\iota, k_\iota(\zeta); \hat{\Omega}_\iota)
$$

$$
- \overline{\frac{\partial K}{\partial \xi}(r_\iota, \overline{k_\iota(z)}; \hat{\Omega}_\iota)} L(r_\iota, k_\iota(\zeta); \hat{\Omega}_\iota) \left] a_\iota \sqrt{k_\iota'(z)} \sqrt{k_\iota'(\zeta)} \right. ,
$$

and hence, using Lemma 14, we see that $DL(w_\iota, z, \zeta; \Omega_\iota)$ is continuous at $t = 0$, i.e., $\partial L(z, \zeta; \Omega_\iota)/\partial t$ is continuous at $t = 0$. This completes the proof of Theorem 8 in the case III.

§7. Proof of Case IV

In this section, we prove Theorem 8 in the case IV. We work only with K; the argument for L is analogous. Let $\Omega \in \mathscr{G}$ and let Γ be a closed analytic arc with the arc-length representation w_ι $(0 \le t \le |\Gamma|)$ such that $\bar{\Omega} \supset \Gamma$ and $\partial\Omega \cap \Gamma = \{w_0\}$. Without loss of generality, we may assume

that $w_0 = 0$.   We show that $\partial K(z, \bar{\zeta}; \Omega_0)/\partial t$ exists and $\partial K(z, \bar{\zeta}; \Omega_t)/\partial t$ is continuous at $t = 0$.   There exists a component $\Lambda \in \mathscr{A}$ of Ω^c with an endpoint 0.   Let h be the conformal mapping from Λ^c onto $[-1, 0]^c$ such that $h(\infty) = \infty$ and $h(0) = 0$.   Then Lemma 9 shows that $h(\Gamma) \in \mathscr{L}_\theta$ for some $-\pi < \theta \le \pi$.   If $\theta = 0$, then the case II yields the required properties.   Thus we may assume that $\theta \ne 0$.   Let w_s^* $(0 \le s \le |h(\Gamma)|)$ be the arc-length representation of $h(\Gamma)$ such that $w_0^* = 0$, and let $s = \hat{s}(t)$ be the number defined by $w_s^* = h(w_t)$.   If the required properties are established for $h(\Omega)$ and $h(\Gamma)$, then

$$\frac{\partial K}{\partial t}(z, \bar{\zeta}; \Omega_t) = \frac{\partial K}{\partial s}(h(z), \overline{h(\zeta)}; h(\Omega) - \{w_x^*; 0 \le x \le s\})$$

$$\times h'(w_t)\frac{dw_t}{dt}\left(\frac{dw_s^*}{ds}\right)^{-1}\sqrt{h'(z)}\,\overline{\sqrt{h'(\zeta)}}\quad (0 \le t \le |\Gamma|)\,.$$

Lemma 9 shows that $h'(w_t)(dw_t/dt)(dw_s^*/ds)^{-1}$ is right-continuous at $t = 0$, which yields that $\partial K(z, \bar{\zeta}; \Omega_0)/\partial t$ exists and $\partial K(z, \bar{\zeta}; \Omega_t)/\partial t$ is right-continuous at $t = 0$.   Thus, from the beginning, we may assume that $\Lambda = [-1, 0]$ and $\Gamma \in \mathscr{L}_\theta$, $\theta \ne 0$.   We put $\Lambda_t = [-1, 0] \cup \Gamma_t$ $(0 \le t \le |\Gamma|)$.   Here are some lemmas necessary for the proof.

LEMMA 15.   *For an analytic function P in* **D**, *we put*

$$U_s = \{\tau_s(y); 0 \le y \le 1\}, \qquad \tau_s(y) = ie^{i\theta/2}y + \sqrt{s}\,y^2 P(\sqrt{s}\,y)\,,$$
$$V_s = U_s \cup \overline{U}_s \quad (0 \le s < 1)\,.$$

Then

(43) $$\lim_{s \downarrow 0} \gamma(V_s) = \gamma(V_0)\,.$$

The sequence $\{V_s\}_{0 < s < 1}$ converges to V_0 as $s \downarrow 0$, however, (43) is not obvious.   Take, for example,

$$E_n = \bigcup_{k=0}^{n-1}\left\{\left[\frac{2k}{2n}, \frac{2k+1}{2n}\right] \cup \left(\frac{1+i}{2n} + \left[\frac{2k}{2n}, \frac{2k+1}{2n}\right]\right)\right\} \quad (n = 1, 2, \cdots)\,.$$

This sequence converges to $[0, 1]$ but $\gamma([0, 1]) > \lim_{n \to \infty} \gamma(E_n)$.   (See [18]). In the proof of (43), the connectivity of V_s is important.   Take the Green's function $G_s(z, \infty)$ of V_s^c with pole at ∞ $(0 \le s < 1)$.   Then we see that $\lim_{s \downarrow 0} G_s(z, \infty) = G_0(z, \infty)$.   The Ahlfors function $f(z; V_s^c)$ of V_s is expressed as $-\exp\{-G_s(z, \infty) - iG_s^*(z, \infty)\}$, where $G_s^*(z, \infty)$ is the conjugate multi-valued harmonic function of $G_s(z, \infty)$ such that $G_s^*(\infty, \infty) = 0 \pmod{2\pi}$. Thus $\lim_{s \downarrow 0} f(z; V_s^c) = f(z; V_0^c)$, which gives (43).

LEMMA 16. *Let* V_s, τ_s $(0 \leq s < 1)$ *be the same as in Lemma* 15. *Let*

$$\alpha_s = \lim_{y \downarrow 1} \{q(\tau_s(y); V_s^c) - q(\tau_s(1); V_s^c)\}/\sqrt{y - 1} \quad (0 \leq s < 1),$$

where q *is the function defined in the beginning of* §4. *Then* $\lim_{s \downarrow 0} \alpha_s = \alpha_0$.

To prove this lemma, we write simply $\hat{q}_s(z) = q(z; V_s^c)$ $(0 \leq s < 1)$. Lemma 15 shows that $\{\hat{q}_s\}_{0 < s < 1}$ locally uniformly converges to $\hat{q}_0$ as $s \downarrow 0$, and hence $\hat{q}_s(S(0, 2)) \subset \overline{D(0, M)} - D(0, 1/M)$ $(0 < s < 1)$ for some $M > 1$. Since $\hat{q}_s(V_s) = S(0, 1)$ and $\hat{q}_s$ does not take 0 in V_s^c, we have $\hat{q}_s(\overline{D(0, 2)} - V_s)$ $\subset \overline{D(0, M)} - D(0, 1/M)$. There exists $\varepsilon_0 > 0$ such that the ranges of a semi-disk $D(0, \varepsilon_0) \cap \{\text{Re } z > 0\}$ by analytic functions $\tau_s(1 + z^2)$ $(0 < s < 1)$ are contained in $\overline{D(0, 2)}$. Form $l_s(z) = \hat{q}_s \circ \tau_s(1 + z^2)$. Since $l_s(i[-\varepsilon_0, \varepsilon_0])$ $\subset S(0, 1)$ and the range of the semi-disk by $\hat{q}_s$ is contained in a ring $\overline{D(0, M)} - D(0, 1/M)$, we may regard l_s as an analytic function in $D(0, \varepsilon_0)$ whose range is contained in the ring. Thus $\{l_s\}_{0 < s < 1}$ is a normal family of analytic functions in $D(0, \varepsilon_0)$. Lemma 15 shows that $\{\hat{q}_s\}_{0 < s < 1}$ converges to $\hat{q}_0$ in $D(\varepsilon_0/2, \varepsilon_0/4)$, which yields that $\{l_s\}_{0 < s < 1}$ converges to l_0 in $D(0, \varepsilon_0)$. Thus $\lim_{s \downarrow 0} l_s'(0) = l_0'(0)$. Since

$$\hat{q}_s \circ \tau_s(1 + \delta) = \hat{q}_s \circ \tau_s(1) + \alpha_s \sqrt{\delta} + o(\sqrt{\delta}) \quad (\delta \downarrow 0),$$

we have

$$l_s(z) = l_s(0) + \alpha_s z + o(|z|) \quad (z \to 0),$$

and hence $\alpha_s = l_s'(0)$ $(0 \leq s < 1)$. Thus

$$\lim_{s \downarrow 0} \alpha_s = \lim_{s \downarrow 0} l_s'(0) = l_0'(0) = \alpha_0.$$

LEMMA 17. *Let*

$$\beta_t = \lim_{u \downarrow t} \{q(w_u; \Lambda_t^c) - q(w_t; \Lambda_t^c)\}/\sqrt{u - t} \quad (0 < t \leq |\Gamma|).$$

Then $\lim_{t \downarrow 0} \beta_t$ *exists and the value is common for all* $\Gamma \in \mathscr{L}_\theta$, *i.e., the value depends only on* θ. *If* $\theta = \pi$, *then* $\lim_{t \downarrow 0} \beta_t = 0$.

To prove this lemma, we write simply $\tilde{q}_t(z) = q(z; \Lambda_t^c)$ $(0 < t \leq |\Gamma|)$. Since $\Gamma \in \mathscr{L}_\theta$, we can express Γ as $e^{i\theta}s + s^{3/2}\hat{P}(\sqrt{s})$, $0 \leq s < r$ in a small disk $D(0, r)$ with an analytic function $\hat{P}$ in $D(0, r^2)$. Dilating the coordinate axes if necessary, we may assume that $r = 1$. Let $\hat{w}_s = e^{i\theta}s + s^{3/2}\hat{P}(\sqrt{s})$ $(0 \leq s < 1)$. This is a parametric representation of $\Gamma \cap D$ different, in general, from w_t. Define $t = \tilde{t}(s)$ by $w_t = \hat{w}_s$. Then dt/ds is

right-continuous at $s = 0$ and $\tilde{t}'(0) = 1$. We have $\hat{\beta}_s = \beta_t \sqrt{dt/ds}$ with

$$\hat{\beta}_s = \lim_{\varepsilon \downarrow 0} \{\tilde{q}_t(\hat{w}_{s+\varepsilon}) - \tilde{q}_t(\hat{w}_s)\}/\sqrt{\varepsilon} \quad (t = \tilde{t}(s)).$$

Thus it is sufficient to show that $\{\hat{\beta}_s\}_{0 < s < 1}$ converges to a quantity depending only on θ. Take a conformal mapping

$$Q(z) = i\,\frac{q(z; [-1, 0]^c) - 1}{q(z; [-1, 0]^c) + 1}$$

from $[-1, 0]^c$ onto the upper halfplane $\mathbf{C}_+$. Then $Q(0) = 0$ and $Q(\infty) = i$. We can write $Q(z) = i\sqrt{z} + zP^*(\sqrt{z})$ with an analytic function P^* in $\mathbf{D}$, and hence we can write

$$i(\hat{w}_s)^{1/2} + \hat{w}_s P^*((\hat{w}_s)^{1/2}) = ie^{i\theta/2}\sqrt{s} + sP(\sqrt{s})$$

with an analytic function P in $\mathbf{D}$. Thus $Q(\Gamma \cap \mathbf{D})$ is expressed as $ie^{i\theta/2}y + y^2 P(y)$, $0 \le y < 1$. Let U_s, V_s, τ_s, α_s, $\hat{q}_s = q(\,\cdot\,; V_s^c)$ $(0 \le s < 1)$ be the same as in Lemmas 15 and 16. Then $Q(\Gamma_t) = \sqrt{s}\,U_s$ $(t = \tilde{t}(s), (0 \le s < 1))$. Since $\hat{q}_s'(\infty) > 0$ and V_s is symmetric with respect to the real line $\mathbf{R}$, $\hat{q}_s$ is a conformal mapping from $\mathbf{C}_+ - U_s$ onto $\mathbf{C}_+ - \{\mathbf{C}_+ \cap \overline{\mathbf{D}}\}$, and hence

$$m_s(z) = \frac{1}{2}\{\hat{q}_s(Q(z)/\sqrt{s}) + \hat{q}_s(Q(z)/\sqrt{s})^{-1}\}$$

is a conformal mapping from Λ_t^c $(t = \tilde{t}(s))$ onto $\mathbf{C}_+$. Thus

$$\{m_s(z) - \overline{m_s(\infty)}\}/\{m_s(z) - m_s(\infty)\}$$

is a conformal mapping from Λ_t^c onto $\overline{\mathbf{D}}^c$, and hence

$$\tilde{q}_t(z) = e^{i\phi_s}\{m_s(z) - \overline{m_s(\infty)}\}/\{m_s(z) - m_s(\infty)\} \quad (t = \tilde{t}(s))$$

for some $0 \le \phi_s < 2\pi$. Since $Q(\hat{w}_{s+\varepsilon})/\sqrt{s} = \tau_s(\sqrt{1 + (\varepsilon/s)})$, we have

$$\tilde{q}_t(\hat{w}_{s+\varepsilon}) - \tilde{q}_t(\hat{w}_s) = -e^{i\phi_s}\frac{\{m_s(\infty) - \overline{m_s(\infty)}\}\{m_s(\hat{w}_{s+\varepsilon}) - m_s(\hat{w}_s)\}}{\{m_s(\hat{w}_{s+\varepsilon}) - m_s(\infty)\}\{m_s(\hat{w}_s) - m_s(\infty)\}}$$

$$= -e^{i\phi_s}\frac{\{m_s(\infty) - \overline{m_s(\infty)}\}\{1 - [\hat{q}_s \circ \tau_s(\sqrt{1 + (\varepsilon/s)})\hat{q}_s \circ \tau_s(1)]^{-1}\}}{2\{m_s(\hat{w}_{s+\varepsilon}) - m_s(\infty)\}\{m_s(\hat{w}_s) - m_s(\infty)\}}$$

$$\times \{\hat{q}_s \circ \tau_s(\sqrt{1 + (\varepsilon/s)}) - \hat{q}_s \circ \tau_s(1)\} \quad (t = \tilde{t}(s)).$$

Thus

$$(44) \qquad \hat{\beta}_s = -e^{i\phi_s}\frac{\alpha_s\{m_s(\infty) - \overline{m_s(\infty)}\}\{1 - \hat{q}_s \circ \tau_s(1)^{-2}\}}{2\sqrt{2s}\{m_s(\hat{w}_s) - m_s(\infty)\}^2}.$$

To study the right-hand side, we write

$$(45) \qquad \hat{q}_s(\zeta) = \gamma(V_s)^{-1}\zeta + \sum_{n=0}^{\infty} a_{s,n}\zeta^{-n} \quad (\zeta \to \infty).$$

Then $|a_{s,n}| \le M^n$ $(n = 0, 1, \cdots; 0 < s < 1)$ for some $M > 1$. Lemma 15 shows that

$$\lim_{s \downarrow 0} \sqrt{s}\, m_s(\infty) = \lim_{s \downarrow 0} \frac{\sqrt{s}}{2}\{\hat{q}_s(i/\sqrt{s}) + \hat{q}_s(i/\sqrt{s})^{-1}\} = \frac{i}{2}\gamma(V_0)^{-1}.$$

Since $Q(z) = i - i/(2z) + O(|z|^{-2})$ $(z \to \infty)$, we have

$$\hat{q}_s(Q(z)/\sqrt{s}) = \gamma(V_s)^{-1}(i/\sqrt{s})\left\{1 - \frac{1}{2z} + O(|z|^{-2})\right\}$$

$$+ \sum_{n=0}^{\infty} a_{s,n}(-i)^n s^{n/2}\left\{1 - \frac{1}{2z} + O(|z|^{-2})\right\}^{-n}$$

$$= \hat{q}_s(i/\sqrt{s}) - \left\{\frac{i}{2\sqrt{s}}\gamma(V_s)^{-1} - \sum_{n=1}^{\infty}\frac{n}{2}a_{s,n}(-i)^n s^{n/2}\right\}\frac{1}{z} + O(|z|^{-2})$$

$$(z \to \infty),$$

and hence

$$(46) \qquad \tilde{q}_t'(\infty) = e^{i\phi_s}\{m_s(\infty) - \overline{m_s(\infty)}\}\lim_{z \to \infty}\frac{1}{z\{m_s(z) - m_s(\infty)\}}$$

$$= 2e^{i\phi_s}\frac{m_s(\infty) - \overline{m_s(\infty)}}{1 - \hat{q}_s(i/\sqrt{s})^{-2}}\lim_{z \to \infty}\frac{1}{z\{\hat{q}_s(Q(z)/\sqrt{s}) - \hat{q}_s(i/\sqrt{s})\}}$$

$$= -2e^{i\phi_s}\frac{m_s(\infty) - \overline{m_s(\infty)}}{1 - \hat{q}_s(i/\sqrt{s})^{-2}}$$

$$\times \left\{\frac{i}{2\sqrt{s}}\gamma(V_s)^{-1} - \sum_{n=1}^{\infty}\frac{n}{2}a_{s,n}(-i)^n s^{n/2}\right\}^{-1} \quad (t = \tilde{t}(s)).$$

Since $\lim_{t \downarrow 0}\tilde{q}_t'(\infty) = 4$ and $\lim_{s \downarrow 0}\sqrt{s}\,m_s(\infty) = i\gamma(V_0)^{-1}/2$, we have $\lim_{s \downarrow 0} e^{i\phi_s} = -1$. Consequently, Lemma 16 and (44) yield

$$\lim_{s \downarrow 0}\hat{\beta}_s = -\sqrt{2}\,i\alpha_0\gamma(V_0)\{1 - \hat{q}_0 \circ \tau_0(1)^{-2}\}.$$

The expression in the right-hand side shows that $\lim_{s \downarrow 0}\hat{\beta}_s$ depends only on θ, i.e., $\lim_{t \downarrow 0}\beta_t$ depends only on θ. If $\theta = \pi$, then $\hat{q}_0 \circ \tau_0(1) = -1$, which gives $\lim_{s \downarrow 0}\hat{\beta}_s = 0$, i.e., $\lim_{t \downarrow 0}\beta_t = 0$.

LEMMA 18. *Let n_t be the conformal mapping from Λ_t^c onto $[-1, \delta_t]^c$ $(\delta_t > 0)$ such that $n_t(\infty) = \infty$, $|n_t'(\infty)| = 1$ and $n_t(w_t) = \delta_t$ $(0 < t \le |\Gamma|)$. Then*

$$(47) \qquad \lim_{t \downarrow 0} |n_t'(w_t)| = \lim_{t \downarrow 0} \delta_t/t = \left(\frac{\pi - \theta}{\pi + \theta} \right)^{\theta/\pi}.$$

Suppose that $0 < \theta < \pi$. We begin by showing that $\lim_{t \downarrow 0} \delta_t/t$ exists and depends only on θ. Recall $\tilde{q}_t(z) = q(z; \Lambda_t^c)$ $(0 \leq t \leq |\Gamma|)$. The function $\tilde{q}_t(w_t)^{-1}\tilde{q}_t(z) + \tilde{q}_t(w_t)\tilde{q}_t(z)^{-1}$ maps Λ_t^c onto $[-2, 2]^c$ so that the image of w_t is 2. Thus

$$n_t(z) = \gamma(\Lambda_t)\{\tilde{q}_t(w_t)^{-1}\tilde{q}_t(z) + \tilde{q}_t(w_t)\tilde{q}_t(z)^{-1} + 2\} - 1,$$

which shows that $\delta_t = 4\gamma(\Lambda_t) - 1$. Since $\tilde{q}_t'(\infty) = 1/\gamma(\Lambda_t)$, (46) shows that

$$\gamma(\Lambda_t) = \frac{|1 - \hat{q}_s(i/\sqrt{s})^{-2}|}{2|m_s(\infty) - \overline{m_s(\infty)}|} \left| \frac{i}{2\sqrt{s}} \gamma(V_s)^{-1} - \sum_{n=1}^{\infty} \frac{n}{2} a_{s,n}(-i)^n s^{n/2} \right| \quad (t = \tilde{t}(s)),$$

where $\hat{q}_s$, m_s, $\tilde{t}(s)$ are the functions in the proof of Lemma 17. Recall (45). Since $\hat{q}_s'(\infty) > 0$ and V_s is symmetric with respect to $\mathbf{R}$, $a_{s,n}$ $(n = 0, 1)$ are real-valued, and hence

$$\sqrt{s} |m_s(\infty) - \overline{m_s(\infty)}|$$
$$= \frac{\sqrt{s}}{2} \left| \hat{q}_s(i/\sqrt{s}) - \overline{\hat{q}_s(i/\sqrt{s})} + \hat{q}_s(i/\sqrt{s})^{-1} - \overline{\hat{q}_s(i/\sqrt{s})^{-1}} \right|$$
$$= \gamma(V_s)^{-1} - \{a_{s,1} + \gamma(V_s)\}s + o(s) \quad (s \downarrow 0).$$

Thus

$$\gamma(\Lambda_t) = \frac{1}{4} \frac{\{1 + \gamma(V_s)^2 s\}\{\gamma(V_s)^{-1} + a_{s,1}s\}}{\gamma(V_s)^{-1} - \{a_{s,1} + \gamma(V_s)\}s} + o(s)$$
$$= \frac{1}{4} + \frac{1}{2}\gamma(V_s)\{\gamma(V_s) + a_{s,1}\}s + o(s) \quad (t = \tilde{t}(s), s \downarrow 0).$$

Since $\{\hat{q}_s\}_{0<s<1}$ converges to $\hat{q}_0$, we have $\lim_{s \downarrow 0} a_{s,1} = a_{0,1}$. Consequently,

$$\lim_{t \downarrow 0} \delta_t/t = \lim_{t \downarrow 0} \{4\gamma(\Lambda_t) - 1\}/t$$
$$= \lim_{s \downarrow 0} \frac{2\gamma(V_s)\{\gamma(V_s) + a_{s,1}\}}{\tilde{t}'(s)} = 2\gamma(V_0)\{\gamma(V_0) + a_{0,1}\}.$$

The last expression shows that $\lim_{t \downarrow 0} \delta_t/t$ depends only on θ. Next we show that $\lim_{t \downarrow 0} |n_t'(w_t)|$ exists and depends only on θ. Since

$$n_t(w_u) - n_t(w_t) = \gamma(V_t)\tilde{q}_t(w_u)^{-1}\tilde{q}_t(w_t)^{-1}\{\tilde{q}_t(w_u) - \tilde{q}_t(w_t)\}^2 \quad (0 < t < u),$$

we have

$$n_t'(w_t) = \left(\frac{dw_t}{dt} \right)^{-1} \lim_{u \downarrow t} \frac{n_t(w_u) - n_t(w_t)}{u - t} = \gamma(V_t)\left(\frac{dw_t}{dt} \right)^{-1} \tilde{q}_t(w_t)^{-2}\beta_t^2.$$

Thus

(48) $$\lim_{t \downarrow 0} |n_t'(w_t)| = \gamma(V_0) \lim_{t \downarrow 0} |\beta_t|^2 \,.$$

Lemma 17 shows that $\lim_{t \downarrow 0} |n_t'(w_t)|$ depends only on θ. Since both $\lim_{t \downarrow 0} |n_t'(w_t)|$ and $\lim_{t \downarrow 0} \delta_t/t$ depend only on θ, it is sufficient to prove (47) in the case $\Gamma = e^{i\theta}[0, 1]$. For $0 < \varepsilon < 1$, we take the Schwarz-Christoffel transformation

$$\rho_\varepsilon(z) = \frac{1}{z}(z - e^{-i\varepsilon})^{1+\phi}(z - e^{i\varepsilon})^{1-\phi} \quad (\phi = \theta/\pi)$$

from $\overline{\mathbf{D}}^c$ onto a domain F_ε^c of form

$$e^{i\psi_\varepsilon}F_\varepsilon = [-\sigma_\varepsilon, 0] \cup (e^{i\theta}[0, \eta_\varepsilon])$$

for some triple $(\psi_\varepsilon, \sigma_\varepsilon, \eta_\varepsilon)$ of non-negative numbers. Let $\mathfrak{a}$ be the number in $(-\varepsilon, \varepsilon)$ such that $\left. \dfrac{\partial}{\partial x} |\rho_\varepsilon(e^{ix})| \right|_{x=\mathfrak{a}} = 0$. Then

$$(1 + \phi) \tan \frac{\varepsilon - \mathfrak{a}}{2} = (1 - \phi) \tan \frac{\varepsilon + \mathfrak{a}}{2} \,,$$

and hence $\mathfrak{a}$ is analytic in a neighborhood of 0 with respect to ε and $\mathfrak{a} = \phi\varepsilon + O(\varepsilon^3)$ $(\varepsilon \downarrow 0)$. Since $\rho_\varepsilon(e^{\pm i\varepsilon}) = 0$, we have

$$\eta_\varepsilon = |\rho_\varepsilon(e^{i\mathfrak{a}})| = 4 \sin^{1+\phi} \frac{\varepsilon + \mathfrak{a}}{2} \sin^{1-\phi} \frac{\varepsilon - \mathfrak{a}}{2} \,,$$

and hence η_ε is analytic in a neighborhood of 0 with respect to ε and

$$\eta_\varepsilon = (1 + \phi)^{1+\phi}(1 - \phi)^{1-\phi}\varepsilon^2 + O(\varepsilon^4) \quad (\varepsilon \downarrow 0) \,.$$

Let $\mathfrak{b}$ be the number in $(-\varepsilon, \varepsilon)$ such that $\left. \dfrac{\partial}{\partial x} |\rho_\varepsilon(-e^{ix})| \right|_{x=\mathfrak{b}} = 0$. Then

$$(1 + \phi) \tan \frac{\varepsilon + \mathfrak{b}}{2} = (1 - \phi) \tan \frac{\varepsilon - \mathfrak{b}}{2} \,,$$

and hence $\mathfrak{b}$ is analytic in a neighborhood of 0 with respect to ε and $\mathfrak{b} = -\phi\varepsilon + O(\varepsilon^3)$ $(\varepsilon \downarrow 0)$. We have

$$\sigma_\varepsilon = |\rho_\varepsilon(-e^{i\mathfrak{b}})| = 4 \cos^{1+\phi} \frac{\varepsilon + \mathfrak{b}}{2} \cos^{1-\phi} \frac{\varepsilon - \mathfrak{b}}{2} \,,$$

and hence σ_ε is analytic in a neighborhood of 0 with respect to ε and

$$\sigma_\varepsilon = 4 - (1 + \phi)(1 - \phi)\varepsilon^2 + O(\varepsilon^4) \quad (\varepsilon \downarrow 0).$$

Considering a conformal mapping $e^{i\psi_\varepsilon}\sigma_\varepsilon^{-1}\rho_\varepsilon(z)$ from $\bar{D}^c$ onto $\{[-1, 0] \cup (e^{i\theta}[0, \eta_\varepsilon \sigma_\varepsilon^{-1}])\}^c$, we obtain

$$\gamma([-1, 0] \cup (e^{i\theta}[0, \eta_\varepsilon \sigma_\varepsilon^{-1}])) = \sigma_\varepsilon^{-1}.$$

Given a small number $t > 0$, we define $\varepsilon = \varepsilon(t)$ by

$$t = \eta_\varepsilon \sigma_\varepsilon^{-1} = \frac{1}{4}(1 + \phi)^{1+\phi}(1 - \phi)^{1-\phi}\varepsilon^2 + \cdots.$$

Since

$$\sigma_\varepsilon^{-1} = \frac{1}{4} + \frac{(1 + \phi)(1 - \phi)}{16}\varepsilon^2 + \cdots$$

$$= \frac{1}{4} + \frac{1}{4}\left(\frac{1 - \phi}{1 + \phi}\right)^\phi t + o(t) \quad (t \downarrow 0),$$

we have

$$\lim_{t \downarrow 0} \delta_t/t = 4 \lim_{t \downarrow 0} \frac{1}{t}\{\gamma(\Lambda_t) - \gamma([-1, 0])\} = \left(\frac{1 - \phi}{1 + \phi}\right)^\phi,$$

which shows the second equality in (47). Since δ_t is analytic in a neighborhood of 0 with respect to t, we have

(49) $$\lim_{t \downarrow 0} \frac{d\delta_t}{dt} = 4 \lim_{t \downarrow 0} \frac{\partial \gamma}{\partial t}(\Lambda_t) = \left(\frac{1 - \phi}{1 + \phi}\right)^\phi.$$

We can write

$$n_t(w_u) = \delta_t + n_t'(w_t)(w_u - w_t) + O(|w_u - w_t|^{3/2})$$

$$= \delta_t + n_t'(w_t)\frac{dw_t}{dt}(u - t) + O((u - t)^{3/2}) \quad (u \downarrow t)$$

with $n_t'(w_t)dw_t/dt > 0$. Since $|n_t'(\infty)| = 1$, Lemma 11 shows that

$$\gamma(\Lambda_u) - \gamma(\Lambda_t) = \gamma(n_t(\Lambda_u)) - \gamma([-1, \delta_t])$$

$$= \frac{1}{4}|n_t'(w_t)|(u - t) + o(u - t) \quad (u \downarrow t),$$

which gives that $d\delta_t/dt = |n_t'(w_t)|$. Combined with (49), the first equality in (47) follows. In the same manner as above, we obtain (47) in the case where $-\pi < \theta < 0$. Suppose that $\theta = \pi$. Since $\Gamma \in \mathscr{L}_\pi$, we have

$\Lambda_t \subset \{x + iy; \ -1 \le x \le 0, \ |y| \le Ct^{3/2}\}$ $(t \downarrow 0)$ for some constant C. Since $\gamma(\{x + iy; \ -1 \le x \le 0, \ |y| \le Ct^{3/2}\}) = 1/4 + o(t)$ $(t \downarrow 0)$, we have $\lim_{t \downarrow 0} \delta_t / t = 0$. Since $\lim_{t \downarrow 0} \beta_t = 0$, (48) yields that $\lim_{t \downarrow 0} |n_t'(w_t)| = 0$. This completes the proof of Lemma 18.

Now we return to the proof of the existence of $\partial K(z, \zeta; \Omega_0)/\partial t$ and the continuity of $\partial K(z, \zeta; \Omega_t)/\partial t$ at $t = 0$ (in the case where $\Lambda = [-1, 0]$ and $\Gamma \in \mathscr{L}_\theta, \theta \ne 0$.) Let n_t be the conformal mapping in Lemma 18 $(0 < t \le |\Gamma|)$. Then

$$\frac{\partial K}{\partial t}(z, \zeta; \Omega_t) = \frac{\partial K}{\partial \varepsilon}(n_t(z), \overline{n_t(\zeta)}; n_t(\Omega_{t+\varepsilon}))\Big|_{\varepsilon=0} \sqrt{n_t'(z)} \, \overline{\sqrt{n_t'(\zeta)}}$$

$$= \frac{\partial K}{\partial \varepsilon}(n_t(z), \overline{n_t(\zeta)}; n_t(\Omega_t) - [0, \varepsilon])\Big|_{\varepsilon=\delta_t} |n_t'(w_t)| \sqrt{n_t'(z)} \, \overline{\sqrt{n_t'(\zeta)}}$$

$$(0 < t \le |\Gamma|).$$

The argument in the case I shows that

$$\lim_{t \downarrow 0} \frac{\partial K}{\partial \varepsilon}(n_t(z), \overline{n_t(\zeta)}; n_t(\Omega_t) - [0, \varepsilon])\Big|_{\varepsilon=\delta_t} = \frac{\partial K}{\partial \varepsilon}(z, \zeta; \Omega - [0, \varepsilon])\Big|_{\varepsilon=0},$$

and hence Lemma 18 yields that

$$\lim_{t \downarrow 0} \frac{\partial K}{\partial t}(z, \zeta; \Omega_t) = \left(\frac{\pi - \theta}{\pi + \theta}\right)^{\theta/\pi} \frac{\partial K}{\partial \varepsilon}(z, \zeta; \Omega - [0, \varepsilon])\Big|_{\varepsilon=0}.$$

Let n_t^* be the conformal mapping from Λ_t^c onto $[-1, \delta_t]^c$ such that $n_t^*(\infty) = \infty$ and $n_t^{*\prime}(\infty) = 1$ $(0 < t \le |\Gamma|)$. (Note that $n_t^*(w_t) \ne \delta_t$.) Lemma 12 shows that $n_t^{*\prime}(z) = o(t)$ $(t \downarrow 0, z \in \Omega - \Gamma)$. Thus

$$K(z, \zeta; \Omega_t) = K(n_t^*(z), \overline{n_t^*(\zeta)}; n_t^*(\Omega_t)) \sqrt{n_t^{*\prime}(z)} \, \overline{\sqrt{n_t^{*\prime}(\zeta)}}$$

$$= K(z, \bar{\zeta}; \Omega - [0, \delta_t]) + o(t)$$

$$= K(z, \bar{\zeta}; \Omega) + \delta_t \frac{\partial K}{\partial \varepsilon}(z, \bar{\zeta}; \Omega - [0, \varepsilon])\Big|_{\varepsilon=0} + o(t) \quad (t \downarrow 0).$$

By Lemma 18, we have

$$\frac{\partial K}{\partial t}(z, \bar{\zeta}; \Omega_0) = \left(\frac{\pi - \theta}{\pi + \theta}\right)^{\theta/\pi} \frac{\partial K}{\partial \varepsilon}(z, \bar{\zeta}; \Omega - [0, \varepsilon])\Big|_{\varepsilon=0}.$$

Consequently, $\partial K(z, \zeta; \Omega_0)/\partial t$ exists and $\partial K(z, \bar{\zeta}; \Omega_t)/\partial t$ is continuous at $t = 0$. This completes the proof of Theorem 8 in the case IV.

Remark 19. We easily see that $\lim_{t \downarrow 0} Dc(w_t, \infty; \Omega_0)$ $(= Dc(w_0, \infty; \Omega_0)$,

say) exists even if $w_0 \in \partial \Omega$. We here note that the right-derivative $\partial c(\infty; \Omega_0)/\partial t$ at $t = 0$ is not, in general, equal to $Dc(w_0, \infty; \Omega_0)/4$. Take $\Omega = [-1, 0]^c$ and $\Gamma = e^{i\theta}[0, 1]$ $(0 < \theta < \pi)$, for example. Then the second equality in (47) shows that

$$\frac{\partial c}{\partial t}(\infty; \Omega_0) = \frac{1}{4}\left(\frac{\pi - \theta}{\pi + \theta}\right)^{\theta/\pi}.$$

We have

$$Dc(w_t, \infty; \Omega_0) = \cos\left\{\frac{1}{2}\operatorname{Im}\int_{-1}^{0}\frac{dx}{x - w_t}\right\} = \cos\left\{\frac{1}{2}\psi(w_t)\right\},$$

where $\psi(w_t)$ is the angle at w_t of the triangle with vertexes -1, 0, w_t. (See Proposition 24.) Letting t tend to 0, we obtain

$$Dc(w_0, \infty; \Omega_0) = \cos\frac{\theta}{2}.$$

Since $\{(\pi - \theta)/(\pi + \theta)\}^{\theta/\pi} \neq \cos(\theta/2)$, we have $\partial c(\infty; \Omega_0)/\partial t \neq Dc(w_0, \infty; \Omega_0)/4$.

§ 8. Application to simply-connected domains

In this section, we mainly study $D^2c(w, z, \zeta; \Omega)$ for simply-connected domains Ω. Prior to the study, we give some remarks to $Dc(z, \infty; \Omega)$ and $D^2c(z, \zeta, \infty; \Omega)$ for general multiply-connected domains Ω. Let h be a conformal mapping from a domain Ω containing ∞ onto a radial slit domain [21, p. 335] Ω^* such that $h(\infty) = \infty$, $|h'(\infty)| = 1$. Then $Dc(z_0, \infty; \Omega)$ $\leq Dc(0, \infty; \Omega^*)$, where z_0 is the point in Ω such that $h(z_0) = 0$. In fact, Rengel's inequality [33, p. 393] shows that $|h'(z_0)| \leq 1$. Since $Dc(z, \zeta; \Omega)$ $\times |dz||d\zeta|$ is conformally invariant, we obtain the required inequality. If Ω^* is a circular slit domain [21, p. 335], we have $Dc(z_0, \infty; \Omega) \geq Dc(0, \infty; \Omega^*)$. Thus radial slit domains and circular slit domains are very important to estimate $Dc(z, \infty; \Omega)$. A simple calculation shows that $Dc(z, \infty; \Omega)$ is superharmonic in Ω and, if Ω is bounded by analytic Jordan curves, then $Dc(z, \infty; \Omega) = 0$ on $\partial \Omega$. The behaviour of $D^2c(z, \zeta, \infty; \Omega)$ near $z = \zeta$ is as follows. Recall the definition of $D^2c(z, \zeta, \infty; \Omega)$. We can rewrite this quantity as

$$D^2c(z, \zeta, \infty; \Omega) = 2\operatorname{Re}[\{-\phi(z; \Omega)g(\zeta; \Omega) + g(z; \Omega)\phi(\zeta; \Omega)\}\overline{L(z, \zeta; \Omega)} - \{\phi(z; \Omega)\overline{\phi(\zeta; \Omega)} - g(z; \Omega)\overline{g(\zeta; \Omega)}\}\overline{K(z, \bar{\zeta}; \Omega)}].$$

Since the Ahlfors function $f(z; \Omega)$ of Ω^c equals $g(z; \Omega)/\phi(z; \Omega)$, we have

$$\lim_{\varepsilon \downarrow 0} D^2 c(\zeta + \varepsilon e^{i\theta}, \zeta, \infty ; \Omega) = 2\operatorname{Re}\Big[\lim_{\varepsilon \downarrow 0}\{-\phi(\zeta + \varepsilon e^{i\theta}; \Omega)g(\zeta; \Omega)$$

$$+ g(\zeta + \varepsilon e^{i\theta}; \Omega)\phi(\zeta; \Omega)\}\frac{e^{i\theta}}{\varepsilon} - \{|\phi(\zeta; \Omega)|^2 - |g(\zeta; \Omega)|^2\}\overline{K(\zeta, \zeta; \Omega)}\Big]$$

$$= 2\operatorname{Re}[e^{2i\theta}\{-\phi'(\zeta; \Omega)g(\zeta; \Omega) + g'(\zeta; \Omega)\phi(\zeta; \Omega)\}$$

$$- |\phi(\zeta; \Omega)|^2\{1 - |f(\zeta; \Omega)|^2\}K(\zeta, \bar{\zeta}; \Omega)]$$

$$= 2|\phi(\zeta; \Omega)|^2\{1 - |f(\zeta; \Omega)|^2\}$$

$$\times \operatorname{Re}\Big(e^{2i\theta}\frac{\phi(\zeta; \Omega)}{\overline{\phi(\zeta; \Omega)}}\frac{f'(\zeta; \Omega)}{1 - |f(\zeta; \Omega)|^2} - K(\zeta, \bar{\zeta}; \Omega)\Big)$$

$$(= D_\theta^2 c(\zeta, \zeta, \infty ; \Omega), \text{ say}) \quad (0 \le \theta < 2\pi).$$

Note that $K(\zeta, \bar{\zeta}; \Omega)$ is the supremum of $|f'(\zeta)|$ over all $f \in H^\infty(\Omega)$, $\|f\|_{H^\infty} \le 1$. The function $\{f(z; \Omega) - f(\zeta; \Omega)\}/\{1 - f(z; \Omega)\overline{f(\zeta; \Omega)}\}$ belongs to $H^\infty(\Omega)$, the norm is equal to 1 and the derivative at $z = \zeta$ is equal to $f'(\zeta; \Omega)$ $\times \{1 - |f(\zeta; \Omega)|^2\}^{-1}$. Thus $D_\theta^2 c(\zeta, \zeta, \infty ; \Omega) \le 0$. Since $D_\theta^2 c(\zeta, \zeta, \infty ; \Omega)$ is not a constant with respect to θ, $D^2 c(z, \zeta, \infty ; \Omega)$ cannot be extended as a function in $(z, \zeta) \in \Omega \times \Omega$. If Ω is simply-connected, then there uniquely exists $0 \le \theta < 2\pi$ such that $D_\theta^2 c(\zeta, \zeta, \infty ; \Omega) = 0$. In the case where Ω is simply-connected, it is easy to study $D^2 c(w, z, \zeta ; \Omega)$. We have

PROPOSITION 20. *If Ω is simply-connected, then $D^2 c(w, z, \zeta ; \Omega) \le 0$. In particular,*

$$(50) \quad D^2 c(0, z, \zeta ; \mathbf{D}) = -\frac{(1 + |z\zeta|)^4 |1 + z\zeta|^2}{(1 - |z|^2)(1 - |\zeta|^2)|z\zeta|^2 |z - \zeta|^2}$$

$$\times \left\{\Big(\frac{|z| + |\zeta|}{1 + |z\zeta|}\Big)^2 - \Big|\frac{z + \zeta}{1 + z\bar{\zeta}}\Big|^2\right\}\left\{\Big(\frac{|z| + |\zeta|}{1 + |z\zeta|}\Big)^2 - \Big|\frac{z - \zeta}{1 - z\bar{\zeta}}\Big|^2\right\}.$$

The equality $D^2 c(0, z, \zeta ; \mathbf{D}) = 0$ holds if and only if $z\zeta$ is real.

The hyperbolic distance in $\mathbf{D}$ is defined by

$$d(z, \zeta) = \operatorname{arctanh}\Big|\frac{z - \zeta}{1 - z\bar{\zeta}}\Big| \quad (z, \zeta \in \mathbf{D}).$$

We have

$$d(z, 0) + d(0, \zeta) = \operatorname{arctanh}\frac{|z| + |\zeta|}{1 + |z\zeta|}.$$

Thus the expression in the right-hand side of (50) shows that $D^2 c$ is closely related to the triangle inequality with respect to the hyperbolic distance. The proof of this proposition is as follows. By Proposition 2,

we may assume that $\Omega = \mathbf{D}$ and $w = 0$. Since

$$K(z, \zeta; \mathbf{D}) = \frac{1}{1 - z\bar{\zeta}}, \quad L(z, \zeta; \mathbf{D}) = \frac{1}{z - \zeta} \quad [21, \text{p. } 391],$$

we have

$$D^2 c(w, z, \zeta; \mathbf{D}) = 2\operatorname{Re}\left[\left\{\frac{1}{w - z}\frac{1}{1 - \bar{w}\zeta} - \frac{1}{1 - \bar{w}z}\frac{1}{w - \zeta}\right\}\frac{1}{\bar{z} - \bar{\zeta}}\right.$$

$$\left. - \left\{\frac{1}{w - z}\frac{1}{\bar{w} - \bar{\zeta}} - \frac{1}{1 - \bar{w}z}\frac{1}{1 - w\bar{\zeta}}\right\}\frac{1}{1 - \bar{z}\zeta}\right],$$

and hence

$$D^2 c(0, z, \zeta; \mathbf{D}) = 2\operatorname{Re}\left\{\frac{z - \zeta}{z\zeta(\bar{z} - \bar{\zeta})} - \frac{1 - z\bar{\zeta}}{z\bar{\zeta}(1 - \bar{z}\zeta)}\right\}$$

$$= \frac{1}{|z\zeta|^2|z - \zeta|^2}\{(|z|^2 - |\zeta|^2)^2 - |z - \zeta|^2(|z|^2 + |\zeta|^2)\}$$

$$- \frac{1}{|z\zeta|^2|1 - z\bar{\zeta}|^2}\{(1 - |z\zeta|^2)^2 - |1 - z\bar{\zeta}|^2(1 + |z\zeta|^2)\}$$

$$= \frac{1}{|z\zeta|^2}\left\{\frac{(|z|^2 - |\zeta|^2)^2}{|z - \zeta|^2} - \frac{(1 - |z\zeta|^2)^2}{|1 - z\bar{\zeta}|^2} + (1 - |z|^2)(1 - |\zeta|^2)\right\}$$

$$= \frac{1}{|z\zeta|^2}\left\{\frac{(|z|^2 - |\zeta|^2)^2}{|z - \zeta|^2} - \frac{(1 - |z\zeta|^2)^2}{|1 - z\bar{\zeta}|^2} - \frac{(|z|^2 - |\zeta|^2)^2}{(|z| - |\zeta|)^2} + \frac{(1 - |z\zeta|^2)^2}{(1 - |z\zeta|)^2}\right\}$$

$$= \frac{(|z|^2 - |\zeta|^2)^2}{|z\zeta|^2}\left\{\frac{1}{|z - \zeta|^2} - \frac{1}{(|z| - |\zeta|)^2}\right\}$$

$$- \frac{(1 - |z\zeta|^2)^2}{|z\zeta|^2}\left\{\frac{1}{|1 - z\bar{\zeta}|^2} - \frac{1}{(1 - |z\zeta|)^2}\right\}$$

$$= -\frac{(|z| + |\zeta|)^2(2|z\zeta| - z\bar{\zeta} - \bar{z}\zeta)}{|z\zeta|^2|z - \zeta|^2} + \frac{(1 + |z\zeta|)^2(2|z\zeta| - z\bar{\zeta} - \bar{z}\zeta)}{|z\zeta|^2|1 - z\bar{\zeta}|^2}$$

$$= -\frac{(2|z\zeta| - z\bar{\zeta} - \bar{z}\zeta)(1 + |z\zeta|)^2}{|z\zeta|^2|z - \zeta|^2}\left\{\left(\frac{|z| + |\zeta|}{1 + |z\zeta|}\right)^2 - \left|\frac{z - \zeta}{1 - z\bar{\zeta}}\right|^2\right\}.$$

A calculation shows that

$$\left(\frac{|z| + |\zeta|}{1 + |z\zeta|}\right)^2 - \left|\frac{z + \zeta}{1 + z\bar{\zeta}}\right|^2 = \frac{(2|z\zeta| - z\bar{\zeta} - \bar{z}\zeta)(1 - |z|^2)(1 - |\zeta|^2)}{(1 + |z\zeta|)^2|1 + z\bar{\zeta}|^2}, \text{ i.e.,}$$

$$2|z\zeta| - z\bar{\zeta} - \bar{z}\zeta = \frac{(1 + |z\zeta|)^2|1 + z\bar{\zeta}|^2}{(1 - |z|^2)(1 - |\zeta|^2)}\left\{\left(\frac{|z| + |\zeta|}{1 + |z\zeta|}\right)^2 - \left|\frac{z + \zeta}{1 + z\bar{\zeta}}\right|^2\right\}.$$

Replacing $2|z\zeta| - z\bar{\zeta} - \bar{z}\zeta$ by the expression in the right-hand side, we obtain (50). Equality (50) shows that $D^2 c(0, z, \zeta; \mathbf{D}) = 0$ if and only if

"either $\dfrac{|z| + |\zeta|}{1 + |z\zeta|} = \left| \dfrac{z + \zeta}{1 + z\zeta} \right|$ or $\dfrac{|z| + |\zeta|}{1 + |z\zeta|} = \left| \dfrac{z - \zeta}{1 - z\zeta} \right|$ ".

Thus $D^2 c(0, z, \zeta; \mathbf{D}) = 0$ if and only if $z\zeta$ is real. This completes the proof of Proposition 20.

COROLLARY 21. *Let* $\mathrm{Cap}(\cdot)$ *denote the logarithmic capacity* [35, p. 132]. *Then for any two bounded continua* A *and* B *such that* $A \cap B \neq \emptyset$,

$$\mathrm{Cap}(A \cup B) \leq \mathrm{Cap}(A) + \mathrm{Cap}(B).$$

Note that $\mathrm{Cap}(\cdot)$ is not subadditive. If E is connected, then $\mathrm{Cap}(E) = \gamma(E)$ [35, p. 132]. Hence it is sufficient to show that $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$. Without loss of generality, we may assume that A, B are bounded by Jordan curves, $A^c, B^c \in \mathscr{G}$ and $\partial A \cap \partial B$ consists of finite points. Let $w_0 \in (\partial \Omega_{A \cup B}) \cap (\partial A \cap \partial B)$, where $\Omega_{A \cup B}$ is the component of $(A \cup B)^c$ containing ∞. For $0 < \varepsilon < |\partial A|$, A_ε denotes an arc with an endpoint w_0 such that $A_\varepsilon \subset \partial A$ and $|A_\varepsilon| = |\partial A| - \varepsilon$. We define B_ε in the same manner. If $\gamma(A_\varepsilon \cup B_\varepsilon) \leq \gamma(A_\varepsilon) + \gamma(B_\varepsilon)$ for all sufficiently small numbers $\varepsilon > 0$, then the required inequality is immediately deduced. Thus, from the beginning, we may assume that A, B are arcs in $\mathscr{A}$ with a common endpoint w_0 contained in $\partial \Omega_{A \cup B}$ and $A \cap B$ consists of finite points. We define an arc-length representation ζ_t ($0 \leq t \leq |B|$) of B so that $\zeta_0 = w_0$. Let $B_t = \{\zeta_s; 0 \leq s \leq t\}$ ($0 \leq t \leq |B|$) and

$$I = \{0 \leq t \leq |B|; \zeta_t \in A \cap B \text{ or } \zeta_t \notin \partial \Omega_{A \cup B_t}\}.$$

We define a function on $[0, |B|]$ by

$$D_0 c(\zeta_t, \infty; \Omega_{A \cup B_t}) = \begin{cases} Dc(\zeta_t, \infty; \Omega_{A \cup B_t}) & (t \in [0, |B|] - I) \\ 0 & (t \in I). \end{cases}$$

Then $D_0 c(\zeta_t, \infty; \Omega_{A \cup B_t})$ is bounded on $[0, |B|]$. Since $\gamma(A \cup B_t) = \gamma(\Omega^c_{A \cup B_t})$, we have

$$\frac{\partial \gamma}{\partial t}(A \cup B_t) = 0 = \frac{1}{4} D_0 c(\zeta_t, \infty; \Omega_{A \cup B_t}) \quad (\zeta_t \notin \partial \Omega_{A \cup B_t}),$$

and hence

$$\gamma(A \cup B) = \gamma(A) + \frac{1}{4} \int_0^{|B|} D_0 c(\zeta_t, \infty; \Omega_{A \cup B_t}) dt.$$

We show that

$$(51) \quad D_0 c(\zeta_t, \infty; \Omega_{A \cup B_t}) \leq D c(\zeta_t, \infty; B_t^c) \quad (t \in [0, |B|]) \,.$$

If $t \in I$, (51) evidently holds. Suppose that $t \notin I$. Given $\varepsilon > 0$, we can choose $\zeta_t^* \in \Omega_{A \cup B_t}$ so that

$$|D c(\zeta_t, \infty; \Omega_{A \cup B_t}) - D c(\zeta_t^*, \infty; \Omega_{A \cup B_t})| < \varepsilon \,,$$
$$|D c(\zeta_t, \infty; B_t^c) - D c(\zeta_t^*, \infty; B_t^c)| < \varepsilon \,.$$

We define an arc-length representation z_s $(0 \leq s \leq |A|)$ of A so that $z_0 = \zeta_0$. Let $A_s = \{z_x; 0 \leq x \leq s\}$ $(0 \leq s \leq |A|)$ and

$$J = \{0 \leq s \leq |A|; z_s \in A \cap B \text{ or } z_s \notin \partial \Omega_{A_s \cup B_t}\} \,.$$

We define a function on $[0, |A|]$ by

$$D_0^2 c(z_s, \zeta_t^*, \infty; \Omega_{A_s \cup B_t}) = \begin{cases} D^2 c(z_s, \zeta_t^*, \infty; \Omega_{A_s \cup B_t}) & (s \in [0, |A|] - J) \\ 0 & (s \in J) \,. \end{cases}$$

Since

$$\frac{\partial}{\partial s} D c(\zeta_t^*, \infty; \Omega_{A_s \cup B_t}) = 0 \quad (z_s \notin \partial \Omega_{A_s \cup B_t}) \,,$$

we have

$$D c(\zeta_t^*, \infty; \Omega_{A \cup B_t}) = D c(\zeta_t^*, \infty; B_t^c) + \frac{1}{4} \int_0^{|A|} D_0^2 c(z_s, \zeta_t^*, \infty; \Omega_{A_s \cup B_t}) ds \,.$$

Since $\Omega_{A_s \cup B_t}$ is simply-connected, Proposition 20 shows that $D_0^2 c(z_s, \zeta_t^*, \infty; \Omega_{A_s \cup B_t}) \leq 0$, and hence $D c(\zeta_t^*, \infty; \Omega_{A \cup B_t}) \leq D c(\zeta_t^*, \infty; B_t^c)$. Thus

$$D c(\zeta_t, \infty; \Omega_{A \cup B_t}) \leq D c(\zeta_t, \infty; B_t^c) + 2\varepsilon \,.$$

Since $\varepsilon > 0$ is arbitrary, we obtain (51). Inequality (51) shows that

$$\gamma(A \cup B) \leq \gamma(A) + \frac{1}{4} \int_0^{|B|} D c(\zeta_t, \infty; B_t^c) dt = \gamma(A) + \gamma(B) \,.$$

This completes the proof.

Here is another expression of $D^2 c$ from the point of view of Löwner's D. E. for simply-connected domains [33, p. 387].

PROPOSITION 22. *Let Γ be an arc in $\mathscr{A}$ with the arc-length representation w_t $(0 \leq t \leq |\Gamma|)$ and let p_t be the conformal mapping from E_t^c onto $\overline{\mathbf{D}}^c$ such that $p_t(\infty) = \infty$ and $p_t(w_t) = 1$. Then*

$$(52) \quad D^2 c(w_t, z, \infty; \Gamma_t^c) = - \frac{4}{\gamma(\Gamma_t)} \frac{\partial \gamma}{\partial t}(\Gamma_t) \lambda_t(z) D c(z, \infty; \Gamma_t^c) \quad (0 < t \leq |\Gamma|) \,,$$

where $\lambda_t(z) = \{\mathrm{Im}\, p_t(z)\}^2/|1 - p_t(z)|^4$.

COROLLARY 23. *Let Γ and λ_t ($0 \leq t \leq |\Gamma|$) be the same as above. Then*

(53) $\quad \gamma(\Gamma) = \dfrac{1}{4} \displaystyle\int_0^{|\Gamma|} \exp\left\{ - \int_0^t \dfrac{1}{\gamma(\Gamma_s)} \dfrac{\partial \gamma}{\partial s} (\Gamma_s) \lambda_s(w_t)\, ds \right\} dt \,.$

Since $Dc(z, \infty; \Gamma_t^c) \geq 0$, (52) also shows that $D^2 c(w_t, z, \infty; \Gamma_t^c) \leq 0$. Let $q_t^\dagger(z) = q(z; \Gamma_t^c)$ ($0 \leq t \leq |\Gamma|$). Then $q_t^{\dagger\prime}(\infty) = 1/\gamma(\Gamma_t)$. Since

$$-\frac{1}{i} \gamma(\Gamma_t) \frac{q_t^{\dagger\prime}(z)}{q_t^\dagger(z)}\, dz = -\frac{1}{i} \gamma(\Gamma_t)\, d \log q_t^\dagger(z)$$

$$= -\gamma(\Gamma_t)\, d \arg q_t^\dagger(z) \geq 0 \quad \text{on } \partial \Gamma_t^c,$$

we have

(54) $\quad g(z; \Gamma_t^c) = -\sqrt{\gamma(\Gamma_t) q_t^{\dagger\prime}(z)}/q_t^\dagger(z), \quad \phi(z; \Gamma_t^c) = \sqrt{\gamma(\Gamma_t) q_t^{\dagger\prime}(z)}\,,$

where a branch of $\sqrt{}$ is chosen so that $\sqrt{q_t^{\dagger\prime}(\infty)} = 1/\sqrt{\gamma(\Gamma_t)}$. Thus

(55) $\quad Dc(z, \infty; \Gamma_t^c) = \gamma(\Gamma_t)|q_t^{\dagger\prime}(z)|\{1 - |q_t^\dagger(z)|^{-2}\}\,.$

Choosing a suitable parametric representation of Löwner's D. E., we obtain

(56) $\quad \dfrac{\partial q_t^\dagger}{\partial t}(z) = \dfrac{1}{\gamma(\Gamma_t)} \dfrac{\partial \gamma}{\partial t}(\Gamma_t) q_t^\dagger(z) \dfrac{1 + \kappa_t^\dagger q_t^\dagger(z)}{1 - \kappa_t^\dagger q_t^\dagger(z)} \quad (\kappa_t^\dagger = \overline{q_t^\dagger(w_t)})\,.$

In fact, for a pair $t < u$, there exists a conformal mapping $\Phi_{t,u}$ from $\overline{\mathbf{D}}^c$ onto a subdomain of $\overline{\mathbf{D}}^c$ such that $\Phi_{t,u}(\infty) = \infty$, $\Phi_{t,u}'(\infty) > 0$ and $q_t^\dagger = \Phi_{t,u} \circ q_u^\dagger$. Poisson's formula shows that

$$\log \frac{q_t^\dagger}{q_u^\dagger} = \log \frac{\Phi_{t,u}(q_u^\dagger)}{q_u^\dagger} = -\frac{1}{2\pi} \int_{\psi_{t,u}}^{\psi_{t,u}'} \log |\Phi_{t,u}(e^{i\psi})| \frac{e^{i\psi} + q_u^\dagger}{e^{i\psi} - q_u^\dagger}\, d\psi \,,$$

where $(\psi_{t,u}, \psi_{t,u}') = \{0 \leq \psi < 2\pi; |\Phi_{t,u}(e^{i\psi})| > 1\}$. Analogously,

$$\log \frac{\gamma(\Gamma_u)}{\gamma(\Gamma_t)} = \log \frac{q_t^{\dagger\prime}(\infty)}{q_u^{\dagger\prime}(\infty)} = \frac{1}{2\pi} \int_{\psi_{t,u}}^{\psi_{t,u}'} \log |\Phi_{t,u}(e^{i\psi})|\, d\psi \,.$$

Note that $\lim_{u \downarrow t} \psi_{t,u} = \lim_{u \downarrow t} \psi_{t,u}' = q_t^\dagger(w_t)$. Letting u tend to t, we obtain (56). Equality (55) shows that

$$\frac{\partial}{\partial t} Dc(\,\cdot\,, \infty; \Gamma_t^c) = \frac{\partial \gamma}{\partial t}(\Gamma_t)|q_t^{\dagger\prime}|\{1 - |q_t^\dagger|^{-2}\}$$

$$+ \gamma(\Gamma_t) \frac{\partial |q_t^{\dagger\prime}|}{\partial t}\{1 - |q_t^\dagger|^{-2}\} + 2\gamma(\Gamma_t)|q_t^{\dagger\prime}||q_t^\dagger|^{-3} \frac{\partial |q_t^\dagger|}{\partial t} \,.$$

Equality (56) yields that

$$\frac{\partial |q_t^\dagger|}{\partial t} = \frac{1}{\gamma(\Gamma_t)} \frac{\partial \gamma}{\partial t}(\Gamma_t)|q_t^\dagger| \frac{1 - |q_t^\dagger|^2}{|1 - \kappa_t^\dagger q_t^\dagger|^2},$$

$$\frac{\partial |q_t^{\dagger\prime}|}{\partial t} = \frac{1}{\gamma(\Gamma_t)} \frac{\partial \gamma}{\partial t}(\Gamma_t)|q_t^{\dagger\prime}| \mathrm{Re}\, \frac{1 + 2\kappa_t^\dagger q_t^\dagger - \kappa_t^{\dagger 2} q_t^{\dagger 2}}{(1 - \kappa_t^\dagger q_t^\dagger)^2},$$

and hence

$$\frac{\partial}{\partial t} Dc(\cdot, \infty; \Gamma_t^c) = \frac{\partial \gamma}{\partial t}(\Gamma_t)|q_t^{\dagger\prime}|\{1 - |q_t^\dagger|^{-2}\}$$

$$+ \frac{\partial \gamma}{\partial t}(\Gamma_t)|q_t^{\dagger\prime}|\{1 - |q_t^\dagger|^{-2}\}\mathrm{Re}\, \frac{1 + 2\kappa_t^\dagger q_t^\dagger - \kappa_t^{\dagger 2} q_t^{\dagger 2}}{(1 - \kappa_t^\dagger q_t^\dagger)^2}$$

$$+ 2\frac{\partial \gamma}{\partial t}(\Gamma_t)|q_t^{\dagger\prime}||q_t^\dagger|^{-2} \frac{1 - |q_t^\dagger|^2}{|1 - \kappa_t^\dagger q_t^\dagger|^2}$$

$$= \frac{\partial \gamma}{\partial t}(\Gamma_t)|q_t^{\dagger\prime}|\{1 - |q_t^\dagger|^{-2}\}\left(1 + \mathrm{Re}\, \frac{1 + 2\kappa_t^\dagger q_t^\dagger - \kappa_t^{\dagger 2} q_t^{\dagger 2}}{(1 - \kappa_t^\dagger q_t^\dagger)^2} - \frac{2}{|1 - \kappa_t^\dagger q_t^\dagger|^2}\right)$$

$$= \frac{2}{\gamma(\Gamma_t)} \frac{\partial \gamma}{\partial t}(\Gamma_t)Dc(\cdot, \infty; \Gamma_t^c)\left(\mathrm{Re}\, \frac{1}{(1 - \kappa_t^\dagger q_t^\dagger)^2} - \frac{1}{|1 - \kappa_t^\dagger q_t^\dagger|^2}\right)$$

$$= -\frac{4}{\gamma(\Gamma_t)} \frac{\partial \gamma}{\partial t}(\Gamma_t)Dc(\cdot, \infty; \Gamma_t^c)\frac{(\mathrm{Im}\, \kappa_t^\dagger q_t^\dagger)^2}{|1 - \kappa_t^\dagger q_t^\dagger|^4}.$$

Since $p_t = \kappa_t^\dagger q_t^\dagger$ and $\partial Dc(z, \infty; \Gamma_t^c)/\partial t = D^2 c(w_t, z, \infty; \Gamma_t^c)/4$, we obtain (52). Combined with Theorem 1, (52) immediately yields (53).

It is interesting to compare our variational formula with Löwner's D. E.. We here point out that (56) is deduced from Theorem 8. Recall (54). Since $q_t^\dagger(z)^{-1} = -g(z; \Gamma_t^c)\phi(z; \Gamma_t^c)^{-1}$, Theorem 8 shows that

$$q_t^\dagger(z)^{-2}\frac{\partial q_t^\dagger}{\partial t}(z) = \left\{\frac{\partial g}{\partial t}(z; \Gamma_t^c) + q_t^\dagger(z)^{-1}\frac{\partial \phi}{\partial t}(z; \Gamma_t^c)\right\}\phi(z; \Gamma_t^c)^{-1}$$

$$= -\frac{1}{4}\{DK(w_t, z, \infty; \Gamma_t^c) + q_t^\dagger(z)^{-1}DL(w_t, z, \infty; \Gamma_t^c)\}\phi(z; \Gamma_t^c)^{-1}.$$

Since

$$K(w, \bar{z}; \Gamma_t^c) = -\frac{1}{1 - q_t^\dagger(w)\overline{q_t^\dagger(z)}}\sqrt{q_t^{\dagger\prime}(w)}\overline{\sqrt{q_t^{\dagger\prime}(z)}},$$

$$L(w, z; \Gamma_t^c) = \frac{1}{q_t^\dagger(w) - q_t^\dagger(z)}\sqrt{q_t^{\dagger\prime}(w)}\sqrt{q_t^{\dagger\prime}(z)},$$

we have

$$- \frac{1}{4}\{DK(w, z, \infty; \Gamma_t^c) + q_t^\dagger(z)^{-1}DL(w, z, \infty; \Gamma_t^c)\}\phi(z; \Gamma_t^c)^{-1}$$

$$= \frac{1}{4}(L(w, z; \Gamma_t^c)\overline{\phi(w; \Gamma_t^c)} - \overline{K(w, \bar{z}; \Gamma_t^c)}g(w; \Gamma_t^c)$$

$$+ q_t^\dagger(z)^{-1}\{L(w, z; \Gamma_t^c)\overline{g(w; \Gamma_t^c)} - \overline{K(w, \bar{z}; \Gamma_t^c)}\phi(w; \Gamma_t^c)\})\phi(z; \Gamma_t^c)^{-1}$$

$$= \frac{1}{4}|q_t^{\dagger\prime}(w)|\left(\frac{1}{q_t^\dagger(w) - q_t^\dagger(z)} - \frac{1}{q_t^\dagger(w)\{1 - \overline{q_t^\dagger(w)}q_t^\dagger(z)\}}\right.$$

$$\left. - \frac{1}{\overline{q_t^\dagger(w)}q_t^\dagger(z)\{q_t^\dagger(w) - q_t^\dagger(z)\}} + \frac{1}{q_t^\dagger(z)\{1 - \overline{q_t^\dagger(w)}q_t^\dagger(z)\}}\right)$$

$$= \frac{1}{4}|q_t^{\dagger\prime}(w)|\left(\frac{-1 + \overline{q_t^\dagger(w)}q_t^\dagger(z)}{\overline{q_t^\dagger(w)}q_t^\dagger(z)\{q_t^\dagger(w) - q_t^\dagger(z)\}} + \frac{q_t^\dagger(w) - q_t^\dagger(z)}{q_t^\dagger(w)q_t^\dagger(z)\{1 - \overline{q_t^\dagger(w)}q_t^\dagger(z)\}}\right)$$

$$= \frac{|q_t^{\dagger\prime}(w)|}{4|q_t^\dagger(w)|^2 q_t^\dagger(z)\{q_t^\dagger(w) - q_t^\dagger(z)\}\{1 - \overline{q_t^\dagger(w)}q_t^\dagger(z)\}}$$

$$\times (-q_t^\dagger(w)\{1 - \overline{q_t^\dagger(w)}q_t^\dagger(z)\}^2 + \overline{q_t^\dagger(w)}\{q_t^\dagger(w) - q_t^\dagger(z)\}^2)$$

$$= \frac{|q_t^{\dagger\prime}(w)|\{|q_t^\dagger(w)|^2 - 1\}\{q_t^\dagger(w)^2 - |q_t^\dagger(w)|^2 q_t^\dagger(z)^2\}}{4q_t^\dagger(w)|q_t^\dagger(w)|^2 q_t^\dagger(z)\{q_t^\dagger(w) - q_t^\dagger(z)\}\{1 - \overline{q_t^\dagger(w)}q_t^\dagger(z)\}} \;.$$

Put $w = w_u$ $(0 < t < u)$. Letting u tend to t, we have

$$\frac{\partial q_t^\dagger}{\partial t}(z) = \gamma_t^\dagger q_t^\dagger(z)\frac{1 + \overline{q_t^\dagger(w_t)}q_t^\dagger(z)}{1 - \overline{q_t^\dagger(w_t)}q_t^\dagger(z)}$$

with $\gamma_t^\dagger = \lim_{u \downarrow t} |q_t^{\dagger\prime}(w_u)|\{|q_t^{\dagger\prime}(w_u)|^2 - 1\}/4$. Since $q_t^{\dagger\prime}(\infty) = 1/\gamma(\Gamma_t)$, this equality shows that

$$- \gamma(\Gamma_t)^{-2}\frac{\partial \gamma}{\partial t}(\Gamma_t) = -\gamma_t^\dagger \gamma(\Gamma_t)^{-1}, \quad \text{i.e., } \gamma_t^\dagger = \gamma(\Gamma_t)^{-1}\frac{\partial \gamma}{\partial t}(\Gamma_t) \,.$$

Thus we obtain (56).

§9. Application to doubly-connected domains

In this section, we study $D^2c(w, z, \zeta; \Omega)$ for doubly-connected domains Ω. We begin by showing some examples of the computation of D^2c. Let $R_\rho = \{\rho < |z| < 1\}$ $(0 < \rho < 1)$.   Then

$$K(z, \bar{\zeta}; R_\rho) = \sum_{n=-\infty}^{\infty} \frac{(z\bar{\zeta})^n}{1 + \rho^{2n+1}} ,$$

$$L(z, \zeta; R_\rho) = \frac{1}{z - \zeta} + \sum_{n=0}^{\infty}\frac{\rho^{2n+1}(z^{2n+1} - \zeta^{2n+1})}{(z\zeta)^{n+1}(1 + \rho^{2n+1})} \quad [21, \text{p. } 391],$$

Thus, in the case of R_ρ, we can write D^2c, explicitly.   Using Jacobian

elliptic functions [12], we can rewrite D^2c as follows. Define a modulus $0 < k < 1$ of elliptic functions by $\rho = e^{-\pi K'/K}$, where $K = K(k)$ is the complete elliptic integral of the first kind and $K' = K(\sqrt{1-k^2})$. Let sn and dn denote the Jacobian elliptic functions with modulus k. Thus

$$(57) \quad D^2c(w, z, \zeta; R_\rho) = \frac{2K^3}{\pi^3|wz\zeta|}$$

$$\times \operatorname{Im}\left[\left(\frac{dn(\xi-u)}{sn(\xi-u)}\frac{dn(\bar{\xi}-v)}{sn(\bar{\xi}-v)} - \frac{dn(\bar{\xi}-u)}{sn(\bar{\xi}-u)}\frac{dn(\xi-v)}{sn(\xi-v)}\right)\frac{dn(\bar{u}-\bar{v})}{sn(\bar{u}-\bar{v})}\right.$$

$$\left. - \left(\frac{dn(\xi-u)}{sn(\xi-u)}\frac{dn(\bar{\xi}-\bar{v})}{sn(\bar{\xi}-\bar{v})} - \frac{dn(\bar{\xi}-u)}{sn(\bar{\xi}-u)}\frac{dn(\xi-\bar{v})}{sn(\xi-\bar{v})}\right)\frac{dn(\bar{u}-v)}{sn(\bar{u}-v)}\right],$$

$$\xi = \frac{K}{i\pi}\log w, \quad u = \frac{K}{i\pi}\log z, \quad v = \frac{K}{i\pi}\log\zeta,$$

where a branch of log is chosen so that $\log x \geq 0$ $(x \geq 1)$. In fact, put

$$K^*(z, \bar{\zeta}) = -\frac{K}{i\pi}\frac{dn(u-\bar{v})}{sn(u-\bar{v})}\frac{1}{\sqrt{z}\sqrt{\bar{\zeta}}},$$

$$L^*(z, \zeta) = \frac{K}{i\pi}\frac{dn(u-v)}{sn(u-v)}\frac{1}{\sqrt{z}\sqrt{\zeta}}.$$

Since $sn(u + 2K) = -sn\,u$, $dn(u + 2K) = dn\,u$ and $\sqrt{e^{2\pi i}} = -1$, these functions are single-valued in R_ρ as functions of z. Since $sn\,u = u + o(u)$, $dn\,u = 1 + o(u)$ $(u \to 0)$, we have $L^*(z, \zeta) = 1/(z - \zeta) + O(1)$ $(z \to \zeta)$. If $|z| = 1$, then u is real-valued, and hence $L^*(z, \zeta)\sqrt{z} = \overline{K^*(z, \bar{\zeta})\sqrt{z}}$. Since $(1/i)(|z|/z)\,dz = |dz|$ on $|z| = 1$, we have $(1/i)L^*(z, \zeta)\,dz = \overline{K^*(z, \bar{\zeta})}|dz|$ on $|z| = 1$. If $|z| = \rho$, then $u = iK' + (K/\pi)\arg z$. Since $sn(u + iK') = sn(u - iK')$, $dn(u + iK') = -dn(u - iK')$, we have $L^*(z, \zeta)\sqrt{z} = -\overline{K^*(z, \bar{\zeta})\sqrt{z}}$ on $|z| = \rho$. Since $(1/i)(|z|/z)\,dz = -|dz|$ on $|z| = \rho$, we have $(1/i)L^*(z, \zeta)\,dz = \overline{K^*(z, \bar{\zeta})}|dz|$ on $|z| = \rho$. Thus $K^*(z, \bar{\zeta}) = K(z, \bar{\zeta}; R_\rho)$ and $L^*(z, \zeta) = L(z, \zeta; R_\rho)$, which gives (57). Let Ω_ρ be a radial slit domain which is conformally equivalent to R_ρ. Then

$$Q_{\rho,z,\zeta}(w) = \exp\{\mathscr{N}_\rho(w, \zeta) - \mathscr{N}_\rho(w, z)\}$$

$$= \frac{w-z}{w-\zeta}\exp\left(-\sum_{n=1}^{\infty}\frac{X_n(w) + Y_n(w)\rho^{2n}}{n(1-\rho^{2n})}\right)$$

$$X_n(w) = (\bar{z}w)^n - (\bar{\zeta}w)^n,$$

$$Y_n(w) = (w/z)^n - (w/\zeta)^n - (z/w)^n + (\zeta/w)^n$$

is a conformal mapping from R_ρ onto Ω_ρ such that $Q_{\rho,z,\zeta}(z) = 0$ and $Q_{\rho,z,\zeta}(\zeta) = \infty$, where $\mathscr{N}_\rho(w, \xi)$ is a multi-valued analytic function in

$R_\rho - \{\xi\}$ whose real part is the Neumann function with pole at ξ [21, p. 377]. Hence, in the case of Ω_ρ also, we can write D^2c explicitly. Here is an example of D^2c which is expressed by elementary functions.

PROPOSITION 24.   Let $\Omega = \{\bigcup_{k=1}^{n}[a_k, b_k]\}^c$ $(a_1 < b_1 < \cdots < a_n < b_n)$.   Then

$$(58) \quad D^2c(w, z, \zeta; \Omega) = \frac{1}{4|M(w)M(z)M(\zeta)|}$$

$$\times \mathrm{Re}\Bigg[\left(\frac{M(w) + \overline{M(z)}}{w - z} \frac{\overline{M(w)} - M(\zeta)}{\overline{w} - \zeta}\right.$$

$$\left. - \frac{\overline{M(w)} - M(z)}{\overline{w} - z} \frac{M(w) + M(\zeta)}{w - \zeta}\right) \frac{\overline{M(z)} + \overline{M(\zeta)}}{\overline{z} - \overline{\zeta}}$$

$$- \left(\frac{M(w) + M(z)}{w - z} \frac{\overline{M(w)} + \overline{M(\zeta)}}{\overline{w} - \overline{\zeta}}\right.$$

$$\left. - \frac{\overline{M(w)} - M(z)}{\overline{w} - z} \frac{M(w) - \overline{M(\zeta)}}{w - \overline{\zeta}}\right) \frac{\overline{M(z)} - M(\zeta)}{\overline{z} - \zeta}\Bigg],$$

$$M(\xi) = \prod_{k=1}^{n} \sqrt{(b_k - \xi)/(a_k - \xi)} \quad (\xi = w, z, \zeta).$$

Note that $\sqrt{M(z)}$ is single-valued.   Since

$$K(z, \overline{\zeta}; \Omega) = \frac{1}{z - \overline{\zeta}} \sinh(h_z - \overline{h}_\zeta), \quad L(z, \zeta; \Omega) = \frac{1}{z - \zeta} \cosh(h_z - h_\zeta),$$

$$h_\xi = \frac{1}{4} \sum_{k=1}^{n} \int_{a_k}^{b_k} \frac{1}{x - \xi} dx \quad (\xi = z, \zeta) \quad ([3], [20]),$$

we have

$$D^2c(w, z, \zeta; \Omega)$$

$$= 2\mathrm{Re}\Bigg[\frac{1}{(w-z)(\overline{w}-\zeta)(\overline{z}-\overline{\zeta})} \cosh(h_w - h_z)\sinh(\overline{h}_w - h_\zeta)\cosh(\overline{h}_z - \overline{h}_\zeta)$$

$$- \frac{1}{(\overline{w}-z)(w-\zeta)(\overline{z}-\overline{\zeta})} \sinh(\overline{h}_w - h_z)\cosh(h_w - h_\zeta)\cosh(\overline{h}_z - \overline{h}_\zeta)$$

$$- \frac{1}{(w-z)(\overline{w}-\overline{\zeta})(\overline{z}-\zeta)} \cosh(h_w - h_z)\cosh(\overline{h}_w - \overline{h}_\zeta)\sinh(\overline{h}_z - h_\zeta)$$

$$+ \frac{1}{(\overline{w}-z)(w-\overline{\zeta})(\overline{z}-\zeta)} \sinh(\overline{h}_w - h_z)\sinh(h_w - \overline{h}_\zeta)\sinh(\overline{h}_z - h_\zeta)\Bigg]$$

$$= \frac{1}{4}\mathrm{Re}\Bigg[\frac{1}{(w-z)(\overline{w}-\zeta)(\overline{z}-\overline{\zeta})}$$

$$\times \left(\frac{\sqrt{M(w)}}{\sqrt{M(z)}} + \frac{\sqrt{M(z)}}{\sqrt{M(w)}}\right)\left(\frac{\overline{(\sqrt{M(w)})}}{\sqrt{M(\zeta)}} - \frac{\sqrt{M(\zeta)}}{\overline{(\sqrt{M(w)})}}\right)\left(\frac{\overline{(\sqrt{M(z)})}}{\overline{(\sqrt{M(\zeta)})}} + \frac{\overline{(\sqrt{M(\zeta)})}}{\overline{(\sqrt{M(z)})}}\right)$$

$$- \frac{1}{(\overline{w}-z)(w-\zeta)(\overline{z}-\overline{\zeta})}$$

$$\times \left(\frac{(\overline{\sqrt{M(w)}})}{\sqrt{M(z)}} - \frac{\sqrt{M(z)}}{(\overline{\sqrt{M(w)}})} \right) \left(\frac{\sqrt{M(w)}}{\sqrt{M(\zeta)}} + \frac{\sqrt{M(z)}}{\sqrt{M(w)}} \right) \left(\frac{(\overline{\sqrt{M(z)}})}{(\overline{\sqrt{M(\zeta)}})} + \frac{(\overline{\sqrt{M(\zeta)}})}{(\overline{\sqrt{M(z)}})} \right)$$

$$- \frac{1}{(w-z)(\overline{w}-\overline{\zeta})(\overline{z}-\zeta)}$$

$$\times \left(\frac{\sqrt{M(w)}}{\sqrt{M(z)}} + \frac{\sqrt{M(z)}}{\sqrt{M(w)}} \right) \left(\frac{(\overline{\sqrt{M(w)}})}{(\overline{\sqrt{M(\zeta)}})} + \frac{(\overline{\sqrt{M(\zeta)}})}{(\overline{\sqrt{M(w)}})} \right) \left(\frac{(\overline{\sqrt{M(z)}})}{\sqrt{M(\zeta)}} - \frac{\sqrt{M(\zeta)}}{(\overline{\sqrt{M(z)}})} \right)$$

$$+ \frac{1}{(\overline{w}-z)(w-\zeta)(\overline{z}-\zeta)}$$

$$\times \left(\frac{(\overline{\sqrt{M(w)}})}{\sqrt{M(z)}} - \frac{\sqrt{M(z)}}{(\overline{\sqrt{M(w)}})} \right) \left(\frac{\overline{\sqrt{M(w)}}}{(\overline{\sqrt{M(\zeta)}})} - \frac{(\overline{\sqrt{M(\zeta)}})}{\sqrt{M(w)}} \right) \left(\frac{(\overline{\sqrt{M(z)}})}{\sqrt{M(\zeta)}} - \frac{\sqrt{M(\zeta)}}{(\overline{\sqrt{M(z)}})} \right) \Bigg] .$$

which yields (58).

Note that any triply-connected domain is conformally equivalent to a radial slit ring [33, p. 413] which is equivalent to a domain of form in this proposition. Thus (58) is applicable to simply, doubly, and triply connected domains.

There remains to prove Theorem 3 in the case where Ω is doubly-connected. There are two methods of the proof of this theorem; a geometric method by (58) and an analytic method by theta functions [12] and (57). In this note, we use the first method. We shall see that the required inequality is deduced from an inequality of Möbius type in elementary geometry. It is sufficient to show that

(59) $D^2 c(x, is, -it; E^c) \leq 0$

for all $s > 0$, $t > 0$, $x \in \mathbf{R} - E$ and closed sets E of form:

$$E = [a, b] \cup [a', b'] \quad (a < b < a' < b') ,$$

$$E = (-\infty, b'] \cup [a, b] \cup [a', \infty] \quad (b' < a < b < a') .$$

In fact , once (59) has been shown, the required inequality is deduced as follows. By Proposition 2, we may assume that Ω is a ring R_ρ. Since $D^2 c(w, z, \zeta; R_\rho)$ is continuous in $R_\rho \times R_\rho \times R_\rho - \{w \neq z,\ w \neq \zeta,\ z \neq \zeta\}$, we may assume that any straight line passing through 0 contains at most one point in $\{w, z, \zeta\}$. Rotating the coordinate axes if necessary, we may assume that $w \in \mathbf{R}$. We may also assume that $\operatorname{Im} z > 0$ and $\operatorname{Im} \zeta < 0$; this

is possible because $D^2c(w, z, \zeta; R_\rho)$ is invariant for any permutation of a triple (w, z, ζ). Using elliptic functions, we can define a conformal mapping f from R_ρ onto a domain F^c of form $F = [a'', b''] \cup [a^*, b^*] \subset \mathbf{R}$ so that $f(w) \in \mathbf{R}$, $\operatorname{Im} f(z) > 0$ and $\operatorname{Im} f(\zeta) < 0$. Take a linear transformation g such that $g \circ f(z)/i > 0$, $g \circ f(\zeta)/i < 0$ and $g(\mathbf{R} \cup \{\infty\}) = \mathbf{R} \cup \{\infty\}$. Then $g(F)$ is a union of two continua on $\mathbf{R} \cup \{\infty\}$. Thus we obtain a conformal mapping $g \circ f$ from R_ρ onto $E^c = g(F)^c$ of type (59). We have $g \circ f(w) \in \mathbf{R} \cup \{\infty\}$. Choosing a point in R_ρ sufficiently near to w if necessary, we may assume that $g \circ f(w) \neq \infty$. Applying (59) to $x = g \circ f(w)$, $s = g \circ f(z)/i$ and $t = -g \circ f(\zeta)/i$, we obtain $D^2c(w, z, \zeta; R_\rho) \leq 0$, by Proposition 2.

To prove (59), we assume, for a while, that $E = [a, b] \cup [a', b']$ ($a < b < 0 < a' < b'$). Applying (58) to $\Omega = E^c$ and $w = x \in \mathbf{R} - E$, we have

$$
\begin{aligned}
D^2c(x, z, \zeta; E^c) = {}& \frac{1}{4|M(x)M(z)M(\zeta)|} \operatorname{Re}\left[\frac{\overline{M(z)} + \overline{M(\zeta)}}{(x - z)(x - \zeta)(\bar{z} - \zeta)} \right. \\
& \times \{(M(x) + M(z))(M(x) - M(\zeta)) - (M(x) - M(z))(M(x) + M(\zeta))\} \\
& - \frac{\overline{M(z)} - M(\zeta)}{(x - z)(x - \bar{\zeta})(\bar{z} - \zeta)} \\
& \left. \times \{(M(x) + M(z))(M(x) + \overline{M(\zeta)}) - (M(x) - M(z))(M(x) - \overline{M(\zeta)})\} \right] \\
= {}& \frac{1}{2|M(z)M(\zeta)||(x - z)(x - \zeta)|^2} \\
& \times \operatorname{Re}\left[\frac{(x - \bar{z})(x - \bar{\zeta})}{\bar{z} - \zeta} (\overline{M(z)} + \overline{M(\zeta)})(M(z) - M(\zeta)) \right. \\
& \left. - \frac{(x - \bar{z})(x - \zeta)}{\bar{z} - \zeta} (\overline{M(z)} - M(\zeta))(M(z) + \overline{M(\zeta)}) \right] \\
= {}& \frac{Ax^2 + Bx + C}{2|M(z)M(\zeta)||(x - z)(x - \zeta)|^2}
\end{aligned}
$$

with

$$
\begin{aligned}
A = {}& \operatorname{Re}\left\{ \frac{1}{\bar{z} - \zeta} (\overline{M(z)} + \overline{M(\zeta)})(M(z) - M(\zeta)) \right. \\
& \left. - \frac{1}{\bar{z} - \zeta} (\overline{M(z)} - M(\zeta))(M(z) + \overline{M(\zeta)}) \right\}, \\
B = {}& \operatorname{Re}\left\{ -\frac{\bar{z} + \bar{\zeta}}{\bar{z} - \zeta} (\overline{M(z)} + \overline{M(\zeta)})(M(z) - M(\zeta)) \right. \\
& \left. + \frac{\bar{z} + \zeta}{\bar{z} - \zeta} (\overline{M(z)} - M(\zeta))(M(z) + \overline{M(\zeta)}) \right\},
\end{aligned}
$$

$$C = \mathrm{Re}\left\{\frac{\overline{z\zeta}}{\bar{z} - \zeta}\, (\overline{M(z)} + \overline{M(\zeta)})(M(z) - M(\zeta)) \right.$$
$$\left. - \frac{\bar{z}\zeta}{\bar{z} - \zeta}\, (\overline{M(z)} - M(\zeta))(M(z) + \overline{M(\zeta)})\right\}.$$

(Here we used $M(x) > 0$ $(x \in \mathbf{R} - E)$.)   Since $Ax^2 + Bx + C$ is quadratic, it is sufficient to show that

(60) $$B^2 - 4AC < 0, \quad A < 0.$$

First we show $B^2 - 4AC < 0$.   To express $B^2 - 4AC$ in terms of geometry, we write

$$r_\xi e^{i\theta\xi} = M(\xi)^2 = \frac{(b - \xi)(b' - \xi)}{(a - \xi)(a' - \xi)} \quad (r_\xi > 0,\ |\theta_\xi| < \pi,\ \xi = z, \zeta).$$

Then, putting $z = is$, $\zeta = -it$, we have

$$A = -2\sqrt{r_z r_\zeta}\left\{\frac{1}{s + t}\sin\frac{\theta_z - \theta_\zeta}{2} + \frac{1}{s - t}\sin\frac{\theta_z + \theta_\zeta}{2}\right\}$$

$$B = -\frac{s - t}{s + t}(r_z - r_\zeta) + \frac{s + t}{s - t}(r_z - r_\zeta)$$

$$= \frac{4st}{(s + t)(s - t)}\sqrt{r_z r_\zeta}\{\sqrt{r_z/r_\zeta} - \sqrt{r_\zeta/r_z}\},$$

$$C = -2\sqrt{r_z r_\zeta}\left\{\frac{st}{s + t}\sin\frac{\theta_z - \theta_\zeta}{2} - \frac{st}{s - t}\sin\frac{\theta_z + \theta_\zeta}{2}\right\},$$

and hence

$$\frac{B^2 - 4AC}{16 r_z r_\zeta} = \left\{\frac{st}{(s + t)(s - t)}\right\}^2\{r_z r_\zeta^{-1} + r_z^{-1} r_\zeta - 2\}$$

$$- \left\{\frac{1}{s + t}\sin\frac{\theta_z - \theta_\zeta}{2} + \frac{1}{s - t}\sin\frac{\theta_z + \theta_\zeta}{2}\right\}$$

$$\times \left\{\frac{st}{s + t}\sin\frac{\theta_z - \theta_\zeta}{2} - \frac{st}{s - t}\sin\frac{\theta_z + \theta_\zeta}{2}\right\}$$

$$= \left\{\frac{st}{(s + t)(s - t)}\right\}^2\{r_z r_\zeta^{-1} + r_z^{-1} r_\zeta - 2\}$$

$$- \frac{st}{2(s + t)^2}\{1 - \cos(\theta_z - \theta_\zeta)\} + \frac{st}{2(s - t)^2}\{1 - \cos(\theta_z + \theta_\zeta)\}$$

$$= \left\{\frac{st}{(s + t)(s - t)}\right\}^2\left\{r_z r_\zeta^{-1} + r_z^{-1} r_\zeta - 2\cos\theta_z \cos\theta_\zeta + \frac{s^2 + t^2}{st}\sin\theta_z \sin\theta_\zeta\right\}$$

$$= \left\{\frac{st}{(s + t)(s - t)}\right\}^2 D, \quad \text{say.}$$

Note that $0 < \theta_z < \pi$, $-\pi < \theta_\zeta < 0$. Let R, Q, R', Q', r, q, r', q', α, α', β, β' be the positive numbers in the figure in § 1. Then

$$D = \frac{QQ'rr'}{RR'qq'} + \frac{RR'qq'}{QQ'rr'} - 2\cos(\alpha + \alpha')\cos(\beta + \beta')$$

$$- \frac{s^2 + t^2}{st}\sin(\alpha + \alpha')\sin(\beta + \beta').$$

A calculation yields that

(61) $$D = -\frac{1}{RQR'Q'rqr'q'}(b - a)(b' - a')(b' - a)(a' - b)(s^2 - t^2)^2.$$

Thus $B^2 - 4AC < 0$. For the sake of completeness, we show our computation of D. We have

$$\frac{QQ'rr'}{RR'qq'} + \frac{RR'qq'}{QQ'rr'} = \frac{Q^2Q'^2r^2r'^2 + R^2R'^2q^2q'^2}{RQR'Q'rqr'q'}$$

$$= \frac{1}{RQR'Q'rqr'q'}\{(s^2 + b^2)(s^2 + b'^2)(t^2 + a^2)(t^2 + a'^2)$$

$$+ (s^2 + a^2)(s^2 + a'^2)(t^2 + b^2)(t^2 + b'^2)\}$$

$$= \frac{1}{RQR'Q'rqr'q'}\{2s^4t^4 + c_1(s^4t^2 + s^2t^4) + c_2(s^4 + t^4)$$

$$+ c_3s^2t^2 + c_4(s^2 + t^2) + 2a^2b^2a'^2b'^2\}$$

with

$$c_1 = a^2 + b^2 + a'^2 + b'^2, \qquad c_2 = a^2a'^2 + b^2b'^2,$$

$$c_3 = 2(a^2 + a'^2)(b^2 + b'^2), \qquad c_4 = a^2a'^2(b^2 + b'^2) + (a^2 + a'^2)b^2b'^2.$$

We have

$$2\cos(\alpha + \alpha')\cos(\beta + \beta') + \frac{s^2 + t^2}{st}\sin(\alpha + \alpha')\sin(\beta + \beta')$$

$$= 2\{\cos\alpha\cos\alpha' - \sin\alpha\sin\alpha'\}\{\cos\beta\cos\beta' - \sin\beta\sin\beta'\}$$

$$+ \frac{s^2 + t^2}{st}\{\sin\alpha\cos\alpha' + \cos\alpha\sin\alpha'\}\{\sin\beta\cos\beta' + \cos\beta\sin\beta'\}$$

$$= 2\left\{\frac{(s^2 + ab)(s^2 + a'b')}{RQR'Q'} - \frac{s^2(b - a)(b' - a')}{RQR'Q'}\right\}$$

$$\times \left\{\frac{(t^2 + ab)(t^2 + a'b')}{rqr'q'} - \frac{t^2(b - a)(b' - a')}{rqr'q'}\right\}$$

$$+ \frac{s^2 + t^2}{st}\left\{\frac{s(b - a)(s^2 + a'b')}{RQR'Q'} + \frac{(s^2 + ab)s(b' - a')}{RQR'Q'}\right\}$$

$$\times \left\{ \frac{t(b-a)(t^2+a'b')}{rqr'q'} + \frac{(t^2+ab)t(b'-a')}{rqr'q'} \right\}$$

$$= \frac{1}{RQR'Q'rqr'q'} [2\{(s^2+ab)(s^2+a'b') - s^2(b-a)(b'-a')\}$$

$$\times \{(t^2+ab)(t^2+a'b') - t^2(b-a)(b'-a')\}$$

$$+ (s^2+t^2)\{(b-a)(s^2+a'b') + (s^2+ab)(b'-a')\}$$

$$\times \{(b-a)(t^2+a'b') + (t^2+ab)(b'-a')\}]$$

$$= \frac{1}{RQR'Q'rqr'q'} \{2s^4t^4 + d_1(s^4t^2 + s^2t^4) + d_2(s^4+t^4) + d_3 s^2 t^2$$

$$+ d_4(s^2+t^2) + 2a^2b^2a'^2b'^2\}$$

with

$$d_1 = 2\{ab + a'b' - (b-a)(b'-a')\} + (b-a+b'-a')^2 ,$$

$$d_2 = 2aba'b' + (b-a+b'-a')\{(b-a)a'b' + ab(b'-a')\} ,$$

$$d_3 = 2\{ab + a'b' - (b-a)(b'-a')\}^2$$
$$+ 2(b-a+b'-a')\{(b-a)a'b' + ab(b'-a')\} ,$$

$$d_4 = 2aba'b'\{ab + a'b' - (b-a)(b'-a')\} + \{(b-a)a'b' + ab(b'-a')\}^2 .$$

Since

$$d_1 = c_1, \quad d_4 = c_4, \quad d_2 - c_2 = (b-a)(b'-a')(b'-a)(a'-b) ,$$

$$d_3 - c_3 = -2(b-a)(b'-a')(b'-a)(a'-b) ,$$

we have (61). Next we show $A < 0$. Since

$$0 \le B^2 < 4AC = 16 r_z r_\zeta st \left\{ \frac{1}{(s+t)^2} \sin^2 \frac{\theta_z - \theta_\zeta}{2} - \frac{1}{(s-t)^2} \sin^2 \frac{\theta_z + \theta_\zeta}{2} \right\} ,$$

we have

$$\left| \frac{1}{s-t} \sin \frac{\theta_z + \theta_\zeta}{2} \right| < \frac{1}{s+t} \sin \frac{\theta_z - \theta_\zeta}{2} \quad (-\pi < \theta_\zeta < 0 < \theta_z < \pi),$$

and hence

$$A = -2\sqrt{r_z r_\zeta} \left\{ \frac{1}{s+t} \sin \frac{\theta_z - \theta_\zeta}{2} + \frac{1}{s-t} \sin \frac{\theta_z + \theta_\zeta}{2} \right\} < 0 .$$

Thus (60) holds, which yields (59) ($a < b < 0 < a' < b'$). In the above argument, the signs of a, b, a', b' are not essential. We have (59) as long as $a < b < a' < b'$. In the case where $E = (-\infty, b'] \cup [a, b] \cup [a', \infty]$ ($b' < a < b < a'$), we apply (58) to $\Omega = \{[-y, b'] \cup [a, b] \cup [a', y]\}^c$ ($y >$

$|a'| + |b'|$). Letting y tend to ∞, we obtain

$$D^2c(x, z, \zeta; E^c) = \frac{A'x^2 + B'x + C'}{2|M(z)M(\zeta)||(x - z)(x - \zeta)|^2},$$

where $M(\xi)$ ($\xi = z, \zeta$), A', B', C' are defined, in the same manner as above, by

$$M(\xi) = \lim_{y \uparrow \infty} \left(\frac{(b' - \xi)(b - \xi)(y - \xi)}{(-y - \xi)(a - \xi)(a' - \xi)} \right)^{1/2},$$

$$r_\xi e^{i\theta_\xi} = -\frac{(b' - \xi)(b - \xi)}{(a - \xi)(a' - \xi)} \quad (r_\xi > 0, |\theta_\xi| < \pi, \xi = z, \zeta).$$

Note that $B'^2 - 4A'C'$ is equal to the discriminant with respect to $[-a', -b] \cup [-a, -b']$, $z = is$, $\zeta = -it$. Thus $B'^2 - 4A'C' < 0$. Since $A'C' > 0$ and $0 < (\theta_z - \theta_\zeta)/2 < \pi$, we have, in the same manner as above, $A' < 0$. Thus (59) holds for $b' < a < b < a'$. This completes the proof of $D^2c(\cdot, \cdot, \cdot; \Omega) \leq 0$ for doubly-connected domains Ω. Combined with Proposition 20, Theorem 3 follows.

Using Theorem 3, we can study $\gamma(A) + \gamma(B) - \gamma(A \cup B)$ for two continua A, B with a quantitative estimate. Moreover, let A be a union of two continua and let B a continuum such that $A \cap B \neq \emptyset$. Then, in the same manner as in §8, we have $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$, by Theorem 3.

Remark 25. The function $c(\infty; \Omega_t)$ is not, in general, concave. Take $\Omega = [-1, 0]$ and $\Gamma = iy + [0, 1]$ ($y > 0$), for example. Then Proposition 24 shows that

$$Dc(w_0, \infty; \Omega_0) = \left(\frac{1 + y/\sqrt{1 + y^2}}{2} \right)^{1/2}$$

Choose $y > 0$ sufficienty small so that $c(\infty; \Omega_1) \geq 0.49$ and $Dc(w_0, \infty; \Omega_0) \leq 0.7$. Then, if $Dc(w_t, \infty; \Omega_t) \leq Dc(w_0, \infty; \Omega_0)$ ($0 \leq t \leq 1$), we have

$$0.49 \leq c(\infty; \Omega_1) \leq c(\infty; \Omega_0) + \frac{1}{4} Dc(w_0, \infty; \Omega_0) \leq \frac{1.7}{4} \leq 0.43,$$

which is a contradiction. Thus $\partial c(\infty; \Omega_t)/\partial t = Dc(w_t, \infty; \Omega_t)/4$ is not decreasing.

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Department of Mathematics
School of Science
Nagoya University
Chikusa-ku, Nagoya, 464-01
Japan