

## RANKIN-SELBERG METHOD FOR SIEGEL CUSP FORMS

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### Introduction

Let  $G_n$  (resp.  $\Gamma_n$ ) be the real symplectic (resp. Siegel modular) group of degree  $n$ . The Siegel cusp form is a holomorphic function on the Siegel upper half plane which satisfies functional equations relative to  $\Gamma_n$  and vanishes at the cusps. For an integer  $r$ ,  $1 \leq r \leq n$ , there exists a maximal parabolic subgroup  $P_r$  of  $G_n$  defined by

$$P_r = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_n \mid a_{21} = c_{21} = 0, c_{11} = 0, c_{12} = 0 \right\},$$

in which we decompose an  $n \times n$  matrix  $x$  into  $r \times r$ ,  $r \times (n-r)$ ,  $(n-r) \times r$  and  $(n-r) \times (n-r)$  submatrices  $x_{11}$ ,  $x_{12}$ ,  $x_{21}$  and  $x_{22}$ , respectively. Let  $F$  and  $H$  be Siegel cusp forms of the same weight  $l$ . For any half-integral positive definite symmetric matrix  $S$  of size  $r$ , we denote by  $f_s$  and  $h_s$  the  $S$ -th Fourier-Jacobi coefficients relative to  $P_r$  of  $F$  and  $H$ , respectively. Then they are Jacobi cusp forms of weight  $l$  and index  $S$  and we denote their Petersson inner product by  $(f_s, h_s)$ . Consider a Dirichlet series defined by

$$D_r(F, H : s) = \sum_{S/\sim} \frac{1}{\varepsilon(S)} \frac{(f_s, h_s)}{(\det S)^s},$$

in which the summation is taken over the set of equivalence classes of  $S$  and  $\varepsilon(S)$  denotes the order of its automorphism group. This is an obvious generalization of the symmetric square for the elliptic cusp forms ([8]). Our main objective is to show that the Rankin-Selberg method is applicable to the study of the analytic properties of  $D_r(F, H : s)$ .

We remark that, in the special case where  $r = n$ , this type of Dirichlet series has been examined by Maass [5] for  $n = 2$  and by Kurokawa for general  $n$  (unpublished). Also Kohnen-Skoruppa [4] recently investigated

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the case where  $n = 2$  and  $r = 1$ . Among other things, they showed that if  $F = H$  is in the Maass space and is a common eigen function of the Hecke operators, then  $D_1(F, F; s)$  has Euler product and, up to some elementary facts, coincides with Andrianov's spinor zeta function [1].

Now we give a brief account of the paper. In Section 1, we collect standard facts about Fourier-Jacobi expansion of the Siegel modular forms. In Section 2, following Kalinin [3] we closely examine the Eisenstein series  $E_r(s; g)$  for the symplectic group. It is a function on  $\mathbf{C} \times G_n$  and is a non-holomorphic automorphic form of weight zero with respect to  $g$ . We show that, as a function on  $\mathbf{C}$ , it can be continued meromorphically to the entire complex plane and satisfies a functional equation (Theorem 2.2). In a special case where  $r = 1$ , it has a nice holomorphy property (Theorem 2.3). In Section 3, we calculate the Petersson inner product  $(FE_r, H)$ . It turns out that, up to some elementary factors, it is equal to a translate of  $D_r(F, H; s)$  (Theorem 3.2). Then, applying the Rankin-Selberg method, we get analytic continuation and a functional equation for  $D_r(F, H; s)$  (Theorem 3.4).

*Notation.* As usual we denote by  $\mathbf{Z}$ ,  $\mathbf{Q}$ ,  $\mathbf{R}$  and  $\mathbf{C}$  the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. For an associative ring  $A$  with identity element,  $A^\times$  denotes the group of invertible elements of  $A$ .

We denote by  $M_{m,n}$  the set of  $m \times n$  matrices. We put  $M_m = M_{m,m}$ . If  $x$  is a matrix,  ${}^t x$ ,  $\det(x)$  and  $\text{tr}(x)$  stand for its transpose, determinant and trace, respectively. The identity and zero matrix in  $M_m$  are denoted by  $1_m$  and  $0_m$ , respectively. If  $x_1, \dots, x_r$  are square matrices,  $\text{diag}(x_1, \dots, x_r)$  denotes the matrix with  $x_1, \dots, x_r$  in the diagonal blocks and zero matrices in all other blocks.

For an algebraic group  $\mathbf{G}$  defined over  $\mathbf{Q}$  and a commutative ring  $A$ , we denote by  $\mathbf{G}(A)$  the group of  $A$ -valued points of  $\mathbf{G}$ .

We put  $\text{Sym}_m = \{S \in M_m \mid {}^t S = S\}$ . For  $S \in \text{Sym}_m$  and  $x \in M_{m,n}$ , we write  $S[x] = {}^t x S x$ . Two symmetric matrices  $S, T \in \text{Sym}_m(\mathbf{Q})$  are said equivalent and written as  $S \sim T$ , if there exists  $g \in GL_m(\mathbf{Z})$  such that  $S[g] = T$ .

The symplectic group  $Sp_n$  of degree  $n$  is defined by

$$Sp_n = \{g \in M_{2n} \mid {}^t g J_n g = J_n\},$$

in which  $J_n = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}$ . The Siegel upper half plane  $H_n$  of degree  $n$

is the set of symmetric matrices  $\tau = \text{Sym}_n(\mathbb{C})$  with positive definite imaginary parts  $\text{Im}(\tau) > 0$ .

For a real number  $x$ , we denote by  $[x]$  the largest integer such that  $[x] \leq x$ . For a complex number  $s$ , we write  $\mathbf{e}(s) = e^{2\pi i s}$ . We also write  $\xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$ , in which  $\Gamma$  denotes the gamma function and  $\zeta$  denotes the Riemann zeta function.

## 1. Preliminaries

The purpose of this section is to summarize those items that we shall need in the following. Let us start at the Siegel cusp forms. Let  $G_n$  be the symplectic group of degree  $n$ . We put  $G_n = G_n(\mathbb{R})$  and  $\Gamma_n = G_n(\mathbb{Z})$ . Then  $G_n$  operates transitively on the Siegel upper half plane, namely for any  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  in  $G_n$  and  $\tau$  in  $H_n$ , we define

$$g\langle\tau\rangle = (a\tau + b)(c\tau + d)^{-1},$$

and the canonical automorphic factor is given by

$$j(g, \tau) = c\tau + d.$$

The isotropy subgroup  $K$  at  $\tau_0 = i1_n$  is a maximal compact subgroup of  $G_n$ . Let us fix a natural number  $l$  and consider a function  $F$  on  $G_n$  which satisfies the functional equation

$$(S1) \quad F(\gamma g k) = \det j(k, \tau_0)^{-l} F(g),$$

for all  $\gamma$  in  $\Gamma_n$  and  $k$  in  $K$ . For any function  $F$  on  $G_n$  which satisfies (S1), we put

$$F^o(\tau) = \det j(g_\tau, \tau_0)^l F(g_\tau),$$

in which for any  $\tau$  in  $H_n$  we take an element  $g_\tau$  in  $G_n$  such that  $g_\tau \langle \tau_0 \rangle = \tau$ . Then  $F^o$  does not depend on the choice of  $g_\tau$  and defines a function on  $H_n$ . For a function  $F$  on  $G_n$  satisfying (S1), we consider the following conditions.

(S2) The associated function  $F^o$  on  $H_n$  is holomorphic.

(S3) The function  $F$  is bounded on  $G_n$ .

The functions on  $G_n$  which satisfy the conditions (S1), (S2) and (S3) are called the Siegel cusp forms of weight  $l$ , and we denote by  $S(l)$  the

totality of such functions. We also define the Petersson inner product on  $S(l)$  by

$$(F_1, F_2) = \int_{\Gamma_n/G_n} F_1(g) \overline{F_2(g)} dg,$$

in which  $dg$  denotes the Haar measure on  $G_n$ .

Secondly we shall briefly recall the basic facts about the Jacobi forms. For more details, we refer to Murase [7]. Let  $m$  and  $r$  be natural numbers. For  $h = (\lambda, \mu, \kappa) \in M_{r,m} \times M_{r,m} \times \text{Sym}_r$  and  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{G}_m$ , we put

$$(h, g) = \begin{bmatrix} 1_r & 0 & \kappa & \mu \\ 0 & 1_m & {}^t\mu & 0 \\ 0 & 0 & 1_r & 0 \\ 0 & 0 & 0 & 1_m \end{bmatrix} \times \begin{bmatrix} 1_r & \lambda & 0 & 0 \\ 0 & 1_m & 0 & 0 \\ 0 & 0 & 1_r & 0 \\ 0 & 0 & -{}^t\lambda & 1_m \end{bmatrix} \times \begin{bmatrix} 1_r & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1_r & 0 \\ 0 & c & 0 & d \end{bmatrix}.$$

Then  $\mathbf{G}_{m,r} = \{(h, g) | h \in M_{r,m} \times M_{r,m} \times \text{Sym}_r, g \in \mathbf{G}_m\}$  forms a  $\mathbf{Q}$ -algebraic subgroup of  $\mathbf{G}_{m+r}$ , and it is a semi-direct product of the Heisenberg group  $\mathbf{H}_{m,r} = \{(h, 1_{2m}) | h \in M_{r,m} \times M_{r,m} \times \text{Sym}_r\}$  and  $\mathbf{G}_m$ . Note that the center of  $\mathbf{G}_{m,r}$  is  $\mathbf{Z}_{m,r} = \{((0, 0, \kappa), 1_{2m}) | \kappa \in \text{Sym}_r\}$ . For simplicity, we write  $hg$  for each element  $(h, g)$  of  $\mathbf{G}_{m,r}$ . Let  $D_{m,r}$  denote the complex domain  $H_m \times M_{r,m}(\mathbf{C})$ . Then  $G_{m,r} = \mathbf{G}_{m,r}(\mathbf{R})$  acts on  $D_{m,r}$  transitively by

$$\eta \langle Z \rangle = (g \langle \tau \rangle, zj(g, \tau)^{-1} + \lambda g \langle \tau \rangle + \mu),$$

in which  $\eta = (\lambda, \mu, \kappa)g \in G_{m,r}$  and  $Z = (\tau, z) \in D_{m,r}$ . The stabilizer of  $Z_0 = (\tau_0, 0) \in D_{m,r}$  in  $G_{m,r}$  coincides with  $\mathbf{Z}_{m,r}(\mathbf{R})K$ . We shall fix a natural number  $l$  and a half-integral positive definite symmetric matrix  $S$  of size  $r$ . The automorphic factor  $J_{l,s}: G_{m,r} \times D_{m,r} \rightarrow \mathbf{C}^\times$  of weight  $l$  and index  $S$  is defined by

$$J_{l,s}(\eta, Z) = \det j(g, \tau)^l J_s(\eta, Z),$$

where for  $\eta = (\lambda, \mu, \kappa)g \in G_{m,r}$ ,  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and  $Z = (\tau, z) \in D_{m,r}$  we put

$$J_s(\eta, Z) = \mathbf{e}(-\text{tr}(S\kappa) + \text{tr}(S[z]j(g, \tau)^{-l}c) - 2\text{tr}({}^t\lambda S z j(g, \tau)^{-1}) - \text{tr}(S[\lambda]g \langle \tau \rangle)).$$

We also define a character  $\psi_s$  of  $\text{Sym}_r(\mathbf{R})/\text{Sym}_r(\mathbf{Z})$  by

$$\psi_s(\kappa) = \mathbf{e}(\text{tr}(S\kappa)).$$

Let  $f$  be a function on  $G_{m,r}$  satisfying

$$(J1) \quad f((0, 0, \kappa)\gamma\eta k) = \det j(k, \tau_0)^{-l} \psi_S(\kappa) f(\eta)$$

for  $\kappa \in \text{Sym}_r(\mathbf{R})$ ,  $\gamma \in G_{m,r}(\mathbf{Z})$  and  $k \in K$ . For each  $Z \in D_{m,r}$ , take an element  $\eta_z \in G_{m,r}$  so that  $\eta_z \langle Z_0 \rangle = Z$  and put

$$f^o(Z) = f(\eta_z) J_{l,s}(\eta_z, Z).$$

Then  $f^o(Z)$  does not depend on the choice of  $\eta_z$  and defines a function on  $D_{m,r}$ .

Let  $S(l, S)$  be the space of functions  $f$  on  $G_{m,r}$  satisfying the following conditions (J2) and (J3) as well as (J1).

(J2) The associated function  $f^o$  is holomorphic on  $D_{m,r}$ .

(J3) The function  $f$  is bounded on  $G_{m,r}$ .

Each element of  $S(l, S)$  is called a Jacobi cusp form of weight  $l$  and index  $S$ . The Petersson inner product is defined by

$$(f_1, f_2) = \int_{G_{m,r}(\mathbf{Z}) \backslash G_{m,r}} f_1(\eta) \overline{f_2(\eta)} d\eta.$$

Finally let us explain about Fourier-Jacobi expansions of automorphic forms relative to a parabolic subgroup. Take integers  $r, n$  such that  $1 \leq r \leq n$  and put  $m = n - r$ . Then we have the maximal parabolic subgroup  $P_r$  of  $G_n$  defined by (see Section 2)

$$P_r = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_n \mid a_{21} = 0, c_{11} = 0, c_{12} = 0, c_{21} = 0, d_{12} = 0 \right\},$$

in which  $a, b, c$ , and  $d$  are  $n \times n$  matrices and decompose an  $n \times n$  matrix  $x$  into  $r \times r$ ,  $r \times m$ ,  $m \times r$  and  $m \times m$  blocks  $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ . We shall always consider  $G_{m,r}$  as a subgroup of  $P_r$ . For any element  $w$  in  $GL(r, \mathbf{R})$ , we define  $\tilde{w} = \text{diag}(w, 1_m, {}^t w^{-1}, 1_m)$ . Then any element in  $P_r$  can be written uniquely as  $\eta \tilde{w}$ , where  $\eta \in G_{m,r}$  and  $w \in GL(r, \mathbf{R})$ . Let  $F$  be a Siegel cusp form of weight  $l$  for  $\Gamma_n$ . For any positive definite half-integral matrix  $S \in \text{Sym}_r(\mathbf{Q})$ , we define a function  $f_s$  on  $G_{m,r}$  by

$$f_s(\eta) = \int_{\text{Sym}_r(\mathbf{R}) / \text{Sym}_r(\mathbf{Z})} F((0, 0, x)\eta) \mathbf{e}(-\text{tr}(S(i1_m + x))) dx.$$

Then  $f_s$  is a Jacobi cusp form of weight  $l$  and index  $S$  for  $\Gamma_m$  and we

call it the  $S$ -th Fourier-Jacobi coefficient of  $F$  relative to  $P_r$ . The Fourier-Jacobi expansion of  $F$  relative to  $P_r$  is given by

$$F(\eta\tilde{w}) = \sum_{S>0} \mathbf{e}(i \operatorname{tr}(S[w])) (\det w)^l f_S(\eta),$$

in which the summation is taken over the set of positive definite half-integral symmetric matrices  $S \in \operatorname{Sym}_r(\mathbf{Q})$ . We note that, by the uniqueness of the Fourier-Jacobi expansion we have

$$f_{\iota_{usu}}((\lambda, \mu, \kappa)g) = (\det u)^l f_S((u\lambda, u\mu, u\kappa^t u)g)$$

for all  $S > 0$ ,  $u \in GL(r, \mathbf{Z})$  and  $(\lambda, \mu, \kappa)g \in G_{m,r}$ .

In terms of the associated functions  $F^o$  and  $f_S^o$  with  $F$  and  $f_S$ , the Fourier-Jacobi expansion may be written as

$$F^o = \sum_{S>0} f_S^o(\tau_{22}, \tau_{12}) \mathbf{e}(\operatorname{tr}(S\tau_{11})),$$

in which we decompose  $\tau \in H_n$  into blocks  $\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix}$  with  $\tau_{11} \in H_r$ ,  $\tau_{12} \in M_{r,m}(\mathbf{C})$  and  $\tau_{22} \in H_m$ .

## § 2. Eisenstein series

This section is devoted to a discussion of the Eisenstein series for the symplectic group. Since we essentially follow Kalinin [3], and since many of the statements can be proved in the similar way as [3], we omit most of the proofs.

As in the previous section, let  $G_n$  be the real symplectic group of degree  $n$  and let  $\Gamma_n$  be the Siegel modular group in  $G_n$ . Since we fix  $n$  all through this section, for simplicity we drop the index  $n$  and write just  $G$  and  $\Gamma$  for example. Let  $\mathfrak{g}$  be the Lie algebra of  $G$ . We denote by  $e_{ij}$ ,  $(i, j = 1, \dots, 2n)$  the matrix unit of size  $2n$ , and put  $h_i = e_{ii} - e_{i+n, i+n}$  for  $1 \leq i \leq n$ . Then the Lie subalgebra  $\alpha$  spanned by  $h_i$ ,  $(1 \leq i \leq n)$  is a Cartan subalgebra of  $\mathfrak{g}$ . In the dual vector space  $\alpha^*$  we choose basis  $\varepsilon_i$ ,  $(1 \leq i \leq n)$  which is dual to  $h_i$ . As a system of positive roots relative to the Cartan subalgebra  $\alpha$ , we may choose the set

$$\Sigma = \{2\varepsilon_i (1 \leq i \leq n), \varepsilon_i \pm \varepsilon_j (1 \leq i < j \leq n)\}.$$

With this choice of order, the set of simple roots is given by

$$\Sigma^o = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} (1 \leq i \leq n-1), \alpha_n = 2\varepsilon_n\}.$$

The Weyl group  $W$  is generated by the orthogonal reflections  $w_{a_i}$  for  $1 \leq i \leq n$ . We set

$$\mathfrak{p} = \mathfrak{a} + \sum_{\alpha \in \Sigma} \mathfrak{n}_\alpha,$$

in which  $\mathfrak{n}_\alpha$  is the root subspace corresponding to  $\alpha$ . Then  $(\mathfrak{p}, \mathfrak{a})$  is a Borel pair in  $\mathfrak{g}$  in the sense of [2]. Let  $(P, A)$  be the Borel pair in  $G$  corresponding to  $(\mathfrak{p}, \mathfrak{a})$ , and let  $P = UAM$  be its Langlands decomposition. Let  $K$  be a maximal compact subgroup of  $G$ . Then we have  $G = PK = UAMK$ . Therefore any element  $g$  in  $G$  can be written as  $g = uamk$ , with  $u \in U$ ,  $a \in A$ ,  $m \in M$  and  $k \in K$ , and the  $A$ -part  $a$  is uniquely determined. We denote it by  $a(g)$ . Let  $\mathfrak{a}_\mathbb{C}^*$  be the dual of the complexified vector space  $\mathfrak{a}_\mathbb{C} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{a}$ . For any  $\lambda$  in  $\mathfrak{a}_\mathbb{C}^*$  and for any  $a$  in  $A$ , we put

$$\omega_\lambda(a) = e^{\lambda(\log a)},$$

in which  $\log$  denotes the inverse of the exponential map of  $\mathfrak{a}$  to  $A$ . We introduce coordinates on  $\mathfrak{a}_\mathbb{C}^*$  as follows. We set for  $1 \leq i \leq n$ ,

$$\bar{\omega}_i = \varepsilon_1 + \cdots + \varepsilon_i.$$

Note that  $\bar{\omega}_i$ ,  $i = 1, \dots, n$  are the fundamental weights. For  $(z_1, \dots, z_n) \in \mathbb{C}^n$  we set

$$\lambda(z_1, \dots, z_n) = \sum_{i=1}^n z_i \bar{\omega}_i.$$

In terms of these coordinates the vector  $\lambda(1, \dots, 1)$  is the half-sum  $\rho$  of the positive roots.

Now we define the Eisenstein series associated to the constant function on  $M$ . For any  $z = (z_1, \dots, z_n) \in \mathbb{C}^n$  and for any  $g \in G$ , we set

$$\begin{aligned} E(z; g) &= E(\lambda(z); g) \\ &= \sum_{\gamma \in I \cap P \backslash \Gamma} \omega_{\lambda(z) + \rho}(a(\gamma g)). \end{aligned}$$

We remark that from the general theory of the Eisenstein series,  $E(z; g)$  is holomorphic for  $\operatorname{Re}(z_i) > 1$ ,  $1 \leq i \leq n$ . Let us fix an integer  $r$  such that  $1 \leq r \leq n$ . We set

$$\tilde{E}_r(z_r; g) = \operatorname{Res}_{z_{n=1}} \cdots \widehat{\operatorname{Res}_{z_{r=1}}} \cdots \operatorname{Res}_{z_{1=1}} E(z_1, \dots, z_n; g),$$

in which we take residues at  $z_i = 1$ ,  $1 \leq i < n$  except at  $z_r = 1$ .

We shall need another type of Eisenstein series. We know that for

any subset  $F_r = \Sigma^\circ - \{\alpha_r\}$  of  $\Sigma^\circ$ , there exists a parabolic pair  $(\mathfrak{p}_r, \mathfrak{a}_r)$  such that  $\mathfrak{p}_r \supset \mathfrak{p}$  and  $\mathfrak{a}_r \subset \mathfrak{a}$ . In particular, by definition we have

$$\begin{aligned}\mathfrak{a}_r &= \{H \in \mathfrak{a} \mid \alpha_i(H) = 0 \text{ for } i \neq r\} \\ &= \mathbf{R} \cdot \left( \sum_{i=1}^r h_i \right).\end{aligned}$$

We denote by  $\Sigma_r$ , the set of elements  $\alpha \in \Sigma$  which are not identically equal to zero on  $\mathfrak{a}_r$ , and we set

$$\mathfrak{n}_r = \sum_{\alpha \in \Sigma_r} \mathfrak{n}_\alpha.$$

Then we have

$$\mathfrak{p}_r = \mathfrak{z}(\mathfrak{a}_r) + \mathfrak{n}_r,$$

in which  $\mathfrak{z}(\mathfrak{a}_r)$  is the centralizer of  $\mathfrak{a}_r$  in  $\mathfrak{g}$ . Let  $(P_r, A_r)$  be the parabolic pair in  $G$  corresponding to  $(\mathfrak{p}_r, \mathfrak{a}_r)$ . Take a Langlands decomposition  $P_r = U_r A_r M_r$  of  $P_r$ . Then we have  $G = P_r K = U_r A_r M_r K$ , and for any  $g$  in  $G$  we denote by  $a_r(g)$  the  $A_r$ -part of  $g$ .

For any  $s \in \mathbf{C}$  and  $g \in G$ , we define

$$E_r(s; g) = \sum_{\gamma \in \Gamma \cap P_r \backslash \Gamma} \varphi_s^{(r)}(\gamma g),$$

where we write  $\varphi_s^{(r)}(g) = \omega_{2s, \overline{\mathfrak{a}_r}}(a_r(g))$ . It follows from the general theory of the Eisenstein series that the sum in the right hand side converges absolutely for  $\operatorname{Re}(s) > n - (r - 1)/2$ . The relation between the two Eisenstein series  $\tilde{E}_r$  and  $E_r$  is given by the following

**LEMMA 2.1.** *There exists a domain  $V \subset \{s \in \mathbf{C} \mid \operatorname{Re}(s) > n - (r - 1)/2\}$  such that for all  $s \in V$*

$$E_r(s; g) = c \cdot \tilde{E}_r(2s - 2n + r; g),$$

in which  $c$  is a non-zero constant given by

$$c = \prod_{j=1}^{n-r} \xi(2j) \prod_{j=2}^r \xi(j).$$

**THEOREM 2.2.** *Let*

$$\mathcal{E}_r(s; g) = \prod_{i=1}^r \xi(2s + 1 - i) \prod_{i=1}^{\lceil r/2 \rceil} \xi(4s - 2n + 2r - 2i) \cdot E_r(s; g).$$

*For any  $g \in G$  the function  $\mathcal{E}_r(s; g)$  is meromorphic in  $s$  on the entire complex plane and holomorphic for  $\operatorname{Re}(s) > (2n - r + 1)/2$ . It satisfies a*



*functional equation*

$$\mathcal{E}_r(s; g) = \mathcal{E}_r\left(\frac{2n - r + 1}{2} - s; g\right).$$

*It has a simple pole at  $s = n - (r - 1)/2$  with residue*

$$\frac{1}{2} \prod_{j=2}^r \xi(j) \prod_{j=1}^{\lceil r/2 \rceil} \xi(2n - 2r + 2j + 1).$$

*Proof.* In the Weyl group  $W$  consider an element  $w$  for which we have  $w\varepsilon_j = \varepsilon_{r+1-j}$  for  $1 \leq j \leq r$ . Then our theorem follows from the functional equation of the Eisenstein series  $E(\lambda(z); g)$  for  $w$ . For more details see the proof of [3] Theorem 2'. Q.E.D.

If  $r \geq 2n - 2r + 1$ , then cancellations of elementary factors occur and we can replace  $\mathcal{E}_r(s; g)$  by

$$\prod_{i=1}^{2n-2r+1} \xi(2s + 1 - i) \prod_{i=1}^{\lceil r/2 \rceil} \xi(4s - 2n + 2r - 2i) \cdot E_r(s; g).$$

Of course the residue at  $s = n - (r - 1)/2$  would be

$$\frac{1}{2} \prod_{j=2}^{2n-2r+1} \xi(j) \prod_{j=1}^{\lceil r/2 \rceil} \xi(2n - 2r + 2j + 1).$$

By definition  $E_r(s; g)$  is right  $K$ -invariant as a function on  $G$ . Hence it may be considered as a function on the Siegel upper half plane. We define a function  $E_r^o(s; \tau)$  on  $H_n$  by

$$E_r^o(s; g\langle i1_n \rangle) = E_r(s; g),$$

for all  $g \in G$ . If we put  $\tau = g\langle i1_n \rangle$ , we have

$$\varphi_s^{(r)}(g) = \left( \frac{\det \operatorname{Im}(\tau)}{\det \operatorname{Im}(\tau_{22})} \right)^s,$$

in which we decompose  $\tau$  into blocks  $\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix}$  with  $\tau_{11} \in H_r$ ,  $\tau_{22} \in H_{n-r}$ . Therefore we have another expression for  $E_r^o(s; \tau)$ :

$$E_r^o(s; \tau) = \sum_{\gamma \in \Gamma \cap P_r \backslash \Gamma} \left( \frac{\det \operatorname{Im}(\gamma \langle \tau \rangle)}{\det \operatorname{Im}(\gamma \langle \tau \rangle)_{22}} \right)^s.$$

All the statements about  $E_r$  in this section are easily reformulated in terms of  $E_r^o$ .

In the special where  $r = 1$ , we can say much more. For another extreme case where  $r = n$ , see [3] Theorem 2'.

THEOREM 2.3. *Let*

$$\mathcal{E}_1(s; g) = \xi(2s)E_1(s; g).$$

*For any  $g \in G$ , the function  $\mathcal{E}_1(s; g)$  is holomorphic in  $s$  on the entire complex plane except for simple poles at  $s = n$  and  $s = 0$  with residues  $\frac{1}{2}$ ,  $-\frac{1}{2}$ , respectively. It satisfies the functional equation*

$$\mathcal{E}_1(s; g) = \mathcal{E}_1(n - s; g).$$

*Proof.* All we have to do is to prove that  $\mathcal{E}_1(s; g)$  is holomorphic in the half plane  $0 < \text{Re}(s)$  except for simple pole at  $s = n$ . For that purpose it suffices to consider the constant term  $\mathcal{E}_{1,P}(s; g)$  is the Fourier expansion of  $\mathcal{E}_1(s; g)$  relative to the Borel subgroup  $P$  (see [3] Lemma 2.3). It is easy to see that for any  $a \in A$  and  $m \in M$  we have

$$\mathcal{E}_{1,P}(s; am) = \xi(2s) \sum_{w\alpha_i \langle 0, i \rangle 1} \tilde{c}(w; s) \omega_{w\lambda(2s-2n+1, 1, \dots, 1) + \rho}(a),$$

where the summation is taken over the set of  $w \in W$  such that  $w\alpha_i < 0$  for  $i > 1$ ,

$$\begin{aligned} \tilde{c}(w; s) &= \prod_{\alpha \in \Sigma', w\alpha < 0} c(\alpha, s), \\ \Sigma' &= \{2\varepsilon_1, \varepsilon_1 \pm \varepsilon_j, 1 < j \leq n\}, \end{aligned}$$

and

$$c(\alpha, s) = \begin{cases} \frac{\xi(2s - n)}{\xi(2s - n + 1)} & \text{if } \alpha = 2\varepsilon_1 \\ \frac{\xi(2s - 2n + j - 1)}{\xi(2s - 2n + j)} & \text{if } \alpha = \varepsilon_1 - \varepsilon_j \\ \frac{\xi(2s - j + 1)}{\xi(2s - j + 2)} & \text{if } \alpha = \varepsilon_1 + \varepsilon_j. \end{cases}$$

Now consider an element  $w$  in  $W$  such that  $w\alpha_i < 0$  for  $2 \leq i \leq n$ . If  $w\alpha_1 < 0$ , then such  $w = w_0$  is unique and  $w_0\alpha < 0$  for all  $\alpha \in \Sigma$ . Therefore in this case we have

$$\tilde{c}(w_0; s) = \frac{\xi(2s - 2n + 1)}{\xi(2s)}.$$

So let us assume that  $w\alpha_1 = w(\varepsilon_1 - \varepsilon_2) > 0$ . If  $w(2\varepsilon_1) < 0$ , then  $w(\varepsilon_1 + \varepsilon_j)$

$< 0$  for all  $1 < j \leq n$ . Suppose that  $1 \leq j < k \leq n$ . Since  $\varepsilon_1 - \varepsilon_k = (\varepsilon_1 - \varepsilon_j) + (\varepsilon_j - \varepsilon_k)$ , it is easy to see that if  $w(\varepsilon_1 - \varepsilon_k) > 0$ , then  $w(\varepsilon_1 - \varepsilon_j) > 0$ . Take the largest integer  $k$  such that  $w(\varepsilon_1 - \varepsilon_k) > 0$ , then we have

$$\tilde{c}(w: s) = \frac{\xi(2s - 2n + k)}{\xi(2s)}.$$

We note that the above condition determines the signatures of  $w\alpha$  for all positive roots  $\alpha$ , so such an element  $u_k$  in  $W$  is unique. Actually it is given by  $u_k = w_{k-1} \cdots w_1 w_0$ , where for  $1 \leq i \leq k$ ,  $w_i$  denotes the reflection defined by the simple root  $\alpha_i$ .

On the other hand, if  $w(2\varepsilon_1) > 0$  then  $w(\varepsilon_1 - \varepsilon_j) > 0$  for all  $j$ . Similarly take the largest integer  $k$  such that  $w(\varepsilon_1 + \varepsilon_k) < 0$ , then we have

$$\tilde{c}(w: s) = \frac{\xi(2s - k + 1)}{\xi(2s)}.$$

Therefore we know that the singularities of  $\mathcal{E}_{1,P}(s: g)$  for  $\text{Re}(s) \geq n/2$  are at most simple poles at  $s = (n + j)/2$ ,  $0 \leq j \leq n$ . An easy calculation shows that

$$\begin{aligned} u_k \lambda(1 - k, 1, \dots, 1) &= u_{k+1} \lambda(1 - k, 1, \dots, 1) \\ &= \bar{\omega}_k - \rho. \end{aligned}$$

Since  $\xi(s)$  has simple poles at  $s = 1$  and  $s = 0$  with residues 1 and  $-1$  respectively, it follows that  $\mathcal{E}_{1,P}$  is holomorphic at  $s = n - k/2$ ,  $1 \leq k < n$ . Similarly, by considering the element  $w_n \cdots w_1 w_0$ , we can show that  $\mathcal{E}_{1,P}$  is holomorphic at  $s = n/2$ . On the other hand, the functional equation shows that  $\mathcal{E}_{1,P}$  is holomorphic for  $0 < \text{Re}(s) < n$  as well. Q.E.D.

### § 3. Rankin-Selberg convolution

Let  $F$  and  $H$  be Siegel cusp forms of weight  $l$  for  $\Gamma_n$ . We fix an integer  $r$ ,  $1 \leq r \leq n$ , and consider the parabolic subgroup  $P_r$ . For any positive definite half-integral matrix  $S \in \text{Sym}_r(\mathbf{Q})$ , we denote by  $f_s$  and  $h_s$  the  $S$ -th Fourier-Jacobi coefficient relative to  $P_r$  of  $F$  and  $H$ , respectively (see Section 1). We shall consider a Dirichlet series defined by

$$D_r(F, H: s) = \sum_{S \sim} \frac{1}{\varepsilon(S)} \frac{(f_s, h_s)}{(\det S)^s},$$

in which the summation is taken over the set of representatives of the  $GL(r, \mathbf{Z})$ -equivalence class of positive definite half-integral symmetric

matrices and, for any such  $S$ ,  $\varepsilon(S)$  denotes the order of its automorphism group.

LEMMA 3.1. *The series*

$$\sum_{S/\sim} \frac{1}{\varepsilon(S)} \frac{(f_s, h_s)}{(\det S)^s}$$

*converges absolutely for  $\operatorname{Re}(s) > l + (r + 1)/2$  and represents a holomorphic function there,*

*Proof.* Since  $F$  and  $H$  are cusp forms we have

$$|f_s| \leq c_F \cdot (\det S)^{l/2}, \quad |h_s| \leq c_H \cdot (\det S)^{l/2},$$

in which  $c_F$  and  $c_H$  are constants depending only on  $F$  and  $H$ , respectively. Therefore we have  $|(f_s, h_s)| \leq c \cdot \det S^l$ , with a positive constant  $c$ . On the other hand it is well known that the series

$$\sum_{S/\sim} \frac{1}{\varepsilon(S)} \frac{1}{(\det S)^s}$$

is absolutely convergent for  $\operatorname{Re}(s) > (r + 1)/2$  (see [9]).

Q.E.D.

It is a general philosophy due to Rankin and Selberg, that the analytic properties of  $D_r(F, H; s)$  follow from those of the Eisenstein series  $E_r$  via the convolution  $(FE_r(s; *), H)$ .

THEOREM 3.2. *For  $\operatorname{Re}(s) > n - l - (r - 1)/2$ , we have*

$$\begin{aligned} (FE_r(s; *), H) &= c \cdot (4\pi)^{-r(s+l-n+(r-1)/2)} \prod_{k=1}^r \Gamma\left(s + l - n + \frac{k-1}{2}\right) \\ &\quad \cdot D_r\left(F, H; s + l - n + \frac{r-1}{2}\right), \end{aligned}$$

*with a positive constant  $c$ .*

*Proof.* Since  $E_r(s; *)$  is an automorphic form in the sense of [2], and since  $F$  and  $H$  are cusp forms, the integral  $(FE_r(s; *), H)$  converges absolutely if  $\operatorname{Re}(s)$  is sufficiently large. It follows from the definition that

$$\begin{aligned} (FE_r(s; *), H) &= \int_{I \backslash G} F(g) E_r(s; g) \overline{H(g)} dg \\ &= \int_{I \cap P_r \backslash G} F(g) \varphi_s^{(r)}(g) \overline{H(g)} dg. \end{aligned}$$

Since  $G = P_r K$ , we can normalize the Haar measures on  $G$ ,  $P_r$ , and  $K$  so that

$$dg = dp dk,$$

where  $dk$  is the Haar measure on  $K$  such that  $\int_K dk = 1$  and  $dp$  is a left Haar measure on  $P_r$ . The integrand  $F(g)\varphi_s^{(r)}(g)\overline{H(g)}$  is  $K$ -invariant on the right, therefore we have

$$(FE_r(s: *), H) = \int_{\Gamma \cap P_r \backslash P_r} F(p)\varphi_s^{(r)}(p)\overline{H(p)}dp.$$

Let  $P_r = U_r A_r M_r$  be the Langlands decomposition of  $P_r$ . It is well known that a left Haar measure  $dp$  on  $P_r$  is given by

$$dp = e^{-2\rho_r(a)} du da dm,$$

in which  $2\rho_r = (2n - r + 1)\bar{\omega}_r$  is the sum of roots in  $\Sigma_r$  (see Section 2) and  $du$ ,  $da$ ,  $dm$  represent Haar measures on  $U_r$ ,  $A_r$  and  $M_r$ , respectively. We shall change our notation slightly. Write an element  $p$  in  $P_r$  in the form  $p = \eta \tilde{w}$  in which  $\eta \in G_{n-r, r}$ ,  $w \in GL(r, \mathbf{R})$  and  $\tilde{w} = \text{diag}(w, 1_{n-r}, {}^t w^{-1}, 1_{n-r})$ . Then, in terms of the new coordinates, we have

$$dp = |\det w|^{-2n+r-1} d\eta dw,$$

in which  $d\eta$  and  $dw$  are the Haar measures on  $G_{n-r, r}$  and  $GL(r, \mathbf{R})$ , respectively. Also by definition we get  $\varphi_s^{(r)}(\eta \tilde{w}) = |\det w|^{2s}$ . Substitute the Fourier-Jacobi expansions into the integrand. Concerning about the Petersson inner product of Jacobi forms, we remark that  $(f_T, h_T) = (f_S, h_S)$  if  $T$  and  $S$  are equivalent and  $(f_T, h_S) = 0$  if  $T \neq S$ . Therefore we have

$$\begin{aligned} (FE_r(s: *), H) \\ = \sum_{S \sim} \frac{1}{\varepsilon(S)} \int_{GL(r, \mathbf{R})} |\det w|^{2s+2l-2n+r-1} e^{-4\pi l r(S[w])} dw \int_{G_{n-r, r}(\mathbf{Z}) \backslash G_{n-r, r}} f_S(\eta) \overline{h_S(\eta)} d\eta. \end{aligned}$$

Then our theorem follows from the following lemma.

**LEMMA 3.3 ([6]).** *Let  $S$  be a positive definite symmetric matrix of degree  $r$ . Then we have for  $\text{Re}(s) > r - 1$*

$$\int_{GL(r, \mathbf{R})} |\det w|^s e^{-4\pi l r(S[w])} dw = c_r \cdot (\det S)^{-s/2} (4\pi)^{-(r/2)s} \prod_{k=1}^r \Gamma\left(\frac{s}{2} - \frac{k-1}{2}\right),$$

where  $c_r$  is a positive constant depending on the normalization of the Haar measure.

Combining Theorem 2.2 and Theorem 3.2 we obtain the following

THEOREM 3.4. *Let*

$$\mathcal{D}_r(F, H; s) = \left( F\mathcal{E}_r\left(s + n - l - \frac{r-1}{2} : *\right), H \right).$$

*Then  $\mathcal{D}_r(F, H; s)$  can be continued meromorphically to the entire complex plane and holomorphic for  $\operatorname{Re}(s) > l$ . It has a simple pole at  $s = l$ . It satisfies a functional equation*

$$\mathcal{D}_r(F, H; s) = \mathcal{D}_r\left(F, H; 2l - n + \frac{r-1}{2} - s\right).$$

*Remark.* Note that  $\mathcal{D}_r(F, H; s)$  is a constant multiple of

$$(4\pi)^{-rs} \prod_{k=1}^r \Gamma\left(s - \frac{k-1}{2}\right) \xi(2s - 2l + 2n + 2 - r - k) \\ \times \prod_{i=1}^{\lceil r/2 \rceil} \xi(4s - 4l + 2n + 2 - 2i) \times D_r(F, H; s).$$

In the special case where  $r = 1$ , we have a better result by Theorem 2.3.

THEOREM 3.5. *Assume that  $r = 1$ . Then  $\mathcal{D}_1(F, H; s)$  is holomorphic on  $\mathbb{C}$  except for simple poles at  $s = l$  and  $s = l - n$ . The residue at  $s = l$  is  $\frac{1}{2}(F, H)$ . It satisfies the functional equation*

$$\mathcal{D}_1(F, H; s) = \mathcal{D}_1(F, H; 2l - n - s).$$

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