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BOOLEAN VALUED INTERPRETATION OF BANACH SPACE THEORY AND MODULE STRUCTURES OF VON NEUMANN ALGEBRAS

Dedicated to Professor Tosiyuki Tugué on his sixtieth birthday

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§1. Introduction

Recently, systematic applications of the Scott-Solovay Boolean valued set theory were done by several authors; Takeuti [25, 26, 27, 28, 29, 30], Nishimura [13, 14] Jech [8] and Ozawa [15, 16, 17, 18, 19, 20] in analysis and Smith [23], Eda [2, 3] in algebra. This approach seems to be providing us with a new and powerful machinery in analysis and algebra. In the present paper, we shall study Banach space objects in the Scott-Solovay Boolean valued universe and provide some useful transfer principles from theorems of Banach spaces to theorems of Banach modules over commutative AW*-algebras. The obtained machinery will be applied to resolve some problems concerning the module structures of von Neumann algebras.

Since Sakai [21] succeeded in characterizing von Neumann algebras by their Banach space structures, the structure of the predual space was an intrinsic tool for studying von Neumann algebras. However, it was recognized that there are certain limitations of this tool for the case with non-trivial center as pointed out by Halpern [5, p. 183]. In this connection, he stated in [32] that the dual spaces of C*-algebras are too small to characterize intrinsically the different type of algebras and cited the following results as evidence: For a C*-algebra A, the set $\{\alpha f \mid f$ is a pure state of A, $0 \leq \alpha \leq 1\}$ is weak*-compact if and only if A is CCR, A has a Hausdorff structure space, and A modulo the closure of the ideal of elements with continuous trace is commutative (Glimm [33, Theorem 6]): The pure state space of A is equal to the state space of A if and

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only if A is NGCR and the ideal (0) is prime (Tomiyama and Takasaki [34]). He proposed the investigation of module structures over the center as an intrinsic tool instead of Banach space structures. In fact, he showed that a von Neumann algebra is of type I if and only if it is the second dual module of some Banach module over the center [5, Theorem 9] and that an AW*-algebra can be embedded in a type I AW*-algebra as its own bicommutant with the same center if and only if it is the dual module of some Banach module over the center [6, Theorem 7]. These results are pertinent generalizations of the well-known facts concerning factors and Banach spaces. Further, he succeeded in characterizing CCR, GCR and NGCR algebras in terms of the topology of the dual module over the center of enveloping von Neumann algebras [32]. However, this approach requires somewhat cumbersome tasks to build a theory of Banach modules parallel with Banach space theory; see [5] and [32]. Thus, it will be much desirable to obtain more direct methods which transfer theorems of Banach spaces to theorems of Banach modules. The purpose of this paper is to establish such transfer principles using the methods of mathematical logic and Boolean valued set theory.

In Section 2, we present main results concerning module structures of von Neumann algebras and AW*-algebras without any invoking of Boolean valued set theory. We shall present the following generalizations of Halpern's results cited above: Let Z be a commutative AW*-algebra. A C*-algebra can be embedded in a type I AW*-algebra with center Z as its own bicommutant if and only if it contains Z in the center as a unital C*-subalgebra and it is the Z-dual of some normed Z-module. We call such C*-algebras Z-embeddable. A C*-algebra with center Z is a type I AW*-algebra if and only if it is the second Z-dual of some normed Z-module. A C*-algebra with center Z is a finite type I AW*-algebra if and only if it is the Z-dual of itself.

Since the predual of a von Neumann algebra (i.e., a $\boldsymbol{C}$-embeddable C*-algebra) is unique in the category of Banach spaces, the problem of the uniqueness of the Z-preduals of Z-embeddable C*-algebras arises naturally. To resolve this problem we introduce the concept of Kaplansky-Banach modules following the spirit of Kaplansky's AW*-module [11]. Then we obtain that the Z-preduals of Z-embeddable C*-algebras are unique in the category of Kaplansky-Banach modules. This suggests that the precise analogue of a Banach space among Z-modules is a

Kaplansky-Banach module.

Let B be the complete Boolean algebra of projections in Z. In Section 3, we give necessary preliminaries on the Scott-Solovay Boolean valued universe $V^{(B)}$ of set theory. In Section 4, we construct a functor from the category of Banach spaces in $V^{(B)}$ to the category of Kaplansky-Banach modules. In Section 5, we construct its adjoint functor. Eventually, we show that these two categories are equivalent. In Section 6, we examine the subcategory of C*-algebras in $V^{(B)}$ and show that, for a unital C*-algebra A which contains Z as a unital C*-subalgebra of the center, the following three conditions are equivalent: (1) A can be embedded in a C*-algebra C in $V^{(B)}$ (in such a manner that A is a C*-subalgebra of the bounded global section algebra of C). (2) Every element x of A has the smallest projection e in B such that $ex=x$. (3) A can be embedded in a type I AW*-algebra with the center Z as a C*-subalgebra containing the center Z. This result improves the characterization of C*-algebras in $V^{(B)}$ previously obtained by Takeuti [30]. In Section 7, we study the Z-duals of normed Z-modules. Applying these transfer principles, we prove in Section 8 the results concerning module structures of von Neumann algebras and AW*-algebras presented in Section 2.

§ 2. Main results in applications

Let Z be a commutative AW*-algebra; denote by $\|\cdot\|_\infty$ the norm of Z. Let B be the complete Boolean algebra of all projections in Z. Let X be a unital Z-module. In this paper, every Z-module will be assumed to be unital. Then X has a linear space structure over $\boldsymbol{C}$ by defining the scalar multiplication as $\alpha x=(\alpha 1)x$ for any $\alpha \in \boldsymbol{C}$ and $x \in X$. A Z-module X with norm $\|\cdot\|$ will be called a *normed Z-module*, if $\|ax\| \leq \|a\|_\infty \|x\|$ for every $a \in Z$ and $x \in X$. If a normed Z-module is a Banach space it will be called a Banach Z-module

Obviously, a C*-algebra which contains Z as a unital C*-subalgebra of the center is a Banach Z-module.

Let X, Y be two normod Z-modules. Denote by $\mathrm{Hom}_Z(X, Y)$ the space of all bounded Z-linear maps from X into Y. We shall write $X^\# = \mathrm{Hom}_Z(X, Z)$, We shall call $X^\#$ the *Z-dual* of X. An element of $X^\#$ will be called a *Z-functional* on X. We say that X and Y are *isometrically Z-isomorphic* if there exists a surjective bounded Z-linear map $T \in$

$\mathrm{Hom}_Z(X, Y)$ such that $\|Tx\| = \|x\|$ for all $x \in X$ and write $X \cong Y$. We say that a C*-algebra A which contains Z as a unital C*-subalgebra of the center is *Z-dual* if there is a normed Z-module X such that $A \cong X^{\#}$, *Z-bidual* if there is a normed Z-module X such that $A \cong X^{\#\#}$ and *Z-self-dual* if $A \cong A^{\#}$.

Our main purpose is to characterize those C*-algebras that are Z-dual, Z-bidual and Z-self-dual, respectively.

A C*-algebra A will be called *Z-embeddable* if there is a type I AW*-algebra L with center Z and a *-monomorphism $\pi: A \to L$ such that $\pi(A) = \pi(A)''$, where $\pi(A)''$ stands for the bicommutant of $\pi(A)$ in L. In this case, A is an AW*-algebra which contains Z as a unital AW*-subalgebra of the center. For the detailed account of Z-embeddable C*-algebras, we refer the reader to [19].

The first application of our machinery is the following.

THEOREM A. *Let Z be a commutative AW^*-algebra and let A be a C^*-algebra which contains Z as a unital C^*-subalgebra of the center. Then A is Z-dual if and only if it is Z-embeddable.*

The second application of our machinery is the following theorem, which generalizes [5; Theorem 9] due to Halpern to the case of AW*-algebras.

THEOREM B. *Let Z be a commutative AW^*-algebra and let A be a C^*-algebra with center Z. Then A is Z-bidual if and only if it is a type* I *AW^*-algebra.*

The Z-self-dual C*-algebras with center Z are characterized as follows.

THEOREM C. *Let Z be a commutative AW^*-algebra and let A be a C^*-algebra with center Z. Then A is Z-self-dual if and only if it is a finite type* I *AW^*-algebra.*

Theorem A is reduced to Sakai's characterization of von Neumann algebras [21] for the case $Z = \boldsymbol{C}$ and it was proved for the case that Z is the center of A, by Halpern [6; Theorem 7]. However, this result is far from the uniqueness of the predual X even if we restrict X in the category of Banach Z-modules. What is the proper category for which the uniqueness holds? Intuitively, the solution is the precise analogue of Banach spaces for the scalars Z. One of our motivations in the following sections is to establish this 'precise analogue' using the methods of mathematical logic. Now, we shall present our solution.

A function $\|\cdot\|_Z\colon X \to Z$ is called a *Z-valued norm* on X if it satisfies the following conditions:

(N1) $\|x+y\|_Z \leq \|x\|_Z + \|y\|_Z$,

(N2) $\|ax\|_Z = |a|\|x\|_Z$,

(N3) $\|x\|_Z \geq 0$, and $\|x\|_Z = 0$ only if $x = 0$,

for all $x, y \in X$ and $a \in Z$, where $|a|$ stands for the absolute value of a in Z. A Z-module with a Z-valued norm will be called a *Z-normed Z-module*. The Z-valued norm $\|\cdot\|_Z$ defines a scalar valued norm $\|\cdot\|$ on X by the relation $\|x\| = \|\|x\|_Z\|_\infty$ for all $x \in X$, which will be called the *induced norm* from $\|\cdot\|_Z$. By the relation $\|ax\| = \|\|ax\|_Z\|_\infty = \||a|\|x\|_Z\|_\infty \leq \|a\|_\infty\|x\|$ for all $a \in Z$ and $x \in X$, every Z-normed Z-module is a normed Z-module with its induced norm.

We say that a unital C*-algebra A which contains Z as a unital C*-subalgebra of the center is *Z-scalable* if, for any $x \in A$, there is the largest projection e in Z such that $ex = 0$. By [1, p. 14, Proposition 6], every AW*-algebra which contains Z as an AW*-subalgebra of the center is Z-scalable. Every Z-scalable C*-algebra A is also Z-normed Z-module (Theorem 6.1). The Z-valued norm is defined by

$$\|x\|_Z = \inf\{a \in Z \mid x^*x \leq a^2,\ a \geq 0\}$$

for all $x \in A$ and its induced norm coincides with its original norm.

Every C*-subalgebra of a type I AW*-algebra with center Z which contains Z is a Z-scalable C*-algebra and every Z-scalable C^*-algebra has such an embedding in a type I AW*-algebra (Theorem 6.5).

A family $\{b_i\}$ of elements of B is called a *partition of unity* of B if $b_i b_j = 0$ for $i \neq j$, and $\sup_i b_i = 1$. A Z-normed Z-module X will be called a *Kaplansky Z-module* if it enjoys the following property:

(K1) Let $\{b_i\}$ be a partition of unity of B, and $\{x_i\}$ a bounded family in X; then there exists in X an element x with $b_i x = b_i x_i$ for all i.

If $b_i x = 0$ for all i, then $b_i\|x\|_Z = 0$ for all i and hence $x = 0$. It follows that the element x of (K1) is unique, and we shall write $x = \sum_i b_i x_i$. A Kaplansky Z-module X will be called a *Kaplansky-Banach Z-module* if

X is a Banach space with respect to the scalar valued norm $\|\cdot\|$ induced from the Z-valued norm $\|\cdot\|_Z$.

An AW*-algebra which contains Z as a unital AW*-subalgebra of the center is a Kaplansky-Banach Z-module [1, p. 53, Proposition 2].

In [11], Kaplansky introduced AW*-modules. In this paper, we shall call them *Kaplansky-Hilbert modules*. A Z-valued norm $\|\cdot\|_Z$ on a Kaplansky-Hilbert Z-module H is defined by the relation $\|x\|_Z = \langle x|x\rangle_Z^{1/2}$ for all $x \in H$, where $\langle\cdot|\cdot\rangle_Z$ is the Z-valued inner product on H. Then we have $\|\langle x|x\rangle_Z\|_\infty^{1/2} = \|\langle x|x\rangle_Z^{1/2}\|_\infty = \|\|x\|_Z\|_\infty$ for all $x \in H$, by the C*-condition of the norm on Z. Thus it is easy to see that H is a Kaplansky-Banach Z-module.

A Z-functional f on a Z-embeddable AW*-algebra A will be called *positive* if $f(x^*x) \geq 0$ for all $x \in A$. A positive Z-functional f on A will be called *normal* if, for any uniformly bounded increasing directed family $\{a_i\}$ of positive elements in A, $f(\sup_i a_i) = \sup_i f(a_i)$. Denote by $A_\#$ the set of all Z-functionals f on a Z-embeddable AW*-algebra A such that there are four positive normal Z-functionals f_1, f_2, f_3, f_4 such that $f = f_1 - f_2 + i(f_3 - f_4)$. A bounded Z-linear map $T\colon X \to Y$ from a Z-normed Z-module X to a Z-normed Z-module Y will be called a *Z-isometry* if $\|Tx\|_Z = \|x\|_Z$ for all $x \in X$. We say that X is *Z-isometrically Z-isomorphic* to Y if there is a surjective Z-linear Z-isometry $T \in \mathrm{Hom}_Z(X, Y)$, and write $X \simeq_Z Y$. Then for any Z-normed Z-modules X and Y, if $X \cong Y$ then $X \cong_Z Y$ (Proposition 7.5).

THEOREM D. *Let A be a Z-embeddable C*-algebra. Then $A_\#$ is a Kaplansky-Banach Z-module and A is the Z-dual of $A_\#$. If A is the Z-dual of another Kaplansky-Banach Z-module X then X is Z-isometrically Z-isomorphic to $A_\#$.*

Thus we have reached the uniqueness of the predual of a Z-embeddable C*-algebra. Let A be a von Neumann algebra. Then A is Z-embeddable for any AW*-subalgebra Z of the center of A. In fact, we have the following.

THEOREM E. *Let A be a Z_0-embeddable C*-algebra for an AW*-subalgebra Z_0 of the center of A. Then A is Z-embeddable for every AW*-subalgebra of the center with $Z_0 \subseteq Z$.*

Thus our results will illustrate much the module structures of von Neumann algebras.

§ 3. Preliminaries on Boolean valued analysis

Throughout this paper B denotes a complete Boolean algebra. Let Ω be the Stone representation space of B. Denote by Z the commutative AW*-algebra $C(\Omega)$ of all complex-valued continuous functions on Ω. Then the set of all projections in Z is a complete Boolean algebra under the natural ordering of Z which is isomorphic to B. Thus we may assume that B is the complete Boolean algebra of projections in Z. The symbol 0 signifies both the least element of B and the zero of Z. The symbol 1 signifies both the greatest element of B and the unit of Z. For $b, c \in B$, $b \vee c$ denotes the supremum, $b \wedge c$ or bc denotes the infimum, $1 - b$ denotes the complement of b, and $b \Rightarrow c = (1-b) \vee c$.

For each ordinal α, let

$$V_\alpha^{(B)} = \{u \mid u\colon \operatorname{dom}(u) \longrightarrow B \text{ and } \operatorname{dom}(u) \subseteq \bigcup_{\beta<\alpha} V_\beta^{(B)}\}.$$

The *Scott-Solovay Boolean valued universe* $V^{(B)}$ is defined by

$$V^{(B)} = \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{(B)},$$

where On is the class of all ordinals. An element of $V^{(B)}$ will be called a *B-valued set*. The language which describes $V^{(B)}$ is the language of set theory augmented by all B-valued sets as constant symbols and denoted by $L(V^{(B)})$.

To each statement ϕ in $L(V^{(B)})$ we assign a B-valued truth value $[\![\phi]\!]$ by the following recursive rules:

(1) $[\![u \in v]\!] = \sup_{y \in \operatorname{dom}(v)}(v(y) \wedge [\![u = y]\!])$,

(2) $[\![u = v]\!] = \inf_{x \in \operatorname{dom}(u)}(u(x) \Rightarrow [\![x \in v]\!]) \wedge \inf_{y \in \operatorname{dom}(v)}(v(y) \Rightarrow [\![y \in u]\!])$,

(3) $[\![\neg\phi]\!] = 1 - [\![\phi]\!]$,

(4) $[\![\phi_1 \wedge \phi_2]\!] = [\![\phi_1]\!] \wedge [\![\phi_2]\!]$,

(5) $[\![\phi_1 \vee \phi_2]\!] = [\![\phi_1]\!] \vee [\![\phi_2]\!]$,

(6) $[\![(\forall x)\phi(x)]\!] = \inf_{u \in V^{(B)}}[\![\phi(u)]\!]$,

(7) $[\![(\exists x)\phi(x)]\!] = \sup_{u \in V^{(B)}}[\![\phi(u)]\!]$.

If a statement ϕ is inferred from a statement ψ in the first-order predicate calculus then $[\![\psi]\!] \leq [\![\phi]\!]$. The basic theorem of the Scott-Solovay Boolean valued set theory is the following [31, Theorem 13.12, Theorem 14.25].

THEOREM 3.1 (Scott-Solovay). *If ϕ is a theorem of* ZFC, *then* $[\![\phi]\!] = 1$.

We say that a statement ϕ in $L(V^{(B)})$ holds in $V^{(B)}$, if $[\![\phi]\!] = 1$. From the above theorem, every theorem of ZFC holds in $V^{(B)}$, and hence every theorem of mathematics based on ZFC does.

The original universe V of ZFC can be embedded in $V^{(B)}$ by the following operation $^\vee$ defined by the $\in$-recursion: For each $y \in V$, $\check{y} = \{\check{x} \mid x \in y\} \times \{1\}$. Then we have the following [31, Corollary 13.19].

THEOREM 3.2. *If $\phi(x_1, \cdots, x_n)$ is a bounded formula in $L(V^{(B)})$ then for $u_1, \cdots, u_n \in V$,*

$$\phi(u_1, \cdots, u_n) \text{ if and only if } [\![\phi(\check{u}_1, \cdots, \check{u}_n)]\!] = 1\,.$$

The following theorem is called the maximum principle [31, Theorem 16.2].

THEOREM 3.3. *If $\phi(x)$ is a formula in $L(V^{(B)})$ then there is some $u \in V^{(B)}$ such that $[\![\phi(u)]\!] = [\![(\exists x)\phi(x)]\!]$.*

We say that an element $u \in V^{(B)}$, satisfying some property, exists *uniquely* if there is another $u' \in V^{(B)}$ satisfying the same property then $[\![u = u']\!] = 1$.

Let $\{b_i\}$ be a partition of unity and let $\{u_i\}$ be a family of B-valued sets with a common index set. Then there is a unique element $u \in V^{(B)}$ such that $[\![u = u_i]\!] \geq b_i$ for any i. We denote this u by $\sum_i u_i b_i$.

A B-valued set $u \in V^{(B)}$ is called *definite* if $u(x) = 1$ for all $x \in \operatorname{dom}(u)$. If $u \in V^{(B)}$ is definite then $[\![x \in u]\!] = 1$ for all $x \in \operatorname{dom}(u)$. The *global section set* $u^{(B)}$ of a *B-valued* set $u \in V^{(B)}$ is defined by

$$u^{(B)} = \{x' \mid [\![x \in u]\!] = 1\}\,,$$

where x' is some representative from the equivalence class $\{y \in V^{(B)} \mid [\![x = y]\!] = 1\}$. In the sequel, we shall omit the symbol $'$ in x', conventionally. Although $\{y \in V^{(B)} \mid [\![x = y]\!] = 1\}$ is a proper class, we can avoid the use of a uniform choice function in the selection $x \mapsto x'$ by considering instead the subset $\{y \in V^{(B)} \mid [\![x = y]\!] = 1$ and y is of the least rank with $[\![x = y]\!] = 1\}$ (cf. [25, p. 14]). If $u \in V^{(B)}$ is nonempty in $V^{(B)}$, i.e., $[\![u \neq \emptyset]\!] = 1$, then

$$[\![u = u^{(B)} \times \{1\}]\!] = 1\,.$$

If $u \in V^{(B)}$ is definite, then

$$u^{(B)} = \{\sum_i u_i b_i \mid \{u_i\} \text{ is a family in } \mathrm{dom}(u) \text{ and } \{b_i\} \text{ is a partition of unity of } B\}.$$

The following theorems are useful for manipulation of Boolean values ([31, Theorem 13.13], [25, p. 14]).

THEOREM 3.4. *For any formula $\phi(x)$ in $L(V^{(B)})$ and $u \in V^{(B)}$, we have*
(1) $[\![(\forall x \in u)\phi(x)]\!] = \inf_{x \in \mathrm{dom}(u)}(u(x) \Rightarrow [\![\phi(x)]\!])$,
(2) $[\![(\exists x \in u)\phi(x)]\!] = \sup_{x \in \mathrm{dom}(u)}(u(x) \wedge [\![\phi(x)]\!])$.

THEOREM 3.5. *For any formula $\phi(x)$ in $L(V^{(B)})$ and nonempty set u in $V^{(B)}$, we have*:
(1) $[\![(\forall x \in u)\phi(x)]\!] = 1$ *if and only if, for any* $x \in u^{(B)}$, $[\![\phi(x)]\!] = 1$.
(2) $[\![(\exists x \in u)\phi(x)]\!] = 1$ *if and only if there is some* $x \in u^{(B)}$ *such that* $[\![\phi(x)]\!] = 1$.
If u is definite then:
(3) $[\![(\forall x \in u)\phi(x)]\!] = 1$ *if for any* $x \in \mathrm{dom}(u)$, $[\![\phi(x)]\!] = 1$.

Let d be a subset of $V^{(B)}$, a function $f: d \to V^{(B)}$ is called *extensional* if $[\![x = y]\!] \leq [\![f(x) = f(y)]\!]$ for all $x, y \in d$. Functions in $V^{(B)}$ are characterized as follows [25, Proposition 4.2, p. 22].

THEOREM 3.6. *Let $u, v \in V^{(B)}$ be definite. The relation*

$$[\![f(x) = g(x)]\!] = 1$$

for all $x \in \mathrm{dom}(u)$ sets up a one-to-one correspondence between all functions f from u to v in $V^{(B)}$, i.e., $[\![f: u \to v]\!] = 1$, and all extensional functions g: $\mathrm{dom}(u) \to v^{(B)}$.

Since our metalanguage manipulating the language $L(V^{(B)})$ and the model $V^{(B)}$ is also based on set theory, the situation is sometimes very confusing. For instance, the symbol $\boldsymbol{C}$ will be used to denote the complex number field both in the language $L(V^{(B)})$ and in our metalanguage. To avoid the confusion, we shall use the following notational convention. Let $F(x_1, \cdots, x_n)$ be an n-ary function symbol which may be introduced in $L(V^{(B)})$ by a definition. Let $u_1, \cdots, u_n \in V^{(B)}$. By the maximum principle, there is a unique $u \in V^{(B)}$ such that $[\![u = F(u_1, \cdots, u_n)]\!] = 1$. We shall denote one of such u by $F(u_1, \cdots, u_n)_B$ and call it the *interpretation* of $F(u_1, \cdots, u_n)$. The global section set of $F(u_1, \cdots, u_n)_B$ will be denoted by $F(u_1, \cdots, u_n)^{(B)}$. For example, $\boldsymbol{C}$ is a 0-ary function symbol standing for the complex number field, C_B is the complex number field in $V^{(B)}$, and $\boldsymbol{C}^{(B)}$ is the global section set of the complex number field in $V^{(B)}$.

For any $u, v \in V^{(B)}$, define $\{u, v\}_B$ and $\langle u, v\rangle_B$ as follows: $\{u, v\}_B = \{\check{u}, \check{v}\} \times \{1\}$, $\langle u, v\rangle_B = \{\{u, u\}_B, \{u, v\}_B\}_B$. Then $[\![\langle u, v\rangle_B = \langle u, v\rangle]\!] = 1$ [30. Theorem 14.14]. For any $u, v \in V^{(B)}$, define $(u \times v)_B \in V^{(B)}$ as follows:

$$\operatorname{dom}((u \times v)_B) = \{\langle x, y\rangle_B \mid \langle x, y\rangle \in \operatorname{dom}(u) \times \operatorname{dom}(v)\},$$
$$(u \times v)_B(\langle x, y\rangle_B) = [\![x \in u]\!] \wedge [\![y \in v]\!],$$

for all $\langle x, y\rangle \in \operatorname{dom}(u) \times \operatorname{dom}(v)$. Then $[\![(u \times v)_B = u \times v]\!] = 1$ [23, p. 285]. Let d, e be subsets of $V^{(B)}$. A function f: $d \times e \to V^{(B)}$ is called *extensional* if $[\![x = x']\!] \wedge [\![y = y']\!] \le [\![f(x, y) = f(x', y')]\!]$ for all $\langle x, y\rangle$, $\langle x', y'\rangle \in d \times e$.

THEOREM 3.7. *Let $u, v, w \in V^{(B)}$ be definite and let f be an extensional function from* $\operatorname{dom}(u) \times \operatorname{dom}(v)$ *to* $\operatorname{dom}(w)$. *Then there exists some $g \in V^{(B)}$ such that*

$$[\![g\colon u \times v \to w]\!] = 1 \quad \textit{and} \quad [\![g(\langle x, y\rangle_B) = f(x, y)]\!] = 1,$$

for all $\langle x, y\rangle \in \operatorname{dom}(u) \times \operatorname{dom}(v)$.

Proof. Since u and v are definite, $(u \times v)_B$ is also definite. Let h be a function from $\operatorname{dom}((u \times v)_B)$ to $\operatorname{dom}(w)$ such that $h(\langle x, y\rangle_B) = f(x, y)$ for all $\langle x, y\rangle \in \operatorname{dom}(u) \times \operatorname{dom}(v)$. From Theorem 3.6, we have only to show that h is extensional. Let $\langle x, y\rangle_B$, $\langle x', y'\rangle_B$ be in $\operatorname{dom}((u \times v)_B)$. Then $\langle x, y\rangle$, $\langle x', y'\rangle \in \operatorname{dom}(u) \times \operatorname{dom}(v)$ and hence we have

$$\begin{aligned}
[\![\langle x, y\rangle_B = \langle x', y'\rangle_B]\!] &= [\![x = x']\!] \wedge [\![y = y']\!] \\
&\le [\![f(x, y) = f(x', y')]\!] \\
&= [\![h(\langle x, y\rangle_B) = h(\langle x', y'\rangle_B)]\!] .
\end{aligned}$$

Thus h is extensional. Q.E.D.

Denote by $\boldsymbol{N}$ the set of all natural numbers and by $\boldsymbol{Q}$ the rational number field. Then we have $[\![\check{\boldsymbol{N}} = \boldsymbol{N}_B]\!] = 1$ and $[\![\check{\boldsymbol{Q}} = \boldsymbol{Q}_B]\!] = 1$. Moreover, the rational number r in $V^{(B)}$ is $\check{r}$ and the correspondence preserves the usual operation of numbers [25, p. 11]. The situation is much different for the real number field $\boldsymbol{R}$ and the complex number field $\boldsymbol{C}$. Denote by $\hat{\boldsymbol{C}}$ the bounded part of the global section set $\boldsymbol{C}^{(B)}$ of $\boldsymbol{C}$ in $V^{(B)}$, i.e.,

$$\hat{\boldsymbol{C}} = \{a \in \boldsymbol{C}^{(B)} \mid (\exists K \in \boldsymbol{R})[\![|a|_B < \check{K}]\!] = 1\},$$

where $|a|_B$ is the absolute value of a in $V^{(B)}$. Then $\hat{C}$ has a natural commutative AW*-algebra structure which is *-isomorphic to Z [17, Theorem 3.5]. Denote by $x \mapsto \tilde{x}$ this *-isomorphism from Z onto $\hat{C}$. Then we have the following: Let $\alpha \in C$ and $u, x, y \in Z$.

1) $[\![C_B = \{\tilde{u} \mid u \in Z\} \times \{1\}]\!] = 1$.

2) $[\![\tilde{u} = \check{\alpha}]\!] = 1$ if and only if $u = \alpha 1$.

3) $[\![\tilde{u} = \tilde{x} + \tilde{y}]\!] = 1$ if and only if $u = x + y$.

4) $[\![\tilde{u} = \tilde{x}\tilde{y}]\!] = 1$ if and only if $u = xy$.

5) $[\![\tilde{u} = (\tilde{x})^*]\!] = 1$ (where $*$ is the complex conjugate) if and only if $u = x^*$ (where * is the involution of Z).

6) $[\![\tilde{u} \in R_B]\!] = 1$ if and only if u is self-adjoint.

7) $[\![\tilde{x} \leq \tilde{y}]\!] = 1$ if and only if $x \leq y$.

8) $[\![\tilde{x} = \tilde{y}]\!] = \sup\{b \in B \mid bx = by\}$.

Thus, in the sequel we shall always identify $x \in Z$ with $\tilde{x} \in \hat{C}$. Under this identification we have

1) $[\![C_B = Z \times \{1\}]\!] = 1$,

2) $[\![x = y]\!] = \sup\{b \in B \mid bx = by\}$, for all $x, y \in Z$.

From the above, we have also $[\![\{0, 1\}_B = B \times \{1\}]\!] = 1$ and $\{0, 1\}^{(B)} = B$.

§ 4. Banach spaces in $V^{(B)}$

Let $\langle X, +, \cdot, \|\cdot\|_B \rangle$ be a normed linear space in $V^{(B)}$. Let $X^{(B)}$ be the global section set of X, i.e.,

$$X^{(B)} = \{u \in V^{(B)} \mid [\![u \in X]\!] = 1\}.$$

The *bounded part* $\hat{X}$ *of* $X^{(B)}$ is defined by

$$\hat{X} = \{u \in X^{(B)} \mid (\exists K \in R)[\![\|u\|_B \leq \check{K}]\!] = 1\}.$$

It is easy to see that, $\hat{X}$ is the set of all $x \in X^{(B)}$ such that $\|x\|_B \in Z$. By [16, Lemma 3.1, p. 594], we have $[\![\hat{X} \times \{1\} = X]\!] = 1$.

The *bounded global section module* $\langle \hat{X}, +, \cdot, \|\cdot\|_Z \rangle$ (or simply denoted by $\hat{X}$) of X is defined as follows:

1) For every $x, y \in \hat{X}$, the sum u of x and y is defined as the unique element $u \in \hat{X}$ such that $[\![u = x + y]\!] = 1$, which is also denoted by $x + y$.

2) For every $x \in \hat{X}$ and $a \in Z$, the product u of a and x is defined as the unique element $u \in \hat{X}$ such that $[\![u = a \cdot x]\!] = 1$, which is also denoted by $a \cdot x$ or ax.

3) For every $x \in \hat{X}$, $\|x\|_Z = \|x\|_B$.

THEOREM 4.1. *Let X be a normed linear space in $V^{(B)}$. Then the bounded global section module $\hat{X}$ is a Kaplansky Z-module. Further, $\hat{X}$ is a Kaplansky-Banach Z-module if and only if X is a Banach space in $V^{(B)}$.*

Proof. It is easy to see that $\hat{X}$ is a Z-normed Z-module (see [30, Proposition 1.1, p. 208], for the similar discussions). Let $\{b_i\}$ be a partition of unity of B, and $\{x_i\}$ a bounded family in $\hat{X}$. Let $x = \sum_i x_i b_i \in V^{(B)}$. Then $[\![x = x_i \in X]\!] \geq b_i$ for all i and hence $x \in X^{(B)}$ and $\|x\|_B \leq \sup_i \|x_i\|$. Thus $x \in \hat{X}$. For any, i, we have $[\![b_i = 1]\!] = b_i$ and $[\![b_i = 0]\!] = 1 - b_i$, and hence

$$\begin{aligned}[\![b_i x = b_i x_i]\!] &\geq [\![(b_i = 1 \wedge x = x_i) \vee b_i = 0]\!] \\ &= ([\![b_i = 1]\!] \wedge [\![x = x_i]\!]) \vee [\![b_i = 0]\!] \\ &= b_i \vee (1 - b_i) \\ &= 1 .\end{aligned}$$

It follows that $b_i x = b_i x_i$ for all i. Thus $\hat{X}$ is a Kaplansky Z-module. The last part of the assertions follows from [30, Proposition 1.2, p. 208]. Q.E.D.

Let X, Y be two normed linear spaces in $V^{(B)}$. Consider the normed linear space $L(X, Y)_B$ of all bounded linear maps $T: X \to Y$ in $V^{(B)}$. Denote by $L(X, Y)^{(B)}$ the global section set of $L(X, Y)_B$ and by $L(X, Y)^\wedge$ its bounded global section module. Let $T \in L(X, Y)^{(B)}$. Denote by $\|T\|_B$ the operator bound of T in $V^{(B)}$. Then it is easy to see that

$$\|T\|_B = \inf\{a \in \boldsymbol{R}^{(B)} \mid \|Tx\|_B \leq a\|x\|_B \text{ for all } x \in X^{(B)}\} .$$

Let $T^{(B)}$ be the extensional map $T^{(B)}: X^{(B)} \to Y^{(B)}$ such that $[\![T^{(B)}(x) = T(x)]\!] = 1$ for all $x \in X^{(B)}$. Denote by $\hat{T}$ the restriction of $T^{(B)}$ on $\hat{X}$.

Let X, Y be two Z-normed Z-modules. A Z-linear map $T: X \to Y$ is called *Z-bounded* if there is some $a \in Z$ such that $\|Tx\|_Z \leq a\|x\|_Z$ for all $x \in X$. The *Z-bound* $\|T\|_Z$ of T is defined by

$$\|T\|_Z = \inf\{a \in Z \mid \|Tx\|_Z \leq a\|x\|_Z \text{ for all } x \in X\} .$$

LEMMA 4.2. *Let X, Y be two Z-normed Z-modules. Then a Z-linear map $T: X \to Y$ is Z-bounded if and only if it is bounded.*

Proof. Suppose that T is Z-bounded and $\|Tx\|_Z \leq a\|x\|_Z$ for $a \in Z$ and $x \in X$. Then we have $\|Tx\|_Z \leq \|a\|_\infty \|x\| 1$ so that $\|Tx\| \leq \|a\|_\infty \|x\|$.

Thus T is bounded. Conversely, suppose that T is bounded. Let $\varepsilon > 0$ and let $x \in X$. Let $y = x/(\varepsilon 1 + \|x\|_Z)$. Then $\|y\| = \|\|y\|_Z\|_\infty = \|\|x\|_Z/(\varepsilon 1 + \|x\|_Z)\|_\infty \leq 1$, and by the boundedness of T, we have

$$\begin{aligned}\|Tx\|_Z &= (\varepsilon 1 + \|x\|_Z)\|Ty\|_Z \\ &\leq \|Ty\|(\varepsilon 1 + \|x\|_Z) \\ &\leq \|T\|(\varepsilon 1 + \|x\|_Z)\,.\end{aligned}$$

Since ε is arbitrary, we have $\|Tx\|_Z \leq \|T\|\|x\|_Z$. Thus T is Z-bounded.

Q.E.D.

THEOREM 4.3. *Let X, Y be normed linear spaces in $V^{(B)}$. For any $T \in L(X, Y)^\wedge$, $\hat{T}$ is a bounded Z-linear map from $\hat{X}$ into $\hat{Y}$. The correspondence $T \mapsto \hat{T}$ sets up a one-to-one correspondence between $L(X, Y)^\wedge$ and* $\mathrm{Hom}_Z(\hat{X}, \hat{Y})$ *satisfying*:

(1) $(aT + bS)^\wedge(x) = a\hat{T}(x) + b\hat{S}(x)$, *for all* $T, S \in L(X, Y)^\wedge$, $a, b \in Z$ *and* $x \in \hat{X}$.

(2) $\|T\|_B = \|\hat{T}\|_Z$,

for all $T \in L(X, Y)^\wedge$. Moreover, for any three normed linear spaces X, Y, W in $V^{(B)}$, and $S \in L(X, Y)^\wedge$, $T \in L(Y, W)^\wedge$, we have

(3) $(TS)^\wedge(x) = \hat{T}\hat{S}(x)$,

for all $x \in \hat{X}$.

Proof. Let $T \in L(X, Y)^\wedge$. By the relation $\|Tx\|_B \leq \|T\|_B\|x\|_B$, $\hat{T}(x) \in \hat{Y}$ for all $x \in \hat{X}$. Thus obviously, $\hat{T}$ is a Z-bounded Z-linear map from $\hat{X}$ into $\hat{Y}$ and it is bounded by Lemma 4.2. Let $S \in \mathrm{Hom}_Z(\hat{X}, \hat{Y})$. Then by [16, Lemma 2.3, p. 593], S is extensional. Thus there is some $T \in V^{(B)}$ such that $[\![T\colon X \to Y]\!] = 1$ and $[\![Tx = Sx]\!] = 1$ for all $x \in \hat{X}$. By Lemma 4.2, S is Z-bounded. Since $[\![Z \times \{1\} = \boldsymbol{C}_B]\!] = 1$, we can easily check that $[\![T \in L(X, Y)_B]\!] = 1$. We have

$$\|Tx\|_B = \|Sx\|_B = \|Sx\|_Z \leq \|S\|_Z\|x\|_B\,,$$

for all $x \in \hat{X}$. Since $[\![\hat{X} \times \{1\} = X]\!] = 1$, we have

$$[\![\forall x \in X(\|Tx\|_B \leq \|S\|_Z\,|x\|_B)]\!] = 1\,,$$

whence $T \in L(X, Y)^\wedge$. Thus we have shown that for any $S \in \mathrm{Hom}_Z(\hat{X}, \hat{Y})$, there is some $T \in L(X, Y)^\wedge$ such that $\hat{T} = S$. By the relation $[\![\hat{X} \times \{1\} = X]\!] = 1$, $\hat{T} = \hat{S}$ for $T, S \in L(X, Y)^\wedge$, then $[\![\forall x \in X(Tx = Sx)]\!] = 1$ so that $[\![T = S]\!] = 1$. Therefore, the correspondence $T \mapsto \hat{T}$ is a one-to-one correspondence between $L(X, L)^\wedge$ and $\mathrm{Hom}_Z(\hat{X}, \hat{Y})$. The rest of the assertions can be checked by a straightforward verification. Q.E.D.

§ 5. Construction of Banach spaces in $V^{(B)}$

Denote by $\mathbf{Norm}_\infty^{(B)}$ the category of normed linear spaces in $V^{(B)}$ and bounded linear maps T in $V^{(B)}$ with $\|T\|_B \in Z$: An object is a B-valued set X such that $[\![X$ is a normed linear space$]\!] = 1$, an arrow $T: X \to Y$ is a B-valued set T such that

$$[\![T \text{ is a bounded linear transformation from } X \text{ to } Y]\!] = 1,$$

and such that $\|T\|_B \in Z$. The composition of arrows is the function composition in $V^{(B)}$. Then a hom-set of $\mathbf{Norm}_\infty^{(B)}$ is $L(X, Y)^\wedge$ for objects X, Y. Denote by Z-**Kaplansky** the category of Kaplansky Z-modules and bounded Z-linear maps. In the preceding section, we have constructed a functor $X \mapsto \hat{X}$, $T \mapsto \hat{T}$ from $\mathbf{Norm}_\infty^{(B)}$ to Z-**Kaplansky**. In the following, we construct its adjoint functor. Eventually, it will be shown that this pair of adjoint functors is an equivalence of these two categories.

THEOREM 5.1. *Let $\langle X, +, \cdot, \|\cdot\|_Z\rangle$ be a Z-normed Z-module. For any $x \in X$, define $\tilde{x} \in V^{(B)}$ by*

$$\operatorname{dom}(\tilde{x}) = \{\check{y} \mid y \in X\},$$
$$\tilde{x}(\check{y}) = [\![\|x - y\|_Z = 0]\!].$$

Then $[\![\tilde{x} \subseteq \check{X}]\!] = 1$ for all $x \in X$ and the correspondence $x \mapsto \tilde{x}$ is bijective in the sense that $x = y$ if and only if $[\![\tilde{x} = \tilde{y}]\!] = 1$ for all $x, y \in X$. Define $\tilde{X} \in V^{(B)}$ by

$$\tilde{X} = \{\tilde{x} \mid x \in X\} \times \{1\}.$$

Then there is a unique normed linear space structure $\langle \tilde{X}, +, \cdot, \|\cdot\|_B\rangle$ on $\tilde{X}$ in $V^{(B)}$ such that

$$[\![\tilde{x} + \tilde{y} = (x + y)^\sim]\!] = 1,\ [\![\tilde{a} \cdot \tilde{x} = (a \cdot x)^\sim]\!] = 1 \text{ and } [\![\|\tilde{x}\|_B = \|x\|_Z]\!] = 1,$$

for all $x, y \in X$ and $a \in Z$.

Proof. Let $x \in X$. Since $\operatorname{dom}(\tilde{x}) = \operatorname{dom}(\check{X})$, we have

$$\begin{aligned}
[\![\tilde{x} \subseteq \check{X}]\!] &= [\![\forall y \in \tilde{x}(y \in \tilde{x} \Rightarrow y \in \check{X})]\!] \\
&= \inf_{y \in \operatorname{dom}(\tilde{x})} (\tilde{x}(y) \Rightarrow [\![y \in \check{X}]\!]) \\
&= \inf_{y \in X} (\tilde{x}(\check{y}) \Rightarrow [\![\check{y} \in \check{X}]\!]) \\
&= 1.
\end{aligned}$$

Thus $[\![\tilde{x} \subseteq \check{X}]\!] = 1$ for all $x \in X$. Consider the function $\langle \check{x}, \check{y} \rangle \mapsto \|x - y\|_Z$ from $\mathrm{dom}(\check{X}) \times \mathrm{dom}(\check{X})$ to Z. Then it is obviously extensional, and hence from Theorem 3.7 there is a function $d \in V^{(B)}$ such that $[\![d\colon \check{X} \times \check{X} \to C_B]\!] = 1$ and $[\![d(\check{x}, \check{y}) = \|x - y\|_Z]\!] = 1$ for all $x, y \in X$. By the properties of the Z-valued norm, it is easy to see that $[\![d$ is a semi-metric on $\check{X}]\!] = 1$. By interpreting the property of a semi-metric that

$$\{\forall z \in \check{X}(d(x, z) = 0 \Leftrightarrow d(y, z) = 0)\} \Leftrightarrow d(x, y) = 0\,,$$

we have

$$\inf_{z \in X} [\![d(\check{x}, \check{z}) = 0 \Leftrightarrow d(\check{y}, \check{z}) = 0]\!] = [\![d(\check{x}, \check{y}) = 0]\!]\,,$$

for all $x, y \in X$. Thus we have

$$\begin{aligned} [\![\tilde{x} = \tilde{y}]\!] &= [\![\forall z \in \check{X}(z \in \tilde{x} \Leftrightarrow z \in \tilde{y})]\!] \\ &= \inf_{z \in X} (\tilde{x}(\check{z}) \Leftrightarrow \tilde{y}(\check{z})) \\ &= \inf_{z \in X} [\![d(\check{x}, \check{z}) = 0 \Leftrightarrow d(\check{y}, \check{z}) = 0]\!] \\ &= [\![d(\check{x}, \check{y}) = 0]\!] \\ &= [\![\|x - y\|_Z = 0]\!]\,, \end{aligned}$$

for all $x, y \in X$. Thus, if $\tilde{x} = \tilde{y}$ then $\|x - y\|_Z = 0$ so that $x = y$. It follows that the correspondence $x \mapsto \tilde{x}$ is bijective. Let $x, x', y, y' \in X$. Then

$$\begin{aligned} \|(x + y) - (x' + y')\|_Z &= \|(x - x') + (y - y')\|_Z \\ &\leq \|x - x'\|_Z + \|y - y'\|_Z\,. \end{aligned}$$

It follows that

$$\begin{aligned} [\![\tilde{x} = \tilde{x}' \wedge \tilde{y} = \tilde{y}']\!] &= [\![\|x - x'\|_Z = 0 \wedge \|y - y'\|_Z = 0]\!] \\ &= [\![\|x - x'\|_Z + \|y - y'\|_Z = 0]\!] \\ &\leq [\![\|(x + y) - (x' + y')\|_Z = 0]\!] \\ &= [\![(x + y)^\sim = (x' + y')^\sim]\!]\,. \end{aligned}$$

Consequently, the function $\langle \tilde{x}, \tilde{y} \rangle \mapsto (x + y)^\sim$ from $\mathrm{dom}(\tilde{X}) \times \mathrm{dom}(\tilde{X})$ to $\mathrm{dom}(\tilde{X})$ is extensional so that from Theorem 3.7 there is a function $+$ in $V^{(B)}$ such that $[\![+\colon \tilde{X} \times \tilde{X} \to \tilde{X}]\!] = 1$ and $[\![\tilde{x} + \tilde{y} = (x + y)^\sim]\!] = 1$ for all $x, y \in X$. Let $a, a' \in Z$ and $x, x' \in X$. Then

$$\|a' \cdot x' - a \cdot x\|_Z \leq |a'| \|x' - x\|_Z + |a - a'| \|x\|_Z\,.$$

It follows from the similar manipulations that

$$[\![a = a' \wedge \tilde{x} = \tilde{x}']\!] \leq [\![(a' \cdot x')^{\sim} = (a \cdot x)^{\sim}]\!] .$$

Consequently, the function $\langle a,\tilde{x}\rangle \mapsto (a \cdot x)^{\sim}$ from $Z \times \mathrm{dom}(\tilde{X})$ to $\mathrm{dom}(\tilde{X})$ is extensional so that there is a function $\cdot$ in $V^{(B)}$ such that $[\![\cdot : C_B \times \tilde{X} \to \tilde{X}]\!] = 1$ and $[\![a \cdot \tilde{x} = (a \cdot x)^{\sim}]\!] = 1$ for all $a \in Z$ and $x \in X$. Let x, $x' \in X$. Then

$$| \|x\|_Z - \|x'\|_Z | \leq \|x - x'\|_Z ,$$

and hence we have

$$[\![x = x']\!] \leq [\![x_Z = \|x'\|_Z]\!] .$$

Consequently, the function $\tilde{x} \mapsto \|x\|_Z$ from $\mathrm{dom}(\tilde{X})$ to Z is extensional, so that there is a function $\|\cdot\|_B$ in $V^{(B)}$ such that $[\![\|\cdot\|_B \colon \tilde{X} \to \boldsymbol{C}]\!] = 1$ and $[\![\|\tilde{x}\|_B = \|x\|_Z]\!] = 1$ for all $x \in X$. Now it is a matter of straightforward verification that the structure $\langle \tilde{X}, +, \cdot, \|\cdot\|_B\rangle$ is a normed linear space in $V^{(B)}$ and that it is uniquely determined. Q.E.D.

Let X be a Z-normed Z-module. The normed linear space $\tilde{X}$ constructed by Theorem 5.1 will be called the *Boolean embedding* of X into $V^{(B)}$.

THEOREM 5.2. *Let X be a Z-normed Z-module and let $\tilde{X}$ be the Boolean embedding of X into $V^{(B)}$. Then the relation $[\![J_X(x) = \tilde{x}]\!] = 1$ for all $x \in X$ sets up a Z-linear Z-isometry J_X from X into $(\tilde{X})^\wedge$. Further, J_X is surjective if and only if X is a Kaplansky Z-module.*

Proof. By Theorem 5.1, $x = y$ if and only if $[\![\tilde{x} = \tilde{y}]\!] = 1$ for all x, $y \in X$, and hence the relation $[\![J_X(x) = \tilde{x}]\!] = 1$ for all $x \in X$ defines an injection J_X from X into $\tilde{X}^{(B)}$. For any $x \in X$, we have $[\![\|J_X(x)\|_B = \|\tilde{x}\|_B = \|x\|_Z]\!] = 1$ by Theorem 5.1, so that $\|J_X(x)\|_B \in Z$, and hence $J_X(x) \in (\tilde{X})^\wedge$. Now it is easy to check that J_X is a Z-linear Z-isometry from X into $(\tilde{X})^\wedge$. If J_X is surjective then $X \cong_Z (\tilde{X})^\wedge$ and hence X is a Kaplansky Z-module by Theorem 4.1. Conversely, suppose that X is a Kaplansky Z-module. Let $x \in (\tilde{X})^\wedge$. Since $\tilde{X} = \{\tilde{x} \mid x \in X\} \times \{1\}$, there is a family $\{x_i\}$ in X and a partition $\{b_i\}$ of unity with a common index set such that $[\![x = \tilde{x}_i]\!] \geq b_i$ or equivalently $[\![b_i x = (b_i x_i)^{\sim}]\!] = 1$ for all i. Since $x \in (\tilde{X})^\wedge$, there is some $K \in \boldsymbol{R}$ such that $[\![\|x\|_B \leq \check{K}]\!] = 1$ and hence $[\![\|(b_i x_i)^{\sim}\|_B \leq \check{K}]\!] = 1$ for all i. It follows that $\{b_i x_i\}$ is a bounded family in X. Since X is a

Kaplansky Z-module, there is some $y \in X$ such that $b_i y = b_i x_i$ for all i. We have $b_i J_X(y) = J_X(b_i y) = J_X(b_i x_i)$ and hence $[\![b_i J_X(y) = (b_i x_i)^\sim = b_i x]\!] = 1$ for all i. Consequently, we have $[\![J_X(y) = x]\!] = 1$. Thus, J_X is surjective. Q.E.D.

We shall call J_X the *embedding map* of X.

THEOREM 5.3. *Let X be a Z-normed Z-module and let Y be a normed linear space in $V^{(B)}$. For every $T \in \mathrm{Hom}_Z(X, \hat{Y})$, there is a unique $S \in L(\tilde{X}, Y)^\wedge$ such that $T = \hat{S} \circ J_X$.*

Proof. Let $T \in \mathrm{Hom}_Z(X, \hat{Y})$. Since $[\![\hat{Y} \times \{1\} = Y]\!] = 1$, we can assume without any loss of generality that $\mathrm{dom}(Y) = \hat{Y}$. By the bijective correspondence $x \mapsto \tilde{x}$, we can define T': $\mathrm{dom}(\tilde{X}) \to \mathrm{dom}(Y)$ by $T'\tilde{x} = Tx$ for all $x \in X$. Let $x, y \in X$. Then $\|Tx - Ty\|_Z \leq \|T\| \|x - y\|_Z$ so that

$$\begin{aligned}[\![\tilde{x} = \tilde{y}]\!] &= [\![\|x = y\|_Z = 0]\!] \\ &\leq [\![\|Tx - Ty\|_Z = 0]\!] \\ &= [\![Tx = Ty]\!] \\ &= [\![T'\tilde{x} = T'\tilde{y}]\!] .\end{aligned}$$

It follows that T' is extensional so that there is some $S \in V^{(B)}$ such that $[\![S\colon \tilde{X} \to Y]\!] = 1$ and $[\![Tx = S\tilde{x}]\!] = 1$ for all $x \in X$. Since $[\![\hat{X} \times \{1\} = X]\!] = 1$, such an S is unique in $V^{(B)}$. By the similar arguments as in the proof of Theorem 4.3, we have $S \in L(\tilde{X}, Y)^\wedge$. By the relation $[\![Tx = S\tilde{x}]\!] = 1$ for all $x \in X$, we obtain that $T = \hat{S} \circ J_X$. Q.E.D.

THEOREM 5.4. *Let X, Y be two Z-normed Z-modules and let $T \in \mathrm{Hom}_Z(X, Y)$. Then there is a unique $\tilde{T} \in V^{(B)}$ such that $\tilde{T} \in L(\tilde{X}, \tilde{Y})^\wedge$, $[\![\tilde{T}(\tilde{x}) = (Tx)^\sim]\!] = 1$ for all $x \in X$ and $\|\tilde{T}\|_B = \|T\|_Z$.*

Proof. The assertion follows from the similar arguments as in the proof of Theorem 5.3. Q.E.D.

Let X, Y be two Z-normed Z-modules and let $T \in \mathrm{Hom}_Z(X, Y)$. The bounded linear map $\tilde{T}$: $\tilde{X} \to \tilde{Y}$ in $V^{(B)}$ obtained by Theorem 5.4 will be called the *Boolean embedding* of T into $V^{(B)}$.

Denote by Z-**Norm** the category of Z-normed Z-modules and bounded Z-linear maps. Then Z-**Kaplansky** is a full subcategory of Z-**Norm**. Now we can summarize the functorial properties of the Boolean embedding as follows.

THEOREM 5.5. *The Boolean embedding* $E: X \mapsto \tilde{X}$, $E: T \mapsto \tilde{T}$ *is a functor from* Z-**Norm** *to* $\mathbf{Norm}_\infty^{(B)}$ *such that*

(1) $E(aT + bS) = aE(T) + bE(S)$,

(2) $\|E(T)\|_B = \|T\|_Z$,

for all $a, b \in Z$ *and* $T, S \in \mathrm{Hom}_Z(X, Y)$ *and* $X, Y \in Z$-**Norm**. *The functor* E *is a left adjoint functor of the functor* $R: X \mapsto \hat{X}$, $R: T \mapsto \hat{T}$ *from* $\mathbf{Norm}_\infty^{(B)}$ *to* Z-**Norm** *constructed in Section* 4. *The corresponding natural transformation* $1 \dot{\to} RE$ *on* Z-**Norm** *is* $\{J_X \mid X \in Z\text{-}\mathbf{Norm}\}$ *obtained in Theorem* 5.2. *If we restrict* E *to* Z-**Kaplansky** *then this adjoint pair establishes an equivalence between* Z-**Kaplansky** *and* $\mathbf{Norm}_\infty^{(B)}$.

Proof. Immediate consequences from our results in Sections 4 and 5; see [12, Theorem IV. 1.2, p. 81; Theorem IV. 4.1, p. 91] for the consequences from category theory. Q.E.D.

By the above theorem, the correspondence $RE: X \mapsto (\tilde{X})^\wedge$, $RE: T \mapsto (\tilde{T})^\wedge$ is a functor from Z-**Norm** to Z-**Kaplansky** and the embedding map J_X is a universal arrow from X to this functor RE. Thus we have the following corollary.

COROLLARY 5.6. *Let X be a Z-normed Z-module. Then the Kaplansky Z-module $(\tilde{X})^\wedge$ is a unique Kaplansky Z-module up to Z-isometric Z-isomorphism such that, for any Kaplansky Z-module Y, every* $T \in \mathrm{Hom}_Z((\tilde{X})^\wedge, Y)$ *is obtained from a unique* $S \in \mathrm{Hom}_Z(X, Y)$ *by the relation* $T = S \circ J_X$. *The correspondence* $S \mapsto S \circ J_X$ *is a Z-isometric Z-isomorphism from* $\mathrm{Hom}_Z(X, Y)$ *onto* $\mathrm{Hom}_Z(\tilde{X}^\wedge, Y)$.

For any normed linear space X, denote by $\mathrm{UB}(X)$ the unit ball of X. For a normed linear space X in $V^{(B)}$, $\mathrm{UB}(X)_B$ will stand for the unit ball of X in $V^{(B)}$. In this case, denote by $\mathrm{UB}(X)^{(B)}$ the global section set of $\mathrm{UB}(X)_B$, i.e., $\mathrm{UB}(X)^{(B)} = \{x \in V^{(B)} \mid [\![x \in \mathrm{UB}(X)_B]\!] = 1\}$. The following theorem shows that the concept of the unit ball is preserved by the Boolean embedding.

THEOREM 5.7. (1) *For any normed linear space X in* $V^{(B)}$, $\mathrm{UB}(X)^{(B)} = \mathrm{UB}(\tilde{X})$.

(2) *For any Kaplansky Z-module X,* $J_X(\mathrm{UB}(X)) = \mathrm{UB}(\hat{X})^{(B)}$.

Proof. (1) Let $x \in \mathrm{UB}(X)^{(B)}$. Then $\|x\|_B \le 1$ and hence $x \in \hat{X}$. Since $\|x\| = \|\|x\|_B\|_\infty \le 1$, we have $x \in \mathrm{UB}(\hat{X})$. Let $x \in \mathrm{UB}(\tilde{X})$. Then $x \in X^{(B)}$ and $\|x\|_B \le \|x\|1 \le 1$, so that $x \in \mathrm{UB}(X)^{(B)}$.

(2) From Theorem 5.2, J_X is a Z-isometric Z-isomorphism from X onto $(\tilde{X})^\wedge$ and hence $J_X(\mathrm{UB}(X)) = \mathrm{UB}((\tilde{X})^\wedge)$. From (1), we have $J_X(\mathrm{UB}(X)) = \mathrm{UB}(\tilde{X})^{(B)}$. Q.E.D.

THEOREM 5.8. *Let X be a Kaplansky Z-module and let $\phi(x)$ be a formula in $L(V^{(B)})$. Then we have:*

(1) $[\![(\forall x \in \mathrm{UB}(\tilde{X})_B)\phi(x)]\!] = 1$ *if and only if for all* $u \in \mathrm{UB}(X)$, $[\![\phi(\tilde{u})]\!] = 1$.

(2) $[\![(\exists x \in \mathrm{UB}(\tilde{X})_B)\phi(x)]\!] = 1$ *if and only if there is some* $u \in \mathrm{UB}(X)$ *such that* $[\![\phi(\tilde{u})]\!] = 1$.

Proof. Immediate consequences from Theorem 5.7 and Theorem 3.5. Q.E.D.

§ 6. Boolean embeddings of C*-algebras

Consider a unital C*-algebra A in $V^{(B)}$ and its bounded global section module $\hat{A}$. By the property $[\![(\forall a \in C)(\forall x \in A)(a1_A)x = x(a1_A)]\!] = 1$ and the identification $\hat{C} = Z$, we have $(a1_A)x = x(a1_A)$ for all $a \in Z$ and $x \in \hat{A}$, where 1_A is the unit of A in $V^{(B)}$. Let 1_Z be the unit of Z. Then by the identification $\hat{C} = Z$, 1_Z is the numeral one in $V^{(B)}$. By the property $[\![(\forall x \in A)1_Z x = x]\!] = 1$, we have $1_Z 1_A = 1_A$. Thus the set of all elements of the form $a1_A$ for $a \in Z$ will be identified with Z and hence we can see that $\hat{A}$ contains Z as a unital C*-subalgebra of the center.

Let $x \in \hat{A}$ and $b \in \{0, 1\}^{(B)} = B$. Then $[\![b = 0 \vee b = 1]\!] = 1$. Suppose $bx = 0$. Then $[\![bx = 0]\!] = 1$ and, by the property $[\![bx = 0 \wedge b = 1 \Rightarrow x = 0]\!] = 1$, we have $b = [\![b = 1]\!] \leq [\![x = 0]\!]$. Thus if $bx = 0$ then $b \leq [\![x = 0]\!]$. Further, if $b = [\![x = 0]\!]$, we have $b = [\![x = 0]\!] \leq [\![bx = 0]\!]$, and $1 - b = [\![b = 0]\!] \leq [\![bx = 0]\!]$, and hence $[\![bx = 0]\!] = 1$, that is $bx = 0$. Thus $[\![x = 0]\!]$ is the maximum element of the set $\{b \in B \mid bx = 0\}$.

We say that a unital C*-algebra A which contains Z as a unital C*-subalgebra of the center is *Z-scalable* if, for any $x \in A$, there is the largest projection e in Z such that $ex = 0$. It should be noted that e is the largest projection in Z such that $ex = 0$ if and only if $1 - e$ is the smallest projection in Z such that $(1 - e)x = x$. In the case that Z is the center of A such a projection $1 - e$ is usually called the central cover of x. For the above observations, the bounded global section module of a C*-algebra in $V^{(B)}$ is Z-scalable. By [1, p. 14, Proposition 6], every AW*-algebra which contains Z as an AW*-subalgebra of the center is Z-scalable.

From now onward, let A be a unital C*-algebra which contains Z as a unital C*-subalgebra of the center. Then A has a natural Z-module structure.

THEOREM 6.1. *Let A be a unital C*-algebra which contains Z in the center as a unital C*-subalgebra. For any $x \in A$, define $\|x\|_Z$ by*

$$\|x\|_Z = \inf\{a \in Z \mid x^*x \leq a^2,\ a \geq 0\}.$$

Then $\|x\|_Z \geq 0$ and $\|\cdot\|_Z$ has the following properties:

(1) $\|\|x\|_Z\|_\infty \leq \|x\|$,

(2) $\|x+y\|_Z \leq \|x\|_Z + \|y\|_Z$,

(3) $\|ax\|_Z = |a|\|x\|_Z$,

(4) $\|xy\|_Z \leq \|x\|_Z\|y\|_Z$,

(5) $\|x^*x\|_Z = \|x\|_Z^2$,

for all $x, y \in A$. Moreover, if A is Z-scalable, it satisfies the following properties:

(6) $\|\|x\|_Z\|_\infty = \|x\|$,

(7) $\|x\|_Z = 0$ *only if* $x = 0$,

for all $x \in A$, and hence $\|\cdot\|_Z$ is a Z-valued norm on A.

Proof. Since Z is a commutative AW*-algebra, the infimum in the definition of $\|x\|_Z$ always exists and obviously $\|x\|_Z \geq 0$.

(1) By the relation $x^*x \leq \|x^*x\|1 = \|x\|^2 1$, we have $\|x\|_Z \leq \|x\|1$ so that $\|\|x\|_Z\|_\infty \leq \|x\|$.

(2) By a faithful *-representation, we can assume that A is a C*-algebra of bounded operators on a Hilbert space H. Since Z is in the center of A, we have $A \subseteq Z'$, where $'$ stands for the commutant in $L(H)$. Since Z'' is an abelian von Neumann algebra, there is a localizable measure space (Γ, μ) and a *-isomorphism λ from Z'' onto $L^\infty(\Gamma, \mu)$. Let $\xi, \eta \in H$. The linear functional $\lambda(a) \mapsto \langle a\xi \mid \eta\rangle$ on $L^\infty(\Gamma, \mu)$ is bounded and completely additive on projections, and thus by the Radon-Nikodym theorem there is a μ-integrable function $F(\xi, \eta)$ uniquely in $L^1(\Gamma, \mu)$ such that

$$\langle a\xi \mid \eta\rangle = \int_\Gamma \lambda(a)(\gamma)F(\xi, \eta)(\gamma)\mu(d\gamma),$$

for all $a \in Z''$. Then it is easy to check that the map $\langle \xi, \eta\rangle \mapsto F(\xi, \eta)$ from $H \times H$ to $L^1(\Gamma, \mu)$ has the following properties: For any $x \in Z'$ and $a, b \in Z''$, and $\xi, \eta, \zeta \in H$, (F1) $F(a\xi + b\eta, \zeta) = \lambda(a)F(\xi, \zeta) + \lambda(b)F(\eta, \zeta)$,

(F2) $F(\xi, \eta)^* = F(\eta, \xi)$,

(F3) $F(\xi, \xi) \geq 0$ and $F(\xi, \xi) = 0$ only if $\xi = 0$,

(F4) $F(x\xi, \eta) = F(\xi, x^*\eta)$.

Let $G(\xi) = F(\xi, \xi)^{1/2}$ for all $\xi \in H$. Then $G(\xi) \in L^2(\Gamma, \mu)$ and the map $\xi \in H \mapsto G(\xi)$ has the following properties [15, Lemma 4.1]: For any $a \in Z''$ and $\xi, \eta \in H$,

(G1) $G(\xi + \eta) \leq G(\xi) + G(\eta)$,

(G2) $G(a\xi) = |\lambda(a)|G(\xi)$.

Now we shall claim the following: For any $x, y \in Z'$,

(G3) $x^*x \leq y^*y$ if and only if $G(x\xi) \leq G(y\xi)$ for all $\xi \in H$.

If $x^*x \leq y^*y$ then $\langle ax^*x\xi | \xi\rangle \leq \langle ay^*y\xi | \xi\rangle$ for all $a \in Z''$ with $a \geq 0$ and $\xi \in H$, and hence $F(x^*x\xi, \xi) \leq F(y^*y\xi, \xi)$ so that $G(x\xi) \leq G(y\xi)$ by (F4). Conversely, if $G(x\xi) \leq G(y\xi)$ then it is easy to see that $\langle x^*x\xi | \xi\rangle \leq \langle y^*y\xi | \xi\rangle$. Thus property (G3) is concluded. To prove (2), let $x, y \in A$ and $a, b \in Z$ be such that $x^*x \leq a^2$, $y^*y \leq b^2$ and $a, b \geq 0$. Then by properties (G1)–(G3), we have, for all $\xi \in H$,

$$\begin{aligned} G((x + y)\xi) &\leq G(x\xi) + G(y\xi) \leq G(a\xi) + G(b\xi) \\ &= \lambda(a)G(\xi) + \lambda(b)G(\xi) = \lambda(a + b)G(\xi) \\ &= G((a + b)\xi), \end{aligned}$$

so that $(x + y)^*(x + y) \leq (a + b)^2$. It follows that $\|x + y\|_Z \leq \|x\|_Z + \|y\|_Z$.

(3) Let $x = b_1x_1 + b_2x_2$ with $x_1, x_2 \in A$ and $b_1, b_2 \in B$ such that $b_1b_2 = 0$. Let a, a_1, a_2 be positive elements of Z such that $x^*x \leq a^2$, $x_1^*x_1 \leq a_1^2$ and $x_2^*x_2 \leq a_2^2$. Then

$$x^*x = b_1x_1^*x_1 + b_2x_2^*x_2 \leq b_1a_1^2 + b_2a_2^2 = (b_1a_1 + b_2a_2)^2 ,$$

so that $\|x\|_Z \leq b_1a_1 + b_2a_2$, and hence $\|x\|_Z \leq b_1\|x_1\|_Z + b_2\|x_2\|_Z$ by taking the infima of a_1 and a_2 in Z. On the other hand, we have

$$\begin{aligned} x_1^*x_1 &\leq b_1x_1^*x_1 + (1 - b_1)x_1^*x_1 + b_2x_2^*x_2 = x^*x + (1 - b_1)x_1^*x_1 \\ &\leq a^2 + (1 + b_1)a_1^2. \end{aligned}$$

It follows that $\|x_1\|_Z^2 \leq a^2 + (1 - b_1)a_1^2$ so that $\|x_1\|_Z^2 \leq \|x\|_Z^2 + (1 - b_1)\|x_1\|_Z^2$. Thus, by multiplying the both sides by b_1, $b_1\|x_1\|_Z^2 \leq b_1\|x\|_Z^2$ and hence $b_1\|x_1\|_Z \leq b_1\|x_2\|_Z$. Similarly, $b_2\|x_2\|_Z \leq b_2\|x\|_Z$ and thus $b_1\|x_1\|_Z + b_2\|x_2\|_Z \leq \|x\|_Z$. Therefore, we have shown $\|x\|_Z = b_1\|x_1\|_Z + b_2\|x_2\|_Z$. By induction, for any pairwise orthogonal projections $b_1, b_2, \cdots, b_n \in B$ and x_1,

$x_2, \cdots, x_n \in A$, we have $\|\sum_{i=1}^n b_i x_i\|_Z = \sum_{i=1}^n b_i \|x_i\|_Z$. Now we shall prove the general case. Let $a \in Z$ and $x \in A$. Let ε be an arbitrary positive number. Since Z is generated by projections [1, p. 45, Proposition 1], there is an element $a' \in Z$ of the form $a' = \sum_{i=1}^n \alpha_i b_i$ with $\alpha_i \in C$ and $b_i \in B$ such that $\|a - a'\|_\infty \leq \varepsilon$. Obviously, we can assume that b_i's are pairwise orthogonal. Thus we have

$$\begin{aligned}\|a'x\|_Z &= \left\|\sum_{i=1}^n b_i(\alpha_i x)\right\|_Z = \sum_{i=1}^n b_i \|\alpha_i x\|_Z = \sum_{i=1}^n |\alpha_i| b_i \|x\|_Z \\ &= |a'| \|x\|_Z .\end{aligned}$$

By the triangular inequality $|a| - |a'| \leq |a - a'|$ and (1), we have

$$\begin{aligned}\||a|\|x_Z - |a'|\|x\|_Z\|_\infty &\leq \||a| - |a'|\|_\infty \|\|x\|_Z\|_\infty \\ &\leq \|a - a'\|_\infty \|x\| \\ &\leq \varepsilon \|x\| .\end{aligned}$$

By (1) and (2) we have

$$\begin{aligned}\|\|ax\|_Z - \|a'x\|_Z\|_\infty &\leq \|\|ax - a'x\|_Z\|_\infty \\ &\leq \|\|ax - a'x\| \leq \|a - a'\|\|x\| \\ &\leq \varepsilon\|x\| .\end{aligned}$$

Thus we have

$$\begin{aligned}\|\|ax\|_Z - |a|\|x\|_Z\|_\infty &= \|\|ax\|_Z - \|a'x\|_Z + |a'|\|x\|_Z - |a|\|x\|_Z\|_\infty \\ &\leq \|\|ax\|_Z - \|a'x\|_Z\|_\infty + \||a'|\|x\|_Z - |a|\|x\|_Z\|_\infty \\ &\leq 2\varepsilon\|x\| .\end{aligned}$$

Therefore, we have proved $\|ax\|_Z = |a|\|x\|_Z$.

(4) Let $x, y \in A$ and let a_1 and a_2 be positive element of Z such that $x^*x \leq a_1^2$ and $y^*y \leq a_2^2$. Then we have

$$(xy)^*(xy) = y^*(x^*x)y \leq y^*a_1^2 y \leq a_1(y^*y)a_1 \leq (a_1a_2)^2 .$$

It follows that $\|xy\|_Z \leq a_1a_2$ and hence $\|xy\|_Z \leq \|x\|_Z\|y\|_Z$.

(5) Let $x \in A$ and $a \in Z$ with $a \geq 0$. Since Z is in the center of A, $x^*x \leq a^2$ iff $(x^*x)^*(x^*x) \leq (a^2)^2$. Thus the conclusion follows from the order isomorphic property of $a \mapsto a^2$ for the positive elements of Z.

(6) From (1) it suffices to show that $x^*x \leq \|x\|_Z^2$ for all $x \in A$. Let $x \in A$ and let Γ be the maximal ideal space of the commutative C^*-algebra $C^*(Z \cup \{x^*x\})$ generated by Z and x^*x. By the Gelfand transform, $C^*(Z \cup \{x^*x\})$ is $*$-isomorphic to $C(\Gamma)$, and we shall simply write $a(\gamma)$ for

the value at $\gamma \in \Gamma$ of the Gelfand transform of $a \in C^*(Z \cup \{x^*x\})$. Obviously, it suffices to show that, for any $\gamma \in \Gamma$ and positive number α, if $x^*x(\gamma) > \alpha$ then $\|x\|_Z^2(\gamma) \geq \alpha$. Let $\alpha > 0$ and set $S = \{\gamma \in \Gamma \mid x^*x(\gamma) > \alpha\}$. By Z-scalability of A, there is a projection $c \in B$ such that

$$c = \max\{b \in B \mid b[x^*x - \alpha 1]^+ = 0\},$$

where $[\cdot]^+$ stands for the positive part. Let $U = \{\gamma \in \Gamma \mid c(\gamma) = 1\}$. If $\gamma \in S$ then $[x^*x - \alpha 1]^+(\gamma) \neq 0$, hence $c(\gamma) = 0$ so that $\gamma \in \Gamma \backslash U$. It follows that $S \subseteq \Gamma \backslash U$. On the other hand, let $a \in Z$ be such that $a \geq 0$ and $x^*x \leq a^2$ and let

$$d = \max\{b \in B \mid b[a^2 - \alpha 1]^+ = 0\}.$$

By the relation $x^*x \leq a^2$, we have $[x^*x - \alpha 1]^+ \leq [a^2 - \alpha 1]^+$ so that $d \leq c$. Now let Ω be the maximal ideal space of Z and consider the Gelfand transform of Z. Then Ω is a Stonean space in which the closure of every open set is clopen. Let $V = \{\omega \in \Omega \mid d(\omega) = 1\}$. By taking complement, we have

$$1 - d = \min\{b \in B \mid b[a^2 - \alpha 1]^+ = [a^2 - \alpha 1]^+\},$$

and hence $\Omega \backslash V$ is the smallest clopen subset of Ω such that $\{\omega \in \Omega \mid a^2(\omega) > \alpha\} \subseteq \Omega \backslash V$. It follows that $\Omega \backslash V$ is the closure of $\{\omega \in \Omega \mid a^2(\omega) > \alpha\}$, which is clopen since Ω is Stonean. Thus we have $\Omega \backslash V \subseteq \{\omega \in \Omega \mid a^2(\omega) \geq \alpha\}$, and hence $\alpha(1 - d) \leq a^2$. By the relation $d \leq c$, we have $\alpha(1 - c) \leq a^2$. Thus we have shown that, for any $a \in Z$ with $a \geq 0$ and $x^*x \leq a^2$, we have $\sqrt{\alpha}(1 - c) \leq a$. Since $\sqrt{\alpha}(1 - c) \in Z$, we conclude that $\sqrt{\alpha}(1 - c) \leq \|x\|_Z$. Thus $\|x\|_Z^2(\gamma) \geq \alpha$ for all $\gamma \in \Gamma \backslash U$, but $S \subseteq \Gamma \backslash U$, and therefore we have proved that $\|x\|_Z^2(\gamma) \geq \alpha$ for all $\gamma \in \Gamma$ with $x^*x(\gamma) > \alpha$. Q.E.D.

Let A be a Z-scalable C*-algebra. From Theorem 6.1, A is a Z-normed Z-module. Thus we can construct the Boolean embedding $\tilde{A}$ of A by Theorem 5.1. A normed *-algebra A will be called a *pre-C*-algebra* if $\|x^*x\| = \|x\|^2$ for all $x \in A$. Obviously, the metric completion of a pre-C*-algebra is a C*-algebra.

THEOREM 6.2. *Let A be a Z-scalable C*-algebra. Then the Boolean embedding $\tilde{A}$ of A is a unital pre-C*-algebra in $V^{(B)}$, where the product operation $\times$ and the involution $*$ satisfy that $[\![\tilde{x} \times \tilde{y} = (xy)^\sim]\!] = 1$ and $[\![(\tilde{x})^* = (x^*)^\sim]\!] = 1$ for all $x, y \in A$.*

Proof. By Theorem 5.1, $\tilde{A}$ is a normed linear space in $V^{(B)}$. Let $x, x', y, y' \in A$. Then

$$\|xy - x'y'\|_Z \leq \|x\|_Z \|y - y'\|_Z + \|y'\|_Z \|x - x'\|_Z .$$

By the similar argument as in the proof of Theorem 5.1, the function $\langle \tilde{x}, \tilde{y} \rangle \mapsto (x \times y)^{\sim}$ from $\mathrm{dom}(\tilde{A}) \times \mathrm{dom}(\tilde{A})$ to $\mathrm{dom}(\tilde{A})$ is extensional, so that there is a function $\times$ in $V^{(B)}$ such that $[\![\times : \tilde{A} \times \tilde{A} \to \tilde{A}]\!] = 1$ and $[\![\tilde{x} \times \tilde{y} = (xy)^{\sim}]\!] = 1$ for all $x, y \in A$. Similarly, there is a function $*$ in $V^{(B)}$ such that $[\![*\colon \tilde{A} \to \tilde{A}]\!] = 1$ and $[\![(\tilde{x})^* = (x^*)^{\sim}]\!] = 1$ for all $x \in A$. With these operations, it is easy to see that $\tilde{A}$ is a *-algebra with unit $\tilde{1}$. By Theorem 6.1, it is easy to see that $\tilde{A}$ is a pre-C*-algebra. Q.E.D.

A Z-scalable C*-algebra A is called a unital Z-C*-algebra if its Z-normed Z-module structure is a Kaplansky-Banach Z-module, i.e., it satisfies condition (K1). From [1, p. 53, Proposition 2], every AW*-algebra which contains Z as a unital AW*-subalgebra of the center is a unital Z-C*-algebra.

Let $\langle A, +, \cdot, \times, *, \|\cdot\|_B \rangle$ be a unital C*-algebra in $V^{(B)}$. The bounded global section algebra $\langle \hat{A}, +, \cdot, \times, \|\cdot\|_Z, \|\cdot\| \rangle$ (or simply denoted by $\hat{A}$) of A is defined as follows:

1) The partial structure $\langle \hat{A}, +, \cdot, \|\cdot\|_Z \rangle$ is the bounded global section module of the Banach space structure $\langle A, +, \cdot, \|\cdot\|_B \rangle$ in $V^{(B)}$.

2) For every pair $x, y \in \hat{A}$, the product u of x and y is defined as the unique element $u \in \hat{A}$ such that $[\![u = x \times y]\!] = 1$, which is also denoted by $x \times y$ or xy.

3) For every $x \in \hat{A}$, the adjoint u of x is defined as the unique element $u \in \hat{A}$ such that $[\![u = x^*]\!] = 1$, which is also denoted by x^*.

4) For every $x \in \hat{A}$, the scalar valued norm $\|x\|$ is defined by $\|x\| = \|\|x\|_Z\|_\infty$.

THEOREM 6.3. *Let A be a unital C*-algebra in $V^{(B)}$. Then the bounded global section algebra $\hat{A}$ is a unital Z-C*-algebra. Conversely, for any unital Z-C*-algebra A, the Boolean embedding $\tilde{A}$ is a unital C*-algebra in $V^{(B)}$ such that $(\tilde{A})^\wedge$ is Z-linearly *-isomorphic to A.*

Proof. The first part of the assertion can be checked easily. The second part follows from Theorem 5.2 and Theorem 6.2: For the similar result, compare with [30, Theorem 1.1, p. 214] and [20, Theorem 2].

Q.E.D.

General descriptions of non-unital C*-algebras in $V^{(B)}$ are obtained in [30].

PROPOSITION 6.4. *Let A be a Z-scalable C*-algebra. The Boolean embedding $\tilde{A}$ has the following properties*:

(1) *For any $x \in A$, x is a self-adjoint (partial isometry, unitary, projection) element if and only if $[\![\tilde{x}$ is a self-adjoint (partial isometry, unitary, projection) element$]\!] = 1$.*

(2) *For any $x, y \in A$, $x \leq y$ if and only if $[\![\tilde{x} \leq \tilde{y}]\!] = 1$.*

Further, if A is a unital Z-C-algebra then we have the following*:

(3) *For every self-adjoint (positive) $x \in \mathrm{UB}(A)_B$ in $V^{(B)}$, there is a self-adjoint (positive) $y \in \mathrm{UB}(A)$ such that $[\![x = \tilde{y}]\!] = 1$.*

(4) *For every projection (partial isometry, unitary) $x \in \tilde{A}$ in $V^{(B)}$ there is a projection (partial isometry, unitary) $y \in A$ such that $[\![x = \tilde{y}]\!] = 1$.*

Proof. Immediate consequences from Theorem 6.2; assertions (3) and (4) follows from Theorem 5.7. Q.E.D.

Now we have the following characterization of Z-scalable C*-algebra.

THEOREM 6.5. *Let A be a C*-algebra which contains Z as a unital C*-subalgebra of the center. Then the following conditions (1)–(7) are all equivalent.*

(1) *A is Z-scalable.*

(2) *There is an AW*-algebra L of type I with center Z and a *-monomorphism π: $A \to L$ such that $Z = \pi(Z) \subseteq \pi(A) \subseteq L$.*

(3) *For any $x \in A$, there is a projection P of norm one from A onto Z such that $x^*x \leq P(x^*x)$.*

(4) *For any $x \in A$, $x^*x \leq \|x\|_Z^2$.*

(5) *For any $x \in A$, $\|x\| \leq \|\|x\|_Z\|_\infty$.*

(6) *For any $x \in A$, $\|x\|_Z = 0$ only if $x = 0$, and hence $\|\cdot\|_Z$ is a Z-valued norm on A.*

(7) *For any bounded family $\{x_i\}$ in A and partition of unity $\{b_i\}$ in B with a common index set I, the relation $b_i y = b_i x_i$ for all $i \in I$ holds for at most one element $y \in A$ (if such a y exists).*

Proof. (1) $\Rightarrow$ (2): Suppose that A is Z-scalable. By Theorem 6.2, the Boolean embedding $\tilde{A}$ of A is a pre-C*-algebra in $V^{(B)}$ and hence there is a C*-algebra C in $V^{(B)}$ which is the metric completion in $V^{(B)}$ of $\tilde{A}$. By Theorem 5.2 it is easy to see that there is a Z-linear *-monomorphism

$h: A \to \hat{C}$ such that $h(x) = \tilde{x}$ for all $x \in A$. By the theory of C*-algebras, $[\![$There is a Hilbert space H and a *-monomorphism $j: C \to L(H)]\!] = 1$. Then $[\![\|j\|_B = 1]\!] = 1$ so that $j \in L(C, L(H))^\wedge$ and it is easy to check that $\hat{j}: \hat{C} \to L(H)^\wedge$ is a Z-linear *-monomorphism (cf. Theorem 4.3). From [17, Theorem 4.1] $\hat{H}$ is a Kaplansky-Hilbert module over Z and $L(H)^\wedge$ is identical with the AW*-algebra $\mathrm{End}_Z(\hat{H})$ of type I with the center Z. Now it is obvious that the composition $\pi = \hat{j} \circ h$ is a Z-linear *-monomorphism from A to $\mathrm{End}_Z(\hat{H})$. By Z-linearity and the property $\pi(1) = 1$, it is obvious that $\pi(Z) = Z$.

(2) $\Rightarrow$ (3): Suppose (2). Then we can assume without any loss of generality that A itself is an AW*-algebra of type I. Since every AW*-algebra is a Z-C*-algebra, from Theorem 6.3 we can assume that A is the bounded global section algebra $\hat{C}$ of a unital C*-algebra C in $V^{(B)}$. Let $x \in A = \hat{C}$. By the Hahn-Banach theorem of states of C*-algebras, $[\![$There is a positive linear functional f on C such that $\|f\|_B = 1$, $f(1) = 1$ and that $x^*x \le \|x^*x\|_B 1 = f(x^*x)1]\!] = 1$. Then it is easy to see that $\hat{f}$ is a positive Z-linear map from $\hat{C}$ to Z such that $\hat{f}(1) = 1$ and that $x^*x \le \hat{f}(x^*x)1$. Thus, by setting $P(y) = \hat{f}(y)1$ for all $y \in \hat{C}$, P is a projection from $\hat{C}$ onto the center Z of $\hat{C}$ such that $x^*x \le P(x^*x)$. In order to see that P is of norm one, let $y \in \hat{C}$. Then we have

$$\begin{aligned} \|P(y)\| &= \|\hat{f}(y)1\| = \|\hat{f}(y)\|_\infty = \||\hat{f}(y)|\|_\infty \\ &\le \|\|f\|_B\|y\|_B\|_\infty = \|\|y\|_B\|_\infty \\ &= \|y\|. \end{aligned}$$

Thus P is of norm one.

(3) $\Rightarrow$ (4): Let $x \in A$ and $a \in Z$ satisfy $x^*x \le a^2$ and $a \ge 0$. Let P be a projection of norm one from A onto Z such that $x^*x \le P(x^*x)$. Then, by the Tomiyama theorem [24, p. 131, Theorem 3.4], P is a positive linear map and hence we have $x^*x \le P(x^*x) \le P(a^2) = a^2$. Since $P(x^*x) \in Z$, we have $x^*x \le P(x^*x) \le \|x\|_Z^2$.

(4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (6) are obvious.

(6) $\Rightarrow$ (7): This is obvious. For if $b_i y = 0$ for all i, then $b_i\|y\|_Z = \|b_i y\|_Z = 0$ so that $\|y\|_Z = 0$, and hence $y = 0$ from assumption (6).

(7) $\Rightarrow$ (1): Let $x \in A$. Suppose (7). Let $\{b_i\}$ be a maximal pairwise orthogonal family of projections in B such that $b_i x = 0$ for all i. Let $c = \sup\{b_i\}$. Then $\{b_i\} \cup \{1 - c\}$ is a partition of unity in B and cx satisfies $b_i(cx) = 0$ for all i and $(1 - c)(cx) = 0$. By assumption (7), we have

$cx = 0$. If $d \in B$ satisfies $dx = 0$ then $(1 - c)d = 0$ by maximality of $\{b_i\}$ so that $d = cd$. Thus $c = \max\{b \in B \mid bx = 0\}$. Q.E.D.

An example of a C*-algebra which contains Z as a unital C*-subalgebra of the center but is not Z-scalable is obtained as follows. Suppose that B is the complete Boolean algebra of regular open subsets of the unit interval $[0, 1]$ and $Z = C(\Omega)$ where Ω is the Stone representation space of B. Let A be the commutative W^*-algebra $l^\infty(\Omega)$ of bounded functions on Ω. Then A contains Z as a unital C*-subalgebra. Since Z has no minimal projections, for any minimal projection e of $l^\infty(\Omega)$, there is no largest projection $b \in B$ such that $be = 0$. Thus A is not Z-scalalbe. From a similar consideration, it is easy to see that, from any C*-algebra A which contains Z as a unital C*-subalgebra of the center, the universal atomic representation π of A always produces a C*-algebra $\pi(A)''$ which is not Z-scalable, provided that B is not totally atomic (cf. [24, p. 176]).

Theorem 6.1 generalizes Takeuti's result [30, p. 212, Proposition 3] in weakening his assumptions. Our proofs of statements (2) and (3) of Theorem 6.1 are alternative proofs of the corresponding statements in his result and the proofs of Theorem 6.1 (6) and Theorem 6.5 [(7) $\Rightarrow$ (1)] will compensate for his omission of a proof of the statement $x^*x \leq \|x\|_Z^2$ for all $x \in A$.

§ 7. Dual modules of Banach modules

For a normed linear space X in $V^{(B)}$, denote by X^* the dual space of X in $V^{(B)}$, i.e., $X^* = L(X, C)_B$.

LEMMA 7.1. *For any normed linear space X in $V^{(B)}$, $(X^*)^\wedge \cong (\hat{X})^\#$.*

Proof. It follows from Theorem 4.3 that $L(X, C)^\wedge$ is Z-isometrically Z-isomorphic to $\mathrm{Hom}_Z(\hat{X}, \hat{C})$. Since $\hat{C} = Z$, we have

$$(X^*)^\wedge = L(X, C)^\wedge \cong \mathrm{Hom}_Z(\hat{X}, Z) = (\hat{X})^\# . \qquad \text{Q.E.D.}$$

Let X be a normed Z-module. For any $f \in X^\#$ we can define a Z-module action by $af(x) = a(f(x))$, for all $a \in Z$ and $x \in X$. Then $X^\#$ is a Z-module.

LEMMA 7.2. *Let X be a normed Z-module. For any $f \in X^\#$ define $\|f\|_Z$ by*

$$\|f\|_Z = \sup\{|f(x)| \mid \|x\| \leq 1,\ x \in X\} .$$

Then $X^{\#}$ is a Kaplansky-Banach Z-module with Z-valued norm $\|\cdot\|_Z$. Further, we have

$$\| \|f\|_Z \|_\infty = \|f\|,$$

for all $f \in X^{\#}$, where $\|f\| = \sup\{\|f(x)\|_\infty \mid \|x\| \leq 1,\ x \in X\}$.

Proof. Since $\|f(x)\|_\infty \leq \|f\|$ for all $x \in X$ with $\|x\| \leq 1$, the supremum $\|f\|_Z$ always exists. It is a matter of routine verification to check that $\|\cdot\|_Z$ is a Z-valued norm and $\| \|f\|_Z \|_\infty = \|f\|$ for all $f \in X^{\#}$. To check condition (K1), let $\{b_i\}$ be a partition of unity in B and $\{f_i\}$ a bounded family in $X^{\#}$. Let f be a function from X to Z such that $f(x) = \sum_i b_i f_i(x)$ for all $x \in X$. Then $\|f\| \leq \sup_i \|f_i\|$ and hence it is easy to see that $f \in X^{\#}$. Thus (K1) holds. Since $\| \|f\|_Z \|_\infty = \|f\|$ for all $f \in X^{\#}$ it follows from a standard argument that $X^{\#}$ is a Banach space. Thus $X^{\#}$ is a Kaplansky-Banach Z-module with Z-valued norm $\|\cdot\|_Z$. Q.E.D.

LEMMA 7.3. *Let X be a Z-normed Z-module. Then for any $f \in X^{\#}$*

$$\|f\|_Z = \inf\{a \in Z \mid |f(x)| \leq a\|x\|_Z,\ a \geq 0,\ x \in X\}.$$

Proof. Since $\|x\| \leq 1$ if and only if $\|x\|_Z \leq 1$ for all $x \in X$, the assertion follows from a standard argument. Q.E.D.

LEMMA 7.4. *Let X be a Z-normed Z-module. For any $x \in X$, we have*

$$\|x\|_Z = \sup\{|f(x)| \mid \|f\| \leq 1,\ f \in X^{\#}\}.$$

Proof. Let $\tilde{X}$ be the Boolean embedding of X into $V^{(B)}$. Let $x \in X$. Then by the Hahn-Banach theorem

$$[\![\exists f \in \tilde{X}^*(f(\tilde{x}) = \|\tilde{x}\|_B \wedge \|f\|_B = 1)]\!] = 1.$$

Thus by Lemma 7.1, there is some $f \in X^{\#}$ such that $f(x) = \|x\|_B$ and $\|f\| = 1$. For any $f \in X^{\#}$ if $\|f\| \leq 1$ then $\|f\|_Z \leq 1$ and $|f(x)| \leq \|x\|_Z$. Therefore the assertion follows. Q.E.D.

PROPOSITION 7.5. *Let X, Y be two Z-normed Z-modules. Every isometric Z-isomorphism T: $X \to Y$ is Z-isometric.*

Proof. Let $x \in X$. Since the correspondence $f \mapsto f \circ T$ is an isometric Z-isomorphism from $Y^{\#}$ onto $X^{\#}$ we have from Lemma 7.4,

$$\begin{aligned} \|Tx\|_Z &= \sup\{|f(Tx)| \mid \|f\| \leq 1,\ f \in Y^{\#}\} \\ &= \sup\{|g(x)| \mid \|g\| \leq 1,\ g \in X^{\#}\} \\ &= \|x\|_Z. \end{aligned}$$

Q.E.D.

By the above result, in the sequel, we shall not distinguish the relation $X \cong_Z Y$ from $X \cong Y$ for Z-normed Z-modules X, Y.

LEMMA 7.6. *Let $X^{\#}$ be the Z-dual of a normed Z-module X. Then there is a Z-normed Z-module Y such that $X^{\#}$ is Z-isometrically Z-isomorphic to $Y^{\#}$.*

Proof (cf. [5, p. 188]). Let X' be the reduced module X/K with the norm $\|x+K\| = \inf\{\|x+k\| \mid k \in K\}$, where $K = \bigcap \{\text{kernel of } f \mid f \in X^{\#}\}$. It is standard to show that $X^{\#}$ is Z-isometrically Z-isomorphic to $X'^{\#}$ under the correspondence $f \mapsto f'$ where $f'(x+K) = f(x)$ for all $x \in X$. By replacing X by X' if necessary, we can assume that $K = \{0\}$. For each $x \in X$, the function $f \mapsto f(x)$ defines an element F_x of $X^{\#\#}$. The correspondence $x \mapsto F_x$ is a Z-isomorphism of X onto a submodule Y of the Z-module $X^{\#\#}$ with $\|F_x\| \leq \|x\|$. Then Y is a Z-normed Z-module. Setting $f'(x) = f(F_x)$ for all $f \in Y^{\#}$ and $x \in X$, we obtain a Z-functional f' of X. It then follows that

$$\begin{aligned}\|f\|_Z &= \sup\{|f(F_x)| \mid \|F_x\| \leq 1,\ F_x \in Y\} \\ &\geq \sup\{|f'(x)| \mid \|x\| \leq 1,\ x \in X\} \\ &= \|f'\|_Z .\end{aligned}$$

By Lemma 7.3, for any $F \in X^{\#\#}$ with $\|F\| \leq 1$,

$$\begin{aligned}|F(f')| &\leq \|F\|_Z \|f'\|_Z \leq \|F\| \|f'\|_Z \\ &\leq \|f'\|_Z .\end{aligned}$$

Consequently,

$$\begin{aligned}\|f'\|_Z &\geq \sup\{|F(f')| \mid \|F\| \leq 1,\ F \in X^{\#\#}\} \\ &\geq \sup\{|f(F_x)| \mid \|F_x\| < 1,\ F_x \in Y\} \\ &= \|f\|_Z ,\end{aligned}$$

whence $\|f'\|_Z = \|f\|_Z$. Therefore, the correspondence $f \mapsto f'$ is a Z-isometric Z-isomorphism from $Y^{\#}$ onto $X^{\#}$. Q.E.D.

§ 8. Proofs of Theorems A–E

In this section, we shall prove Theorems A–E presented in Section 2, using those results obtained in the preceding sections.

THEOREM A. *Let Z be a commutative AW^*-algebra and let A be a C^*-algebra which contains Z as a unital C^*-subalgebra of the center.*

Then A is Z-dual if and only if it is Z-embeddable.

Proof. Suppose that A is Z-embeddable. Then A can be identified with a Z-von Neumann algebra on a faithful Kaplansky-Hilbert Z-module H. By [19, Theorem 2.3], there is a von Neumann algebra $\tilde{A}$ on a Hilbert space $\tilde{H}$ in $V^{(B)}$, such that $(\tilde{A})^\wedge \cong A$. Since $\tilde{A}$ is a von Neumann algebra in $V^{(B)}$, there is a Banach space Y in $V^{(B)}$ such that $\tilde{A}$ is isometrically isomorphic to Y^*. By Theorem 5.5 and Lemma 7.1, A is Z-isometrically Z-isomorphic to the Z-dual $(\hat{Y})^\#$ of the Kaplansky-Banach Z-module $\hat{Y}$. Conversely, suppose that A is the Z-dual of a normed Z-module. Then by Lemma 7.6, we may assume that A is the Z-dual $X^\#$ of a Z-normed Z-module X. Then $\tilde{A}$ is the dual of a normed linear space $\tilde{X}$ in $V^{(B)}$. Thus $\tilde{A}$ is a Banach space in $V^{(B)}$ and, by Theorem 6.2, $\tilde{A}$ is a C*-algebra which is a dual space in $V^{(B)}$, and hence it can be regarded as a von Neumann algebra in $V^{(B)}$. By [19, Theorem 2.3], $(\tilde{A})^\wedge$ is a Z-von Neumann algebra so that it is Z-embeddable. Since A is a Z-dual, it follows from Lemma 7.2 that it is a Kaplansky-Banach module. By Theorem 5.2, $A \cong (\tilde{A})^\wedge$, and hence A is Z-embeddable. Q.E.D.

THEOREM B. *Let Z be a commutative AW^*-algebra and let A be a C^*-algebra with center Z. Then A is Z-bidual if and only if it is a type I AW^*-algebra.*

Proof. Suppose that A is a type I AW*-algebra. Then by [19, Theorem 2.3, Theorem 3.1], there is a type I factor $\tilde{A}$ in $V^{(B)}$ such that A is Z-linearly *-isomorphic to $(\tilde{A})^\wedge$. Since every type I factor is the second dual of some Banach space, there is some Banach space X in $V^{(B)}$ such that $[\![A \cong X^{**}]\!] = 1$. Thus we have $A \cong (\tilde{A})^\wedge \cong (X^{**})^\wedge \cong (\hat{X})^{\#\#}$ by Lemma 7.1, and hence A is the second Z-dual of a normed Z-module $\hat{X}$. Conversely, suppose that A is the second Z-dual $X^{\#\#}$ of a normed Z-module X. By Lemma 7.6, we may assume that X is a Z-normed Z-module. Let $\tilde{A}$ and $\tilde{X}$ be their Boolean embeddings. Since $A \cong X^{\#\#}$, it follows from Theorem 5.4 and Lemma 7.1 that $[\![\tilde{A} \cong \tilde{X}^{**}]\!] = 1$. Thus by Theorem 6.2, $\tilde{A}$ is a W*-algebra in $V^{(B)}$. Since the center of A is Z, $\tilde{A}$ is a W*-factor in $V^{(B)}$ by [19, Theorem 2.3]. Since every second dual C*-algebra has a minimal projection, $\tilde{A}$ is a type I factor. By [19, Theorem 3.1], $(\tilde{A})^\wedge$ is a type I AW*-algebra. Since $A \cong X^{\#\#}$, A is a Kaplansky-Banach Z-module, it follows from Theorem 5.2 that $A \cong (\tilde{A})^\wedge$. Therefore A is a type I AW*-algebra. Q.E.D.

THEOREM C. *Let Z be a commutative AW^*-algebra and let A be a C^*-algebra with center Z. Then A is Z-self-dual if and only if it is a finite type* I *AW^*-algebra.*

Proof. Suppose that A is a finite type I AW*-algebra. Then by [20, Theorem 5, Theorem 7], there is a finite type I factor $\tilde{A}$ in $V^{(B)}$ such that A is Z-linearly *-isomorphic to $(\tilde{A})^\wedge$. Since every finite type I factor is self-dual, we have $[\![\tilde{A} \cong (\tilde{A})^*]\!] = 1$. Thus we have $A \cong (\tilde{A})^\wedge \cong ((\tilde{A})^*)^\wedge \cong ((\tilde{A})^\wedge)^\# \cong A^\#$. It follows that A is Z-self-dual. Conversely, suppose that A is Z-self-dual. Then A is Z-bidual and hence A is a type I AW*-algebra. Thus by [19, Theorem 2.3] there is a type I AW*-factor $\tilde{A}$ in $V^{(B)}$ such that $A \cong (\tilde{A})^\wedge$. Since A is Z-self-dual, we have $[\![\tilde{A} \cong (\tilde{A})^*]\!] = 1$. It follows that $[\![\tilde{A}$ is finite$]\!] = 1$. By [20, Theorem 7], we can conclude that A is finite. Q.E.D.

LEMMA 8.1. *Let A be a unital Z-C^*-algebra and let D be a non-empty increasing directed subset of* $\mathrm{UB}(\tilde{A})_B$ *in $V^{(B)}$. Let $\hat{D} \subseteq A$ be such that $\hat{D} = \{x \in A \mid [\![\tilde{x} \in D]\!] = 1\}$. Then we have the following*:

(1) *$\hat{D}$ is an increasing directed subset of* $\mathrm{UB}(A)$.

(2) *If $u = \sup \hat{D}$ then $[\![\tilde{u} = \sup D]\!] = 1$.*

(3) *If $[\![u = \sup D]\!] = 1$ then there is some $v \in \mathrm{UB}(A)$ such that $v = \sup \hat{D}$ and that $[\![u = \tilde{v}]\!] = 1$.*

Proof. Let $x \in D^{(B)}$, i.e., $[\![x \in D]\!] = 1$. Since $D^{(B)} \subseteq \mathrm{UB}(\tilde{A})^{(B)}$, it follows from Theorem 5.7, that there is some $y \in \mathrm{UB}(A)$ such that $[\![x = \tilde{y}]\!] = 1$. Then obviously $y \in \hat{D}$. From Theorem 5.7, we can conclude that the correspondence $x \mapsto \tilde{x}$ is a bijection from $\hat{D}$ to $D^{(B)}$ such that $x \leq y$ if and only if $[\![\tilde{x} \leq \tilde{y}]\!] = 1$. Thus it is easy to see that $\hat{D}$ is an increasing directed subset of $\mathrm{UB}(A)$ and hence (1) holds. Now it is a matter of routine verification that (2) and (3) follows from the order preserving nature of the bijective correspondence $x \mapsto \tilde{x}$. Q.E.D.

LEMMA 8.2. *Let A be a unital Z-C^*-algebra and let D be a non-empty increasing directed subset of* $\mathrm{UB}(A)$. *Let $\tilde{D} \in V^{(B)}$ be such that $\tilde{D} = \{\tilde{x} \mid x \in D\} \times \{1\}$. Then we have the following*:

(1) *$[\![\tilde{D}$ is an increasing directed subset of* $\mathrm{UB}(\tilde{A})_B]\!] = 1$.

(2) *If $u = \sup D$ then $[\![\tilde{u} = \sup \tilde{D}]\!] = 1$.*

(3) *If $[\![u = \sup \tilde{D}]\!] = 1$ then there is some $v \in \mathrm{UB}(A)$ such that $v = \sup D$ and that $[\![u = \tilde{v}]\!] = 1$.*

Proof. Obviously, $[\![\tilde{D} \subseteq \mathrm{UB}(\tilde{A})_B]\!] = 1$. Since D is an increasing directed set, for any $x, y \in D$, there is some $z \in D$ such that $x \leq z$, $y \leq z$. From Proposition 6.4, we have

$$\inf_{x,y \in D} \sup_{z \in D} [\![\tilde{x} \leq \tilde{z} \wedge \tilde{y} \leq \tilde{z}]\!] = 1 .$$

It follows that

$$[\![(\forall x, y \in \tilde{D})(\exists z \in \tilde{D}) x \leq z \wedge y \leq z]\!] = 1 ,$$

and hence it is easy to see that $\tilde{D}$ is an increasing directed subset of $\mathrm{UB}(\tilde{A})_B$ in $V^{(B)}$. Thus (1) holds. Suppose that $u = \sup D$. Then $u \in \mathrm{UB}(A)$ and obviously $\tilde{u}$ is an upper bound of $\tilde{D}$. Let $x \in \mathrm{UB}(\tilde{A})^{(B)}$ be an upper bound of $\tilde{D}$ in $V^{(B)}$. By Theorem 5.7, there is some $y \in \mathrm{UB}(A)$ such that $[\![x = \tilde{y}]\!] = 1$. It is easily seen that y is an upper bound of D so that $u \leq y$. It follows that $[\![\tilde{u} \leq x]\!] = 1$ and hence $[\![\tilde{u} = \sup \tilde{D}]\!] = 1$. Thus (2) holds. Suppose that $[\![u = \sup \tilde{D}]\!] = 1$. Then $[\![u \in \mathrm{UB}(\tilde{A})_B]\!] = 1$. From Theorem 5.7, there is some $v \in \mathrm{UB}(A)$ such that $[\![\tilde{v} = \sup \tilde{D}]\!] = 1$. For any $x \in D$, we have $[\![\tilde{x} \leq \tilde{v}]\!] = 1$ so that $x \leq v$, and hence v is an upper bound of D. Let $x \in \mathrm{UB}(A)$ be an upper bound of D. Then $\tilde{x}$ is an upper bound of $\tilde{D}$ in $V^{(B)}$ and hence $[\![\tilde{v} \leq \tilde{x}]\!] = 1$ so that $v \leq x$. It follows that $v = \sup D$. Thus (3) holds. Q.E.D.

LEMMA 8.3. *Let A be an Z-embeddable C^*-algebra. For any normal positive Z-functional f on A, $\tilde{f}$ is a normal positive linear functional on $\tilde{A}$ in $V^{(B)}$. Every normal positive linear functional on $\tilde{A}$ in $V^{(B)}$ which is Z-bounded arises in this way.*

Proof. Let f be a normal positive Z-functional on A. Then it is easy to see that $\tilde{f}$ is a positive linear functional on $\tilde{A}$ in $V^{(B)}$. To see that $\tilde{f}$ is normal in $V^{(B)}$, let D be a non-empty increasing directed subset of the unit ball of $\tilde{A}$ in $V^{(B)}$. Let $\hat{D}$ be such that $\hat{D} = \{p \in A \mid [\![\tilde{p} \in D]\!] = 1\}$. Then by Lemma 8.1, $\hat{D}$ is a non-empty increasing directed subset of the unit ball of A and $[\![(\sup \hat{D})^\sim = \sup D]\!] = 1$. By the normality of f, we have $f(\sup \hat{D}) = \sup \{f(p) \mid p \in \hat{D}\}$. It is easy to see that

$$[\![\sup \{\tilde{f}(p) \mid p \in D\} = \sup \{f(p) \mid p \in \hat{D}\}^\sim]\!] = 1 .$$

It follows that

$$[\![\tilde{f}(\sup D) = \sup \{\tilde{f}(p) \mid p \in D\}]\!] = 1 .$$

Thus $\tilde{f}$ is normal in $V^{(B)}$. Conversely let g be a normal positive linear functional on $\tilde{A}$ in $V^{(B)}$ such that $\|g\|_B \in Z$. Then there is a positive Z-functional f on $\tilde{A}$ such that $[\![g = \tilde{f}]\!] = 1$. Let D be an increasing directed subset of the unit ball of A and let $\tilde{D}$ be such that $\tilde{D} = \{\tilde{x} \mid x \in D\} \times \{1\}$. By Lemma 8.2, $\tilde{D}$ is an increasing directed subset of the unit ball of $\tilde{A}$ in $V^{(B)}$ and that $[\![(\sup D)^{\sim} = \sup \tilde{D}]\!] = 1$. Since $\tilde{f}$ is normal in $V^{(B)}$, we have $[\![\tilde{f}(\sup \tilde{D}) = \sup\{\tilde{f}(x) \mid x \in \tilde{D}\}]\!] = 1$. It is easy to see that $f(\sup D)$ is an upper bound of the set $f(D) = \{f(x) \mid x \in D\}$. Let u be an upper bound of $f(D)$. Then for any $x \in f(D)$, $[\![\tilde{x} \leq \tilde{u}]\!] = 1$ and hence $[\![\sup\{\tilde{f}(x) \mid x \in \tilde{D}\} \leq \tilde{u}]\!] = 1$. By the preceding arguments, we have $[\![\tilde{f}((\sup D)^{\sim}) \leq \tilde{u}]\!] = 1$. Thus we have $f(\sup D) \leq u$. It follows that $\sup f(D) = f(\sup D)$ so that f is normal. Q.E.D.

THEOREM D. *Let A be a Z-embeddable* C**-algebra. Then $A_\#$ is a Kaplansky-Banach Z-module and A is the Z-dual of $A_\#$. If A is the Z-dual of another Kaplansky-Banach Z-module X then X is Z-isometrically Z-isomorphic to $A_\#$.*

Proof. Let A be a Z-embeddable C*-algebra. Obviously, $A_\#$ is a Z-submodule of $A^\#$. Consider their Boolean embeddings $(A_\#)^{\sim}$ and $(A^\#)^{\sim}$. Then $[\![(A_\#)^{\sim}$ is a linear subspace of $(A^\#)^{\sim}]\!] = 1$ and $[\![(A^\#)^{\sim} = (\tilde{A})^*]\!] = 1$. By Lemma 8.3, $(A_\#)^{\sim}$ is the linear space generated by all normal positive linear functionals on $\tilde{A}$ in $V^{(B)}$. By a certain theorem of W^*-algebras, we have $[\![(A_\#)^{\sim} = (\tilde{A})_*]\!] = 1$. Now we shall show that $((A_\#)^{\sim})^{\wedge} \cong A_\#$. Let $f \in ((A_\#)^{\sim})^{\wedge}$. Then f is a linear combination of normal positive linear functionals in $V^{(B)}$. By Lemma 8.3, there are four normal positive Z-functionals f_1, f_2, f_3, f_4 such that $[\![f = \tilde{g}]\!] = 1$, where $g = (f_1 - f_2) + i(f_3 - f_4)$. It follows that the correspondence $f \mapsto \tilde{f}$ from $A_\#$ to $((A_\#)^{\sim})^{\wedge}$ is surjective and hence from Theorem 5.2, $((A_\#)^{\sim})^{\wedge} \cong A_\#$. Thus $A_\#$ is a Kaplansky-Banach module and A is the Z-dual of $A_\#$. Suppose that A is the dual of another Kaplansky-Banach Z-module X. Then we have $[\![\tilde{A}$ is the dual of $\tilde{X}]\!] = 1$. By the uniqueness theorem of preduals of W*-algebras, we have $[\![\tilde{X}$ is isometrically isomorphic to $\tilde{A}_*]\!] = 1$. Since X is a Kaplansky-Banach Z-module, the following relations holds:

$$X \cong (\tilde{X})^{\wedge} \cong (\tilde{A}_*)^{\wedge} = ((A_\#)^{\sim})^{\wedge} \cong A_\# .$$

Thus X is Z-isometrically Z-isomorphic to $A_\#$. Q.E.D.

THEOREM E. *Let A be a Z-embeddable C^*-algebra for an AW^*-subalgebra Z of the center of A. Then A is Z-embeddable for every AW^*-*

subalgebra of the center with $Z_0 \subseteq Z$.

Proof. Let A be a Z_0-embeddable C*-algebra for an AW*-subalgebra Z_0 of the center of A. Let Z be an AW*-subalgebra of the center of A such that $Z_0 \subseteq Z$. By the assumption, we can assume that A is an AW*-subalgebra of a type I AW*-algebra L with center Z_0. Let Z' be the commutant of Z in L. Since Z is an AW*-subalgebra of L which contains the center of L, it follows from [19, Theorem 3.1] that Z' is of the same type as Z and hence Z' is a type I AW*-algebra. Since Z is commutative, the center of Z' is Z. Since Z is contained in the center of A, we have $A' \cap Z' = A'$ and that $(A' \cap Z')' \cap Z' = A'' \cap Z' = A \cap Z' = A$ so that the bicommutant of A in Z' is A. Thus A is Z-embeddable. Q.E.D.

COROLLARY 8.4. *Let A be a von Neumann algebra. Then A is Z-embeddable for any AW*-subalgebra Z of the center of A.*

Proof. Since every von Neumann algebra is $\boldsymbol{C}$-embeddable AW*-algebra, the assertion follows immediately from Theorem E. Q.E.D.

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