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THE VOLUMES OF SMALL GEODESIC BALLS AND GENERALIZED CHERN NUMBERS OF KAEHLER MANIFOLDS

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§ 1. Introduction

In this paper we study a connection between global and local properties of Kaehler manifolds, more specifically we study a connection between the volumes of small geodesic balls of a manifold M and some generalized Chern numbers. We use the standard power series expansion for $V_m(r)$.

In Theorem 3.1 we give characterizations of a flat compact Kaehler manifold in terms of the volumes of small geodesic balls and generalized Chern numbers $\omega^{n-1}c_1(M)$ and $\omega^{n-2}c_1^2(M)$. In Theorem 4.1 similar questions for complex space forms are considered. So we prove one particular case of the Conjecture (IV) stated by Gray and Vanhecke [6].

In Section 5 we introduce geodesically-Einstein manifolds and then generalize some well known results about Einstein-Kaehler manifolds. Chen and Ogiue [3] obtained the following inequality for a compact Einstein-Kaehler manifold (M, g)

$$\int_M \{2(n+1)c_2 - nc_1^2\} \wedge \omega^{n-2} \geq 0 .$$

So in Theorem 5.1 we prove that the same inequality also holds for geodesically-Einstein compact Kaehler manifolds. Then, some consequences of this inequality for complex surfaces are given. Also, we give examples of some complex surfaces which admit no geodesically-Einstein Kaehler metrics.

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§ 2. Preliminaries

In this paper we use the notations given in [6] and [3]. Let M be

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an n-dimensional analytic Riemannian manifold. Let $r_o > 0$ be so small that the exponential map $\exp_m$ is a diffeomorphism on a ball of radius r_o in the tangent space M_m. We put

$$S_m(r_0) = \text{volume of } \{\exp_m(x) \mid x \in M_m,\ \|x\| = r_o\},$$
$$V_m(r_0) = \text{volume of } \{\exp_m(x) \mid x \in M_m,\ \|x\| \leq r_o\}.$$

Here we mean the $(n-1)$-dimensional volume for $S_m(r_o)$ and the n-dimensional volume for $V_m(r_o)$.

In [6] it is shown (Theorem 3.3) that for $V_m(r)$ and $S_m(r)$ the following power series expansions hold

$$V_m(r) = \Omega_n r^n(1 - Ar^2 + Br^4 + O(r^6)) \tag{2.1}$$

where

$$A = \frac{\tau}{6(n+2)},$$
$$B = \frac{1}{360(n+2)(n+4)}(-3\|R\|^2 + 8\|\rho\|^2 + 5\tau^2 - 18\Delta\tau)$$

and

$$S_m(r) = C_n r^{n-1}(1 - Cr^2 + Dr^4 + O(r^6)) \tag{2.2}$$

where

$$C = \frac{n+2}{n}A, \qquad D = \frac{n+4}{n}B.$$

(Here Ω_n is the volume of the unit ball in $\boldsymbol{R}^n$ and C_n is the $(n-1)$-dimensional volume of the unit Euclidean sphere S^{n-1}. In this case $C_n = n\Omega_n = n\pi^{n/2}(1/\Gamma(n/2+1))$.)

Suppose that M is a Kaehler manifold of complex dimension n. Let $\theta^1, \cdots, \theta^n$ be a local field of unitary coframes. Then the Kaehler metric is written as $g = \sum (\theta^\alpha \otimes \bar{\theta}^\alpha + \bar{\theta}^\alpha \otimes \theta^\alpha)$ and the fundamental 2-form $\phi(X, Y) = g(X, JY)$ is given by $\phi = \sqrt{-1}\sum \theta^\alpha \wedge \bar{\theta}^\alpha$. Here, in Section 2, we use the ranges $\alpha, \beta, \gamma, \delta, \cdots = 1, \cdots, n$. The form ϕ is closed. The fundamental class ω of M is the de Rham cohomology class determined by ϕ. The curvature tensor R of M is the tensor field with local components $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$. Then the (1, 1)-forms Ω^α_β, defined by $\Omega^\alpha_\beta = \sum R^\alpha_{\beta\gamma\bar{\delta}}\theta^\gamma \wedge \bar{\theta}^\delta$, are closed. The Ricci tensor ρ and the scalar curvature τ are given by $\rho_{\alpha\bar{\beta}} = \sum R_{\alpha\bar{\gamma}\gamma\bar{\beta}}$ and $\tau = 2\sum \rho_{\alpha\bar{\alpha}}$. We denote by $\|R\|$ and $\|\rho\|$ the length of the curvature tensor and the Ricci tensor respectively, so that

$$\|R\|^2 = 4\sum R_{\alpha\bar\beta\gamma\bar\delta}R_{\beta\bar\alpha\delta\bar\gamma} \quad \text{and} \quad \|\rho\|^2 = 2\sum \rho_{\alpha\bar\beta}\rho_{\beta\bar\alpha}\,.$$

We need the following general result.

LEMMA 2.1 ([3]). *Let M be an n-dimensional Kaehler manifold. Then*

$$\frac{n(n+1)}{2}\|R\|^2 \geq 2n\|\rho\|^2 \geq \tau^2\,.$$

The first equality holds if and only if M is a complex space form and the second equality holds if and only if M is Einstein.

We define a closed $2k$-form γ_k by

$$\gamma_k = \frac{(-1)^k}{(2\pi\sqrt{-1})^k k!}\sum \delta^{\alpha_1\cdots\alpha_k}_{\beta_1\cdots\beta_k}\Omega^{\beta_1}_{\alpha_1}\wedge\cdots\wedge\Omega^{\beta_k}_{\alpha_k}\,.$$

It is well known that k-th Chern class c_k is determined by the form γ_k. In particular, the first two Chern forms are given by

$$2\pi\gamma_1 = \sqrt{-1}\sum\Omega^\alpha_\alpha$$

and

$$-8\pi^2\gamma_2 = \sum(\Omega^\alpha_\alpha\wedge\Omega^\beta_\beta - \Omega^\alpha_\beta\wedge\Omega^\beta_\alpha)$$

respectively.

Then we have

$$\gamma_1\wedge\phi^{n-1} = \frac{\tau}{n\pi}\phi^n\,, \tag{2.3}$$

$$\gamma_1^2\wedge\phi^{n-2} = \frac{1}{4n(n-1)\pi^2}(\tau^2 - 2\|\rho\|^2)\phi^n \tag{2.4}$$

and

$$\gamma_2\wedge\phi^{n-2} = \frac{1}{8n(n-1)\pi^2}(\tau^2 - 4\|\rho\|^2 + \|R\|^2)\phi^n\,. \tag{2.5}$$

The generalized Chern numbers $\omega^{n-2}c_1(M)$, $\omega^{n-2}c_1^2(M)$ and $\omega^{n-2}c_2(M)$ are defined by $\int_M\gamma_1\wedge\phi^{n-1}$, $\int_M\gamma_1^2\wedge\phi^{n-2}$, and $\int_M\gamma_2\wedge\phi^{n-2}$ respectively.

§ 3. Characterization of flat Kaehler manifolds

THEOREM 3.1. *Let (M, g, J) be a compact, Kaehler manifolds of complex dimension n. Suppose that generalized Chern numbers $\omega^{n-1}c_1$ and*

$\omega^{n-2}c_1^2$ *are nonnegative. Then, if* M *satisfies one of the following conditions,* (i) *or* (ii),

(i) $V_m(r) \geq \Omega_{2n}r^{2n}$

(ii) $2nV_m(r) \leq rS_m(r)$

M *is biholomorphically covered by* C^n.

Proof. We will show first that $\omega^{n-1}c_1(M) \geq 0$, $\omega^{n-2}c_1^2(M) \geq 0$ and the condition (i) imply the result. Because of (i)

$$\tau \leq 0 \quad \text{on } M. \tag{3.1}$$

Then $\omega^{n-1}c_1(M) \geq 0$, $\omega^{n-2}c_1^2(M) \geq 0$ and the relations (2.3) and (2.4) give

$$\int_M \gamma_1 \wedge \phi^{n-1} = \frac{1}{n\pi}\int_M \tau\phi^n \geq 0 \tag{3.2}$$

and

$$\int_M \gamma_1^2 \wedge \phi^{n-2} = \frac{1}{4n(n-1)\pi^2}\int_M (\tau^2 - 2)\|\rho\|^2)\phi^n \geq 0 . \tag{3.3}$$

Since τ is nonpositive, (3.2) implies $\tau = 0$ on M. Because of (3.3), $\rho = 0$ on M and from (i) we have

$$-3\|R\|^2 + 8\|\rho\|^2 + 5\tau^2 - 18\Delta\tau = -3\|R\|^2 \geq 0 .$$

So $R = 0$ on M and M is biholomorphically covered by C^n.

If we take the condition (ii) instead of (i) the proof will go in a similar way.

COROLLARY 3.1. *Let* M *be a Kaehler manifold as in the Theorem* 3.1. *If the first Chern class* $c_1(M)$ *vanishes and if it satisfies one of the two conditions,* (i) *or* (ii), *then* M *is biholomorphically covered by* C^n.

§4. Characterization of Kaehler spaces of constant holomorphic curvature

Let $M(\mu)$ be a Kaehler manifold with complex dimension n and constant holomorphic sectional curvature $\mu \neq 0$. Then for all $p \in M(\mu)$ the volume function for $M(\mu)$ is given by;

$$V_p(r, \mu) = \frac{(4\pi)^n}{n!\,\mu^n}\left\{\sin\frac{\sqrt{\mu}}{2}r\right\}^{2n}$$

or

$$V_p(r, \mu) = \frac{(4\pi)^n}{n!\,|\mu|^n}\left\{\sinh\frac{\sqrt{|\mu|}}{2}r\right\}^{2n}$$

according to whether $\mu > 0$ or $\mu < 0$ (see [4]). In [6] the following conjecture was stated;

(IV) *Let M be a Kaehler manifold with complex dimension n and suppose that for all $m \in M$ and all sufficiently small $r > 0$, $V_m(r)$ is the same as that of an n-dimensional Kaehler manifold with constant holomorphic sectional curvature μ. Then M has constant holomorphic sectional curvature.*

In the following theorem we will prove one particular case of the conjecture (IV).

THEOREM 3.2. *Let M be a compact Kaehler manifold with complex dimension n, and suppose that for all $m \in M$ and all sufficiently small $r > 0$, $V_m(r)$ is the same as that of an n-dimensional compact Kaehler manifold $M(\mu)$ with constant holomorphic sectional curvature μ. Let ω and ω_μ denote the fundamental classes of M and $M(\mu)$ respectively. If the following conditions*

$$\omega^{n-1}c_1(M) = \omega_\mu^{n-1}c_1(M(\mu)) , \tag{4.1}$$

$$\omega^{n-2}c_1^2(M) \geq \omega_\mu^{n-2}c_1^2(M) \tag{4.2}$$

are satisfied, then M has constant holomorphic sectional curvature μ.

Proof. Let τ_μ, $\|\rho_\mu\|^2$ and $\|R_\mu\|^2$ denote the appropriate functions for $M(\mu)$. Since $V_m(r) = V(r, \mu)$ we have

$$\tau = \tau_\mu \tag{4.3}$$

and

$$3(\|R_\mu\|^2 - \|R\|^2) = 8(\|\rho_\mu\|^2 - \|\rho\|^2) \leq 0 . \tag{4.4}$$

The hypotheses (i) and (ii) imply that

$$\int_M \tau\phi^n = \int_{M(\mu)} \tau_\mu \phi_\mu^n \tag{4.5}$$

and

$$\int_M (\tau^2 - 2\|\rho\|^2)\phi^n \geq \int_{M(\mu)} (\tau_\mu^2 - 2\|\rho_\mu\|^2)\phi_\mu^n . \tag{4.6}$$

For $\mu = 0$, from (4.3), (4.6) and (4.4) it follows that $\tau = \|\rho\| = \|R\| = 0$ on on M. So, in this case M is flat as we want to show. For $\mu \neq 0$ formulas (4.3) and (4.5) imply that

$$\int_M \phi^n = \int_{M(\mu)} \phi_\mu^n .$$

Then, using (4.4) and (4.6), we obtain

$$\int_M \|\rho\|^2 \phi^n \leq \int_M \|\rho_\mu\|^2 \phi^n .$$

This inequality, Lemma 2.1 and (4.4) give

$$\int_M \left(\|R\|^2 - \frac{4}{n+1}\|\rho\|^2\right)\phi^n = \frac{4}{3}\left(\frac{3}{n+1} - 2\right)\int_M (\|\rho_\mu\|^2 - \|\rho\|^2)\phi^n \leq 0 .$$

So $\|R\|^2 = (4/(n+1))\|\rho\|^2$ on M and the required result follows from Lemma 2.1.

COROLLARY 4.1. *Let $(M(\mu), g_\mu, J_\mu)$ be a compact n-dimensional Kaehler manifold with constant holomorphic sectional curvature μ, fundamental 2-class ω_μ and almost complex structure J_μ. Suppose that $(M(\mu), g)$ is a Kaehler manifold with fundamental 2-class ω and almost complex structure J. If*

(i) $V_m(r) \geq V(r, \mu)$ *for all $m \in M(\mu)$ and all sufficiently small $r > 0$,*

(ii) $\omega = \omega_\mu$,

(iii) $J = J_\mu$,

then M has constant holomorphic sectional curvature μ.

§ 5. Geodesically-Einstein Kaehler manifolds

DEFINITION 5.1. Let M and M_ε be Riemannian manifolds of the same dimension. We say that M is *geodesically-Einstein with respect to the Einstein manifold M_ε* if there exists a map f: $M \to M_\varepsilon$ such that

$$V_m(r) = V_{f(m)}(r) \tag{5.1}$$

for all $m \in M$ and for all sufficiently small $r > 0$.

It is to expect that geodesically-Einstein manifolds have some similar properties as Einstein manifolds. So, in this section we establish an inequality between Chern classes of geodesically-Einstein Kaehler manifolds. Also geodesically-Einstein Kaehler surfaces are considered.

THEOREM 5.1. *Let M and M_ε be compact, n-dimensional, $n \geq 2$, Kaehler manifolds as it was supposed in the Definition 5.1. If M is geodesically-Einstein with respect to M_ε, then*

$$\int_M \left\{ \gamma_2 - \frac{n}{2(n+1)} \gamma_1^2 \right\} \Lambda \phi^{n-2} \geq 0 . \tag{5.2}$$

For $n \geq 3$ the equality holds if and only if M is a complex space form. For $n = 2$, if M_ε is a homogeneous manifold, the equality holds if and only if M_ε is a complex space form.

Proof. Let $\|R_\varepsilon\|^2$, $\|\rho_\varepsilon\|^2$ and τ_ε denote the appropriate functions for the Einstein-Kaehler manifold M_ε. Since τ_ε is constant on M_ε, Lemma 2.1, (2.1) and (5.1) imply

$$\tau = \tau_\varepsilon \tag{5.3}$$

and

$$3(\|R\|^2 - \|R_\varepsilon\|^2) = 8(\|\rho\|^2 - \|\rho_\varepsilon\|^2) \geq 0 . \tag{5.4}$$

Thus

$$\begin{aligned} 8n(n-1)\pi^2 & \int_M \left(\gamma_2 - \frac{n}{2(n+1)} \gamma_1^2 \right) \wedge \phi^{n-2} \\ & = \int_M \left(\|R_\varepsilon\|^2 - \frac{4}{n+1} \|\rho_\varepsilon\|^2 \right) \phi^n + \frac{2(n-2)}{3(n+1)} \int_M (\|\rho\|^2 - \|\rho_\varepsilon\|^2) \phi^n \geq 0 . \end{aligned}$$

If the equality holds, then $(n+1)\|R_\varepsilon\|^2 = 4\|\rho_\varepsilon\|^2$ on $f(M)$ and for $n \geq 3$, $\|\rho\|^2 = \|\rho_\varepsilon\|^2$. Then $(n+1)\|R\|^2 = 4\|\rho\|^2$ on M by (5.4). Hence, for $n \geq 3$, M is a complex space form because of Lemma 2.1.

Remark. The proof of this result utilizes only the first three non-trivial terms in the power series expansion of $V_m(r)$.

EXAMPLE. Here we will give example of non-Einstein Kaehler manifold M for which

$$V_m(r) = V(r, M_3) + O(r^{4p+6}) \tag{5.5}$$

holds for all $m \in M$ and all small enough $r > 0$. Here M_3 is a complex space form of complex dimension $2p$, $p \geq 2$, and $V(r, M_3)$ is the volume of a geodesic ball of radius r in M_3. So let M_1 and M_2 be complex space forms of complex dimension p, with scalar curvatures equal to τ_1 and τ_2 respectively. Let M_3 have scalar curvature $\tau_1 + \tau_2$. Suppose that $\tau_2 = a\tau_1$ where $(1-p)(1+4p)a^2 - 2(1+p)(1-4p)a = (p-1)(1+4p)$. Then for $M = M_1 \times M_2$ we have (5.5). Since $\tau_1 \neq \tau_2$, $M_1 \times M_2$ is not an Einstein manifold. Due to last remark inequality (5.2) holds for $M = M_1 \times M_2$.

We consider now the consequence of this theorem for a compact Kaehler surface M which satisfies (5.1). Let χ, σ and a denote its Euler characteristic, Hirzebruch signature and arithmetic genus respectively. Then from the Gauss-Bonnet-Chern theorem, the Hirzebruch signature theorem and the Riemann-Roch-Hirzebruch theorem (see [1], [2], [7] and [8]), we have

$$\chi(M) = \int_M c_2\,,$$
$$\sigma(M) = \frac{1}{3}\int_M (c_1^2 - 2c_2)\,,$$
$$a(M) = \frac{1}{12}\int_M (c_1^2 + c_2)\,.$$

Since

$$\chi(M) - 3a(M) = a(M) - \sigma(M) = \frac{1}{4}\int_M (3c_2 - c_1^2) \geq 0$$

we have the following corollary.

COROLLARY 5.1. *Let M be a compact Kaehler surface satisfying the hypotheses of the Theorem* 5.1. *Then*

(i) $\chi(M) \geq 3a(M)$ *and*

(ii) $a(M) \geq \sigma(M)$.

The equality holds in (i) *or* (ii) *if and only if M_ε has constant holomorphic sectional curvature on $f(M) \subset M_\varepsilon$.*

Remark. This corollary is a generalization of the Theorem 10.4 in [6].

THEOREM 5.2. *Let M be a complex surface. Then any surface $\overline{M}$ obtained from M by blowing up k points of M admits no geodesically-Einstein Kaehler metric whenever either*

$$k < \sigma - a \quad or \quad k < \frac{1}{4}(3\sigma - \chi)$$

where σ, a and χ denote the Hirzebruch signature, the arithmetic genus and the Euler characteristic of M.

Proof. Since the arithmetic genus is a birational invariant, the surfaces M and $\overline{M}$ have the same arithmetic genus. On the other hand, topologically, blowing up a point on a surface is equivalent to attaching

CP^2 with opposite orientation (we denote this by $C\bar{P}^2$). Since $\bar{M}$ is obtained from M by blowing up k points of M, $\bar{M}$ is diffeomorphic to the direct sum $M \# kC\bar{P}^2$. Here $\#$ denotes the direct sum of topological spaces. Since we have

$$\sigma(M \# kC\bar{P}^2) = \sigma(M) - k ,$$

and

$$\chi(M \# kC\bar{P}^2) = \chi(M) + k ,$$

this theorem then follows from Corollary 5.1.

Now we can apply Corollary 5.1 on $M = CP^2 \# n = CP^2 \# \cdots \# CP^2$.

COROLLARY 5.2. *The manifold $M = CP^2 \# n$ does not admit a geodesically-Einstein Kaehler metric for $n > 1$.*

Proof. We have $\sigma(M) = n$ and $\chi(M) = n + 2$. Hence

$$\chi(M) - 3\sigma(M) = -2(n-1) < 0 \qquad \text{for } n > 1 .$$

If the required metric exists, then we obtain a contradiction with Corollary 5.1. We should notice that for even n, M does not admit almost complex structure because $\chi + \sigma$ is not multiple of 4.

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