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ERGODIC PROPERTIES OF THE STEPPING STONE MODEL

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§ 1. Introduction

The purpose of this paper is to discuss some ergodic properties of *the stepping stone model* proposed by Kimura, M. [4] and developed by Weiss, G.H. and Kimura, M. [12]. Our model to be discussed in this paper involves selection in addition to mutation and migration which are dealt with in [4], [12]. Because of the additional factor selection, the stochastic process describing our model becomes complicated and presents particularly interesting profound structure of the random phenomena in question.

To describe the model mathematically, basic terminologies and notations are now introduced. The d-dimensional integer lattice Z^d is denoted by X, which is considered as the set of colonies. We assume that all the colonies have the same population size N. The set of possible states of gene frequency is therefore given by $G = \{k/(2N);\ k = 0, 1, \cdots, 2N\}$. Set

$$S = G^X\,,$$

which is the set of sequences of gene frequencies. The Markov process $\{\boldsymbol{p}(n);\ n \geq 0\}$ on S describing our model is defined by the transition probabilities $Q(\boldsymbol{p}, A)$, prescribed below, in the following manner:

We consider the alleles A_1, A_2, and by the gene frequency we mean frequency of genes of the allele A_1. We assume that the allele A_1 mutates to the allele A_2 and A_2 mutates to A_1 with rates u, v $(0 \leq u, v \leq 1)$, respectively, and that migration into x occurs from colony z with rates $\lambda_{xz}(\sum_z \lambda_{xz} = 1, \lambda_{yz} \geq 0)$, and that selection occurs in any colony x, in which relative fitness of A_1 and A_2 are $1 + s_x/2$ and $1 - s_x/2$ $(-2 < s_x < 2)$, respectively. Then we can define a map H from S into $[0, 1]^X$ by

$$(1.1)\qquad H(\boldsymbol{p})_x = \frac{(1 + s_x/2)\{(1 - u - v)\sum_z \lambda_{xz} p_z + v\}}{s_x\{(1 - u - v)\sum_z \lambda_{xz} p_z + v\} + 1 - s_x/2}\,,$$

$$x \in X, \text{ for } \boldsymbol{p} = \{p_x\}_{x \in X} \in S\,.$$

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Given a finite subset Y of X and integers k_x, $x \in Y$, let A be the cylinder subset of S determined by them; namely $A = \{\boldsymbol{p} = \{p_x\}_x \in S;\ p_x = k_x/(2N);\ x \in Y\}$. Define a transition probability $Q(\boldsymbol{p}, A)$ by

$$Q(\boldsymbol{p}, A) = \prod_{x \in Y} \binom{2N}{k_x} H(\boldsymbol{p})_x^{k_x} (1 - H(\boldsymbol{p})_x)^{2N-k_x} .$$

The Markov process $\{\boldsymbol{p}(n);\ n \geq 0\}$ is defined by an initial probability measure μ and the transition probability $Q(\boldsymbol{p}, A)$. Then we have

$$P_\mu(\boldsymbol{p}(n+1) \in A \,|\, \boldsymbol{p}(n) = \boldsymbol{p}) = Q(\boldsymbol{p}, A) .$$

This means that for a given sequence of the gene frequencies $\boldsymbol{p}$ the genes of the next generation are randomly chosen with the probability $H(\boldsymbol{p})_x$ from the colony x. The probability is one of the important characteristics of the model. The model involving selection has been investigated by Nagylaki, T. [9], [10], Maruyama, T. [7], Itatsu, S. [3], and Shiga, T. and Uchiyama, K. [11], in which the properties of the measures are treated, in terms of Markov process.

We are particularly interested in the case where $u > 0$, $v = 0$, $\{\lambda_{xz}\}$ is homogeneous, $\lambda_{xz} = \lambda_{0,z-x}$, and $s_x = s$ for some constant s. In the following only this case will be considered.

Our first aim is to investigate the asymptotic properties of the right and left extremes of linearly ordered colonies, where A_1 genes survive, starting from some special initial states. First of all, our problem of obtaining limit theorems for the Markov chain $\{\boldsymbol{p}(n) \equiv \{p_x(n)\};\ n \geq 0\}$ is paraphrased to limit theorems for the spin system on state space $\{0, 1\}^{\{1,\cdots,2N\} \times X}$, which has been discussed in the author's paper [3]. After that, we introduce a generalized percolation process defined by independent random variables, which has the same law as the spin system. The process plays an essential role in the proofs of Theorems 1 and 2. In particular, the subadditive ergodic theorem discussed in [1] is ready to be applied with the help of the generalized percolation process.

Our second aim is to investigate the ergodic properties of the stepping stone model, which have been partially discussed in [3]. We will obtain much finer results than in the author's previous paper [3]. The convergence theorem for $\boldsymbol{p}(n)$ with translation invariant initial distributions is clarified and the limit distribution is shown to be a convex combination of two extremal invariant measures.

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§ 2. Theorems

Our main results can be summarized in the following theorem: Theorems 1–3. We assume $d = 1$ in Theorems 1 and 2. Set $r_n = \sup\{x; p_x(n) \neq 0\}$ and $\ell_n = \inf\{x; p(n) \neq 0\}$. We tacitly understand that $r_n = -\infty$ and $\ell_n = +\infty$ if $\{x; p_x(n) \neq 0\} = \emptyset$. Denote the indicator function of a set I by χ_I. Then we have

THEOREM 1. *Assume that*

$$\text{the number of } \{x; \lambda_{0x} \neq 0\} \geq 2 \tag{2.1}$$

and that

$$\textstyle\sum_x \lambda_{0x}|x|^2 < +\infty \,. \tag{2.2}$$

(a) *If $p_x(0) = \chi_{(-\infty,0]}(x)$ for the initial state $\boldsymbol{p}(0) = \{p_x(0)\}_x$, then there exists $\alpha \in [-\infty, \infty)$ such that*

$$\lim_{n\to\infty} r_n/n = \alpha \quad a.s\,.$$

If $p_x(0) = \chi_{[0,\infty)}(x)$, then there exists $\beta \in (-\infty, \infty]$ such that

$$\lim_{n\to\infty} \ell_n/n = \beta \quad a.s\,.$$

(b) *Write $\alpha(s) = \alpha$, and $\beta(s) = \beta$, expressing the dependence on s. Then, for any compact set $K \subset [0, 2)$, there exists $c_K > 0$ such that*

$$\alpha(s) - \alpha(s') \geq c_K(s - s') \quad and \quad -\beta(s) + \beta(s') \geq c_K(s - s')$$

for $s, s' \in K$ satisfying $s > s'$. There exists $s_0 \in (0, 2)$ such that

$$\begin{aligned} &\alpha(s) > \beta(s) \qquad \text{for } s > s_0\,, \\ &\alpha(s) < \beta(s) \qquad \text{for } s < s_0\,, \quad and \quad \alpha(s_0) \geq \beta(s_0)\,. \end{aligned}$$

If the assumption (2.2) is replcaced by

$$\textstyle\sum_x |x|^{3+\theta}\lambda_{0x} < \infty \qquad \text{for some } \theta > 0\,, \tag{2.3}$$

then

$$\alpha(s) = -\infty \quad and \quad \beta(s) = \infty \qquad for\ s < s_0\,.$$

Remark 1. The assumption (2.1) is not an essential restriction, because the case where the number of $\{x; \lambda_{0x} \neq 0\} = 1$ is trivial.

Remark 2. Let p_c be the critical value of the oriented bond percolation. Precisely speaking, $p > p_c$ implies that there exists an infinite open path starting the origin 0 with positive probability and $p < p_c$ implies that, with probability 1, there is not any infinite open path starting 0. Using this constant p_c, we can get an estimate on the constant s_0 in Theorem 1 as follows. Let $x_1 < x_2$ be arbitrary distinct elements of the set $\{x; \lambda_{0x} \neq 0\}$. Then

$$s_0 \leq 2(2p_c - p_c^2 - b)/\{2p_c - p_c^2 - (1 - 4p_c + 2p_c^2)b\}$$

where $b = (1 - u)(1/(2N)) \min\{\lambda_{0x_1}, \lambda_{0x_2}\}$.

Remark 3. The constants $\alpha(s)$, $\beta(s)$ can be estimated as follows.

$$\alpha(s) \geq \{(x_2 - x_1)\alpha_0(p) - x_1 - x_2\}/2\,,$$
$$\beta(s) \leq \{-(x_2 - x_1)\alpha_0(p) - x_1 - x_2\}/2\,,$$

when

$$s > 2(2p - p^2 - b)/\{2p - p^2 - (1 - 4p + 2p^2)b\}\,.$$

Here, x_1, x_2 are the same as in Remark 2, and $\alpha_0(p)$ will be explained in the proof of Theorem 1.

Remark 4. Suppose that the migration matrix $\{\lambda_{xz}\}$ is symmetric. Then we see that

$$\beta(s) = -\alpha(s)\,.$$

Hence, the inequality

$$\alpha(s) > \beta(s)$$

is equivalent to $\alpha(s) > 0$, and the inequality

$$\alpha(s) < \beta(s)$$

holds if $\alpha(s) = -\infty$.

If (2.2) is strengthened by (2.3), then

$$\alpha(s) < \beta(s)$$

holds if and only if $\alpha(s) = -\infty$.

THEOREM 2. *Under the assumption of Theorem* 1, *let the initial state* $p(0)$ *be given by* $p_x(0) = (1/(2N))\chi_{\{0\}}(x)$. *Set*

$$\Omega_0 = \{\liminf_{n\to\infty} r_n/n > \limsup_{n\to\infty} l_n/n\}\,.$$

Then there exists $s_1 \in [s_0, 2)$ *such that* Ω_0 *has a positive probability for* $s > s_1$, *and probability zero for* $s < s_1$.

Let $d \geq 1$, and let $\mathscr{S}$ be all translation invariant measures on S. Let $\mathbf{0}$ and $\mathbf{1}$ be elements of S defined by $p_x(0) = 1$ for all $x \in X$ and $p_x(0) = 1$ for $x \in X$, respectively. Then we have

THEOREM 3. *Suppose the random walk on* Z^d *with transition probabilities* $\{\lambda_{xy}\}$ *is irreducible and aperiodic. Then for any* $\mu \in \mathscr{S}$,

$$\lim_{n\to\infty} \mu Q^n = \alpha\delta_{\mathbf{0}} + (1-\alpha)\nu$$

where $\alpha = \mu\{\mathbf{0}\}$ *and* ν *is the invariant measure given by*

$$\nu = \lim_{n\to\infty} \delta_{\mathbf{1}} Q^n .$$

In particular $\{\mu; \mu Q = \mu\} \cap \mathscr{S}$ *are the convex combinations of* $\delta_{\mathbf{0}}$ *and* ν.

Theorem 3 does not exclude the possibility $\nu = \delta_{\mathbf{0}}$ in general. However the case $\nu \neq \delta_{\mathbf{0}}$ which seems to be of main interest has been discussed in [3]. We have now obtained limit theorems which bear characteristic features of each case. Now we pause to review the result in [3].

Let

$$S = G^X , \qquad X = Z^d ,$$

as mentioned above. Assume that the matrix $\{\lambda_{xz}\}$ is given by

$$\lambda_{xz} = \lambda_{0,z-x} = \begin{cases} m & \text{if } |x-z| = 1 \\ 1-2dm & \text{if } x = z \\ 0 & \text{otherwise}, \end{cases}$$

where $0 < m < 1/(2d)$. Then, there exists $s_c \in [2u(2-u)^{-1}, 2)$ such that the stepping stone model is ergodic for $s < s_c$ and not ergodic for $s > s_c$.

Combining this with Theorem 3, we obtain the next statement: If $s < s_c$, then for any probability measure μ on S

$$\lim_{n\to\infty} \mu Q^n = \delta_{\mathbf{0}} ,$$

while, if $s > s_c$, then the invariant measure ν is distinct from $\delta_{\mathbf{0}}$ and, for any $\mu \in \mathscr{S}$,

$$\lim_{n\to\infty} \mu Q^n = \alpha\delta_{\mathbf{0}} + (1-\alpha)\nu ,$$

where $\alpha = \mu\{\mathbf{0}\}$.

Furthermore, the relation between the critical values s_0 and s_c is

$$s_0 \leq s_c\,,$$

because the inequality $\alpha(s) < 0$ obviously implies the equality

$$\lim_{n\to\infty} \mu Q^n = \delta_0\,.$$

§ 3. Ergodic theorems on discrete time spin systems

For the proofs of Theorems 1, 2 we need to introduce *the discrete time spin systems* defined by Liggett, T [5] and apply the ergodic theorems for them to the stepping stone model. Suppose that W is a countable set and let $\rho_w(\eta)$ be a function on $W \times \{0, 1\}^W$ and satisfy $0 \leq \rho_w(\eta) \leq 1$. The spin system η_n on $\{0, 1\}^W$ corresponding to $\{\rho_w(\eta)\}$ is the discrete time Markov chain defined by the transition law

$$P^\eta[\eta_1(w) = 1, w \in T] = \prod_{w\in T} \rho_w(\eta) \qquad \text{for } T \subset W.$$

Then the next comparison theorem follows (see Lemma 1 in [3]).

LEMMA 1. *Let η_n^1, η_n^2 be spin systems η_n^1, η_n^2 on $\{0, 1\}^W$ corresponding to $\{\rho_w^1(\eta)\}$, $\{\rho_w^2(\eta)\}$. Assume*

$$\rho_w^1(\eta) \leq \rho_w^2(\zeta) \qquad \textit{for } \eta \leq \zeta\,,$$

and

$$\eta_0^1 \leq \eta_0^2\,.$$

Then there exists a spin system $\gamma_n = (\eta_n, \zeta_n)$ on $\{0, 1\}^W \times \{0, 1\}^W$ such that η_n and ζ_n have the same law as η_n^1 and η_n^2, respectively, and that

$$\eta_n \leq \zeta_n \qquad \textit{for } n \geq 0\,.$$

Set $W = \{1, \cdots, 2N\} \times X$, and put

$$\rho_{(j,x)}(\eta) = H(\textstyle\sum_{k=1}^{2N} \eta(k, \cdot)/(2N))_x\,, \qquad \text{for } (j, x) \in W \quad \text{and} \quad \eta \in \{0, 1\}^W\,,$$

where H is the probability given by (1.1). Define the spin system η_n on $\{0, 1\}^W$ with transition law corresponding to ρ. Then by [3], $\xi_n(x) = \sum_{j=1}^{2N} \eta_n(j, x)/(2N)$ is a Markov process with the same law as our stepping stone model $p_x(n)$.

In order to prove the Theorem 1 we need to use the theory of oriented percolation processes. The next Lemma 2 asserts the subadditive ergodic theorem (see [1], [6]) for such processes.

LEMMA 2. *Suppose that* $\{X_{m,n}, m \leq n\}$ *are random variables which satisfy the following properties*:

(a) $X_{0,0} = 0$, $X_{0,n} \leq X_{0,m} + X_{m,n}$ *for* $0 \leq m \leq n$.

(b) $\{X_{(n-1)k,nk}, n \geq 1\}$ *is a stationary process for each* $k \geq 1$.

(c) $\{X_{m,m+k}, k \geq 0\}$ *and* $\{X_{m+1,m+k+1}, k \geq 0\}$ *are the same distribution for each* m.

(d) $EX_{0,1}^{+} < +\infty$ $(X_{0,1}^{+} \equiv \max(X_{0,1}, 0))$.

Let $\alpha_n = EX_{0,1} < +\infty$, *which is well defined by* (a), (b), *and* (d). *Then*

$$\alpha = \lim_{n\to\infty} \frac{\alpha_n}{n} = \inf_{n\geq 1} \frac{\alpha_n}{n} \in [-\infty, \infty) \quad \textit{and} \quad X_\infty = \lim_{n\to\infty} \frac{X_{0,n}}{n}$$

exists a.s., with $-\infty \leq X_\infty < \infty$. *Furthermore,* $EX_\infty = \alpha$. *If* $\alpha > -\infty$, *then*

$$\lim_{n\to\infty} E\left|\frac{X_{0,n}}{n} - X_\infty\right| = 0 .$$

If the stationary process in (b) *are ergodic, then* $X_\infty = \alpha$ *a.s.*

Let $\eta_0(j, x) = \chi_{(-\infty,0]}(x)$. For η_n put

$$r_n = \sup\{x; \eta(j, x) \neq 0 \text{ for some } j\},$$

then we have

LEMMA 3. *Suppose* $\sum_x \lambda_{0x}|x|^2 < \infty$. *Let* $\alpha_n = Er_n < +\infty$, *then*

(a) $\alpha = \lim_{n\to\infty}(\alpha_n/n) = \inf_{n>0}(\alpha_n/n) \in [-\infty, \infty)$,

(b) $\lim_{n\to\infty}(r_n/n) = \alpha$ *a.s., and*

(c) *if* $\alpha > -\infty$, *then* $\lim_{n\to\infty} E|(r_n/n) - \alpha| = 0$.

Proof. Let the function of η

$$1 - \rho_{(j,x)}(\eta) = \frac{(1 - s/2)\{u + (1-u)\sum_{i,z}(\lambda_{xz}/(2N))(1 - \eta(i, z))\}}{s(1-u) - s\{(1-u)\sum_{i,z}(\lambda_{xz}/(2N))(1 - \eta(i, z))\} + 1 - s/2}$$

to express the expansion of power series of $1 - \eta$,

$$1 - \rho_{(j,x)}(\eta) = \sum_B f_{(j,x)}(B) \prod_{(i,y)\in B} (1 - \eta(i, y)), \tag{3.1}$$

where the coefficients $f_{(j,x)}(B)$ are expressed in the form

$$\frac{(1 - s/2)(1 + s/2)}{(1 + s(1/2 - u))^2} \sum_n \sum_{i_1,\cdots,i_n,z_1,\cdots,z_n} (\lambda_{xz_1}\lambda_{xz_2}\cdots\lambda_{xz_n})$$
$$\times (2N)^{-n}(1-u)^n[s/\{1 + s(1/2 - u)\}]^{n-1} \qquad \text{for } B \neq \varnothing,$$

and

$$\frac{(1-s/2)u}{1+s(1/2-u)} \qquad \text{for } B=\varnothing\,.$$

Here the above sum is taken over the all $n \geq 1$, $i_1, \cdots, i_n, z_1 \cdots, z_n$ such that $\{(i_1, z_1), \cdots, (i_n, z_n)\} = B$. Then $\{f_{(j,x)}(B)\}$ satisfies $f_{(j,x)}(B) \geq 0$ for every B. Evaluate both sides of (3.1) at $\eta = \mathbf{0}$. Then $\sum_B f_{(j,x)}(B) = 1$, because $\rho_{(j,k)}(\mathbf{0}) = 0$. Let us define the probability space (Ω, P) and the independent random variables $\{\beta_{w,n}\}_{w\in W, n\geq 1}$ with values on finite subsets of W, by $P(\beta_{w,n} = B) = f_w(B)$ for any finite subset B of W. For $n > 0$, $z, w \in W$ we define the bond $(w, n-1)$ to (z, n) is open by $w \in \beta_{z,n}$. We define the oriented percolation on $W \times [0, \infty)$ by the method which is familiar in the theory of the percolation processes ([1]). For $\xi_0 \in \{0, 1\}^W$, $\xi_n \in \{0, 1\}^W$ is defined by $\xi_n(w) = 1$ if there exists a path of open bonds from $(z, 0)$ to (w, n) for some z with $\xi_0(z) = 1$ and $\xi_n(w) = 0$ otherwise. Then ξ_n is the Markov process with transition probabilities

$$P(\xi_{n+1}(z) = 1 \mid \xi_n) = \sum_B f_z(B)(1 - \prod_{w\in B}(1 - \xi_n(w))) = \rho_z(\xi_n)\,.$$

Therefore ξ_n is subject to the same law as the one with the spin system η_n. We call ξ_n the generalized percolation process (g.p.p., see [2]).

Define $r_{0,n} = \max\{x \in \boldsymbol{Z};$ there exists a path of open bonds from $((i, y), 0)$ to $((j, x), n)$ for some $y \leq 0$ and $i, j\}$ for $0 < n$ and $r_{m,n} = \max\{x \in \boldsymbol{Z};$ there exists a path of open bonds from $((i, y), m)$ to $((j, x), n)$ for some $y \leq r_{0,m}$ and $i, j\} - r_{0,m}$ for $0 < m \leq n$. Then $r_{0,m} = r_m$, and $X_{m,n} = r_{m,n}$ satisfies (a), (b), (c) of Lemma 2. The assumption of (d) is satisfied by following:

$$\begin{aligned} Er_1^+ &= \sum_{x=1}^{\infty} x\left[1 - \left(1 - \frac{(1+s/2)(1-u)\sum_{y\leq -x}\lambda_{0y}}{1 - s/2 + s(1-u)\sum_{y\leq -x}\lambda_{0y}}\right)^{2N}\right] \\ &\quad\times \prod_{z\geq x+1}\left(1 - \frac{(1+s/2)(1-u)\sum_{y\leq -z}\lambda_{0y}}{1 - s/2 + s(1-u)\sum_{y\leq -z}\lambda_{0y}}\right)^{2N} \\ &\leq 2N\frac{1+s/2}{1-s/2}(1-u)\sum_{y<-1}\frac{(-y+1)(-y)}{2}\lambda_{0y} < +\infty\,. \end{aligned}$$

Since r_n is the same law with $r_{0,n}$, Lemma is proved. Q.E.D.

Set $r_n^B = \max\{x \in \boldsymbol{Z}^1;$ there exists a path of open bonds from $((i, y), 0)$ to $((j, x), n)$ for some $(i, y) \in B$ and $j\}$, where $B \subset W$. Then we have

LEMMA 4. *Suppose $B \subset A$, where the number of $\{x \geq 0;\ (i, x) \in A$ for some $i\} < +\infty$, and let C be any finite set. Then*

$$0 \leq r_n^{A\cup C} - r_n^A \leq r_n^{B\cup C} - r_n^B .$$

In particular, for $B \subset \{1, \cdots, 2N\} \times (-\infty, -1]$,

$$E(r_n^{B\cup\{(1,0)\}} - r_n^B) \geq 1/(2N) .$$

Proof. Denote the g.p.p. with the initial state ξ by ξ_n^ξ, then

$$\xi_n^{\xi\vee\eta} = \xi_n^\xi \vee \xi_n^\eta$$

where for ξ, $\eta \in \{0, 1\}^W$, $\xi\vee\eta \in \{0, 1\}^W$ is defined by $(\xi \vee \eta)(w) = \max\{\xi(w), \eta(w)\}$. This implies the first relations of Lemma by the same proof as that of Lemma 2.21 of [6]. For $B \subset \{1, \cdots, 2N\} \times (-\infty, -1]$, we have

$$\begin{aligned} E(r_n^{B\cup\{(1,0)\}} - r_n^B) &\geq E(r_n^{\{1,\cdots,2N\}\times(-\infty,-1]\cup\{(1,0)\}} - r_n^{\{1,\cdots,2N\}\times(-\infty,-1]}) \\ &\geq E(r_n^{\{1,\cdots,2N\}\times(-\infty,-1]\cup\{1,\cdots,i+1\}\times\{0\}} \\ &\quad - r_n^{\{1,\cdots,2N\}\times(-\infty,-1]\cup\{1,\cdots,i\}\times\{0\}}) . \end{aligned}$$

The last term is nonincreasing in i and

$$E(r_n^{\{1,\cdots,2N\}\times(-\infty,0]} - r_n^{\{1,\cdots,2N\}\times(-\infty,-1]}) = 1$$

holds by the translation invariance. Therefore we have

$$E(r_n^{B\cup\{(1,0)\}} - r_n^B) \geq \frac{1}{2N} \qquad \text{Q.E.D.}$$

LEMMA 5. *For the spin system* η_n *with* $\eta_0(j, x) = \chi_{(-\infty,0]}(x)$ *put* $\alpha = \alpha(s)$, *and suppose the number of* $\{x; \lambda_{0x} \neq 0\} \geq 2$. *Then*

$$\alpha(s) - \alpha(s') \geq c(t)(s - s') \qquad 0 \leq s' \leq s \leq t < 2 ,$$

whenever $\alpha(s') > -\infty$, *where*

$$c(t) = \inf_{0\leq s\leq t} c_0(1 - s/2)^{2N-1+(2N)k} \exp\left(-\frac{c_1}{1 - s/2}\right),$$

and c_0, c_1 *are positive constants, and* k *is a positive integer.*

Proof. We will prove the Lemma similarly to the proof of the theorem (3.14) in [1]. Set $\alpha_n = \alpha_n(s)$, then if $s > s'$, for any initial states (η, η') with $\eta \geq \eta'$ we can construct spin systems η_n, η'_n on the same space of the probability measure $P^{\eta,\eta'}$ with parameters s, s' respectively such that $\eta_n \geq \eta'_n$, and set r_n, r'_n as each rightmost points respectively. Then $r_n \geq r'_n$. Let

$$\tau = \inf\{n; r_n > r'_n\} .$$

By Lemma 4

$$E(r_n - r'_n) \geq E(r_n - r'_n;\, \tau \leq n) \geq (1/(2N))P(\tau \leq n)\,.$$

By Markov property

$$P(\tau > n) \leqq \sup_{\eta' \leq \eta, \eta \neq 0} P^{\eta,\eta'}(r_1 = r'_1)P(\tau > n-1)\,.$$

By the assumption there exist x_1, x_2 $(x_1 < x_2)$ such that $\lambda_{0x_1} \neq 0$, $\lambda_{0x_2} \neq 0$. If $x = \sup\{y;\, \eta'(j, y) \neq 0 \text{ for some } j\}$, then

$$P^{\eta,\eta'}(r_1 > r'_1) \geq P^{\eta,\eta'}(r_1 = x - x_1, r'_1 < x - x_1)\,.$$

By translation invariance of the probability law of η_n, we can assume $x = x_1$. Hence

$$\begin{aligned}
(3.2) \qquad P^{\eta,\eta'}(r_1 > r'_1) &\geq P^{\eta,\eta'}((\bigcup_i A_{i,0}) \cap \bigcap_{i,k>0} A^c_{ik} \cap \bigcap_i A'^c_{i,0} \cap \bigcap_{i,k>0} A'^c_{i,}) \\
&= P((\bigcup_i A_{i,0}) \cap \bigcap_i A'^c_{i,0} \cap \bigcap_{i,k>0} A^c_{ik}) \\
&\geq P((A_{10} - A'_{10}) \cap \bigcap_{i \neq 1} A^c_{i,0} \cap \bigcap_{i,k>0} A^c_{ik}) \\
&\geq (\rho_{(1,0)}(\eta) - \rho_{(1,0)}(\eta'))(1 - \rho_{(1,0)}(\eta))^{2N-1} \prod_{y>0} (1 - \rho_{(1,y)}(\eta))^{2N}
\end{aligned}$$

where $A_{ik} = \{\eta_1(i, k) = 1\}$ and $A'_{ik} = \{\eta'_1(i, k) = 0\}$ for any $(i, k) \in W$, because of the construction of the spin system. Let $x_0 < x_1$ be an integer such that

$$\sum_{w \leq x_0} \lambda_{0w} \leq \frac{1}{2}\,.$$

Then for any $y > x_1 - x_0$ we have

$$(3.3) \qquad 1 - \rho_{(1,y)}(\eta) \geq \exp\left(-2\frac{1 + s/2}{1 - s/2}(1 - u) \sum_{w \leq -y + x_1} \lambda_{0w}\right).$$

Since $1 - \rho_{(1,y)}(\eta) \geq (u(1 - s/2))/(1 + s(1/2 - u))$ for any y, from (3.2), (3.3) we can prove

$$\begin{aligned}
P^{\eta,\eta'}(r_1 > r'_1) \geq &\left(\frac{u(1 - s/2)}{1 + s(1/2 - u)}\right)^{2N-1+(2N)(x_1 - x_0)} \\
&\times \exp\left(-4N\frac{1 + s/2}{1 - s/2}(1 - u) \sum_{w < x_0} (x_0 - w)\lambda_{0w}\right) \\
&\times \inf\left\{\frac{(s - s')(1 - \gamma)\gamma}{(s'\gamma + 1 - s'/2)(s\gamma + 1 - s/2)}\,; \right. \\
&\qquad \left. (1 - u)\frac{\lambda_{0x_1}}{2N} \leq \gamma \leq (1 - u)(1 - \lambda_{0x_2})\right\} \\
&\geq c_0(1 - s/2)^{2N-1+(2N)(x_1 - x_0)} \exp\left(-\frac{c_1}{1 - s/2}\right)(s - s')\,,
\end{aligned}$$

where c_0, c_1 are positive constants. Put $k = x_1 - x_0$ and

$$c(t) = \inf_{0 \le s \le t} c_0(1 - s/2)^{2N-1+(2N)k} \exp\left(-\frac{c_1}{1 - s/2}\right),$$

then for $0 \le s' \le s \le t$

$$P(\tau \le n) \ge 1 - (1 - c(t)(s - s'))^n .$$

Therefore the inequality

$$\alpha_n(s) - \alpha_n(s') \ge \frac{1}{2N}[1 - (1 - (c(t)(s - s'))^n]$$

holds. The method of the proof of (3.14) in [1] can therefore be applied. For any $M > 0$

$$\begin{aligned}\alpha_n(s) - \alpha_n(s') &= \sum_{k=1}^{nM}\left[\alpha_n\left(s' + \frac{k(s - s')}{nM}\right) - \alpha\left(s' + \frac{(k - 1)(s - s')}{nM}\right)\right] \\ &\ge \frac{nM}{2N}\left[1 - \left(1 - c(t)\frac{s - s'}{nM}\right)^n\right].\end{aligned}$$

Then

$$\lim_{n\to\infty}\frac{1}{n}(\alpha_n(s) - \alpha_n(s')) \ge \frac{M}{2N}[1 - e^{-c(t)(s-s')/M}] .$$

Let $M \to \infty$ to obtain the desired inequality

$$\alpha(s) - \alpha(s') \ge \frac{c(t)}{2N}(s - s') . \qquad \text{Q.E.D.}$$

LEMMA 6. *Under the assumption of Lemma* 5, *if* $s < 2u(2 - u)^{-1}$, *then* $\alpha(s) = -\infty$ *holds.*

Proof. Put $a = (1 - u)(1 + s/2)(1 - s/2)^{-1}$. Then the condition $s < 2u(2 - u)^{-1}$ implies $a < 1$. Let $\{\lambda_{xy}^{(n)}\}$ be the n-th power of the stochastic matrix $\{\lambda_{xy}\}$, then for any γ $(a < \gamma < 1)$,

$$E(r_n + 2\gamma^{-n})^+ \le E\sum_{x>-2\gamma^{-n}, i}(x + 2\gamma^{-n})\eta_n(i, x) \le \sum_{x>-2\gamma^{-n}} a^n(x + 2\gamma^{-n})\sum_{y\le 0}\lambda_{xy}^{(n)} ,$$

since $E^{\eta}\eta_1(j, x) \le a(1/2N)\sum_{i,y}\lambda_{xy}\eta(i, y)$. By homogeneity of λ_{xy}, we have

$$\begin{aligned}\sum_{x>-2\gamma^{-n}}(x + 2\gamma^{-n})\sum_{y\le 0}\lambda_{xy}^{(n)} &= \sum_{z<2\gamma^{-n}}\lambda_{0z}^{(n)}\sum_{-2\gamma^{-n}<x\le -z}(x + 2\gamma^{-n}) \\ &\le \sum_{z<2\gamma^{-n}}\lambda_{0z}^{(n)}([2\gamma^{-n}] + 1 - z)([2\gamma^{-n}] + 2 - z)/2 .\end{aligned}$$

Let P^0 be the probability measure of the random walk X_n with transition probabilities λ_{xy} and with the initial point 0, then the last term is bounded above by $1 + (3/2)\{[2\gamma^{-n}] + 1 - nE^0X_1\}^2 + (3/2)nE^0(X_1 - E^0X_1)^2 \leq 4\gamma^{-2n}$ for all large enough n. Therefore

$$\lim_{n\to\infty} \gamma^n E(r_n + 2\gamma^{-n})^+ \leq \lim_{n\to\infty} 4\left(\frac{a}{\gamma}\right)^n = 0 .$$

This implies $P(r_n \geq -\gamma^{-n}) \to 0$ as $n \to \infty$, and $r_n/n \to -\infty$ in law as $n \to \infty$. Because $r_n/n \to \alpha$ a.s. by Lemma 3, we can obtain $\alpha = -\infty$.

Q.E.D.

We use the result on limit Theorems by Corollary 2 of Theorem 1 in Nagaev, S.V. [8] as following:

Let $\xi_1, \xi_2, \cdots, \xi_n, \cdots$ be a sequence of identically distributed independent random variables with distribution function $F(x)$, $E\xi_i = 0$ and variance of $\xi_i = 1$, and let $F_n(x)$ be the distribution function of $\sum_{k=1}^n \xi_k$. If $c_m = E|\xi_i|^m < \infty$, $m > 2$, then

$$1 - F_n(x) < \frac{B_m c_m n}{x^m}$$

for

$$x > 4\sqrt{n \max\left[\log \frac{n^{m/2-1}}{K_m c_m}, 0\right]} ,$$

where B_m is an absolute constant depending only on m, and

$$K_m = 1 + (m+1)^{m+2} e^{-m} .$$

LEMMA 7. *Assume* (2.3) *and suppose* $s < s_0$. *Then there is a positive constant* K *depending only on* s *such that*

$$P^{\chi_A}(\eta_n \neq 0) \leq K|A|n^{-1-\theta} \tag{3.4}$$

for all finite set $A \subset W$, *and* $n \geq 0$.

Proof. Using the additive property of g.p.p., we see that it suffices to prove the following inequality.

$$P^{\chi_{\{(1,0)\}}}(\eta_n \neq 0) \leq Kn^{-1-\theta} .$$

By Lemmas 4, 5, $\alpha(s) < \beta(s)$. Let h be a constant such that $\alpha(s) < h < \beta(s)$. Then by part (a) of Lemma 3, $\alpha_m(s) < mh$ for some $m > 0$. Using the notation in the proof of Lemma 3, recall that

$$\{r_{(n-1)m,nm},\ n \geq 1\}$$

are independent and identically distributed random variables with mean $\alpha_m(s)$. Furthermore, let $\tilde{r}_n = r_n - nh$ and $\tilde{r}_{m,n} = r_{m,n} - (n-m)h$. Then

$$\tilde{r}_{nm} = \tilde{r}_{0,nm} \leq \sum_{k=1}^{n} \tilde{r}_{(k-1)m,km}\,. \tag{3.5}$$

The condition (2.3) implies

$$E((r_1^+)^{2+\theta}) < +\infty\,.$$

Since $\alpha_m(s) < mh$, we have

$$E(\tilde{r}_m \vee (-M)) < 0$$

for sufficiently large M. Hence we get

$$P\Big(\sum_{k=1}^{n} \tilde{r}_{(k-1)m,km} \geq 0\Big) \leq K_1 n^{-1-\theta}$$

by Nagaev's result, where K_1 is a positive constant. Therefore by the relation (3.5)

$$P(\tilde{r}_{nm} \geq 0) \leq K_1 n^{-1-\theta}$$

for all sufficiently large n. Similarly

$$P(\ell_{nm} - nmh \leq 0) \leq K_2 n^{-1-\theta}$$

for all sufficiently large n and for some positive constant K_2.

For $\eta_0 = \chi_{\{(1,0)\}}$, the relation $\{\eta_n \neq 0\} \subset \{\ell_i \leq r_i \text{ for all } i \leq n\}$ implies

$$\begin{aligned} P^{\chi_{\{(1,0)\}}}(\eta_{nm} \neq 0) &\leq P(\ell_i \leq r_i \text{ for all } i \leq nm) \\ &\leq P(\ell_{nm} \leq r_{nm}) \\ &\leq P(\ell_{nm} \leq nmh) + P(r_{nm} \geq nmh) \\ &\leq (K_1 + K_2) n^{-1-\theta}\,. \end{aligned}$$

Since $P^{\chi_{\{(1,0)\}}}(\eta_i \neq 0)$ is monotone in i, it follows that (3.4) holds with $K = m^{1+\theta}(K_1 + K_2)$. Q.E.D.

LEMMA 8. *Assume* (2.3) *and suppose* $s < s_0$. *Then there is a positive constant* ε *so that*

$$P(r_n > -n^{1+\varepsilon}) \longrightarrow 0 \quad as \quad n \longrightarrow \infty\,. \tag{3.6}$$

In particular, $\alpha(s) = -\infty$.

Proof. Write the initial state $\eta_0(j, x) = \chi_{(-\infty,0]}(x)$ which is used in defining r_n as

$$\chi_{(-\infty,0]}(x) = \chi_{(-\infty,-2n^{1+\varepsilon}]}(x) + \chi_{(-2n^{1+\varepsilon},0]}(x)\,.$$

By the additivity property and the translation invariance of the spin system η_n, it follows that

$$P(r_n > -n^{1+\varepsilon}) \leq P(r_n > n^{1+\varepsilon}) + P^{\chi_{[1,2N]\times(-2n^{1+\varepsilon},0]}(x)}(\eta_n \neq 0)\,.$$

By the subadditivity of r_n, for any $\varepsilon \in (0, \theta)$

$$P(r_n > -n^{1+\varepsilon}) \leq P(r_n > n^{1+\varepsilon}) + 2K(2N)n^{1+\varepsilon}n^{-1-\theta}$$

where K is a constant in Lemma 7. By Lemma 3 we have,

$$P(r_n > n^{1+\varepsilon}) \longrightarrow 0\,.$$

Combining the last two inequalities gives (3.6). Q.E.D.

§ 4. Proof of Theorems 1 and 2

Because $\{\sum_{j=1}^{2N}\eta_n(j, x)/(2N)\}$ has same law as the stepping stone model, the statement (a) of Theorem 1 holds by Lemma 3.

Let $\bar{\xi}_n(x)$ and $\xi_n^0(x)$ be the original oriented bond percolation processes on Z^1 satisfying initial states $\chi_{(-\infty,0]}(x)$ and $\chi_{\{0\}}(x)$, respectively (see [1]). Let p be the probability that each bond from (x, n) to $(y, n+1)$ satisfying $y = x \pm 1$ is open.

Proof of Theorem 1. By [1] there exists the critical value $p_c > 0$ such that $\alpha_0(p) \equiv \lim_{n\to\infty} \sup\{x; \bar{\xi}_n(x) \neq 0\}/n > 0$ if $p > p_c$. Define the map G_n from Z^1 to Z^1 by $G_n(x) = \{(x_2 - x_1)x - n(x_1 + x_2)\}/2$, and let $\eta_n^1(j, x)$, $j = 1, \cdots, 2N$ be independent copies of $\bar{\xi}_n(x)$. Note that $\{\eta_n^1(j, x)\}$ is a spin system with transition law corresponding to $\{\rho_{(j,x)}^1(\eta)\}$ where

$$\rho_{(j,x)}^1(\eta) = p\eta(j, x-1) + p\eta(j, x+1) - p^2\eta(j, x-1)\eta(j, x+1)\,.$$

Define the spin system $\tilde{\eta}_n^1$ corresponding to $\{\tilde{\rho}_{(j,x)}^1(\eta)\}$ with $\tilde{\eta}_0^1(j, x) = \chi_{(-\infty,0]}(x)$ and

$$\tilde{\rho}_{(j,x)}^1(\eta) = p\eta(j, x+x_1) + p\eta(j, x+x_2) - p^2\eta(j, x+x_1)\eta(j, x+x_2).$$

Then $\{\tilde{\eta}_n^1(j, G_n(x))\}$ has the same law as $\{\eta_n^1(j, x)\}$. Recall that the spin system $\{\eta_n(j, x)\}$ has the transition law corresponding to $\{\rho_{(j,x)}(\eta)\}$. We will show there exists a positive constant $s_0 \in (0, 2)$ such that for $s > s_0$ the inequality

$$\tilde{\rho}_{(j,x)}^1(\eta) \leq \rho_{(j,x)}(\eta) \tag{4.1}$$

for any $(j, x) \in W$ and $\eta \in \{0, 1\}^W$. Note that

$$
\text{(4.2)} \qquad \rho_{(j,x)}(\eta) \geq \frac{(1 + s/2)(1 - u)(1/(2N)) \min\{\lambda_{0x_1}, \lambda_{x0_2}\}}{1 - s/2 + s(1 - u)(1/(2N)) \min\{\lambda_{0x_1}, \lambda_{0x_2}\}} \times \{\eta(j, x + x_1) + \eta(j, x + x_2) - \eta(j, x + x_1)\eta(j, x + x_2)\}.
$$

Put

$$
b = (1 - u)(1/(2N)) \min\{\lambda_{0x_1}, \lambda_{0x_2}\},
$$

and

$$
f(p) = \{2p - p^2 - b\} \Big/ \left\{\frac{2p - p^2}{2} + \left(\frac{1}{2} - 2p + p^2\right)b\right\}.
$$

Obviously we see that if $s > f(p)$,

$$
p\xi_1 + p\xi_2 - p^2\xi_1\xi_2 < \frac{(1 + s/2)(1 - u)(1/(2N))\{\lambda_{0x_1}\xi_1 + \lambda_{0x_2}\xi_2\}}{1 - s/2 + s(1 - u)(1/(2N))\{\lambda_{0x_1}\xi_1 + \lambda_{0x_2}\xi_2\}}
$$

for any $\xi_1, \xi_2 \in \{0, 1\}$. Hence, by (4.2), if $s > f(p)$, (4.1) holds. The comparison theorem (Lemma 1) and the inequality (4.1) imply that

$$
\text{(4.3)} \qquad \begin{aligned} & P(\sup\{x; \eta_n(j, x) \neq 0 \text{ for some } j\} > a) \\ & \geq P(\sup\{x; \tilde{\eta}_n^1(j, x) \neq 0 \text{ for some } j\} > a) \\ & \geq P(\sup\{G_n(x); \eta_n^1(j, x) \neq 0 \text{ for some } j\} > a). \end{aligned}
$$

Evaluating the both sides of (4.3) at $a = na'$, we get

$$
P(r_n/n > a) \geq P(\max_j \tilde{r}_{j,n}/n > a)
$$

where

$$
\tilde{r}_{j,n} = \sup\{G_n(x); \eta_n^1(j, x) \neq 0\}.
$$

Since the laws of r_n/n and $\max_j \tilde{r}_{j,n}/n$ converge to the laws of constants $\alpha(s)$ and $\{(x_2 - x_1)\alpha_0(p) - x_1 - x_2\}/2$, which may be $-\infty$, we obtain

$$
\alpha(s) \geq \frac{(x_2 - x_1)\alpha_0(p) - x_1 - x_2}{2}
$$

and

$$
\beta(s) \leq \frac{-(x_2 - x_1)\alpha_0(p) - x_1 - x_2}{2}.
$$

Therefore if $s > f(p_c)$, $\alpha(s) > \beta(s)$ holds. Hence, by the fact that $\alpha(s) - \beta(s)$ is strictly increasing if $\alpha(s) > -\infty$ and $\beta(s) < \infty$ in Lemma 5 and that $\alpha(s) = -\infty$ and $\beta(s) = \infty$ if $s < 2u(2-u)^{-1}$ in Lemma 6, the statement (b) of Theorem 1 is proved with $2u(2-u)^{-1} \leq s_0 \leq f(p_c)$.

If we assume (2.3) in place of (2.2), then from Lemmas 7 and 8, we obtain the results

$$\alpha(s) = -\infty \quad \text{and} \quad \beta(s) = \infty \qquad \text{for } s < s_0 . \qquad \text{Q.E.D.}$$

Proof of Theorem 2. By [1], if $p > p_c$, $P(\xi_n^0(x) \neq 0$ for all $n) > 0$ and $\lim_{n\to\infty} \sup\{x; \xi_n^0(x) \neq 0\}/n = \alpha_0(p)$ a.s. on $\{\xi_n^0(x) \neq 0$ for all $n\}$, with initial state $\xi_0^0(x) = \chi_{\{0\}}(x)$ for $x \in \boldsymbol{Z}$. Let $\eta_n^1(1, x)$ be an independent copy of $\xi_n^0(x)$ and let $\eta_n^1(j, x) = 0$, $j = 2, \cdots, 2N$. Note that $\{\eta_n^1(j, x)\}$ is a spin system corresponding to $\{\rho_{(j,x)}^1(\eta)\}$. Define the spin system $\tilde{\eta}_n^1$ corresponding to $\{\tilde{\rho}_{(j,x)}^1(\eta)\}$ with $\tilde{\eta}_0^1(j, x) = \chi_{\{(1,0)\}}(j, x)$. Put

$$\begin{aligned} E_n &= \{x; \eta_n(j, x) \neq 0 \text{ for some } j\} , \\ F_n &= \{x; \eta_n^1(j, x) \neq 0 \text{ for some } j\} , \\ \tilde{F}_n &= \{x; \tilde{\eta}_n^1(j, x) \neq 0 \text{ for some } j\} . \end{aligned}$$

Then by the comparison between η_n and $\tilde{\eta}_n^1$

$$\begin{aligned} &P(\liminf_n (\sup E_n)/n > a \quad \text{and} \quad \limsup_n (\inf E_n)/n < a') \\ &\geq P(\liminf_n (\sup \tilde{F}_n)/n > a \quad \text{and} \quad \limsup_n (\inf \tilde{F}_n)/n < a') \\ &= P(\lim_n (\sup F_n)/n > a \quad \text{and} \quad \lim_n (\inf F_n)/n < a') , \end{aligned}$$

if

$$s > f(p) . \tag{4.4}$$

Therefore, if we put

$$\begin{aligned} A &= \{\liminf_n r_n/n \geq \{(x_2 - x_1)\alpha_0(p) - x_1 - x_2\}/2\} , \\ B &= \{\limsup_n \ell_n/n \leq \{-(x_2 - x_1)\alpha_0(p) - x_1 - x_2\}/2\} , \\ \Omega_1 &= \{\boldsymbol{p}(n) \neq \boldsymbol{0} \text{ for all } n\} , \\ A' &= \{\lim_n \sup\{x; \xi_n(x) \neq 0\}/n = \alpha_0(p)\} , \\ B' &= \{\lim_n \inf\{x; \xi_n(x) \neq 0\}/n = -\alpha_0(p)\} , \end{aligned}$$

and $\Omega_1' = \{\xi_n \neq \boldsymbol{0}$ for all $n\}$, then

$$P_\mu(A \cap B \cap \Omega_1) \geq P(A' \cap B' \cap \Omega_1') \tag{4.5}$$

where $\mu = \delta_{\{\boldsymbol{p}(0)\}}$. Therefore if

$$s > f(p_c) ,$$

we can find $p > p_c$ such that the pair s, p satisfies (4.4), and $P_\mu(\{\liminf r_n/n \geq \alpha'$ and $\limsup \ell_n/n \leq \beta'\} \cap \Omega_1) > 0$ where

$$\alpha' = \{(x_2 - x_1)\alpha_0(p) - x_1 - x_2\}/2 \quad \text{and} \quad \beta' = \{(x_2 - x_1)\alpha_0(p) - x_1 - x_2\}/2 .$$

The inequality $\alpha' > \beta'$ holds, because $\alpha_0(p) > 0$ for $p > p_c$. Hence we have $\Omega_0 \supset \{\liminf r_n/n \geq \alpha'$ and $\limsup \ell_n/n \leq \beta'\} \cap \Omega_1$. Compare the two spin systems $\{\eta_n(j, x)\}$ corresponding to $\{\rho_{(j,x)}(\eta)\}$ satisfying initial states $\chi_{(-\infty,0]}(x)$ and $\chi_{\{(1,0)\}}(j, x)$, to obtain $\limsup_n r_n/n \leq \alpha(s)$ and $\liminf_n \ell_n/n \geq \beta(s)$. Hence, if $s < s_0$, then $P_\mu\{p(n) \neq \mathbf{0}$ for all $n\} = 0$. Thus Theorem 2 is proved. Q.E.D.

§ 5. Proof of Theorem 3

In this section, S denotes the set $\{0, 1\}^W$ which will be a compact metric space with product topology. We shall prove in the way similar to Ch.III, Theorem 5.18 in [6] and Theorem 1.3 in [11]. We will use a Lemma for the mixing properties of the distribution of η_n.

Denote by $C(S)$ the continuous functions on S and by $C_0(S)$ the functions on S depending finitely many coordinates. Then since $\rho_w(\cdot) \in C(S)$ and $C_0(S)$ is dense in $C(S)$, the map Q_0: $f \to E^\eta(f(\eta_1))$ is a contraction from $C(S)$ into $C(S)$ with sup-norm $\|\cdot\|$. For any $a \in \mathbf{Z}^d$ define the shift translation τ_a on S by $\tau_a\eta(j, x) = \eta(j, x + a)$ for any $x \in \mathbf{Z}^d$, and define an operator T_a on $C(S)$ by

$$T_a f(\eta) = f(\tau_a \eta) \qquad \text{for any } \eta \in S .$$

Then we have

LEMMA 9. *For any $f,\ g \in C(S)$, $n \geq 1$,*

$$\limsup_{|a|\to\infty} \sup_{\eta \in S} |E^\eta(f(\eta_n)g(\tau_a\eta_n)) - E^\eta(f(\eta_n))E^\eta(g(\tau_a\eta_n))| = 0 .$$

Proof. The statement of the Lemma is equivalent to

$$\text{(5.1)} \quad \lim_{|a|\to\infty} \|Q_0^n(fT_a g) - Q_0^n(f)Q_0^n(T_a g)\| = 0 \qquad \text{for any } f, g \in C(S),\ n \geq 1 .$$

We will show the equality (5.1) by induction on n. First, we suppose that f and g depend only on $\{\eta(j, x);\ x \in A, j\}$ and $\{\eta(j, x);\ x \in B, j\}$, respectively, with finite A, $B \subset \mathbf{Z}^d$. Let $|a|$ to be large so that $A \cap (\tau_a)^{-1}(B) = \varnothing$. Then

$$E^\eta[f(\eta_1)g(\tau_a\eta_1)] = E^\eta[f(\eta_1)]E^\eta[g(\tau_a\eta_1)]$$

holds. Since $C_0(S)$ is dense in $C(S)$ and the map Q_0 is a contraction from $C(S)$ into $C(S)$, we get the equality (5.1) in the case where $n=1$. Translation invariance of $\rho_{(j,x)}(\eta)$ implies that

$$Q_0 T_a = T_a Q_0 \qquad \text{for any } a \in \boldsymbol{Z}^d .$$

Suppose $n \geq 1$. Assume the statement (4.1) is true for all $k \leq n$. Then by the assumption of induction,

$$\begin{aligned}
&\limsup_{|a|\to\infty} \| Q_0^{n+1}(fT_a g) - Q_0^{n+1}f \cdot Q_0^{n+1}(T_a g)\| \\
&\quad \leq \limsup_{|a|\to\infty} \| Q_0^n(fT_a g) - Q_0^n f \cdot Q_0^n(T_a g)\| \\
&\qquad + \limsup_{|a|\to\infty} \| Q_0(Q_0^n f \cdot T_a Q_0^n g) - Q_0(Q_0^n f)\cdot Q_0(T_a Q_0^n g)\| = 0 . \qquad \text{Q.E.D.}
\end{aligned}$$

Denote by $\mathscr{S}_0$ the set of all translation invariant probability measures on S. Suppose that $\mu \in \mathscr{S}_0$ and that $\mu\{\eta;\, \eta \equiv 0\} = 0$. Denote by P^A the probability measure of the Markov chain A_n on $F \equiv \{$finite subsets on $W\}$ with transition probabilities $\hat{Q}(A, B)$ defined by

$$\prod_{w\in A} [1-\rho_w(\eta)] = \sum_B \hat{Q}(A, B) \prod_{z\in B} [1-\eta(z)] \qquad \text{for any } \eta \in \{0, 1\}^W .$$

then

$$\hat{\mu}_n(A) \equiv E^\mu \prod_{w\in A} [1-\eta_n(w)] = \int E^A[\prod_{w\in A_n} [1-\eta(w)]]\mu(d\eta) . \tag{5.2}$$

Then we need to show that

$$\lim_{n\to\infty} E^\mu \prod_{w\in A} [1-\eta_n(w)] = \lim_{n\to\infty} E^{\delta_1} \prod_{w\in A} [1-\eta_n(w)]$$

for all finite $A \subset W$. This is equivalent to $\lim_{n\to\infty} \sum_{B\neq\phi} P^A(A_n = B)\hat{\mu}(B) = 0$, where $\hat{\mu}(B) = \int \prod_{w\in B} [1-\eta(w)]d\mu$, because $E^\mu \prod_{w\in A} [1-\eta_n(w)] = \sum_B P^A(A_n = B)\hat{\mu}(B)$.

We shall show that for any $\varepsilon > 0$ there exists an m for which

$$\limsup_{k\to\infty} \sup \{\hat{\mu}_m(B);\, |B| = k,\, B \subset W\} \leq \varepsilon \tag{5.3}$$

holds. By Lemma 9, for any $\varepsilon > 0$, C and $n \geq 0$ there is an L depending on ε, C, and n such that

$$E^\eta \prod_{w\in C} [1-\eta_n(w)] \leq \prod_{w\in C} E^\eta[1-\eta_n(w)] + \varepsilon , \tag{5.4}$$

whenever

$$\min \{|x-y|;\, (i, x), (j, y) \in C,\, (i, x) \neq (j, y)\} \geq L . \tag{5.5}$$

By Hödler's inequality and the fact that $\mu \in \mathscr{S}_0$,

$$\int [\prod_{w \in C} E^{\eta}[1 - \eta_n(w)]]\mu(d\eta) \leq \int [E^{\eta}[1 - \eta_n(1, 0)]]^{|C|}\mu(d\eta) .$$

From (5.2) and (5.4), we see that for any B satisfying (5.5)

$$\hat{\mu}_n(C) \leq \varepsilon + \int [E^{\eta}[1 - \eta_n(1, 0)]]^{|C|}\mu(d\eta) . \tag{5.6}$$

Since $\mu\{\eta; \eta \equiv 0\} = 0$, $\{x; \lambda_{0x}^{(n)} \neq 0\} \to \boldsymbol{Z}^d$ as $n \to \infty$, and the relation that

$$R_n \equiv \{w \in W; \text{there exists } C \text{ such that } w \in C \text{ and that } P^{\{(1,0)\}}(A_n = C) > 0\} \supset \{(j, x); \lambda_{0x}^{(n)} \neq 0\} ,$$

holds, we see

$$\mu\{\eta; \eta \equiv 0 \text{ on } R_n\} \longrightarrow 0 , \qquad \text{as} \quad n \longrightarrow \infty .$$

Combining with the relation

$$E^{\eta}[1 - \eta_n(1, 0)] = E^{\{(1,0)\}} \prod_{w \in A_n} [1 - \eta(w)] ,$$

we get that for any $\varepsilon' > 0$ there exists an m depending only on ε' such that

$$\mu\{\eta; E^{\eta}\eta_m(1, 0) > 0\} \geq 1 - \varepsilon'$$

holds. It follows from the Dominated Convergence Theorem that

$$\limsup_{\ell \to \infty} \int [E^{\eta}[1 - \eta_m(1, 0)]]^{\ell}\mu(d\eta) \leq \varepsilon' . \tag{5.7}$$

Now, let ℓ be any positive integer, and let ε be any positive constant. If $|B|$ is ssufficiently large, there is a $C \subset B$ so that $|C| = \ell$ and C satisfies (5.4) and (5.5) for suitably chosen L. Then (5.6) gives

$$\hat{\mu}_m(B) \leq \hat{\mu}_m(C) \leq \varepsilon + \int [E^{\eta}[1 - \eta_m(1, 0)]]^{\ell}\mu(d\eta)$$

so that

$$\limsup_{k \to \infty} \sup \{\hat{\mu}_m(B); |B| = k, B \subset W\} \leq \varepsilon + \int [E^{\eta}[1 - \eta_m(1, 0)]]^{\ell}\mu(d\eta)$$

for every $\varepsilon > 0$ and $\ell \geq 1$. By (5.7), it follows that μ_m satisfies (5.3) for some m.

Next we shall show

$$\lim_{n \to \infty} P^{A}(|A_n| = k) = 0 \qquad \text{for any } k \geq 1 . \tag{5.8}$$

Since

$$P^A(A_{n+1} = \varnothing, A_n \neq \varnothing) = E^A[P^{A_n}(A_1 = \varnothing); A_n \neq \varnothing] = E^A[c_2^{|A_n|}; A_n \neq \varnothing]$$

where $c_2 = (1 - s/2)u/\{1 + s(1/2 - u)\} > 0$, we have

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} c_2^k P^A(|A_n| = k) = \sum_{n=1}^{\infty} P^A[A_{n+1} = \varnothing, A_n \neq \varnothing] \leq 1,$$

therefore we have

$$\sum_{n=1}^{\infty} P^A(|A_n| = k) < +\infty \qquad \text{for } k \geq 1,$$

to obtain (5.8). From (5.3) and (5.8) we get

$$\limsup_{n\to\infty} \sum_{B\neq\varnothing} P^A(A_n = B)\hat{\mu}_m(B) \leq \varepsilon \qquad \text{for any } \varepsilon. \tag{5.9}$$

Note that

$$\begin{aligned}\sum_{B\neq\varnothing} P^A(A_n = B)\hat{\mu}(B) &= \sum_{B\neq\varnothing}\Big(\sum_{C\neq\varnothing} P^A(A_{n-m} = C)P^C(A_m = B)\hat{\mu}(B)\Big)\\ &\leq \sum_{C\neq\varnothing} P^A(A_{n-m} = C)\hat{\mu}_m(C),\end{aligned} \tag{5.10}$$

because $\varnothing$ is a trap of A_n. From (5.9) and (5.10) we have

$$\limsup_{n\to\infty} \sum_{B\neq\varnothing} P^A(A_n = B)\hat{\mu}(B) \leq \varepsilon$$

for any ε. Thus completing the proof. Q.E.D.

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