J. B. Ripoll
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HELICOIDAL MINIMAL SURFACES IN HYPERBOLIC SPACE

JAIME B. RIPOLL

§ 1. Introduction

Denote by H^3 the 3-dimensional hyperbolic space with sectional curvatures equal to -1, and let g be a geodesic in H^3. Let $\{\psi_t\}$ be the translation along g (see § 2) and let $\{\varphi_t\}$ be the one-parameter subgroup of isometries of H^3 whose orbits are circles centered on g. Given any $\alpha \in R$, one can show that $\lambda = \{\lambda_t\} = \{\psi_t \circ \varphi_{\alpha t}\}$ is a one-parameter subgroup of isometries of H^3 (see § 2) which is called a helicoidal group of isometries with angular pitch α. Any surface in H^3 which is λ-invariant is called a helicoidal surface.

In this work we prove some results concerning minimal helicoidal surfaces in H^3. The first one reads:

THEOREM A. *Let $\alpha \in R$, $|\alpha| < 1$. Then, there exists a one-parameter family Σ of complete simply-connected minimal helicoidal surfaces in H^3 with angular pitch α which foliates H^3. Furthermore, any complete helicoidal minimal surface in H^3 with angular pitch $|\alpha| < 1$ is congruent to an element of Σ.*

We have the following corollary (see also [An]):

COROLLARY B. *Any complete helicoidal minimal surface in H^3 with angular pitch $|\alpha| < 1$ is globally stable.*

The family Σ of Theorem 1 allow us to give a characterization of minimal helicoidal surfaces in H^3, as stated below.

Let $S^2(\infty)$ be the Möbius plane, that is, the 2-sphere equipped with the usual conformal structure. Given two points p_1, p_2 in $S^2(\infty)$ and $\alpha \in [0, \pi/2]$, a differentiable curve $\gamma\colon R \to S^2(\infty)$ which makes an angle α with any circle of $S^2(\infty)$ containing p_1 and p_2 is called a loxodromic curve with end points p_1 and p_2 and with path α. By a pair (L_1, L_2) of

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loxodromic curves we mean two distinct loxodromic curves L_1, L_2 with same path and with same end points.

Now recall that $S^2(\infty)$ can be identified with the asymptotic boundary $\partial_\infty H^3$ of the hyperbolic space H^3, the conformal structure of $S^2(\infty)$ being induced by the extended action of $ISO(H^3)$ to $\partial_\infty H^3 = S^2(\infty)$. We prove:

THEOREM C. *Given any pair of loxodromic curves* (L_1, L_2) *in* $S^2(\infty)$ *with path* $\alpha \in [0, \pi/4)$, *there exists one and only one complete properly immersed minimal surface* M^2 *in* H^3 *such that* $\partial_\infty M^2 = L_1 \cup L_2$ *(*M^2 *is congruent to an element of the family* Σ *mentioned in Theorem* 1).

The question of determining an immersion in hyperbolic space with constant mean curvature by its asymptotic boundary was first taken up by do Carmo and Lawson ([doCL]). In ([doCGT]), this idea was improved and it has been remarked there the strong influence of the asymptotic boundary of a complete constant mean curvature surface in H^3 on its global behaviour. In ([LR]), the authors use this idea to characterize catenoids in hyperbolic space and in ([GRR]) is also used to characterize hyperbolic and parabolic surfaces with constant mean curvature in H^3. We observe that these surfaces, together with the helicoidal ones, exhaust the different types of one-parameter subgroup invariant minimal surfaces in H^3 (see classification in [R]). We finally remark that in proving Theorem 2, no regularity at infinity has to be assumed, contrary to what happens with similar Theorems (see Theorems 3.1 and 3.2 of [LR], Theorems 2 and 3 of [doCGT] and Theorems 3.3 and 5.2 of [GRR]).

I want to thank Professor Manfredo P. do Carmo who suggested to me the questions about helicoidal minimal surfaces in H^3.

The results of this paper are part of my doctoral Thesis at IMPA ([R]).

§ 2. Preliminaries

We will use the Lorentzian model for the hyperbolic space H^3, that is,

$$H^3 = \{(x_1, x_2, x_3, x_4) \mid -x_1^2 + x_2^2 + x_3^2 + x_4^2 = -1\},$$

the Riemannian metric of H^3 being induced by the quadratic form

$$q(x) = -x_1^2 + x_2^2 + x_3^2 + x_4^2 \qquad x = (x_1, x_2, x_3, x_4)$$

of R^4.

Observe that

$$\lambda_t = \begin{bmatrix} \cosh t & \sinh t & 0 & 0 \\ \sinh t & \cosh t & 0 & 0 \\ 0 & 0 & \cos \alpha t & -\sin \alpha t \\ 0 & 0 & \sin \alpha t & \cos \alpha t \end{bmatrix}$$

is a one-parameter subgroup of isometries of H^3 since it preserves q, and it is the sum of the translation

$$\psi_t = \begin{bmatrix} \cosh t & \sinh t & 0 & 0 \\ \sinh t & \cosh t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

along the geodesic g: $-x_1^2 + x_2^2 = -1$ plus the rotation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \alpha t & -\sin \alpha t \\ 0 & 0 & \sin \alpha t & \cos \alpha t \end{bmatrix}$$

around g. By analogy to the Euclidean space, $\lambda = \{\lambda_t\}$ will be called a helicoidal subgroup of isometries with angular pitch α.

Let P^2 be any totally geodesic 2-submanifold of H^3 orthogonal to g. Let $\vec{o} = P^2 \cap g$ and define ρ: $P^2 \to R$ by $\rho(p) = d(\vec{o}, p)$, d: Riemannian distance. Set $r = \sinh \rho$.

From now on, we choose a geodesic h in P^2 parametrized by arc length and such that $h(0) = \vec{o}$. Given $p \in P^2 - \{\vec{o}\}$ denote by $\theta(p)$ the oriented angle between $\vec{p}$ and h where $\vec{p}$ is the geodesic segment from $\vec{o}$ to p. $(r(p), \theta(p))$ will be called the polar coordinates of p. Computations show that the metric ds^2 in P^2 is given in polar coordinates by

$$ds^2 = \frac{dr^2}{1 + r^2} + r^2 d\theta^2 . \tag{2.1}$$

It is easy to verify that any orbit of λ intersects P^2 once and just once. Therefore, any λ-invariant surface is generated by a curve in P^2. We have the following proposition:

PROPOSITION 2.1. *Let γ be a curve in P^2 such that $d\gamma/dt \neq 0$ for any t. Assume that γ generates a minimal λ-invariant surface with angular* *pitch α. Then, the polar coordinates $\theta = \theta(t)$ and $r = r(t)$ of γ satisfy the differential equation.*

$$(2.2)\quad (r^2+1)[(1+\alpha^2)r^2+1]\Big(\dot\theta\ddot r - \dot r\ddot\theta - r(r^2+1)\dot\theta^3 - \frac{3r^2+2}{r(r^2+1)}\dot r^2\dot\theta\Big) - (1+\alpha^2)r(r^2+1)^2\dot\theta\Big(\frac{\dot r^2}{r^2+1} + r^2\dot\theta^2\Big) + 2\alpha^2 r\dot\theta(\dot r^2 + r^2(r^2+1)^2\dot\theta^2) = 0\,.$$

If $\|\dot\gamma\| = 1$, then the oriented geodesic curvature k of γ is given by:

$$(2.3)\qquad k = -\frac{(1+\alpha^2)(r^2+1)^2 - 2\alpha^2(\dot r^2 + r^2(r^2+1)^2\dot\theta^2}{[(1+\alpha^2)r^2+1](r^2+1)^{3/2}}\,r^2\dot\theta$$

Proof. Given $p \in P^2$, define $X(p) = (d/ds)[\lambda_s(p)]_{s=0}$ and observe that $\mathscr{B} = \{X(\gamma(t)), d\gamma/dt\}$ is a basis at $\gamma(t)$ of the tangent plane of the surface S generated by γ. Formula (2.2) is therefore obtained by computing the trace of the second fundamental form of S along γ in the basis $\mathscr{B}$. Formula (2.3) is obtained using (2.2) and the formula of the geodesic curvature of a curve in hyperbolic plane. □

§ 3. Description of the helicoidal minimal surfaces

In this section we study equations (2.1), (2.2) and (2.3) to obtain a description of the helicoidal minimal surfaces.

We begin by observing that the geodesics through $\vec{o}$ in P^2 generate minimal surfaces (note that they satisfy $\theta =$ constant). As in Euclidean space these surfaces will be called *helicoids*.

Remark 3.1. Equations (2.1) and (2.2) show that given $p \in P^2$ and $v \in T_p(P^2)$, $\|v\| = 1$, there exists one and only one curve γ in P^2 parametrized by arc length and generating a helicoidal minimal surface with angular pitch α such that $\gamma(0) = p$ and $\dot\gamma(0) = v$.

Any such curve will be called a *solution curve*.

LEMMA 3.2. *Let γ be a solution curve in P^2 such that $\dot r(t_o) = 0$. Let $\tilde h$ be a geodesic in P^2 orthogonal to γ at $\gamma(t_o)$. Then γ is invariant under the reflexion in P^2 with respect to $\tilde h$.*

Proof. Without loss of generality, we may assume $t_o = 0$. Furthermore, since (2.2) independs on θ, we may also assume that $\theta(0) = 0$, $r = r(t)$ and $\theta = \theta(t)$ being the polar coordinates of γ. Let σ be the reflexion on $\tilde{h}$. Then $\tilde{\gamma} = \sigma \circ \gamma$ is given by $\tilde{r}(t) = r(t)$ and $\tilde{\theta}(t) = -\theta(t) + \pi$. Set $\bar{\gamma}(t) = \gamma(-t)$. Therefore, it is easy to verify that the polar coordinates of $\tilde{\gamma}$ and $\bar{\gamma}$ satisfy (2.1) and (2.2). Furthermore, one has $\tilde{\gamma}(0) = \bar{\gamma}(0)$ and $\dot{\tilde{\gamma}}(0) = \dot{\bar{\gamma}}(0)$, that is, $\tilde{\gamma} = \bar{\gamma}$, which proves the Lemma. □

DEFINITION 3.3. Let v be a vector field of P^2 along the geodesic h which is unitary and normal to h.

Given $u \in R$, denote by γ_u the solution curve determined by the initial conditions

$$\gamma_u(0) = h(u)$$
$$\dot{\gamma}_u(0) = v(u)\,.$$

Let $\Gamma = \{\gamma_u\}_{u \in R}$ and $\Sigma = \{S_u\}_{u \in R}$ where S_u is the helicoidal minimal surface generated by γ_u.

Remark 3.4. It follows from the above definition and from Lemma 3.2, that any curve γ_u is invariant with respect to the reflexion on h. Also, using Remark 3.1, one can prove that γ_{-u} coincides with to the reflexion of γ_u on the geodesic through $\vec{o}$ of P^2 orthogonal to h.

LEMMA 3.5. *Any solution curve of P^2, up to a rotation around $\vec{o}$, belongs to Γ.*

Proof. Let γ be a solution curve in P^2 given in polar coordinates by $\theta = \theta(t)$ and $r = r(t)$. We have just to prove that there exists to such that $\dot{r}(t_o) = 0$. By contradiction assume the opposite. Without loss of generality, we may assume that $\lim_{t\to\infty} r(t) = r_o \geq 0$, and we must have $\lim_{t\to\infty} \dot{r} = 0 = \lim_{t\to\infty} \ddot{r}$. If $r_o > 0$, then, from (2.1) $\lim_{t\to\infty} \dot{\theta} = (1/r_o)$. Derivating (2.1) and taking the limit for $t \to \infty$ we see that $\lim_{t\to\infty} \ddot{\theta} = 0$. But then, taking the limit for $t \to \infty$ of (2.2) we obtain

$$(r_o^2 + 1)[(1 + \alpha^2)r_o^2 + 1]\left(-\frac{r_o^2 + 1}{r_o^2}\right) - (1 + \alpha^2)(r_o^2 + 1)^2 + 2\alpha^2(r_o^2 + 1)^2 = 0$$

and, after simplifications,

$$2r_o^2 + 1 = 0$$

contradiction!

If $r_o = 0$, then from (2.1), $\lim_{t\to\infty} \dot{\theta} = \infty$ and $\lim_{t\to\infty} r\dot{\theta} = 1$. Taking the limit for $t \to \infty$ of (2.2), we obtain

$$\lim_{t\to\infty} \frac{r\dot{\theta}\ddot{r} - r\dot{r}\ddot{\theta} - r\dot{\theta}^2 - 2\dot{r}^2\dot{\theta}}{r} = 1 - \alpha^2$$

and then

$$\lim_{t\to\infty} (r\dot{r}\ddot{\theta} + r\dot{\theta}^2 + 2\dot{r}^2\dot{\theta}) = 0.$$

Derivating (2.1), taking the limit for $t \to \infty$, we obtain $\lim_{t\to\infty} (r\dot{r}\ddot{\theta} + \dot{r}^2\dot{\theta}) = 0$, thus

$$0 = \lim_{t\to\infty} (r\dot{\theta}^2 + \dot{r}^2\dot{\theta}) = \lim_{t\to\infty} (r\dot{\theta} + \dot{r}^2)\dot{\theta} = \lim_{t\to\infty} \dot{\theta}$$

contradiction!

THEOREM 3.6. *Any helicoidal minimal surface with angular pitch α is congruent to an element of Σ.*

Proof. Set $\lambda = \{\lambda_t\}_{t\in R}$, and let S be an helicoidal minimal surface with angular pitch α. Up to congruence, we may assume that S is λ-invariant. Hence, it is generated by a curve β in P^2. From Lemma 3.5, there exists a rotation $\tilde{\theta}$ of P^2 around $\vec{o}$ such that $\tilde{\theta}(\beta) \in \Gamma$. Let θ be the extension of $\tilde{\theta}$ to H^3. Then, it is simple to verify that θ commutes with λ. Therefore, one has

$$\theta(S) = \theta(\lambda(\beta)) = \lambda(\tilde{\theta}(\beta)) \in \Sigma\,. \qquad \square$$

Let $h^\perp$ be the geodesic of P^2 containing $\vec{o}$ orthogonal to h.

PROPOSITION 3.7. *Assume $|\alpha| < 1$. Then, any curve of Γ different from γ_o is a concave graph over $h^\perp$.*

Proof. Let $\gamma_u \in \Gamma$, $u \neq 0$, and let $\theta = \theta(t)$ and $r = r(t)$ be the polar coordinates of γ_u. To prove the proposition we show that $\theta = \theta(t)$ is a strictly increasing or strictly decreasing function of t and that the geodesic curvature of γ_u is always positive.

The first statement is obvious since $\dot{\theta}(t_o) = 0$ in some point t_o, then γ_u would be the geodesic $\theta \equiv \theta(t_o)$ and $u = 0$, contradiction.

Since $\dot{r}(0) = 0$, from (2.3), we have

$$k(0) = \frac{(1-\alpha^2)r(0)\sqrt{r^2(0)+1}}{(1+\alpha^2)r^2(0)+1}$$

and, since $|\alpha| < 1$ and $r(0) > 0$, we see that $k(0) > 0$.

By contradiction, assume that $k(t_o) = 0$ in some point t_o. Therefore from (2.3) we obtain, at $t = t_o$,

$$(1+\alpha^2)(r^2+1)^2-2\alpha^2(\dot{r}^2+r^2(r^2+1)^2\dot{\theta}^2)=0$$

hence $\alpha \neq 0$ and

$$\frac{1+\alpha^2}{2\alpha^2}=\frac{\dot{r}^2}{(r^2+1)^2}+r^2\dot{\theta}^2 .$$

From (2.1), we finally obtain

$$\left(\frac{r\dot{r}}{1+r^2}\right)^2=\frac{\alpha^2-1}{2\alpha^2}$$

contradiction!

DEFINITION 3.8. Given $\gamma_u \in \Gamma$, let $\theta=\theta_u(t)$ be the angular coordinate of γ_u. We define the *angle at infinity* of γ_u by $\theta_\infty(u)=\lim_{t\to\infty}\theta_u(t)$.

It follows from Proposition 3.7 that $\theta_\infty(u)\in(0,\pi/2]$ for any $u\in[0,\infty)$.

LEMMA 3.9. *Let $u_1, u_2 \in R$, $0<u_1<u_2$, and let $\theta=\theta_1(t)$ and $\theta=\theta_2(t)$ be the angular coordinates of γ_{u_1} and γ_{u_2}, respectively. Assume $|\alpha|<1$ and $\theta_\infty(u_2)\le\theta_\infty(u_1)$. Then $\gamma_{u_1}\cap\gamma_{u_2}=\emptyset$.*

Proof. By contradiction, assume $\gamma_{u_1}\cap\gamma_{u_2}\neq\emptyset$. Therefore, rotating γ_{u_2} around $\vec{o}$ while keeping fixed γ_{u_1}, there will exist a moment in which γ_{u_1} and γ_{u_2} are tangent. But then, $\gamma_{u_1}=\gamma_{u_2}$, $u_1=u_2$, contradiction! □

Theorem A stated in the introduction is a consequence of the following result (together with Definition 3.3).

THEOREM 3.10. *Assume $|\alpha|<1$. Then the family Γ foliates P^2.*

Proof. If follows from Proposition 3.7, Remark 3.4 and Lemma 3.9 that we have just to prove that $\theta_\infty(u_1)>\theta_\infty(u_2)$ if $0<u_1<u_2$.

Consider the system of differential equations

$$(*)\qquad \begin{aligned}\dot{r}&=\frac{tr(r^2+1)[(1+\alpha^2)r^2+1]}{t^2(4r^2+1+3(1+\alpha^2)r^4)+(r^2+1)^2(2r^2+1)},\\ \dot{\theta}&=\frac{(r^2+1)[(1+\alpha^2)r^2+1]}{t^2(4r^2+1+3(1+\alpha^2)r^4)+(r^2+1)^2(2r^2+1)}.\end{aligned}$$

Assume that $r=r(t)$ and $\theta=\theta(t)$ satisfy (*). Then they verify (2.2). For observe that $\dot{r}/\dot{\theta}=tr$ so that $(d/dt)(\dot{r}/\dot{\theta})=r+t\dot{r}$, that is, $\dot{\theta}\ddot{r}-\ddot{\theta}\dot{r}=\dot{\theta}^2(r+t\dot{r})$ and replace these data in (2.2).

Given $u\in R^+$, let $r=r_u(t)$ and $\theta=\theta_u(t)$ be the solutions of (*) satisfying

$$r_u(0) = u$$
$$\theta_u(0) = 0 .$$

Let α_u be the curve in P^2 given by $\theta = \theta_u(t)$ and $r = r_u(t)$. It follows from the unicity of the solution curves with respect to the initial conditions that α_u is just a reparametrization of γ_u. Now, given $0 < u_1 < u_2 \in R$, we have $r_{u_1}(t) \neq r_{u_2}(t)$ for any t. Since $r_{u_1}(0) = u_1 < u_2 = r_{u_2}(0)$, we see that $r_{u_1}(t) < r_{u_2}(t)$ for any t. It follows from the expression of $\dot{\theta}$ in (*) that $\dot{\theta}_{u_1}(t) > \dot{\theta}_{u_2}(t)$ for any t. Therefore,

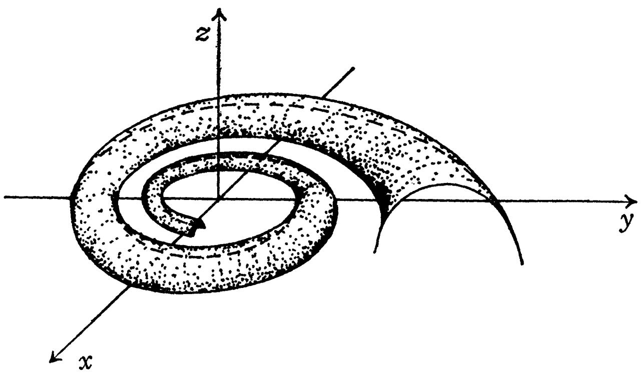
$$\theta_\infty(u_1) = \int_0^\infty \dot{\theta}_{u_1}(t)dt > \int_0^\infty \dot{\theta}_{u_2}(t)dt = \theta_\infty(u_2) ,$$

which proves the theorem.

PICTURE. In what follows we use the half-space model for hyperbolic space, namely

$$H^3 = \{(x, y, z) \mid z > 0\} .$$

Let $\lambda = \{\lambda_t\}$ be the helicoidal group of isometries which leaves invariant the geodesic axis z. We show below a typical surface S_u.



§ 4. Characterization of the helicoidal minimal surfaces

In this section we show that an helicoidal minimal surface is determined by its asymptotic boundary (see [doCL]). For, first we prove a result which relates the action of an helicoidal group on the asymptotic boundary of H^3 and loxodromic curves.

During this section we will use the *half-space model for the hyperbolic space.*

DEFINITION 4.1. Let p_1, p_2 be any two points of $S^2(\infty)$ and $\alpha \in [0, \pi/2]$. A differentiable curve $\gamma\colon R \to S^2(\infty)$ which makes an angle α with any circle of $S^2(\infty)$ containing p_1 and p_2 is called a *loxodromic curve with ending points p_1 and p_2 and path α.*

OBSERVATION 4.2. Let $\lambda = \{\lambda_t\}$ be a helicoidal group of isometries of H^3 which translation pitch α (that is, $\lambda_t = \phi_{\alpha t} \circ \varphi_t$, where $\{\phi_t\}$ is a translation along a geodesic g and $\{\varphi_t\}$ the spherical group fixing g).

Up to conjugation, we may assume that λ leaves invariant the geodesic axis Z (in half-space model). Thus, it is not difficult to see that

$$\lambda_t(X, Y, Z) = e^{\alpha t}\left(\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\begin{bmatrix}X\\ Y\end{bmatrix}, Z\right).$$

PROPOSITION 4.3. *Let γ be a differentiable curve in $S^2(\infty)$. Then, γ is a loxodromic curve if and only if γ is the orbit of some point in $S^2(\infty)$ under the action of an helicoidal group of isometries of H^3.*

Proof. We can identity $S^2(\infty) = \{(X, Y, 0) \mid X, Y \in R\} \cup \{Z = \infty\}$.

Let $\gamma\colon R \to S^2(\infty)$ be a loxodromic curve with ending points p_1, p_2 and path α. Up to a conformal map we may assume that $p_1 = (0, 0, 0)$ and $p_2 = (0, 0, \infty)$. Therefore, the circles connecting p_1 and p_2 are straight lines through the origin of $R^2 = \{(X, Y, 0) \mid X, Y \in R\}$.

Observe that the Euclidean structure of R^2 is compatible with the conformal structure of $S^2(\infty)$. Thus, if $\langle\, ,\rangle$ denotes the usual inner-product in R^2, we must have

$$\frac{\langle \gamma, d\gamma/dt\rangle}{\|\gamma\|\,\|d\gamma/dt\|} \equiv \cos\alpha = c \qquad 0 \le c \le 1\,.$$

If $c = 1$ or $c = 0$ then γ is straight line from p_1 to p_2 or a circle centered on $(0, 0, 0)$, respectively. Therefore, γ is the orbit of a translation (helicoidal group with angular pitch 0) or γ is the orbit of a spherical group (helicoidal group with translation pitch 0), respectively.

Assume that $0 < c < 1$. Setting $\gamma(t) = (X(t), Y(t), 0)$, we obtain

$$\frac{X\dot{X} + Y\dot{Y}}{\sqrt{X^2 + Y^2}\sqrt{\dot{X}^2 + \dot{Y}^2}} = c\,.$$

It is not difficult to show that γ can be described by equations of the type:

$$X(t) = r(t)\cos t$$
$$Y(t) = r(t)\sin t\,.$$

Thus, the above differential equation can be easily integrated, providing

$$\gamma(t) = e^{\beta t}\left(\begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}\right)\begin{pmatrix}e^b\\ 0\end{pmatrix}$$

where b is a constant and $\beta = c/\sqrt{1-c^2}$. This proves the proposition in one direction.

Conversely, given an helicoidal subgroup of isometries $\psi = \{\psi_t\}$, there exists an isometry $g\colon H^3 \to H^3$ such that $\psi_t = g\lambda_t g^{-1}$ (see classification in [R]). The computations above show that the orbits of $\lambda = \{\lambda_t\}$ are loxodromic curves. Thus, given $p \in S^2(\infty)$, we have

$$\psi(p) = \{\psi_t(p)\,|\,t \in R\} = \{g^{-1}\lambda_t(g(p))\,|\,t \in R\} = g^{-1}\lambda(g(p))\,.$$

Since g^{-1} acts conformally in $S^2(\infty)$, $\psi(p)$ is also a loxodromic curve.

DEFINITION 4.4. Two loxodromic curves $L_1, L_2 \subset S^2(\infty)$ having the same path and the same ending points will be called a *pair of loxodromic curves*. Notation: (L_1, L_2).

It follows from Proposition 4.3 that a loxodromic curve L has path α if and only if L is the orbit of an helicoidal group of angular pitch $\beta = \sin\alpha/\cos\alpha$. In particular $0 \le \beta < 1$ if and only if $0 \le \alpha < \pi/4$.

Proof of Theorem C. Up to a conformal map, we may assume that (L_1, L_2) has ending points $(0, 0, 0)$ and $(0, 0, \infty)$. Then (L_1, L_2) are $\{\lambda_t\}$-invariant. This follows from 4.2 and 4.3. Then, it follows from the hypothesis that $\{\lambda_t\}$ has angular pitch α such that $|\alpha| < 1$. Up to a rotation around the Z-axis, we may assume that the points $\{p_1\} = \partial_\infty P^2 \cap L_1$ and $\{p_2\} = \partial_\infty P^2 \cap L_2$ are symmetric with respect to the geodesic h (according to § 2).

Now, it follows from Proposition 3.7 and Definition 3.8 that the map $\theta_\infty\colon [0, \infty) \to (0, \pi/2]$ is continuous and $1-1$. Then, there exists $u_0 \in [0, \infty)$ such that $\partial_\infty\gamma_{u_0} = \{p_1, p_2\}$. Hence, $\partial_\infty S_{u_0} = L_1 \cup L_2$. Clearly, S_{u_0} is unique among the minimal complete helicoidal surfaces λ-invariant.

Let $M \subset H^3$ be a complete properly immersed minimal surface such that $\partial_\infty M = L_1 \cup L_2$.

Let $p_+ = h(+\infty)$ and $p_- = h(-\infty)$. Since $p_+ \notin \partial_\infty M$, there exists a totally geodesic semi-sphere H^2 in H^3 centered on p_+ such that $H^2 \cap M = \emptyset$ and $\partial_\infty M \cap \partial_\infty H^2 = \emptyset$. Hence, since $\partial_\infty M = L_1 \cup L_2$ is $\{\lambda_t\}$-invariant, we have $\lambda_t(\partial_\infty H^2) \cap \partial_\infty M = \emptyset$ for any $t \in R$. It follows from the Tangency Principle (see [doCL]) that $\lambda_t(H^2) \cap M = \emptyset$ for any t. Since $\bigcup_{t \in R} \lambda_t(H^2 \cap P) \subset \bigcup_{t \in R} \lambda_t(H^2)$, it follows that $[\bigcup_{t \in R} \lambda_t(H^2 \cap P^2)] \cap M = \emptyset$.

H^2 and P^2 are totally geodesic submanifolds of H^3, so that $H^2 \cap P^2$ is a geodesic in P^2, say β. Furthermore, since H^2 in centered on $p_+ = h(+\infty)$, β is orthogonal to h. Suppose that $\beta(R) \cap h(R) = \{h(u)\}$. Since the geodesic curvature of γ_u is always positive, we have $\beta(R) \cap \gamma_u(R) = \{h(u)\}$. It follows from the above that $S_u \cap M = [\bigcup_{t \in R} \lambda_t(\gamma_u(R))] \cap M = \emptyset$. Thus, from the Tangency Principle, we obtain $M \cap S_u = \emptyset$ for any $u > u_0$.

Applying the same arguments considering now the point $p_- = h(-\infty)$, we obtain $M \cap S_u = \emptyset$ for any $u < u_0$. Since $S_{u_0} = \lim_{u \to u_0^+} S_u = \lim_{u \to u_0^-} S_u$, we obtain $M = S_{u_0}$. □

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Universidade Federal do Rio Grande do Sul
Instituto de Matemática
Av. Bento Gonçalves, 9500
91.500-Porto Alegre-RS
Brazil