A CONVERGENCE THEOREM FOR RIEMANNIAN MANIFOLDS AND SOME APPLICATIONS

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§ 0. Introduction

The purpose of the present paper is first to reformulate a Lipschitz convergence theorem for Riemannian manifolds originally introduced by Gromov [17] and secondly to give some applications of the theorem to a class of open Riemannian manifolds.

Let $\mathcal{M}(m, \Lambda, I, D)$ denote the class of compact m-dimensional Riemannian manifolds M such that | the sectional curvature K_M of M | $\leq \Lambda^2$, the injectivity radius of $M \geq I > 0$, and the diameter of $M \leq D$. The original Gromov's compactness theorem [17: Theorems 8.25 and 8.28] says that given a sequence $\{M_n\}$ in $\mathcal{M}(m, \Lambda, I, D)$, there exist a subsequence $\{M_{n'}\}$ and a $C^{1,1}$ Riemannian manifold M_∞ such that $\{M_{n'}\}$ converges to M_∞ in the Lipschitz distance (see [17] for the definitions of $C^{1,1}$ Riemannian manifolds and the convergence in the Lipschitz distance). Gromov gave an outline of an argument to justify this theorem. Later, Katsuda [27] worked out Gromov's proof in full detail. On the other hand, very recently, Peters [33] and Greene-Wu [16], independently, have improved the original version as above in a different manner. Both of them make use of harmonic coordinates and the Peter's argument in [32]. Since the Gromov's theorem appeared, some applications have been found by several authors (cf. e.g., [33], [28] and the literature).

In this paper, we shall also reformulate and prove the convergence theorem in our manner. Our proof is similar to the Peters' or Greene-Wu's one, but more transparent than theirs. Our formulation of the convergence theorem is stated as follows:

THEOREM A. *Given a sequence $\{M_n\}_{n=1,2,\ldots}$ in $\mathcal{M}(m, \Lambda, I, D)$, there exist a subsequence $\{M_{n'}\}$ of $\{M_n\}$, a smooth manifold M_∞ and $C^{1,\beta}$ diffeomorphisms*

$\Phi_{n'}\colon M_{n'} \to M_\infty$ $(0 < \beta < 1)$ *such that the pushforward $\Phi_{n'*}g_{n'}$ of the metric $g_{n'}$ of $M_{n'}$ converges to a metric g_∞ of class $C^{1,\beta}$ in the $C^{1,\beta'}$-topology $(0 < \beta' < \beta < 1)$.*

As is pointed out in [33], the regularity of the limit metric is optimal in terms of Hölder conditions.

The second half of this paper is devoted to give some applications of Theorem A (or its proof) to Riemannian manifolds of asymptotically nonnegative curvature. We call a complete connected, noncompact Riemannian manifold *M of asymptotically nonnegative curvature* if the sectional curvature K_M of M satisfies

(H.1) $K_M \geq -k \circ r_o$

where r_o is the distance function to a fixed point o of M and $k(t)$ is a nonnegative, monotone nonincreasing function on $[0, \infty)$ such that the integral $\int^\infty tk(t)dt$ is finite. This class obviously contains the class of Riemannian open manifolds with nonnegative curvature everywhere. From the view point of *geometry at infinity*, it would be natural to study our class rather than the latter class. In [24], we have constructed a metric space $M(\infty)$ associated with a manifold M of asymptotically nonnegative curvature. Let us here explain it briefly (see [24] for details). We call two rays σ and γ of M equivalent if $\mathrm{dis}_M(\sigma(t), \gamma(t))/t$ goes to zero as $t \to \infty$. Define a distance δ_∞ on the equivalence classes by $\delta_\infty([\sigma], [\gamma]) := \lim_{t\to\infty} d_t(\sigma \cap S_t, \gamma \cap S_t)/t$ where S_t denotes the metric sphere around a fixed point of radius t and d_t stands for the inner (or intrinsic) distance on S_t induced from the distance $\mathrm{dis}_M(\ ,\ )$ on M. Then we have a metric space $M(\infty)$ of the equivalence classes of rays with the distance δ_∞ which is independent of the choice of the fixed point (i.e., the center of S_t) and to which a family of scaled metric spheres $\{(1/t)S_t\}$ converges with respect to the Hausdorff distance as t goes to infinity. We note that the complement $M\backslash B_{t_o}$ of a metric ball B_{t_o} with sufficiently large radius t_o is homeomorphic to $S_{t_o} \times (t_o, \infty)$. Actually M is isotopic to B_{t_o} (for large t_o). For simplicity we call a connected component of $M\backslash B_t$ (for large t) *an end* of M and denote it by $\mathscr{E}_\alpha(M)$ $(\alpha = 1, \cdots, \nu(M))$, where $\nu(M)$ is the number of the connected component of $M\backslash B_t$ or S_t (for large t). We write $M_\alpha(\infty)$ for the connected component of $M(\infty)$ corresponding to $\mathscr{E}_\alpha(M)$, so that $\{(1/t)S_t \cap \mathscr{E}_\alpha(M)\}$ converges to $M_\alpha(\infty)$ with respect to the

Hausdorff distance as $t \to \infty$ and then $M_a(\infty)$ turns out to be a compact inner metric space. Since $\mathrm{Vol}_{m-1}(S_t \cap \mathscr{E}_a(M))/t^{m-1}$ tends to a nonnegative constant as $t \to \infty$, let us write $\mathscr{V}ol_{m-1}(M_a(\infty))$ for the limit. Although the metric spheres of M around a fixed point are not smooth in general, we can approximate them by smooth hypersurfaces, e.g., the level hypersurfaces of the Riemannian convolution of the distance function to the fixed point. As is often the case, it is important to know the curvature of such smooth hypersurfaces. In Section 2, we shall prove the following

THEOREM B. *Let M be a manifold of asymptotically nonnegative curvature. Suppose that the sectional curvature K_M of M satisfies*:

(H.2) $$\kappa_M : = \limsup_{t \to \infty} t^2 K(t) < + \infty \, ,$$

where $K(t)$: $= \sup \{$the sectional curvature of M at points x with $\mathrm{dis}_M(o, x) \geq t\}$ and o is a fixed point of M. Let S_t be the metric sphere around o of radius t. Then for large t, there exists a smooth hypersurface S'_t of M which has the following properties:

(i) $(1/t) \max \{\max_{x \in S_t} \mathrm{dis}_M(x, S'_t), \max_{y \in S'_t} \mathrm{dis}_M(S_t, y)\} \to 0$ *as $t \to \infty$.*

(ii) *There is a Lipschitz homeomorphism $\varphi_t \colon S'_t \to S_t$ with*

$$e^{-\varepsilon(t)} \leq \frac{d_t(\varphi_t(x), \varphi_t(y))}{d'_t(x, y)} \leq e^{\varepsilon(t)}$$

where $\varepsilon(t)$ goes to zero as $t \to \infty$ and d_t (resp., d'_t) denotes the inner distance on S_t (resp. S'_t).

(iii) *The second fundamental form α'_t of S'_t is estimated by*

$$\{- (1 + a)\sqrt{\kappa_M} \tan a\sqrt{\kappa_M} - \varepsilon(t)\}g_M \leq t\alpha'_t \leq \left\{1 + \frac{1}{a} + \varepsilon(t)\right\}g_M \, ,$$

where a is a fixed constant with $0 < a < \pi/2\sqrt{\kappa_M}$.

Moreover if $\mathscr{V}ol_{m-1}(M_a(\infty)) > 0$, or equivalently $\lim_{t \to \infty} \mathrm{Vol}_m (B_t \cap \mathscr{E}_a(M))/t^m > 0$ for some end $\mathscr{E}_a(M)$, then one has a smooth approximation $\hat{S}_t$ with (i) and (ii) as above, the second fundamental form $\hat{\alpha}_t$ of which enjoys the following property:

$$(1 - \varepsilon(t))g_M \leq t\hat{\alpha}_t \leq (1 + \varepsilon(t))g_M$$

on $\hat{S}_t \cap \mathscr{E}_a(M)$.

Theorem B says in particular that under the additional condition

(H.2) as above, $M(\infty)$ is the limit (with respect to the Hausdorff distance) of a family of compact $(m - 1)$-dimensional Riemannian manifolds $\{M_t\}$ which are bounded uniformly in diameter and in curvature, and moreover when $\mathscr{V}ol_{m-1}(M_\alpha(\infty)) > 0$ for some α, the volume of the connected component of M_t converging to $M_\alpha(\infty)$ may be assumed to have a positive lower bound uniformly in t. Hence in this case (noncollapsing case), it turns out from Theorem A that $M_\alpha(\infty)$ is a smooth $(m - 1)$-dimensional manifold with $C^{1,\beta}$-metric $(0 < \beta < 1)$. As for the other case (collapsing case), i.e., $\mathscr{V}ol_{m-1}(M_\alpha(\infty)) = 0$, we can apply the theory originated by Gromov and developed by Fukaya (cf. e.g., [12] and the literature) to our situation.

In the remaining sections, we shall show two applications of Theorem B. One of them, Theorem 3.2, is concerning the total curvature of a manifold of asymptotically nonnegative curvature and the other, Theorem 4.1, is on gap phenomena modeled after Euclidean space.

Theorem A is concerning a family of Riemannian manifolds in $\mathscr{M}(m, \Lambda, I, D)$. However a similar result is still valid for a family of Riemannian manifolds in $\mathscr{M}(m, \Lambda, I, \infty)$ (cf. § 1). In this case, we should consider a pair (M, p) of a Riemannian manifold M and a point p of M, and the topology of convergence in Theorem A should be appropriately modified, because $\mathscr{M}(m, \Lambda, I, \infty)$ may contain complete, noncompact Riemannian manifolds. This will be discussed elsewhere.

This paper is a revised version of a part of [23] which was completed while the author was a member of the Mathematical Sciences Research Institute at Berkeley. He greatly appreciates the institute for its hospitality.

§ 1. Proof of Theorem A

The purpose of this section is to prove Theorem A. Before going into the proof, we shall recall first some facts on harmonic coordinates. We refer the reader to e.g., Jost [21] and Greene-Wu [16] for details.

1.1. Let M be a compact Riemannian manifold of dimension m with Riemannian metric g_M. Given a point o of M and a unit tangent vector u at o, we first define the almost linear function $\ell_u(x)$ associated with u as follows: Let $r(x) := \mathrm{dis}_M(o, x)$, $p(x) := \exp_o r(x)u$, $q(x) := \exp_o - r(x)u$ and then set $\ell_u(x) := \{\mathrm{dis}_M(x, q(x))^2 - \mathrm{dis}_M(x, p(x))^2\}/4r(x)$. We assume

from now that the sectional curvature K_M of M satisfies: $|K_M| \leq \Lambda^2$ for a constant Λ and the injectivity radius of M at $o \geq I$ for a constant $I > 0$. Let δ be a number less that min $\{I/2, \pi/4\Lambda\}$ and let $\bar{u}(x)$ be the vector at $x \in B_\delta(o)$ obtained by the parallel translation of u from o to x along the radial geodesic. Then for each $x \in B_\delta(o)$, we have

(i) $|\ell_u(x)| \leq r(x)$,

(ii) $|\text{grad } \ell_u(x) - \bar{u}(x)| \leq 2\Lambda^2 (\sinh 2\Lambda r(x)/\sin 2\Lambda r(x))r^2(x)$,

(iii) $|\nabla^2 \ell_u(x)| \leq \{9\Lambda^2 (\sinh 2\Lambda r(x)/\sin 2\Lambda r(x))\Lambda r(x) \coth \Lambda r(x)\}r(x)$.

Let us take here an orthonormal basis $\{u_1, \cdots, u_m\}$ of T_oM and set $\ell_i := \ell_{u_i}$ ($i = 1, \cdots, m$). Let $h_i: B_\delta(o) \to \mathbf{R}$ ($i = 1, \cdots, m$) be the solution of the Dirichlet problems: $\Delta h_i = 0$ on $B_\delta(o)$ and $h_i = \ell_i$ on $\partial B_\delta(o)$, and define a harmonic map $\mathbf{H}: B_\delta(o) \to \mathbf{R}^m$ by $\mathbf{H}(x) := (h_1(x) - h_1(o), \cdots, h_m(x) - h_m(o))$. Then we have the following

FACT 1.1. *Let m, Λ and I be as before. Then there exists a positive constant $\delta(m, \Lambda, I)$ depending only on m, Λ, and I that given any number $\delta \leq \delta(m, \Lambda, I)$, the above harmonic map $\mathbf{H}: B_\delta(o) \to \mathbf{R}^m$ defines a coordinate system around o which has the following properties:*

(i) $(1 + \eta_0(m, \delta\Lambda))^{-1} r(x) \leq |\mathbf{H}(x)| \leq (1 + \eta_0(m, \delta\Lambda)) r(x)$

(ii) $(1 + \eta_0(m, \delta\Lambda))^{-1} |\xi|^2 \leq \sum_{i,j=1}^m g_{i,j}(x)\xi^i\xi^j \leq (1 + \eta_0(m, \delta\Lambda))|\xi|^2$

(iii) $|g_{i,j}|_{C^{1,\beta}(B_\delta(o))} \leq \eta_1(m, \Lambda, I, \beta)$ $(0 < \beta < 1)$

where we set $g_{i,j}(x) := g_M(\nabla h_i, \nabla h_j)$. For any harmonic function f on $B_\delta(o)$, one has

(iv) $|f|_{C^{2,\beta}(B_{\delta/2}(o))} \leq \eta_2(m, \Lambda, I, \beta) \sup_{B_\delta(o)}|f|$.

Moreover given an integer k and a constant Λ_k, suppose that the norm of the i-th covariant derivatives $(0 \leq i \leq k)$ of the Ricci tensor of M is bounded by Λ_k. Then one has

(v) $|g_{i,j}|_{C^{1+k,\beta}(B_\delta(o))} \leq \eta_3(m, \Lambda, I, \beta, \Lambda_k)$

and for any harmonic function f on $B_\delta(o)$,

(vi) $|f|_{C^{2+k,\beta}(B_{\delta/2}(o))} \leq \eta_4(m, \Lambda, I, \beta, \Lambda_k) \sup_{B_\delta(o)}|f|$.

Here the constants η_i ($i = 1, \cdots, 4$) depend only on the given constants m, Λ, I, β, k, Λ_k as indicated respectively.

1.2. Let us now prove Theorem A in three steps.

Step 1. Let M be a compact Riemannian manifold which belongs to $\mathcal{M}(m, \Lambda, I, D)$. Fix a positive constant δ such that $0 < 4\delta < \delta(m, \Lambda, I)$, where $\delta(m, \Lambda, I)$ is as in Fact 1.1. Then we take a subset Γ of finite points $p_1, \cdots, p_\mu$ of M with $\text{dis}_M(p_i, p_j) > 2\delta$ ($i \neq j$) and $\text{dis}_M(\Gamma, p) \leq 4\delta$ for any

p of M. By the Bishop's comparison theorem, we see that $\mathrm{Vol}_m (B_\delta(p_i))$ $\geq c_1$ for some positive constant c_1 depending only on m, $\varLambda$, and δ, and furthermore $\mathrm{Vol}_m (M) \leq c_2$ for some positive constant c_2 depending only on m, $\varLambda$ and D. Hence we have

$$\mu c_1 \leq \sum_{i=1}^\mu \mathrm{Vol}_m (B_\delta(p_i)) \leq \mathrm{Vol}_m(M) \leq c_2 .$$

This shows that the number μ as above is bounded by a constant μ_δ depending only on m, $\varLambda$, I, D, and δ. In what follows, for the sake of simplicity, we assume that the above μ is equal to μ_δ, by putting $p_i = p_\mu$ for i: $\mu \leq i \leq \mu_\delta$.

Step 2. Set $\delta_0 := \delta(m, \varLambda, I)$, $\eta_0 := \eta_0(m, \delta_0\varLambda)$, $\delta_1 := 10^{-1}(1 + \eta_0)^{-6}\delta_0$, and $\delta_k := (1 + \eta_0)^k\delta_1$ $(2 \leq k \leq 6)$, where $\delta(m, \varLambda, I)$ and $\eta_0(m, \delta_0\varLambda)$ are as in Fact 1.1. We take $\delta_1/4$ as a fixed constant δ as in Step 1 and choose a subset $\varGamma$ of finite points $p_1, \cdots, p_\mu$ $(\mu = \mu_{\delta_1/4})$ of M such that $\mathrm{dis}_M(\varGamma, p) \leq \delta_1$ for any $p \in M$. For each i: $1 \leq i \leq \mu$, we have a harmonic coordinates H_i: $B_{\delta_0}(p_i) \to \boldsymbol{R}^m$ with the properties described in Fact 1.1. Observe first by Fact 1.1 that

$$\boldsymbol{B}^m((1 + \eta_0)^{-1}\delta) \subset H_i(B_\delta(p_i)) \subset \boldsymbol{B}^m((1 + \eta_0)\delta)$$

for any δ: $0 < \delta < \delta_0$, where $\boldsymbol{B}^m(t) := \{v \in \boldsymbol{R}^m : |v| \leq t\}$. Take a smooth function ξ: $[0, \infty) \to [0, \infty)$ such that $\xi(t) \equiv 1$ on $[0, \delta_2]$ and $\xi(t) \equiv 0$ on $[\delta_3, \infty)$. Set $\xi_i := \xi(|H_i|)$ $(i = 1, \cdots, \mu)$. Then each ξ_i is a smooth function on M such that the support of ξ_i is contained in $B_{\delta_6}(p_i)$ and $\xi_i \equiv 1$ on $B_{\delta_5}(p_i)$. Let us now define a smooth map $\mathscr{H}_M$: $M \to \boldsymbol{R}^N$ $(N = (m + 1)\mu)$ by $\mathscr{H}_M := (\xi_1 H_1, \cdots, \xi_\mu H_\mu, \xi_1, \cdots, \xi_\mu)$. Then $\mathscr{H}_M$ turns out to be a smooth embedding of M into a ball $\boldsymbol{B}^N(R_0)$ in $\boldsymbol{R}^N$, where R_0 is a constant depending only on m, $\varLambda$, I, and D. We fix i: $1 \leq i \leq \mu$, say $i = 1$ for simplicity. Then $\mathscr{H}_M(B_{\delta_3}(p_1))$ can be represented as a graph over $H_1(B_{\delta_3}(p_1))$, namely, $\mathscr{H}_M(B_{\delta_3}(p_1)) = \{(x, f_2 F_2, \cdots, f_\mu F_\mu, f_1, \cdots, f_\mu): x = (x_1, \cdots, x_m) \in H_1(B_{\delta_3}(p_1))\}$, where $F_j = H_j \circ H_1^{-1}$ and $f_j = \xi(|F_j|)$. Note here that $H_1(B_{\delta_3}(p_1)) \subset \boldsymbol{B}^m(\delta_2) \subset H_1(B_{\delta_3}(p_1))$. Thus by Fact 1.1, we see that for any β: $0 < \beta < 1$, the $C^{2,\beta}$ norm of F_j and the $C^{2,\beta}$ norm of F_j and the $C^{2,\beta}$ norm of f_j on $\boldsymbol{B}^m(\delta_2)$ are bounded from above by a constant c_3 depending only on m, $\varLambda$, I and β This implies in particular that the length of the second fundamental form α_M of the embedding $\mathscr{H}_M$: $M \to \boldsymbol{B}^N(R_0)$ is bounded by a constant c_4 depending only on m, $\varLambda$, I and β.

Step 3. Let $\{M_n\}$ be a sequence of Riemannian manifolds in

$\mathscr{M}(m, \Lambda, I, D)$. We identity M_n (resp., the metric g_n on M_n) with the image of M_n by the embedding $\mathscr{H}_M$ constructed in Step 2 (resp., $\mathscr{H}_{M_n*}g_n$). Then by the observations in Step 2, we have a subsequence $\{M_{n'}\}$ of $\{M_n\}$ and a compact submanifold M_∞ of class $C^{2,\beta}$ embedded in $B^N(R_0)$ such that $\{M_{n'}\}$ converges to M_∞ in the $C^{2,\beta'}$ topology $(0 < \beta' < \beta < 1)$ as $n' \to \infty$. In what follows, we consider sufficiently large n' so that the projection $\Pi_{n'}: M_{n'} \to M_\infty$ along the normals to M_∞ induces a $C^{2,\beta}$-diffeomorphism from $M_{n'}$ onto M_∞. Hence we have a sequence of $C^{1,\beta}$ metrics $\{\Pi_{n'*}g_{n'}\}$ on M_∞. We claim here that, taking a subsequence $\{\hat{n}\}$ of $\{n'\}$ if necessary,

($\sharp$) $\{\Pi_{\hat{n}*}g_{\hat{n}}\}$ converges to a $C^{1,\beta}$ metric g_∞ on M_∞ in $C^{1,\beta'}$ topology
       $(0 < \beta' < \beta < 1)$ as $\hat{n} \to \infty$.

In fact, let $\{p_{n',i}\}_{i=1,\dots,\mu}$ be as in Step 2 for $M_{n'}$ and fix an index i, say $i = 1$ for simplicity. Moreover let P (resp., $P_{n'}$) be the orthogonal projection from R^N onto $R^m = \{(x_1, \cdots, x_m, 0, \cdots, 0) \in R^N\}$ (resp., the restriction of P to the metric ball $B_{\delta_3}(p_{n',1})$ of $M_{n'}$ around $p_{n',1}$ with radius δ_3). Define a $C^{2,\beta}$ diffeomorphism $\phi_{n'}$ from $B^m(\delta_2)$ into R^m by $\phi_{n'} := P \circ \Pi_{n'} \circ P_{n'}^{-1}$. Then the $C^{2,\beta}$-norm of $\phi_{n'}$ is bounded uniformly in n', and hence $\phi_{n'}$ converges to the identity map in $C^{2,\beta'}$ topology $(0 < \beta' < \beta)$. We set $g_{n';i,j} := (P_{n'*}g_{n'})(\partial/\partial x_i, \partial/\partial x_j)$ $(i,j = 1, \cdots, m)$. Then the $C^{1,\beta}$-norm of $g_{n';i,j}$ on $B^m(\delta_2)$ is bounded uniformly in n'. Hence taking a subsequence $\{n''\}$ of $\{n'\}$ if necessary, we see that as $n'' \to \infty$, $\phi_{n''*}P_{n''*}g_{n''}$ converges to a $C^{1,\beta}$ metric on $B^m(\delta_2)$ in $C^{1,\beta'}$ topology. Then it is easy to derive the above claim ($\sharp$) from this observation. This completes the proof of Theorem A.

1.3. It would be of some use to restate Theorem A as follows:

THEOREM A′. *Let $\{M_n\}$ be a sequence of Riemannian manifolds in $\mathscr{M}(m, \Lambda, I, D)$. Suppose that given a nonnegative integer k, the norm of the i-th covariant derivative $(0 \le i \le k)$ of the Ricci tensor on M_n is bounded uniformly in n by a constant Λ_k. Then there exist a subsequence $\{M_{n'}\}$ of $\{M_n\}$, a smooth manifold M_∞, and $C^{1+k,\beta}$-diffeomorphisms $\Phi_{n'}: M_{n'} \to M_\infty$ $(0 < \beta < 1)$ such that as n' goes to infinity, the pushforward of the metric of $M_{n'}$ by $\Phi_{n'}$ converges to a metric of class $C^{1+k,\beta}$ in $C^{1+k,\beta'}$ topology $(0 < \beta' < \beta < 1)$. Moreover, if in addition to the above assumptions, $\{M_n\}$ is a sequence of Kähler manifolds, i.e., $M_n = (M_n, g_n, J_n)$, then the pushforward of the almost complex structure $J_{n'}$ of $M_{n'}$ converges in $C^{1+k,\beta'}$ topology to an integrable almost complex structure J_∞ on M_∞ which is*

parallel with respect to the limit metric on M_∞.

Proof. The first part is a consequence of the argument of the proof for Theorem A and Fact 1.1 (v). As for the second part, we observe that the $C^{1+k,\beta}$-norm of the almost complex structure J_n on M_n (with respect to the harmonic coordinates described in Fact 1.1) is bounded by a constant depending only on m, Λ, I, D, β and Λ_k, since J_n is parallel with respect to the metric of M_n. Therefore for a subsequence $\{n'\}$ of $\{n\}$, $\{\Phi_{n'*}J_{n'}\}$ converges in $C^{1+k,\beta'}$ topology to an almost complex structure J_∞ of class $C^{1+k,\beta}$ on M_∞ as $n' \to \infty$. Since J_n is integrable, so is J_∞ (cf. [30]) and moreover since J_n is parallel with respect to the metric of M_n, so is J_∞ with respect to the limit metric on M_∞. This completes the proof of Theorem A'.

Let $M_{n'}$, $g_{n'}$, M_∞ and g_∞ be as in Theorem A. In the proof of Theorem A, we used the apriori estimates in Hölder spaces. If we apply the apriori estimates in Sobolev spaces, we see that the components of the limilt metric g_∞, expressed in the harmonic coordinates discribed in Fact 1.1, are contained in the Sobolev spaces $W^{2,p}$ ($p \geq 1$). In particular, the curvature tensor R_∞ of g_∞ are almost everywhere defined (cf. [31] [33: Theorem 5.3]). Moreover the curvature tensor $R_{n'}$ of $g_{n'}$ converges weakly to R_∞ as $n' \to \infty$, namely, for any smooth covariant 4-tensor T, the inner product $\int_{M_{n'}} \langle T, R_{n'} \rangle = \int_{M_{n'}} T^{ijkl} R_{n';ijkl}$ converges to $\int_{M_\infty} \langle T, R_\infty \rangle$. This holds for the Ricci tensors or the scalar curvatures of $g_{n'}$.

Let us now give an application of Theorem A. In order to state it, we need some notations. For a compact Riemannian manifold M of dimension m with metric g_M, we set

$$\mathscr{R}(M) := \frac{1}{\mathrm{Vol}\,(M)^{(m-2)/m}} \int_M \left\| S_M - \frac{c(M)}{m} g_M \right\|,$$

$$\mathscr{K}(M) := \frac{1}{\mathrm{Vol}\,(M)^{(m-2)/m}} \int_M \left\| R_M - \frac{c(M)}{m(m-1)} G_M \right\|,$$

$$\mathscr{L}(M) := \frac{1}{\mathrm{Vol}\,(M)^{(m-3)/m}} \int_M \| \nabla R_M \|,$$

where S_M (resp. R_M) stands for the Ricci tensor of M (resp. the curvature tensor of M), G_M is a covariant 4-tensor defined by $G_M(W, Z, X, Y) = g_M(W, X)g_M(Z, Y) - g_M(Z, X)g_M(Y, W)$, and $c(M)$ denotes the average of the scalar curvature ρ_M of M: $c(M) := \int_M \rho_M / \mathrm{Vol}\,(M)$.

THEOREM 1.1. *Let M be a Riemannian manifold in $\mathcal{M}(m, \Lambda, I, D)$ and δ a positive number. Then there exists a positive constant $\varepsilon(m, \Lambda, I, D; \delta)$ depending only on m, Λ, I, D and δ, such that*

(i) *if $\mathcal{R}(M) < \varepsilon(m, \Lambda, I, D; \delta)$, then M is diffeomorphic to an Einstein manifold M' and the Lipschitz distance between M and M' is less than δ;*

(ii) *if $\mathcal{K}(M) < \varepsilon(m, \Lambda, I, D; \delta)$, then M is diffeomorphic to a space form M' of constant curvature and the Lipschitz distance between M and M' is less than δ;*

(iii) *if $\mathcal{L}(M) < \varepsilon(m, \Lambda, I, D; \delta)$, then M is diffeomorphic to a locally symmetric space M' and the Lipschitz distnce between M and M' is less than δ.*

Theorem 1.1 is a consequence from the following

LEMMA 1.2. *Let $M_{n'}$, $g_{n'}$, M_{∞} and g_{∞} be as in Theorem A.*

(i) *If $\mathcal{R}(M_{n'})$ goes to zero as $n' \to \infty$, then g_{∞} is smooth and define an Einstein metric, i.e., the Ricci tensor S_{∞} of g_{∞} satisfies: $S_{\infty} = (c(M_{\infty})/m)\, g_{\infty}$, where $c(M_{n'})$ converges to $c(M_{\infty})$ as $n' \to \infty$.*

(ii) *If $\mathcal{K}(M_{n'})$ goes to zero as $n' \to \infty$, then g_{∞} is smooth and has constant curvature $c(M_{\infty})/m(m-1)$.*

(iii) *If $\mathcal{L}(M_{n'})$ goes to zero as $n' \to \infty$, then g_{∞} is smocth and M_{∞} is locally symmetric, i.e., the curvature tensor R_{∞} of g_{∞} is parallel.*

Proof. Since $\{c(M_{n'})\}$ is a bounded sequence, we have a subsequence $\{c(M_{\hat{n}})\}$ which converges to a constant c as $\hat{n} \to \infty$. Observe first that the components $g_{n'}^{ij}$ of the induced metric on the cotangent bundle $T^*M_{n'}$, expressed in the harmonic coordinates described in Fact 1.1, satisfies

$$\Delta_{n'} g_{n'}^{ij} = 2 \sum_{k,\ell} g_{n'}^{ik} g_{n'}^{j\ell} S_{n';k\ell} + 2 \sum_{p,q,r,s} g_{n'}^{pq} g_{n'}^{rs} \Gamma_{n';pr}{}^{i} \Gamma_{n';qs}{}^{j}$$

where $\Delta_{n'}$ (resp. $\Gamma_{n';pr}{}^{i}$) stands for the Laplacian of g_{∞} (resp. the Christoffel symbols of g_{∞}). Suppose that $\mathcal{R}(M_{n'})$ goes to zero as $n' \to \infty$. Then it follows that $g_{n'}^{ij}$ satisfies weakly

$$\Delta_{\infty} g_{\infty}^{ij} = \frac{2c}{m} g_{\infty}^{ij} + 2 \sum_{p,q,r,s} g_{\infty}^{pq} g_{\infty}^{rs} \Gamma_{\infty;pr}{}^{i} \Gamma_{\infty;qs}{}^{i} .$$

Hence the standard regularity argument implies that g_{∞}^{ij} is smooth, and furthermore the Ricci tensor S_{∞} satisfies: $S_{\infty} = (c/m)g_{\infty}$. Moreover if $\mathcal{K}(M_{n'})$ goes to zero as $n' \to \infty$, then $\mathcal{K}(M_{\infty}) = 0$ and hence M_{∞} has constant curvature. Finally, let us assume that $\mathcal{L}(M_{n'})$ goes to zero as n'.

Since the covariant derivative $\nabla R_{n'}$ of the curvature tensor $R_{n'}$ of $M_{n'}$ converges weakly to the covariant derivative $\nabla \cdot R_{\infty}$ (in the weak sense) of the curvature tensor R_{∞} of M_{∞} as $n' \to \infty$, we see that $\nabla \cdot R_{\infty} \equiv 0$. Then it follows from the regularity argument that g_{∞} is smooth and R_{∞} is parallel. This completes the proof of Lemma 1.2.

§ 2. Proof of Theorem B

The purpose of this section is to prove Theorem B in Introduction. We shall carry out the proof, based on (the proof of) Theorem A and some facts given in [24]. Throughout this section, let us denote by M a manifold of asymptotically nonnegative curvature. r_p, $B_t(p)$ and $S_t(p)$, respectively, stand for the distance to a point p of M, the metric ball around p of radius t and the metric sphere around p of radius t.

2.1. To begin with, we define a Lipschitz function $F_p\colon M \to R$ associated with a family of metric spheres $\{S_t(p)\}$ around a point p of M by $F_p(x) := \lim_{t \to \infty} t - \mathrm{dis}_M(x, S_t(p))$. In order to prove Theorem B, we have to recall the following two facts:

FACT 2.1 ([24: Lemma 1.4]).

(i) *For any fixed point p of M, $F_p(x)/r_p(x)$ converges to 1 as x goes to infinity. In particular, $F_p\colon M \to R$ is an exhaustion function on M, namely, $\{x \in M\colon F_p(x) \le t\}$ is compact for any $t \in R$.*

(ii) *As $x \in M$ goes to infinity,*

$$\max\{\measuredangle\,(u, v)\colon u, v \in \nabla \cdot r_p(x)\} \to 0\,,$$
$$\max\{\measuredangle\,(u, v)\colon u \in \nabla \cdot r_p(x),\ v \in \nabla \cdot F_p(x)\} \to 0\,,$$

where $\nabla \cdot r_p(x) := \{v \in T_x M\colon |v| = 1,\ t + r_p\,(\exp_x - tv) = r_p(x)\ (0 \le t \le r_p(x))\}$ and $\nabla \cdot F_p := \{v \in T_x M\colon |v| = 1,\ F_p\,(\exp_x tv) - t = F_p(x)\ (t \ge 0)\}$.

FACT 2.2 ([24: Lemma 1.5]). *Let o be the base point in (H.1) and J_k the solution of an equation: $J_k'' + kJ_k = 0$, subject to the initial conditions: $J_k(0) = 0$ and $J_k'(0) = 1$, where k is as in (H.1). Then for any large $t \gg 0$ and small $\varepsilon > 0$, there is a constant $\delta(t, \varepsilon) > 0$ such that the Riemannian mollifier r_δ of $r\,(:= r_o)$ $(0 < \delta \le \delta(t, \varepsilon))$ is well defined and smooth on $B_t(o)$, and it has the following properties:*

(i) $|r - r_\delta| \le \varepsilon$,

(ii) $1 - \varepsilon - \theta_1(r - \varepsilon) \le |\nabla r_\delta| \le 1 + \varepsilon$,

(iii) $1 - \varepsilon \le |\nabla r_\delta|(x)$ *if* $\mathrm{dis}_M(x, \mathscr{E}_0) \ge \varepsilon$,

(iv) $\nabla^2 r_\delta \leq (1 + \varepsilon)(\log J_k)' \circ r_\delta$,

where $\mathscr{E}_0$ stands for the cut locus of M with respect to the base point o and $\theta_1(s) := \max\{\measuredangle(u, v) : u, v \in \nabla \cdot r(x), r(x) \geq s\}$. Moreover the Riemannian mollifier F_δ of $F(:= F_0)$ is also well defined and smooth on $B_t(o)$, and it satisfies:

(v) $|F - F_\delta| \leq \varepsilon$,

(vi) $|\nabla F_\delta - \nabla r_\delta| \leq \varepsilon + \theta_2(r - \varepsilon)$,

(vii) $\nabla^2 F_\delta \geq -(1 + \varepsilon) \displaystyle\int_F^\infty k(s)ds$ on $\{x \in B_t(o) : F(x) > 0\}$ if $k \not\equiv 0$ near $+\infty$;

$$\nabla^2 F_\delta \geq -\varepsilon - \int_F^\infty k(s)ds \text{ on } \{x \in B_t(o) : F(x) > 0\} \text{ if } k \equiv 0 \text{ near } +\infty,$$

where $\theta_2(s) := \max\{\measuredangle(u, v) : u \in \nabla \cdot r(x), v \in \nabla \cdot F(x), r(x) \geq s\}$.

2.2. Let us now give the proof of Theorem B which is devided into three steps. In what follows, we assume that M satisfies (H.2) in Theorem B.

Step 1. Let $F(:= F_0)$ be the Lipschitz function associated with a family of the metric spheres $\{S_t(o)\}$ around a point, say the base point o in (H.1). Let us fix sufficiently large numbers T and T' with $T < T'$ and sufficiently small numbers $\varepsilon, \delta\ (> 0)$ with $\delta \leq \delta(T', \varepsilon)$ as in Fact 2.2. We consider the Riemannian mollifier F_δ of F on $B_{T'}(o)$, and set $\Omega_{\delta,T} := \{x \in M : F_\delta(x) \leq T\}$ and $\Sigma_{\delta,T} := \partial\Omega_{\delta,T}$. For any $x \in \Sigma_{\delta,T}$, we denote by $\eta_x : [0, \infty) \to M$ the geodesic which emanates from x and which is tangent to the outer unit normal of $\Sigma_{\delta,T}$ at x. We define three numbers $\tau_1(x)$, $\tau_2(x)$, and $\tau_3(x) \in (0, \infty]$ associated with η_x as follows: $\tau_1(x) := \inf\{t \in (0, \infty) : \eta_x(t) \in \Omega_{\delta,T}\}$, $\tau_2(x) := \inf\{t \in (0, \infty) : \Sigma_{\delta,T}$ has no focal points along $\eta_{x|[0,t]}\}$, and $\tau_3(x) := \sup\{t \in (0, \infty) : \text{dis}_M(\eta_x(s), \Sigma_{\delta,T}) = s$ for $s \in [0, t]\}$. Clearly, $\tau_3(x) \leq \min\{\tau_1(x)/2, \tau_2(x)\}$. For the sake of simplicity, we assume that k is not constantly equal to zero. Then we have

(2.1) $\tau_3(x) \geq \mu_1(T, \varepsilon) := \dfrac{T - \varepsilon}{\Lambda(T - \varepsilon)} \arctan\left[\dfrac{(1 - \theta(T, \varepsilon))\Lambda(T - \varepsilon)}{(1 + \varepsilon)\lambda(T)}\right]$,

where $\Lambda(t) := t\sqrt{K(t)}$, $K(t) := \sup\{$the sectional curvature of M at points x with $r(x) \geq t\}$, $\lambda(t) := \displaystyle\int_t^\infty k(s)ds$, $\theta(t, \varepsilon) := 2\varepsilon + \theta_1(t - \varepsilon) + \theta_2(t - \varepsilon)$, and θ_i $(i = 1, 2)$ are as in Fact 2.2. Note that

(2.2) $$\dfrac{\mu_1(T, \varepsilon)}{T} \to \dfrac{\pi}{2\sqrt{\kappa_M}}\ (\in (0, \infty])$$

as $\varepsilon \to 0$ and $T \to \infty$. In order to prove (2.1), we first observe that the second fundamental form $\alpha_{\delta,T}$ of $\Sigma_{\delta,T}$ with respect to the outer unit normal satisfies

$$(2.3) \qquad \alpha_{\delta,T} = \frac{1}{|\nabla F_{\delta}|^2} \nabla^2 F_{\delta} \geq -\frac{(1+\varepsilon)}{1-\theta(T,\varepsilon)} \int_T^{\infty} k(u)du \,,$$

because of Fact 2.2 (vi), (vii). Let $J_{T,\varepsilon}$ be the solution of an equation: $J_{T,\varepsilon}'' + (T-\varepsilon)^{-2}\Lambda^2(T-\varepsilon)J_{T,\varepsilon} = 0$, subject to the initial conditions: $J_{T,\varepsilon}(0) = 1$ and $J_{T,\varepsilon}'(0) = -(1+\varepsilon)\lambda(T)/(1-\theta_{\delta}(T,\varepsilon))$, and set $\mu_2(T,\varepsilon) := \inf\{t > 0; J_{T,\varepsilon}(t) = 0\}$. Then $\Sigma_{\delta,T}$ has no focal points along $\eta_{x|[0,t]}$ as long as $\eta_x([0,t]) \subset M \setminus \Omega_{\delta,T}$ and $t < \mu_2(T,\varepsilon)$, because we have (2.3) and the sectional curvature of M is bounded from above by $(T-\varepsilon)^{-2}\Lambda^2(T-\varepsilon)$ on $M \setminus \Omega_{\delta,T}$. Since $\mu_2(T,\varepsilon) \geq \mu_1(T,\varepsilon)$, we have obtained

$$(2.4) \qquad\qquad\qquad \tau_2(x) \geq \mu_1(T,\varepsilon)$$

if $\tau_2(x) \leq \tau_1(x)$. Suppose now that $\min\{\tau_3(x) : x \in \Sigma_{\delta,T}\} < \mu_1(T,\varepsilon)$. Then it turns out from (2.4) that there exists a geodesic $\eta : [0, 2\ell] \to M$ such that $\ell = \min\{\tau_3(x) : x \in \Sigma_{\delta,T}\}$, $\eta(t) = \eta_x(t)$ $(t \in [0,\ell])$ for some $x \in \Sigma_{\delta,T}$, and $\eta(t) = \eta_y(2\ell - t)$ $(t \in [\ell, 2\ell])$ for some $y \in \Sigma_{\delta,T}$. Since we have by Fact 2.2 (vii): $d^2 F_{\delta}(\eta(t))/dt^2 \geq -(1+\varepsilon)\int_T^{\infty} k(s)ds$ on $[0, 2\ell]$, it follows that $(F_{\delta} \circ \eta)'(2\ell) - (F_{\delta} \circ \eta)'(0) \geq -2\ell(1+\varepsilon)\int_T^{\infty} k(s)ds$. This implies that

$$\ell \geq \left[(1+\varepsilon)\int_T^{\infty} k(s)\,ds\right]^{-1} \geq \frac{T}{(1+\varepsilon)\lambda(T)} \,.$$

On the other hand, ℓ is assumed to be less than $\mu_1(T,\varepsilon)$, and hence we have

$$\frac{\Lambda(T-\varepsilon)}{(1+\varepsilon)\lambda(T)} < \arctan \frac{(1-\theta(T,\varepsilon))\Lambda(T-\varepsilon)}{(1+\varepsilon)\lambda(T)} < \frac{(1-\theta(T,\varepsilon))\Lambda(T-\varepsilon)}{(1+\varepsilon)\lambda(T)} \,.$$

This is a contradiction. Thus we have shown (2.1).

  Step 2. Fix positive constants a, b with $a < b < \pi/2\sqrt{\kappa_M} \, (\leq +\infty)$. Then taking a sufficiently large number T and a sufficiently small number ε, we may assume by (2.1) and (2.2) that $\min\{\tau_3(x) : x \in \Sigma_{\delta,T}\} > bT$. Set $\rho_{\delta,T} := \mathrm{dis}_M(\Omega_{\delta,T}, *)$, $\tilde{\Omega}_{\delta,T} := \{x \in M : 0 < \rho_{\delta,T}(x) \leq bT\}$ and $\tilde{\Sigma}_{\delta,T} := \{x \in M : \rho_{\delta,T}(x) = aT\}$. Since $\rho_{\delta,T}$ is smooth on $\tilde{\Omega}_{\delta,T}$, $\tilde{\Sigma}_{\delta,T}$ is also smooth. Moreover it follows from (2.3) that

$$\nabla^2 \rho_{\delta,T} \geq (\log J_{T,\varepsilon})' \circ \rho_{\delta,T}(g_M - d\rho_{\delta,T}^2)$$

on $\tilde{\Omega}_{\delta,T}$, where $J_{T,\varepsilon}$ is as in Step 1. In particular, the second fundamental form $\tilde{\alpha}_{\delta,T}$ of $\tilde{\Sigma}_{\delta,T}$ satisfies

$$(2.5) \qquad \tilde{\alpha}_{\delta,T} \geq \mu_3(T,\varepsilon,a)g_M \ (\mu_3(T,\varepsilon,a) := (\log J_{T,\varepsilon})'(aT)).$$

Note that

$$(2.6) \qquad (1+a)T\mu_3(T,\varepsilon,a) \to -(1+a)\sqrt{\kappa_M}\tan a\sqrt{\kappa_M}$$

as $\varepsilon \to 0$ and $T \to \infty$. In order to get an upper bound of $\tilde{\alpha}_{\delta,T}$, we compare $\rho_{\delta,T}$ with $r\,(=r_o)$. Since

$$\frac{d}{dt}F_\delta(\eta_x(t)) = 1 + \int_0^t \frac{d^2}{ds^2}F_\delta(\eta_x(s))ds$$

$$\geq 1 - (1+\varepsilon)\int_T^\infty k(s)ds \cdot t \qquad \text{(by Fact 2.2 (vii))}$$

$$\geq 1 - (1+\varepsilon)\int_T^\infty sk(s)ds \cdot \frac{t}{T}$$

$$\geq 1 - (1+\varepsilon)v\lambda(T)$$

for any $x \in \Sigma_{\delta,T}$ and $t \in [0, bT]$, we see that

$$(2.7) \qquad \langle \nabla F_\delta, \nabla \rho_{\delta,T}\rangle \geq 1 - (1+\varepsilon)b\lambda(T)$$

on $\tilde{\Omega}_{\delta,T}$. In particular, we have by Facts 2.1 and 2.2

$$(2.8) \quad \min\{\langle\nabla\rho_{\delta,T}, v\rangle : v \in \nabla \cdot r(x), \ x \in \Omega_{\delta,T}\} \geq 1 - (1+\varepsilon)b\lambda(T) - \varepsilon_1(T)$$

where $\varepsilon_1(T)$ goes to zero as $T \to \infty$. Moreover it follows from (2.7) that

$$F_\delta \geq T + \{1 - (1+\varepsilon)s\lambda(T)\}\rho_{\delta,T}$$

and hence

$$(2.9) \qquad r \geq T - \varepsilon + \{1 - (1+\varepsilon)b\lambda(T)\}\rho_{\delta,T}$$

on $\tilde{\Omega}_{\delta,T}$, since $F_\delta \leq F + \varepsilon \leq r + \varepsilon$. On the other hand, by Fact (2.1(i)), we have

$$(2.10) \qquad r \leq \rho_{\delta,T} + (1+\varepsilon_2(T))(T+\varepsilon)$$

where $\varepsilon_2(T)$ goes to zero as $T \to \infty$. It turns out from (2.9) that

$$(2.11) \qquad \nabla^2\rho_{\delta,T} < \left[\frac{1}{\rho_{\delta,T}} + \frac{\lambda(T-\varepsilon)}{(T-\varepsilon)\{1-(1+\varepsilon)b\lambda(T)\}}\right](g_\mu - d\rho_{\delta,T}^2)$$

on $\tilde{\Omega}_{\delta,T}$. In fact, by (2.9), we see that the sectional curvature at $\eta_x(t) \geq -k(r(\eta_x(t))) \geq -k(T - \varepsilon + \{1 - (1+\varepsilon)b\lambda(T)\}t)$. Let $\hat{J}_{T,\varepsilon}$ be the solution of an equation: $\hat{J}''_{T,\varepsilon} - k(T - \varepsilon + \{1 - (1+\varepsilon)b\lambda(T)\}t)\hat{J}_{T,\varepsilon} = 0$, with $\hat{J}_{T,\varepsilon}(0) = 0$ and $\hat{J}'_{T,\varepsilon}(0) = 1$. Then we have

$$\nabla^2 \rho_{\delta,T} < (\log \hat{J}_{T,\varepsilon})' \circ \rho_{\delta,T}(g_M - d\rho_{\delta,T}^2)$$

and

$$(\log \hat{J}_{T,\varepsilon})'(t) = \frac{1}{\hat{J}_{T,\varepsilon}(t)}\left[1 + \int_0^t k(T - \varepsilon + \{1 - (1+\varepsilon)b\lambda(T)\}s)\hat{J}_{T,\varepsilon}(s)ds\right]$$

$$\leq \frac{1}{\hat{J}_{T,\varepsilon}(t)}\left[1 + \hat{J}_{T,\varepsilon}(t)\int_0^t k(T - \varepsilon + \{1 - (1+\varepsilon)b\lambda(T)\}s)ds\right]$$

$$\leq \frac{1}{t} + \frac{\lambda(T - \varepsilon)}{(T - \varepsilon)\{1 - (1+\varepsilon)b\lambda(T)\}} \cdot$$

In particular, (2.11) implies that

$$(2.12) \quad \hat{\alpha}_{\delta,T} < \mu_4(T, \varepsilon, a)g_M \left(\mu_4(T, \varepsilon, a) := \frac{1}{aT} + \frac{\lambda(T - \varepsilon)}{(T - \varepsilon)\{1 - (1+\varepsilon)b\lambda(T)\}}\right).$$

Note that

$$(2.13) \qquad\qquad (1 + a)T\mu_4(T, \varepsilon, a) \to 1 + \frac{1}{a}$$

as $\varepsilon \to 0$ and $T \to \infty$. Finally, let us summarize the results obtained above. For any large number T, we choose sufficiently small $\delta_T > 0$ with $\lim_{T\to\infty}\delta_T = 0$ and set $\tilde{S}_T := \tilde{\Sigma}_{\delta_{\hat{T}},\hat{T}}$ ($\hat{T} := T/(1 + a)$). Then we have

(i) $\max\{\max_{x\in S_T(o)} \text{dis}_M(x, \tilde{S}_T), \max_{y\in\tilde{S}_T} \text{dis}_M(S_T(o), y)\}/T \to 0$
as T goes to infinity (cf. (2.9) and (2.10));

(ii)   the integral curve of $r_{\delta_T}/|\nabla r_{\delta_T}|^2$ defines a Lispchitz homeomorphism φ_T from $\tilde{S}_T$ onto $S_T(o)$ such that

$$e^{-\varepsilon(T)} \leq \frac{d_T(\varphi_T(x), \varphi_T(y))}{\tilde{d}_T(x, y)} \leq e^{\varepsilon(T)},$$

where $\varepsilon(T)$ goes to zero as $T \to \infty$ and $\tilde{d}_T$ (resp., d_T) denotes the inner distance on $\tilde{S}_T$ (resp., $S_T(o)$) (cf. (2.8) and Fact 2.1);

(iii)   the second fundamental form $\tilde{\alpha}_T$ of $\tilde{S}_T$ in M satisfies

$$(1 + a)\sqrt{\kappa_M} \tan a\sqrt{\kappa_M} - \varepsilon(T) \leq T\tilde{\alpha}_T \leq 1 + \frac{1}{a} + \varepsilon(T),$$

where a is a constant with $0 < a < \pi/2\sqrt{\kappa_M}$ (cf. (2.5) and (2.12)). Thus we have proved the first assertion of Theorem B.

Step 3. We are now in the position to prove the second assertion of the theorem, by applying Theorem A to our situation. Suppose that $\mathrm{Vol}_m\,(B_t(o) \cap \mathcal{E}_\alpha(M))/t^m$ is bounded away from zero as $t \to \infty$, for some end $\mathcal{E}_\alpha(M)$ of M. This is equivalent to saying that

$$\mathcal{V}ol_{m-1}(M_\alpha(\infty)) \overset{(\text{def.})}{:=} \lim_{t \to \infty} \mathrm{Vol}_{m-1}(S_t(o) \cap \mathcal{E}_\alpha(M))/t^{m-1}$$

$$= \lim_{t \to \infty} \mathrm{Vol}_{m-1}(\tilde{S}_t \cap \mathcal{E}_\alpha(M))/t^{m-1}$$

$$> 0,$$

where $\{\tilde{S}_t\}$ is as in Step 2. For the sake of simplicity, we assume that M has one end, so that $\mathcal{V}ol_{m-1}(M(\infty))$ is positive. Then if the dimension m of M is greater than or equal to 3, the family $\{(1/t)\tilde{S}_t\}$ of compact Riemannian manifolds is bounded uniformly in curvature, diameter and volume, and hence it turns out from Theorem A that when $m \geq 3$, $M(\infty)$ is a smooth manifold with a Riemannian metric of class $C^{1,\beta}$ ($0 < \beta < 1$), since $M(\infty)$ is the limit of $\{(1/t)\tilde{S}_t\}$ with respect to the Hausdorff distance. In order to construct smooth approximations $\{\tilde{S}_t\}$ stated in the second assertion of Theorem B, we first observe that the injectivity radius of M at a point x is bounded from below by $cr(x)$ for some positive constant c. Actually, this follows from the argument in the proof of the first assertion of the theorem. Secondly, let us consider a family of Riemannian manifolds $\{(1/t)M\}$ and set $A_t(a, b) := \{x \in (1/t)M: b \leq \mathrm{dis}_t\,(o, x) \leq a\}$, where $\mathrm{dis}_t\,(o, x) := (1/t)\,\mathrm{dis}_M\,(o, x)$. Then we have the following

LEMMA 2.3. *Fix two positive numbers a, b with $a > b$. Then for large t, there exists a $C^{2,\beta}$-diffeomorphism Π_t ($0 < \beta < 1$) from $A_t(a, b)$ into the cone $\mathcal{C}(M(\infty))$ over $M(\infty)$, i.e., $\mathcal{C}(M(\infty)) := [0, \infty) \times_{t^2} M(\infty)$ which has the following properties: as t goes to infinity, $\Pi_t(A_t(a, b))$ converges to $[a, b] \times_{t^2} M(\infty)$ and $\Pi_{t*}G_t$ converges to the metric $dt^2 + t^2 g_\infty$ in $C^{1,\beta}$ topology ($0 < \beta' < \beta < 1$), where G_t (resp., g_∞) is the Riemannian metric of $(1/t)M$ (resp., $M(\infty)$).*

The second assertion of Theorem B is an immediate consequence of Lemma 2.3. Actually, (after fixing constants a, b as above with $b < 1 < a$,) we can take the hypersurfaces $\hat{S}_t := \Pi_t^{-1}(\{1\} \times M(\infty))$ as required approximations for the metric spheres $S_t(o)$.

Proof of Lemma 2.3. We first observe that the injectivity radius of $(1/t)M$ on A_t $(10a, b/10)$ is bounded from below by a positive constant c (independent of t) and the sectional curvature K_t of $(1/t)M$ (for large t) is pinched as follows: $-\varepsilon(t) \le K_t \le 10^4 b^{-2} \kappa_M$, where $\varepsilon(t)$ goes to zero as $t \to \infty$. Given a small positive constant $\delta < 10^{-2}b$, we take a finite number of points $x_{t,1}, \cdots, x_{t,\mu}$ of $A_t(4a, b/4)$ (for large t) such that $\mathrm{dis}_t(x_{t,i}, x_{t,j}) \ge 2\delta$ $(i \ne j)$, $\mathrm{dis}_t(x_{t,i}, o) > b/10 + 2\delta$, and the union of the metric balls $B_{t,2\delta}(x_{t,i})$ of $(1/t)M$ around $x_{t,i}$ of radius 2δ covers $A_t(2a, b/2)$. Then the number μ is bounded from above by a positive constant μ_0 which is independent of t. In fact, by setting $\varLambda := 10b^{-1}\sqrt{\kappa_M}$, we have

$$\mu\omega_{m-1}\int_0^\delta \left[\frac{\sin \varLambda u}{\varLambda}\right]^{m-1} du \le \mathrm{Vol}_m\left(\bigcup_{1\le i \le \mu} B_{t,\delta}(x_{t,i})\right)$$

$$\le \frac{1}{t^m}\mathrm{Vol}_m\left(B_{6at}/o\right)$$

$$\le \frac{\omega_{m-1}}{t^m}\int_0^{6at} J_k^{m-1}(u)du \qquad \text{(cf. [24: Lemma 1.2])}$$

$$\le \frac{\omega_{m-1}}{t^m}\int_0^{6at} (J_k'(\infty)u)^{m-1}du$$

$$\le \frac{1}{m}\omega_{m-1}(J_k'(\infty))^{m-1}(6a)^m$$

where J_k is as in Fact 2.2 and $J_k'(\infty) := \lim_{t\to\infty} J_k'(t)\left(< \exp\int_0^\infty tk(t)dt\right)$ (cf. [14: Theorem C])). Thus we can apply the argument of the proof of Theorem A to our situation and obtain the maps $\varPi_t \colon A_t(a, b) \to \mathscr{C}(M(\infty))$ (for large t) mentioned in Lemma 2.3, since $A_t(a, b)$ converges to $[a, b] \times_{t^2} M(\infty)$ with respect to the Hausdorff distance as t goes to infinity. This completes the proof of Lemma 2.3.

2.3. Let us now give two corollaries to Theorem B.

CORLLARY 2.4. *Let M be as in Theorem B. Suppose that the dimension m of M is greater than or equal to 3 and, for some end $\mathscr{E}_a(M)$ of M, $\mathscr{V}ol_{m-1}(M_a(\infty))$ is positive. Then:*

(i) *$M_a(\infty)$ is a compact smooth manifold of dimension $m-1$ with $C^{1,\beta}$ metric $(0 < \beta < 1)$ such that the diameter $\mathrm{diam}(M_a(\infty))$ is not greater than π and the volume $\mathrm{Vol}_{m-1}(M_a(\infty))$ is equal to $\mathscr{V}ol_{m-1}(M_a(\infty))$ and not greater than the volume of a metric ball with radius $\mathrm{diam}(M_a(\infty))$ in the*

sphere of constant curvature 1.

(ii) *If* $\kappa_M < 3$ *and* $M_a(\infty)$ *is simply connected, or if m is odd and* $M_a(\infty)$ *is orientable, then the injectivity radius of* $M_a(\infty)$ *is not less than* $\pi/\sqrt{1 + \kappa_M}$ *and* $\mathrm{Vol}_{m-1}(M_a(\infty))$ *is not less than the volume of a metric ball with radius* $\pi/\sqrt{1 + \kappa_M}$ *in the sphere of constant curvature* $1 + \kappa_M$.

(iii) *If* $\mathrm{diam}(M_a(\infty)) \geq \pi/2$, $\mathrm{Vol}_{m-1}(M_a(\infty))$ *is not less than the volume of a metric ball with radius* $\pi/2\sqrt{1 + \kappa_M}$ *in the sphere of constant curvature* $1 + \kappa_M$.

Proof. Since $M_a(\infty)$ is the limit (in $C^{1,\beta'}$ topology) of a family of compact Riemannian manifolds $\{\hat{S}_t\}$ such that their diameters are uniformly bounded from above, their volumes are uniformly bounded from below by a positive constant, and further the curvature K_t of $\hat{S}_t$ satisfies: $1 - \varepsilon(t) \leq K_t \leq 1 + \kappa_M + \varepsilon(t)$, where $\varepsilon(t)$ goes to zero as $t \to \infty$, the first assertion (resp., the second assertion, the last assertion) turns out to be true because of the Rauch's comparison theorem (resp., the Klingenberg's theorem (cf. [4: Ch. 5]), Lemma 2 in [3]). This completes the proof of Corollary 2.4.

COROLLARY 2.5. *Let M be as in Theorem B. Suppose that the dimension m of M is greater than or equal to 3 and for an end $\mathscr{E}_a(M)$ of M, $\mathscr{V}ol_{m-1}(M_a(\infty))$ is positive. Then $M_a(\infty)$ is isometric to the space form of constant curvature 1, if an integral $\int_{\mathscr{E}_a(M)} |\rho_M|^p (r+1)^{2p-m}$ is finite for some $p \geq 1$, where ρ_M (resp, r) denotes the scalar curvature of M (resp. the distance to a fixed point).*

Proof. In what follows, we assume for the sake of simplicity that M has one end and we keep the same notations as in Lemma 2.3. Suppose that the integral $\int_M |\rho_M|^p (r + 1)^{2p-m}$ is finite for some $p \geq 1$. We observe that this is equivalent to the condition that the integral $\int_M \|R_M\|^p (r + 1)^{2p-m}$ is finite, since $\|R_M\| \leq c_m(|\rho_M| + k \circ r)$ for some constant c_m depending only on m, where R_M denotes the curvature tensor of M and k is as in (H.1). Then it follows that $\int_{A_t(a,b)} \|R_t\|$ converges to zero as $t \to \infty$, where R_t is the curvature tensor of $(1/t)M$. Therefore, applying the same argument of Lemma 1.2 to the family $\{A_t(a, b)\}$, we see that the cone $\mathscr{C}(M(\infty))$ is flat, nemely $M(\infty)$ has constant curvature 1. This completes the proof of

Corollary 2.5.

Remark. In Corollary 5.5, the condition: $\int_{\mathscr{C}_a(M)} |\rho_M|^p (r+1)^{2p-m} < +\infty$

can be replaced by a weaker one: $\liminf t^{2p-m} \int_{A(at,bt)} |\rho_M|^p = 0$, where

$A(at, bt) = \{x \in M: bt \leq r(x) \leq at\}$ $(0 < b < a)$.

Remark 2.6. Let M be a manifold of asymptotically nonnegative curvarure. Then the condition (H.2), namely, the finiteness of κ_M, is obviously equivalent to the condition that the scalar curvature ρ_M of M satisfies: $\sup r^2 \rho_M < +\infty$. Moreover given a nonnegative integer k, suppose that the Ricci tensor S_M of M satisfies: $\|\nabla^i S_M\| \leq c_i/r^{i+2}$ for each i: $0 \leq i \leq k$ and some positive constants c_i. Then according to Theorem A′, Lemma 2.3 can be restated as follows: Fix two positive numbers a, b with $a > b$. Then for large t, there exists a C^{2+k}-diffeomorphism Π_t $(0 < \beta < 1)$ from $A_t(a, b)$ into the cone $\mathscr{C}(M(\infty))$ over $M(\infty)$, i.e., $\mathscr{C}(M(\infty)):= [0, \infty) \times_{t^2} M(\infty)$, which has the following properties: as t goes to infinity, $\Pi_t(A_t(a, b))$ converges to $[a, b] \times_{t^2} M(\infty)$ and $\Pi_{t*} G_t$ converges to the metric $dt^2 + t^2 g_\infty$ in $C^{1+k, \beta'}$ topology $(0 < \beta' < \beta < 1)$, where G_t (resp. g_∞) is the Riemannian metric of $(1/t)M$ (resp. $M(\infty)$).

§ 3. The total curvature of a manifold of asymptotically nonnegative curvature

A well known theorem of Cohn-Vossen [8] states that if the total curvature integral $\int_M K_M$ on a complete oriented two dimensional Riemannian manifold M is absolutely convergent, then $(1/2\pi) \int_M K_M \leq \chi(M)$, where K_M (resp., $\chi(M)$) denotes the Gaussian curvature (resp., the Euler characteristic) of M. Later, several authors, e.g., Huber [19, 20], Finn [11], Maeda [29], Shiohama [35, 36] etc., interpreted the difference: $\chi(M) - (1/2\pi) \int_M K_M$. On the other hand, some authors, e.g., Greene-Wu [13], Poor [34], Walter [38] etc., made their attempts to generalize the Cohn-Vossen's inequality to 4- (or 6-) dimensional Riemannian manifolds of nonnegative sectional curvature (outside a compact set). The purpose of this section is to study some relations between the total curvature and the geometry at infinity of a manifold with asymptotically nonnegative curvature.

Throughout this section, M is an oriented manifold of asymptotically

nonnegative curvature and m denotes the dimension of M.

3.1. To begin with, we shall prove the following

PROPOSITION 4.1. *Let M be as above and suppose $m = 2$, then the following assertions hold:*

(i) *The total curvature $\int_M K_M$ is absolutely convergent.*

(ii) $\chi(M) = (1/2\pi)\int_M K_M + \operatorname{diam}(M(\infty))/\pi.$

(iii) *In the case*: $K_M \geq 0$,

$$2\operatorname{diam}(M(\infty)) = \inf_{p \in M} \operatorname{Length}(\mathscr{R}_p^T) = \inf_{D} \frac{[\operatorname{Length}(\partial D)]^2}{2\operatorname{Area}(D)},$$

where $\mathscr{R}_p^T := \{v \in T_p M : |v| = 1, \exp_p tv \in \mathscr{R}_p$ (the set of all rays starting at p)\} and D ranges over all compact domains of M with smooth boundary ∂D.

Proof. We shall prove the first assertion, referring to [24]. We write K_M^+ (resp., K_M^-) for $\max\{0, K_M\}$ (resp., $\max\{0, -K_M\}$). Then observe first that the integral $\int_M K_M^-$ is finite, because of Hypothesis (H.1) (cf. [24: 1.3]). We fix a point p of M and denote by $\tilde{\mathscr{R}}_p^T$ the part of $\mathscr{R}_p^T$ which consists of a finite number of closed subarcs in the tangent unit circle parametrized by the arc-length θ ($0 \leq \theta \leq 2\pi$). Then for sufficiently large t, we have

$$\chi(M) = \chi(B_t(p))$$
$$= \frac{1}{2\pi}\int_{B_t(p)} K_M + \frac{1}{2\pi}\int_{\tilde{S}_t(p)} \kappa_{g,t} + \delta(t),$$

where $\tilde{S}_t(p) := S_t(p) \cap \exp_p t\tilde{\mathscr{R}}_p^T$, $\kappa_{g,t}$ is the geodesic curvature of $\tilde{S}_t(p)$ and $\delta(t)$ goes to zero as $t \to \infty$ (cf. [24: Lemma 1.2 and Proposition 2.3]). Since $\liminf_{t\to\infty}\int_{\tilde{S}_t(p)} \kappa_{g,t} \geq 0$ (cf. [ibid.: Lemma 1.2 (iii)]), we have

(3.1) $$\int_M |K_M| < +\infty.$$

This proves the assertion (i). To show the second assertion, it is enough to prove that

(3.2) $$\int_{\tilde{S}_t(p)} \kappa_{g,t} \to 2\operatorname{diam}(M(\infty)) = \lim_{t\to\infty} \frac{\operatorname{Length}(S_t(p))}{t}$$

as $t \to \infty$. This is done as follows. For any $v(\theta) = (\sin\theta, \cos\theta) \in \tilde{\mathscr{R}}_p^T$, set

$J_\theta(t) := d\exp_{p|tv(\theta)} t(\partial/\partial\theta)$ (the Jacobi field along the ray $\exp_p tv(\theta)$ with $J_\theta(0)$ $= 0$ and $J'_\theta(0) = \partial/\partial\theta$ at $v(\theta)$). Then it turns out that $J_\theta(t)$ astisfies

$$|J_\theta(t)|'' + K_M(\exp_p tv(\theta))|J_\theta| = 0.$$

In partuculary, we have

$$|J_\theta(t)|' = 1 - \int_0^t K_M(\exp_p sv(\theta))|J_\theta(s)|\,ds.$$

It follows from (3.1) that for almost all $v(\theta)$, the limit $\lambda(\theta)$ of $|J_\theta(t)|'$ as $t \to \infty$ exists and it is given by

$$\lambda(\theta) = 1 - \int_0^\infty K_M(\exp_p sv(\theta))|J_\theta(s)|\,ds.$$

Observe here that $\lambda(\theta) = \lim_{t\to\infty}|J_\theta(t)|/t$ for almost all $v(\theta)$. Therefore we get

$$\int_{\tilde{S}_p(t)} \kappa_{g,t} = \int_{\tilde{\mathscr{R}}_p^T} |J_\theta(t)|'\,d\theta \to \int_{\tilde{\mathscr{R}}_p^T} \lambda(\theta)\,d\theta$$

$$\frac{\text{Length}(\tilde{S}_p(t))}{t} = \int_{\tilde{\mathscr{R}}_p^T} \frac{|J_\theta(t)|}{t}\,d\theta \to \int_{\tilde{\mathscr{R}}_p^T} \lambda(\theta)\,d\theta$$

as $t \to \infty$. This shows (3.2), because $2\,\text{diam}(M(\infty)) = \lim_{t\to\infty}\text{Length}(\tilde{S}_p(t))/t$ (cf. [24: Proposition 2.3]). Moreover it turns out from the above observation that

(3.3) $$2\,\text{diam}(M(\infty)) = \text{Length}(\tilde{\mathscr{R}}_p^T) - \int_{\mathscr{D}_p} K_M,$$

where $\mathscr{D}_p := \{\exp_p tv(\theta): t \geq 0, v(\theta) \in \tilde{\mathscr{R}}_p^T\}$. Let us now suppose that $K_M \geq 0$ everywhere on M. Then we may assume that M is diffeomorphic to $\mathbf{R}^2$, because otherwise, M is a flat cylinder $\mathbf{R} \times S^1$. Taking account of the Toponogov's splitting theorem, we see that for any (small) number $\varepsilon > 0$, there is a point p_ε of M with

(3.4) $$\int_{\mathscr{D}_{p_\varepsilon}} K_M \leq \varepsilon$$

(cf. [29: p. 457]). Now it follows from (3.3) and (3.4) that for any $\varepsilon > 0$,

$$\inf_{p \in M}\text{Length}(\tilde{\mathscr{R}}_p^T) \geq 2\,\text{diam}(M(\infty))$$

$$\geq \text{Length}(\tilde{\mathscr{R}}_p^T) - \varepsilon$$

$$\geq \inf_{p \in M}\text{Length}(\tilde{\mathscr{R}}_p^T) - \varepsilon.$$

Thus we have shown that $2 \operatorname{diam}(M(\infty)) = \inf_{p \in M} \operatorname{Length}(\mathscr{R}_p^T)$. Finally we shall prove that $\operatorname{diam}(M(\infty)) = I_2(M)^2$, where $I_2(M) := \inf \operatorname{Length}(\partial D)/\operatorname{Area}(D)^{1/2}$. This is a consequence of the second assertion (ii) as above and an isoperimetric inequality due to Fiala [10] and Huber [18]. Actually their isoperimetric inequality tells us that, for any simply connected domain D of M,

$$\frac{\operatorname{Length}(\partial D)^2}{4\pi \operatorname{Area}(D)} \geq 1 - \frac{1}{2\pi} \int_D K_M^-.$$

Thus we have

$$\frac{\operatorname{diam}(M(\infty))}{\pi} \geq \frac{I_2(M)^2}{4\pi}$$

$$\geq 1 - \frac{1}{2\pi} \int_M K_M$$

$$= \frac{\operatorname{diam}(M(\infty))}{\pi}.$$

Remarks. (i) The second assertion can be also derived from the first assertion and a result by Shiohama [35] (cf. also Finn [11] and Huber [20]).

(ii) Let M be a manifold of asymptotically nonnegative curvature. Then as is mentioned in Introduction, for each end $\mathscr{E}_\alpha(M)$, we have the limit $\mathscr{V}ol_{m-1}(M_\alpha(\infty))$ of $\operatorname{Vol}_{m-1}(S_t(o) \cap \mathscr{E}_\alpha(M))/t^{m-1}$ as $t \to \infty$, which is independent of the choice of the reference point o. Moreover it is not hard to see that

$$\mathscr{V}ol_{m-1}(M_\alpha(\infty)) = \lim_{t \to \infty} \frac{m \operatorname{Vol}_m(B_t(o) \cap \mathscr{E}_\alpha(M))}{t^m}$$

$$= \lim_{t \to \infty} \frac{[\operatorname{Vol}_{m-1}(S_t(o) \cap \mathscr{E}_\alpha(M))]^m}{[m \operatorname{Vol}_m(B_t(o) \cap \mathscr{E}_\alpha(M))]^{m-1}}.$$

Thus $\mathscr{V}ol_{m-1}(M(\infty)) := \sum_{\alpha=1}^{\nu(M)} \mathscr{V}ol_{m-1}(M_\alpha(\infty))$ is related to an isoperimetric constant $I_m(M)$ defined by

$$I_m(M) := \inf \frac{\operatorname{Vol}_{m-1}(\partial \Omega)}{[\operatorname{Vol}_m(\Omega)]^{(m-1)/m}},$$

where Ω ranges over all bounded domains of M with smooth boundaries. Then, we have

$$I_m(M) \leq m^{(m-1)/m} [\mathscr{V}ol_{m-1}(M(\infty))]^{1/m}.$$

Moreover if the Ricci curvature of M is nonnegative everywhere, we see that

$$I_m(M) \geq C(m) \left[\mathscr{V}ol_{m-1}(M(\infty))\right]^{1/m},$$

where $C(m)$ is a positive constant depending only on m. Actually, in this case, the measure of rays starting at p with respect to the metric on the unit tangent sphere at p is bounded from below by $\mathscr{V}ol_{m-1}(M(\infty))$, so that we can apply a theorem of Croke [7] to our situation.

3.2. As we have seen just above, the situation on M is much simpler when the dimension m is two. However it is more complicated in higher dimensional cases. In the next theorem, we consider the case as in Theorem B.

THEOREM 3.2. *Let M be a manifold of asymptotically nonnegative curvature satisfying the condition (H.2) as in Theorem B. Suppose that the dimension m is greater than or equal to 3. Then the following assertions hold*:

(i) *If the Gauss-Bonnet-Chern integral: $\int_M \Omega_M$ is absolutely convergent, then*

$$\left| \chi(M) - \int_M \Omega_M \right| \leq (1 + \Lambda(m, \kappa_M)) \frac{\mathscr{V}ol_{m-1}(M(\infty))}{\omega_{m-1}},$$

where κ_M is as in (H.2), $\Lambda(m, \kappa_M)$ is a nonnegative constant depending only on m and κ_M, and ω_{m-1} denotes the volume of the unit $(m-1)$-sphere of Euclidean space. Moreover if $\kappa_M = 0$, or $\int_M |\rho_M|^p (r+1)^{2p-m}$ is finite for some $p \geq 1$, then

$$\chi(M) = \int_M \Omega_M + \frac{\mathscr{V}ol_{m-1}(M(\infty))}{\omega_{m-1}}$$

$$= \int_M \Omega_M + \sum_\alpha \frac{1}{\#(\pi_1(M_\alpha(\infty)))},$$

where the summation is taken over the ends $\mathscr{E}_\alpha(M)$ with $\mathscr{V}ol_{m-1}(M_\alpha(\infty)) > 0$.

(ii) *The Gauss-Bonnet-Chern integral $\int_M \Omega_M$ is absolutely convergent, if one of the following conditions holds*:

(ii-a) *m is odd $(\Omega_M: \equiv 0$ by definition$)$.*

(ii-b) $m = 4$.

(ii-c) *The total scalar curvature* $\int_M |\rho_M|^{m/2}$ *is finite.*

(ii-d) *The sectional curvature of M is nonnegative everywhere and M is not diffeomorphic to $\mathbf{R}^m$ (in this case, $\mathscr{V}ol_{m-1}(M(\infty)) = 0$).*

(iii) *In the case: $m = 3, 4$ or 5, one has*

$$\chi(M) - \int_M \Omega_M \geq \frac{\mathscr{V}ol_{m-1}(M(\infty))}{\omega_{m-1}} \geq 0 .$$

COROLLARY 3.3. *Let M be as in Theorem* 3.2 *and suppose that m is odd. Then one has the following assertions:*

(i) $\chi(M) \neq 0 \Rightarrow \mathscr{V}ol_{m-1}(M(\infty)) > 0$.

(ii) $\chi(M) \neq 0 \Longleftrightarrow \mathscr{V}ol_{m-1}(M(\infty)) > 0$, *if $\kappa_M = 0$, or* $\int_M |\rho_M|^p (r+1)^{2p-m}$ *is finite for some $p \geq 1$.*

(iii) $\chi(M) \neq 0 \Longleftrightarrow \mathscr{V}ol_{m-1}(M(\infty)) > 0$ *if $m = 3$ or 5.*

(iv) $\chi(M) \neq 0 \Longleftrightarrow \mathscr{V}ol_{m-1}(M(\infty)) > 0 \Longleftrightarrow M$ *is diffeomorphic to $\mathbf{R}^m$, if M has nonnegative sectional curvature.*

Proof. The assertions (i)–(iii) are direct consequences of Theorem 3.2. The last one is derived from Theorem 3.2 and the structure theorem by Cheeger-Gromoll [6].

COROLLARY 3.4. *Let M be as in Theorem* 3.2. *Suppose that $m = 4$ and*

$$\chi(M) - \frac{1}{2(\sqrt{1 + \kappa_M})^3} < \int_M \Omega_M .$$

Then:

(i) $\mathscr{V}ol_3(M(\infty)) < \mathrm{Vol}_3(B_{r(M)}(1 + \kappa_M))$, *where $r(M) := \pi/2\sqrt{1 + \kappa_M}$ and $B_r(1 + \kappa_M)$ is the metric ball of radius r in the 3-sphere of constant curvature $1 + \kappa_M$.*

(ii) $\mathrm{diam}(M_\alpha(\infty)) < \pi/2$, *if $\mathscr{V}ol_3(M_\alpha(\infty)) > 0$ for an end $\mathscr{E}_\alpha(M)$.*

(iii) *All of the Busemann functions on M are exhaustion, if $\mathscr{V}ol_3(M(\infty)) > 0$ and the sectional curvature is nonnegative.*

Proof. The first assertion is just a consequence of Theorem 3.2. The second one follows from the first one and Corollary 2.4(iii). The last one is derived from the second one and Corollary 4.4 in [24].

Proof of Theorem 3.2.   To begin with, let us recall the definition of the Gauss-Bonnet-Chern integrand Ω_M of M (cf. [7]). Let $\{e_1, \cdots, e_m\}$ be an orthonormal frame over a coordinate neighbourhood of M and $\{\theta^1, \cdots, \theta^m\}$ the dual 1-forms.   Then the structure equations on M are given by

$$d\theta_i = \omega_{ij} \wedge \theta^j, \qquad d\omega_{ij} = \Omega_{ij} + \omega_{ik} \wedge \omega_{kj},$$

where ω_{ij} and Ω_{ij} are, respectively, the connection 1-forms and the curvature 2-forms of M.   The Gauss-Bonnet-Chern form is the m-form on M defined by

$$\Omega_M := \begin{cases} \dfrac{(-1)^n}{2^m \pi^n n!} \sum \varepsilon(i_1, \cdots, i_m)\Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{m-1} i_m} & (m = 2n) \\[2mm] 0 & (m = 2n + 1), \end{cases}$$

where the summation is taken over all permutations $\{i_1, \cdots, i_m\}$ of $\{1, \cdots, m\}$ and $\varepsilon(i_1, \cdots, i_m)$ is the signature.   For an integer $k: 0 \le k \le \frac{1}{2}[m - 1]$, define $(m - 1)$-forms Π_k on the tangent sphere bundle SM of M by

$$\Pi_k(v) = C_{k,m} \sum \varepsilon(i_1, \cdots, i_{m-1})\Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{2k+1} i_{2k}} \wedge \omega_{i_{2k+1} v} \wedge \cdots \wedge \omega_{i_{m-1} v}$$

for $v \in SM$, where the constants $C_{k,m}$ are given by

$$C_{k,m} := \begin{cases} \dfrac{(-1)^k}{\pi^n 2^{n+k} k!} \cdot \dfrac{1}{1 \cdot 3 \cdots (m - 2k - 1)} & (m = 2n) \\[3mm] \dfrac{(-1)^{k+1}}{\pi^n 2^m n!} \begin{bmatrix} n \\ k \end{bmatrix} & (m = 2n + 1). \end{cases}$$

Let D be a compact domain with smooth boundary ∂D.   Then the Gauss-Bonnet-Chern formula of [7] reads

$$(3.5) \qquad \chi(D) = \int_D \Omega_D - \sum_{k=0}^{[(m-1)/2]} \int_{\partial D} \nu^* \Pi_k,$$

where ν denotes the unit outer normal to ∂D.   Let $\{\lambda_i\}_{i=1,\dots,m}$ be the eigenvalues of the second fundamental form α of ∂D (i.e., $\alpha(X, Y) := \langle \nabla_X \nu, Y \rangle$). If we take an orthonormal frame $\{e_1, \cdots, e_m\}$ so that $\nu = e_m$, $\alpha(e_i, e_j) = \lambda_i \delta_{ij}$ $(i, j = 1, \cdots, m - 1)$ along ∂D, then the boundary integrands in (3.5) can be expressed as follows:

$$\nu^* \Pi_k = (-1)^{m-1-2k} C_{k,m} \sum \varepsilon(i_1, \cdots, i_{m-1})\Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{2k-1} i_{2k}}$$
$$\wedge \lambda_{i_{2k+1}} \theta^{i_{2k+1}} \wedge \cdots \wedge \lambda_{i_{m-1}} \theta^{i_{m-1}}.$$

Here it is convenient for us to introduce the following notations: for

$t > 0$, $M^t := (1/t)M$, $B^t := $ the metric ball of M^t around a fixed point of radius 1, $S^t := \partial B^t$, $\Omega^t := $ the Gauss-Bonnet-Chern integrand of M^t, and $\Pi_k^t := $ the $(m-1)$-form on the tangent sphere bundle of M^t ($0 \le k \le [m-1]/2$). Since S^t may not be smooth, we take the smooth approximation $\tilde{S}^t$ for S^t (for large t) as in Theorem B. Then the second fundamental form $\tilde{\alpha}^t$ of $\tilde{S}^t$ satisfies

$$(3.6) \qquad\qquad |\tilde{\alpha}^t| \le c(\kappa_M) + 1$$

for large t, where $c(\kappa_M) > 0$ depends only on κ_M. Moreover in the case that $\mathscr{V}ol_{m-1}(M_a(\infty)) > 0$ for some end $\mathscr{E}_a(M)$, we may assume that

$$(3.7) \qquad\qquad (1 - \varepsilon(t))g_t \le \tilde{\alpha}^t \le (1 + \varepsilon(t))g_t$$

on $\tilde{S}^t \cap \mathscr{E}_a(M)$, where $\varepsilon(t)$ goes to zero as $t \to \infty$ and g_t is the Riemannian metric of M^t. Let us now apply the formula (3.5) to the domains $\tilde{B}^t$ which are bounded by $\tilde{S}^t$. Then since M^t is isotopic to $\tilde{B}^t$, we have

$$\chi(M) = \chi(M^t) = \chi(\tilde{B}^t)$$
$$= \int_{\tilde{B}^t} \Omega^t - \sum_{k=0}^{[(m-1)/2]} \int_{\tilde{S}^t} \nu_t^* \Pi_k^t \,,$$

where ν_t is the outer unit normal to $\tilde{S}^t$. The hypotheses (H.1), (H.2) and (3.6) imply that if the integral $\int_M \Omega_M$ is absolutely convergent, then

$$\left| \chi(M) - \int_M \Omega_M \right| \le (1 + \Lambda(m, \kappa_M)) \cdot \frac{\mathscr{V}ol_{m-1}(M(\infty))}{\omega_{m-1}} \,.$$

Suppose further that $\kappa_M = 0$, or $\int_M |\rho_M|^p (r+1)^{2p-m}$ is finite. Then $M_a(\infty)$ has constant curvature 1 if $\mathscr{V}ol_{m-1}(M_a(\infty))$ is positive, and we have

$$\chi(M) = \int_M \Omega_M + \frac{\mathscr{V}ol_{m-1}(M(\infty))}{\omega_{m-1}} \,,$$
$$= \int_M \Omega_M + \sum_a \frac{1}{\#(\pi_1(M_a(\infty)))} \,,$$

because of (3.7) and the argument in the proof of Corollary 2.5. This proves the first assertion of Theorem 3.2.

Since $\int_M \|\Omega_M\| \le c_m \left(\int_M |\rho_M|^{m/2} + \int_M k \circ r^{m/2} \right)$ for some constant c_m, the integral $\int_M \Omega_M$ is absolutely convergent, if the condition (ii-c) holds. In

the case of (ii-d), M has a soul of dimension $n > 0$ and hence the volume element of M grows like at most r^{m-n} (cf. [22]). This shows that the integral $\int_M \Omega_M$ is absolutely convergent. Let us now examine more closely the cases: $m = 3, 4, 5$. At first, we assume that $m = 4$. In this case, we have

$$\Omega_M = \frac{1}{4\pi^2}\{R_{1212}R_{3434} + R_{1313}R_{2424} + R_{1414}R_{2323} + R_{1234}^2 + R_{1324}^2 + R_{4123}^2\}dv_M$$

where $\Omega_{ij} = \frac{1}{2}R_{jik\ell}\theta^k \wedge \theta^\ell$ and $dv_M := \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4$. If we put $\Omega_M^- := \max\{0, -\Omega_M/dv_M\}$, then we have

$$\Omega_M^- \leq \frac{1}{\pi^2} K(r)k(r)$$

where k and $K(r)$ are as in (H.1) and (H.2), and r is the distance to the base point o in (H.1). Since the volume element dv_M is estimated (within the cut locus of M with respect to the reference point o) by $dv_M \leq J_k(r)^3 dr \wedge d\Theta$ in terms of the polar coordinates (r, Θ) around o, we have

$$\int_M \Omega_M^- dv < +\infty .$$

Thus it is enough to show that

(3.8) $$\frac{\mathscr{V}ol_3(M(\infty))}{\omega_3} \leq \chi(\tilde{B}^t) - \int_{\tilde{B}^t} \Omega^t + \varepsilon(t) ,$$

where $\varepsilon(t)$ goes to zero as $t \to \infty$. (3.8) is derived by the following observations:

$$\lim_{t \to \infty} -\int_{\tilde{S}_t} \nu_t^* \Pi_1^t = \frac{\mathscr{V}ol_3(M(\infty))}{\omega_3} ;$$

$$\liminf_{t \to \infty} -\int_{\tilde{S}^t} \nu_t^* \Pi_1^t = \liminf_{t \to \infty} \frac{1}{4\pi^2}\int_{\tilde{S}^t} \lambda_1^t R_{2323}^t + \lambda_2^t R_{1313}^t + \lambda_3^t R_{1212}^t \geq 0 ,$$

where λ_i^t $(i = 1, 2, 3)$ are the eigenvalues of $\tilde{\alpha}^t$.

We consider next the case: $m = 5$. In this case, we have

$$\lim_{t \to \infty} -\int_{\tilde{S}^t} \nu_t^* \Pi_0^t = \frac{\mathscr{V}ol_4(M(\infty))}{\omega_4} ;$$

$$\liminf_{t \to \infty} -\int_{\tilde{S}^t} \nu_t^* \Pi_1^t$$

$$= \liminf_{t \to \infty} \frac{1}{8\pi^2} \int_{\tilde{S}^t} (\lambda_3^t \lambda_4^t R_{1212}^t + \lambda_2^t \lambda_4^t R_{1313}^t + \lambda_2^t \lambda_3^t R_{1414}^t \lambda_1^t \lambda_4^t R_{2323}^t + \lambda_1^t \lambda_3^t R_{2424}^t + \lambda_1^t \lambda_2^t R_{3434}^t)$$

$$\geq 0 \; ;$$

$$\liminf_{t \to \infty} - \int_{\tilde{S}^t} \nu_t^* \Pi_2^t$$

$$= \liminf_{t \to \infty} \frac{1}{8\pi^2} \int_{\tilde{S}^t} (R_{1212}^t R_{3434}^t + R_{1313}^t R_{2424}^t + R_{1414}^t R_{2323}^t + {R_{1234}^t}^2 + {R_{1324}^t}^2 + {R_{1423}^t}^2)$$

$$\geq 0 \; .$$

These imply that $\chi(M) \geq \mathscr{V}ol_4(M(\infty))/\omega_4$.

Finally we assume that $m = 3$. Then we have

$$\lim_{t \to \infty} - \int_{\tilde{S}^t} \nu_t^* \Pi_0^t = \frac{\mathscr{V}ol_2(M(\infty))}{\omega_2} \; ;$$

$$\liminf_{t \to \infty} - \int_{\tilde{S}^t} \nu_t^* \Pi_1^t = \liminf_{t \to \infty} \frac{1}{4\pi} \int_{\tilde{S}^t} R_{1212}^t \geq 0 \; .$$

These shows that $\chi(M) \geq \mathscr{V}ol_2(M(\infty))/\omega_2$. This completes the proof of Theorem 3.2.

Remark. Let M be a complete, minimal submanifold of dimension m in Euclidean space R^N. Suppose that the total scalar curvature: $\int_M |\alpha_M|^m$ is finite, where α_M denotes the second fundamental form of M. Then Anderson [1] shows that $|\alpha_M| \leq c/r^m$ for some positive constant c, where r is the distance to a fixed point of M. In the case of $m \geq 3$, M satisfies (H.1) and (H.2) with $\kappa_M = 0$ and, since M is a minimal submanifold of Euclidean space, $M_\alpha(\infty)$ is isometric to the unit $(m-1)$-sphere of R^m for each end $\mathscr{E}_\alpha(M)$. Thus it follows from Theorem 3.2 that $\chi(M) = \int_M \Omega_M + \nu(M)$, where $\nu(M)$ stands for the number of the ends of M.

§4. Gap Theorems for asymptotically flat manifolds

A result of Greene-Wu [15] says that a complete, connected, non-compact Riemannian manifold M of dimension m must be isometric to Euclidean space R^m, if M is simply connected at infinity, the sectional curvature K_M is everywhere nonnegative, and moreover (♯): K_M vanishes outside a compact set. Here M is called *simply connected at infinity* if for any compact set C, there is a compact set C' containing C such that $M \backslash C'$ is connected and simply connected. Moreover, Remark in p. 59 of

Ballmann-Gromov-Schreoder [2] states (without proof) that the above fact still holds under a weaker condition: $\mathrm{dis}_M\,(x,\,o)^2 K_M(x)$ goes to zero as $x \in M$ tends to infinity, instead of the condition (♯). We note that this stronger statement was actually proved by Greene-Wu [15] under the additional condition that M possesses a pole o, namely, a point o of M where the exponential map $\exp_o\colon T_o M \to M$ induces a diffeomorphism. (They imposed there a further additional assumption on M when $m = 4$ or 8, but it turns out to be a consequence from the other conditions. See [25, 26] for the reasoning and also more elementary proofs of the gap theorems in [15].) On the other hand, we have similar results for the cases of manifolds with nonpositive curvature (cf. [2: pp. 57–58] [15] [25, 26]).

In this section, we shall prove the following

THEOREM 4.1. *Let M be a manifold of asymptotically nonnegative curvature satisfying* (H.2). *Suppose that the dimension $m \geq 3$ and* $\mathscr{V}ol_{m-1}(M(\infty)) > 0$, *and suppose that* $\kappa_M = 0$, *or* $\int_M |\rho_M|^p(r+1)^{2p-m} < +\infty$ *for some $p \geq 1$. Then M is isometric to Euclidean space R^m if one of the following conditions holds*:

(i) *The sectional curvature is nonnegative everywhere.*

(ii) *The Ricci curvature is nonnegative everywhere and m is odd.*

(iii) *The Ricci curvature is nonnegative everywhere and M is simply connected at infinity.*

(iv) *The sectional curvature is nonpositive everywhere.*

COROLLARY 4.2. *Let M be a manifold of asymptotically nonnegative curvature satisfying* (H.2). *Suppose that the dimension m is odd and the Euler characteristic $\chi(M)$ does not vanish, and suppose that $\kappa_M = 0$, or* $\int_M |\rho_M|^p(r+1)^{2p-m} < +\infty$ *for some $p \geq 1$. Then M is isomorphic to Euclidean space R^m if the Ricci curvature is everywhere nonnegative, or if the sectional curvature is everywhere nonpositive.*

Corollary 4.2 follows from Theorem 4.1 and Corollary 3.3.

Proof of Theorem 4.1. We first observe that if $m \geq 3$ and $\mathscr{V}ol_{m-1}(M_a(\infty)) > 0$ for some end $\mathscr{E}_a(M)$, then $M_a(\infty)$ is isometric to the quotient space of the unit sphere $S^{m-1}(1)$ of R^m by a group Γ_a of isometries of $S^{m-1}(1)$ (cf. Corollary 2.5). Moreover it turns out from this observation, the assumption: $\mathscr{V}ol_{m-1}(M(\infty)) > 0$, and the splitting theorem by Cheeger-Gromoll [5] that if the Ricci curvature of M is nonnegative everywhere,

then $M(\infty)$ is isometric to $S^{m-1}(1)/\Gamma$. On the other hand, we know that if the Ricci curvature of M is nonnegative everywhere and $\mathscr{V}ol_{m-1}(M(\infty))$ $= \omega_{m-1}(: = \mathrm{Vol}_{m-1}(S^{m-1}(1)))$, then M must be isometric to Euclidean space R^m. Thus we have shown Theorem 4.1 under the condition (i) (cf. [6]), (ii) or (iii). It remains to prove the theorem under the condition (iv). Suppose the condition (iv) holds. We note then the following

FACT (cf. [25: Lemma 13]). *Let M be a complete, connected, and noncompact Riemannian manifold of nonpositive curvature. Suppose there is a compact subset C of M such that the fundamental group of a noncompact component of $M \backslash C$ is finite. Then M is simply connected.*

Thus it turns out that M is simply connected, and hence M is diffeomorphic to R^m. This implies that $M(\infty)$ is isometric to $S^{m-1}(1)$, so that M is isometric to R^m, because of the Rauch comparison theorem.

Remark. Let M be as in Theorem B and suppose the dimension is greater than or equal to 3. Then very recently, Uesu [37] has proved that M is isometric to Euclidean space if M is simply connected and M has no focal points.

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