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ON MEYER'S EQUIVALENCE

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§1. Introduction

In the recent years there has been a considerable effort to construct and analyze spaces of test and generalized functionals in infinite dimensional situations, cf. [3, 5, 12, 14] and literature quoted there. In particular Meyer [4, 5] has introduced a certain space of "smooth" functionals on the Wiener space, which was used by Watanabe [14] for an elegant formulation of "Malliavin's calculus" (i.e. he proved a criterion for the existence and regularity of densities of Wiener functionals). This functional space is countably normed and one of its important properties is its algebraic structure. The proof of this property follows from an equivalence of the norms defining the space with a system of norms of Sobolev type [4, 5] (cf. also (1.5), (1.6)).

In this paper we prove the generalization of Meyer's equivalence to *Gaussian spaces*. By this we mean a triple $(\mathscr{N}^*, \mathscr{B}, \mu)$, $\mathscr{N}^*$ being the dual of a separable, nuclear pre-Hilbert space $\mathscr{N}$ with (compatible) scalar product $(\cdot, \cdot)$, $\mathscr{B}$ the topological σ-algebra of $\mathscr{N}^*$ and μ is the Gaussian measure on $\mathscr{B}$ defined by $(\cdot, \cdot)$, i.e.

$$\int_{\mathscr{N}^*} \exp i\langle x, \xi\rangle d\mu(x) = \exp\left(-\frac{1}{2}|\xi|^2\right) \tag{1.1}$$

$\xi \in \mathscr{N}$ and $|\cdot|$ is the norm induced by $(\cdot, \cdot)$, cf. [1]. By $\mathscr{H}$ we shall denote the completion of $\mathscr{N}$ under $|\cdot|$.

A large class of spaces of $\mathscr{N}^*$-test and generalized functionals has been constructed in [7] and, although the above mentioned equivalence will only be proved in one special case, let us review this construction quickly to introduce some notations.

To define spaces of test functionals over $\mathscr{N}^*$ similar to $\mathscr{S}(\boldsymbol{R}^d)$, it would be quite natural to proceed as in the finite dimensional case,

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namely to introduce them as projective limits of chains of Hilbert spaces [1, 9] (recall the construction of $\mathscr{S}(\boldsymbol{R}^d)$ via the Hamiltonian of the harmonic oscillator): Let A be a linear, selfadjoint positive operator on $\mathscr{H}$, then, going through the Fock space formalism [6, 9, 10], we have the linear operator $d\Gamma(A)$ (ess. selfadjoint, positive) on $L^2(\mathscr{N}^*, \mu)$ and we may consider the scalar products

$$(1.2) \qquad (f, g)_{2,m} := \sum_{l=0}^{m} (f, d\Gamma(A)^l g)_{L^2}$$

on a suitable dense subspace (e.g. the polynomials) of $L^2(\mathscr{N}^*, \mu)$. The corresponding completions $L_A^{2,m}$ form a chain of Hilbert spaces, which are continuously embedded into each other, and its projective limit $L_A^{2,\infty}$ can be regarded as a space of test functionals over $\mathscr{N}^*$.

However, as Meyer remarks [5], it is easily seen, that $L_A^{2,\infty}$ is not an algebra. Therefore one considers also the spaces $L_A^{p,m}$, defined as the completions of a suitable dense subspace of $L^2(\mathscr{N}^*, \mu)$ with respect to the norms ($p \geqq 1, m \in \boldsymbol{Z}_+$)

$$(1.3) \qquad \|f\|_{p,m} := \sum_{l=0}^{m} \|d\Gamma(A)^{l/2} f\|_p$$

where $\|\cdot\|_p$ is the norm of $L^p(\mathscr{N}^*, \mu) \equiv L^p$.

Then we can define a space Σ_A as the projective limit of the system $L_A^{p,m}$, where p ranges over $\boldsymbol{N}$ and m over $\boldsymbol{Z}_+$ (the inclusions $L_A^{p,m} \subset L_A^{q,m}$, $p \geqq q$, follow from Hölder's inequality, while $L_A^{p,m+1} \subset L_A^{p,m}$ follows from the hypercontractivity of the semigroup $\exp(-t d\Gamma(A))$ [6, 10], cf. also [7] and [12]).

Let us denote the dual of $L_A^{p,m}$ by $L_A^{-p,-m}$, which is a Banach space with norm $\|\cdot\|_{-p,-m}$. Standard theory [1] implies that Σ_A is a Fréchet space and its dual Σ_A^* is sequentially weakly complete and

$$(1.4) \qquad \Sigma_A^* = \bigcup_{\substack{p \in \boldsymbol{N} \\ m \in \boldsymbol{Z}_+}} L_A^{-p,-m} .$$

The following should be mentioned as an important illustration of our abstract setting:

Choose $\mathscr{N} = \mathscr{S}(\boldsymbol{R})$ and $\mathscr{H} = L^2(\boldsymbol{R}, dt)$, i.e. $(\mathscr{N}^*, \mathscr{B}, \mu)$ is the standard white noise space [3]. Then Wiener functionals can equivalently be described as white noise functionals over $\mathscr{N}^* = \mathscr{S}^*(\boldsymbol{R})$ (cf. [3]).

Next, choose $A = 1$ (identity on $\mathscr{H} = L^2(\boldsymbol{R}, dt)$). Then $d\Gamma(1)$ is the number- or Ornstein-Uhlenbeck operator and Σ is the space of Meyer [4, 5] in a slightly different formulation.

Another typical case would be $\mathscr{N} = \mathscr{S}(\boldsymbol{R}^d)$, $\mathscr{H} = H_{-1}(\boldsymbol{R}^d, dx)$, the Sobolev space of order -1 over $\boldsymbol{R}^d$, so that $(\mathscr{N}^*, \mathscr{B}, \mu)$ describes free Euclidean quantum field theory (of massive bosons) in d dimensions.

Now let $\partial(\eta)$, $\eta \in \mathscr{H}$, denote the annihilation operator on L^2 (with domain $L_1^{2,1}$, s.a.) [9, 10] (cf. also [7, 8]) and let $(e_k; k \in \boldsymbol{Z}_+)$ be a CONS of $\mathscr{H}$ in the domain of the operator A. Then we are looking for an equivalence of norms

$$\| |\nabla_A f| \|_p \sim \| d\Gamma(A)^{1/2} f \|_p \tag{1.5}$$

where

$$|\nabla_A f|^2 := \sum_{k=0}^{\infty} \partial(A^{1/2} e_k) f|^2 \tag{1.6}$$

and $\partial^*(\eta)$ is the L^2-adjoint of $\partial(\eta)$.

Due to the formula

$$\begin{aligned} d\Gamma(A) fg &= f d\Gamma(A) g + g d\Gamma(A) f \\ &\quad - 2 \sum_{k=0}^{\infty} (\partial(A^{1/2} e_k) f)(\partial(A^{1/2} e_k) g) \end{aligned} \tag{1.7}$$

it would then be a matter of induction, as shown in [5, 12], to prove that Σ_A forms an algebra.

Unfortunately, because of technical difficulties, the equivalence (1.5) can here only be proved in the case $A = 1$ ($= \mathrm{id}_{\mathscr{H}}$), although the powerful Littlewood-Paley-Stein (LPS) inequalities entering the proof (cf. also [4]) have been shown in [8] for a broad class of operators A. Equivalence (1.5) for the case $A = 1$ will be proved in the next section.

Finally, in section 3 Watanabe's above mentioned result about the composition of certain test functionals in Σ with tempered distributions is sketched in our setting. This provides a "formulation of Malliavin's calculus" (s.a.) for random variables on Gaussian spaces $(\mathscr{N}^*, \mathscr{B}, \mu)$, which is expected to have applications in fields such as Euclidean quantum field theory.

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§ 2. The equivalence of norms

For the rest of this article we shall assume $A = 1$ ($= \mathrm{id}_{\mathscr{H}}$) and drop all corresponding subscripts. Furthermore we denote the number operator $d\Gamma(1)$ by N.

In this section we prove the equivalence (1.5) of the L^p-norms of $|\nabla f|$ and $N^{1/2}f$. In the finite dimensional case (with Lebesgue measure and N replaced by the Laplacian) the corresponding equivalence follows from the elliptic LPS inequalities [11]. However the noncommutativity of $\partial(\eta)$ and N in our situation destroy such a direct proof via the Riesz transformation as in [11]. Meyer has found a way in [5] how to overcome this difficulty, applying in addition parabolic LPS inequalities. (For the notion of elliptic and parabolic LPS inequalities cf. also [8]). Here we are going to follow his proof rather closely, however the use of stochastic integrals is avoided.

First we introduce a little more notation and prepare two lemmas for later convenience.

Let $\mathscr{H}^{(n)}$ denote the subspace of L^2 isomorphic to $(\mathscr{H}_c)^{\hat{\otimes} n}$ [1] (cf. section 1 of [8]). $\mathscr{I}_n$ denotes the orthogonal projection in L^2 onto $\mathscr{H}^{(n)}$. $\mathscr{P}$ is the algebra of polynomials in the variables $\{\langle x, e_k\rangle,\ k \in \boldsymbol{Z}_+\}$, where $\{e_k,\ k \in \boldsymbol{Z}_+\}$ is a CONS of $\mathscr{H}$ lying in $\mathscr{N}$. Note that $\mathscr{P}$ is dense in all L^p, $1 \leqq p < \infty$.

Sugita proves in [12] the following result, which is essentially a consequence of Nelson's hypercontractivity theorem [6]

LEMMA 2.1. (a) *$\mathscr{I}_n$ extends to a bounded operator on all L^p, $1 < p < \infty$*

(b) *Let $\{P_t, t \in \boldsymbol{R}_+\}$ be the semigroup on L^2 defined by $P_t = \exp(-tN)$. Then for all p, $1 < p < \infty$, and $n \in N$, there exists $c_{p,n} > 0$, so that*

$$\left\| P_t\Big(1 - \sum_{l=0}^{n-1} \mathscr{I}_l\Big) f \right\|_p \leqq c_{p,n} e^{-nt} \|f\|_p \tag{2.1}$$

(c) *For all $n \in \boldsymbol{Z}_+$ the operator*

$$(1 - nN^{-1})^{1/2}\Big(1 - \sum_{l=0}^{n} \mathscr{I}_l\Big)$$

defined on polynomials, extends to a bounded operator on all L^p, $1 < p < \infty$.

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1) $\mathscr{H}_c$ denotes the complexification of $\mathscr{H}$, $\hat{\otimes}$ the symmetric tensor product.

The following lemma is easily verified and its proof omitted.

LEMMA 2.2. *On $\mathscr{P}$ the commutation relations*

$$N^{1/2}\partial(\eta) = \partial(\eta)(N-1)^{1/2}(1-\mathscr{J}_0-\mathscr{J}_1) \tag{2.2}$$

$$P_t\partial(\eta) = e^t\partial(\eta)P_t \tag{2.3}$$

hold for all $\eta \in \mathscr{H}$.

Now we can start proving (1.5) for $A = 1$. For the following let $f \in \mathscr{P} \cap \oplus_{n\geqq 2}\mathscr{H}^{(n)}$ and $p \in (1, \infty)$, unless otherwise stated. Define $g := (N-1)^{1/2}f$ and $g_t := P_t g$.

LEMMA 2.3.

$$\||\nabla f|\|_p \lesssim \left\|\left(\int_0^\infty e^{2t}|\nabla g_t|^2 dt\right)^{1/2}\right\|_p \tag{2.4}$$

Proof. Let $Q_t := \exp(-tN^{1/2})$, $t \in \boldsymbol{R}_+$. We compute

$$\begin{aligned}\partial_t Q_t \partial_k f &= -Q_t N^{1/2}\partial_k f\\ &= -Q_t\partial_k(N-1)^{1/2}f\\ &= -\int_0^\infty P_s\partial_k(N-1)^{1/2}f\nu_t(ds)\\ &= -\int_0^\infty e^s\partial_k g_s\nu_t(ds)\end{aligned}$$

and used Lemma 2.2 and equation (2.3) of [8]. Then

$$\begin{aligned}\sum_k |\partial_t Q_t(\partial_k f)|^2 &\leqq \left(\int_0^\infty e^s|\nabla g_s|\nu_t(ds)\right)^2\\ &\leqq \int_0^\infty e^{2s}|\nabla g_s|^2\nu\ (ds)\end{aligned}$$

where the first inequality follows from taking the l_2-norm under the integral and the second from Schwarz' inequality for the normalized measure $\nu_t(ds)$. Upon integration with tdt, Fubini's theorem and

$$\int_0^\infty \nu_t(ds)tdt = ds\,,$$

we obtain

$$\sum_k G_4(\partial_k f)^2 \leqq \int_0^\infty e^{2s}|\nabla g_s|ds \tag{2.5}$$

and G_4 is the Littlewood-Paley function in section 4, eq. (4.7), of [8].

Since $f \in \mathscr{P} \cap \oplus_{n \geqq 2} \mathscr{H}^{(n)}$ implies that $\int (\partial_k f) d\mu = 0$ and $\sum_k |\partial_k f|^2$ is a finite sum (and hence its root is in all L^p, $p \geqq 1$), we apply Theorem 4.4 of [8], which states that the root of the left hand side of (2.5) has L^p-norm equivalent to the L^p-norm of $|\nabla f|$ (if $1 < p < \infty$), proving (2.4).

LEMMA 2.4.

$$\left\| \left(\int_0^\infty e^{2t} |\nabla g_t|^2 dt \right)^{1/2} \right\|_p \gtrsim \|g\|_p + \int_0^\infty e^t \|g_t\|_p dt \,. \tag{2.6}$$

Proof. Let $h \in \mathscr{P}$. From Lemma 2.4 in [8], we have the bound $(P_t h \equiv h_t)$

$$(P_t |\nabla h_t|^2) \geqq e^{2t} |\nabla h_{2t}|^2$$

so that

$$\begin{aligned} \|h\|_p &\gtrsim \left\| \left(\int_0^\infty (P_t |\nabla h_t|^2) dt \right)^{1/2} \right\|_p \\ &\geqq \left\| \left(\int_0^\infty e^t |\nabla h_t|^2 dt \right)^{1/2} \right\|_p \end{aligned} \tag{2.7}$$

by Theorem A of [8]. Set $h = g_u$, $u \in \boldsymbol{R}_+$.

Then

$$e^{u/2} \|g_u\|_p \gtrsim \left\| \left(\int_u^\infty e^t |\nabla g_t|^2 dt \right)^{1/2} \right\|_p \,. \tag{2.8}$$

Denote

$$H(u, t) := e^{t/2} |\nabla g_t| 1_{(u,\infty)}(t)$$

and $\mathfrak{h} := L^2(\boldsymbol{R}_+, dt)$. Integrating (2.8) with $e^{u/2} du$ we obtain

$$\begin{aligned} \int_0^\infty e^u \|g_u\|_p du &\gtrsim \int_0^\infty e^{u/2} \| \; \|H(u, \cdot)\|_{\mathfrak{h}} \|_p du \\ &\geqq \| \; \| \int_0^\infty e^{u/2} H(u, \cdot) du \|_{\mathfrak{h}} \|_p \\ &= \left\| \left(\int_0^\infty e^t (e^{t/2} - 1)^2 |\nabla g_t|^2 dt \right)^{1/2} \right\|_p \end{aligned}$$

and so

$$\left\| \left(\int_1^\infty e^{2t} |\nabla g_t|^2 dt \right)^{1/2} \right\|_p \lesssim \int_0^\infty e^t \|g_t\|_p dt \,. \tag{2.9}$$

Moreover, from (2.7)

$$\left\| \left(\int_0^1 e^{2t} |\nabla g_t|^2 dt \right)^{1/2} \right\|_p \lesssim \|g\|_p . \tag{2.10}$$

Now inequality (2.6) follows from (2.9), (2.10) and the triangular inequality.

Our next step is

LEMMA 2.5.

$$\||\nabla f|\|_p \lesssim \|g\|_p . \tag{2.11}$$

Proof. In view of Lemmas 2.3 and 2.4, it is sufficient to show that

$$\int_0^\infty e^t \|g_t\|_p dt \lesssim \|g\|_p .$$

But since g is a polynomial in $\oplus_{n\geq 2} \mathscr{H}^{(n)}$, this follows immediately from Lemma 2.1b.

THEOREM 2.6. *Let $f \in L^{p,1}$, $1 < p < \infty$. Then the equivalence of norms* (1.5) *holds in the case $A = \mathrm{id}_{\mathscr{H}}$.*

Proof. By density of $\mathscr{P}$, it is sufficient to prove (1.5) for a polynomial f. Assume first that $f \in \mathscr{P} \cap \oplus_{n\geq 2} \mathscr{H}^{(n)}$. Then Lemma 2.5 provides the bound (recall the definition of g before Lemma 2.3):

$$\begin{aligned} \||\nabla f|\|_p &\lesssim \|(N-1)^{1/2} f\|_p \\ &\leqq \|(1-N^{-1})^{1/2}(1 - J_0 - J_1)\|_{p,p} \|N^{1/2} f\|_p \end{aligned}$$

where $\|\cdot\|_{p,p}$ is the operator norm on L^p. By Lemma 2.1c the first term of the last inequality is finite, so that for $f \in \mathscr{P} \cap \oplus_{n\geq 2} \mathscr{H}^{(n)}$ one side of (1.5) is proved.

General $f \in \mathscr{P}$ is decomposed as $f = f^{(0)} + f^{(1)} + f'$ with $f^{(0)} \in \mathscr{H}^{(0)} = \boldsymbol{C}$, $f^{(1)} \in \mathscr{H}^{(1)} = \mathscr{H}_c$ (the complexification of $\mathscr{H}$) and an elementary computation shows that

$$\||\nabla f^{(1)}|\|_p \lesssim \|N^{1/2} f^{(1)}\|_p = \|f^{(1)}\|_p .$$

Hence

$$\begin{aligned} \||\nabla f|\|_p &\lesssim \|N^{1/2} f^{(1)}\|_p + \|N^{1/2} f'\|_p \\ &\lesssim \|N^{1/2} f\|_p \end{aligned}$$

because the projections J_n are bounded on L^p (Lemma 2.1a).

To obtain the converse inequality, we only have to observe that for $p = 2$ both norms are equal, so that the standard duality argument yields

$$\|N^{1/2}f\|_p \lesssim \||\nabla f|\|_p$$

and the proof is concluded.

Having arrived at Theorem 2.6, the induction arguments of Meyer [5] or Sugita [12] can now be taken over literally for our situation to prove

THEOREM 2.7. *Let $f, g \in L^{2p,2m}$. Then*

$$\|fg\|_{p,2m} \lesssim \|f\|_{2p,2m}\|g\|_{2p,2m}\,. \tag{2.12}$$

In particular Σ forms an algebra.

§ 3. On Watanabe's theorem

In this section we follow essentially Watanabe [14].

Let $\phi \in \mathscr{S}(\boldsymbol{R}^d)$ and $f = (f_1, \cdots, f_d)$ be a map from $\mathscr{N}^*$ to $\boldsymbol{R}^d$ with $f_i \in \Sigma$, $i = 1, 2, \cdots, d$.

Define the $\mathscr{N}^*$-functional

$$(\gamma^{-1})_{ij} := \sum_k (\partial_k f_i)(\partial_k f_j) \tag{3.1}$$

(note that by Theorem 2.7 $(\gamma^{-1})_{ij} \in \Sigma$ for all i, j, $1 \leqq i, j \leqq d$).

We shall need the following

LEMMA 3.1. *Assume that the matrix γ^{-1} is μ-a.e. strictly positive and that γ is in all L^p, $1 \leqq p < \infty$. Then $\phi_{,i} \circ f \in \Sigma$, where $\phi_{,i}$ means the i^{th} partial derivative of the function ϕ, and*

$$\phi_{,i} \circ f = \frac{1}{2} \sum_{j=1}^{d} \gamma_{ij}\{Nf_j \phi \circ f - f_j N\phi \circ f - \phi \circ f Nf_j\}\,. \tag{3.2}$$

Proof. First note that $\gamma^{-1} > 0$ μ-a.e. implies that γ exists μ-a.e. and that $\gamma^{-1} \in \Sigma$ together with $\gamma \in L^p$ for all $p \in [1, \infty)$, the chain rule for ∂_k (the corresponding proof in [7] is easily adapted to the present situation) and Theorem 2.7 imply that $\gamma \in \Sigma$ too.

Next the chain rule for ∂_k yields

$$\partial_k \phi \circ f = \sum_{i=1}^{d} \phi_{,i} \circ f \partial_k f_i \tag{3.3}$$

from which we conclude

$$\sum_k (\partial_k f_j)(\partial_k \phi \circ f) = \sum_i \phi_{,i} \circ f (\gamma^{-1})_{ij} \tag{3.4}$$

and since γ^{-1} is μ-a.e. invertible

$$\phi_{,i}(f) = \sum_{j=1}^{d} \gamma_{ij} \sum_{k} (\partial_k f_j)(\partial_k \phi \circ f)\,.$$

Finally using Lemma 2.2 of [8] (which extends from polynomials to Σ) we find

$$\sum_{k} (\partial_k f_j)(\partial_k \phi \circ f) = \frac{1}{2}(N f_j \phi \circ f - f_j N \phi \circ f - \phi \circ f N f_j)$$

which proves (3.2). Theorem 2.7 implies now that $\phi_{,i} \circ f$ belongs to Σ, since by the chain rule $\phi \circ f$ belongs to Σ for $f_i \in \Sigma$ and $\phi \in \mathscr{S}(\boldsymbol{R}^d)$ and Σ is stable under N and under multiplication.

DEFINITION 3.2. A sequence $\{T_n,\ n \in \boldsymbol{N}\}$ in Σ^* is said to *converge *-strongly* to $T \in \Sigma^*$, if there exist $p \in \boldsymbol{N}$ and $m \in \boldsymbol{Z}_+$, so that $T_n,\ T \in L^{-p,-m}$ for all $n \in \boldsymbol{N}$ and $\|T_n - T\|_{-p,-m}$ converges to zero as $n \to \infty$.

Note that strong-convergence implies strong and weak convergence in Σ^*.

Now we can adopt Watanabe's proof of the following

THEOREM 3.3 (Watanabe). *Let $f = (f_1, \cdots, f_d)$ be a map from $\mathscr{N}^*$ to $\boldsymbol{R}$ with $f_i \in \Sigma$, $i = 1, \cdots, d$. Assume that γ^{-1} is (μ-a.e.) strictly positive with $\gamma \in L^p$ for all $p \in [1, \infty)$. Then there is a unique linear map $\hat{\ }$ from $\mathscr{S}^*(\boldsymbol{R}^d)$ into Σ^* with the properties*

(a) *if $T \in \mathscr{S}(\boldsymbol{R}^d)$, then $\hat{T} = T \circ f$*

(b) *if $T_n \to T$ in $\mathscr{S}^*(\boldsymbol{R}^d)$, $\{T_n,\ n \in \boldsymbol{N}\} \subset \mathscr{S}(\boldsymbol{R}^d)$, then $\hat{T}_n \to \hat{T}$ *-strongly in Σ^*.*

Sketch of the proof. Let $H = -\Delta + |u|^2$ be the Hamiltonian of the harmonic oscillator on $L^2(\boldsymbol{R}^d, du)$. For $\alpha \in \boldsymbol{Z}$, denote by $\mathscr{S}_\alpha$ the Hilbert space obtained as the completion of $\mathscr{S}(\boldsymbol{R}^d)$ under $\|H^{\alpha/2}\cdot\|_{L^2(\boldsymbol{R}^d, du)}$. Then for each $T \in \mathscr{S}^*(\boldsymbol{R}^d)$ there exists $\alpha \in \boldsymbol{Z}$ so that $T \in \mathscr{S}_\alpha$ and a sequence $\{T_n;\ n \in \boldsymbol{N}\}$ in $\mathscr{S}(\boldsymbol{R}^d)$ with $T_n \to T$ in $\mathscr{S}_\alpha$ [9]. Furthermore for $\beta \in \boldsymbol{Z}_+$ large enough

$$H^{-\beta} T_n \longrightarrow H^{-\beta} T$$

in the uniform topology on $\boldsymbol{R}^d$. Let $g \in \Sigma$.

Then

$$\langle T_n \circ f - T_{n'} \circ f, g \rangle = \langle (H^\beta (H^{-\beta} T_n - H^{-\beta} T_{n'})) \circ f, g \rangle\,.$$

Note that $H^{-\beta}T_n - H^{-\beta}T_{n'} \in \mathscr{S}(\boldsymbol{R}^d)$ and the action of H^β on this test function composed with f may be computed by (3.2) and is therefore expressed by a finite sum of terms involving multiplications with γ_{ij}'s, f_i's and operations N. Selfadjointness of N yields then

$$\langle T_n \circ f - T_{n'} \circ f, g \rangle = \langle (H^{-\beta}T_n - H^{-\beta}T_{n'}) \circ f, l(g) \rangle$$

where l is the corresponding action on g and by Lemma 1.1 and Theorem 2.7, $g \in \Sigma$ implies $l(g) \in \Sigma$. Hence there exist $p \in \boldsymbol{N}$ and $m \in \boldsymbol{Z}_+$, so that $\|l(g)\|_{p,m}$ is finite. Since $\{H^{-\beta}T_n,\ n \in \boldsymbol{N}\}$ is Cauchy in the sup-norm, it follows that $\{T_n \circ f,\ n \in \boldsymbol{N}\}$ is Cauchy *-strongly in Σ^*. The sequentially weak completeness of Σ^* implies the existence of $\hat{T} \in \Sigma^*$ as the limit of this sequence. Uniqueness follows by analogous arguments and the theorem is proved.

Similarly one can show [14]

THEOREM 3.4. *Let $u \to T_u$ be a C^k-mapping from $\boldsymbol{R}^d$ into $\mathscr{S}^*(\boldsymbol{R}^d)$, then $\hat{T}_u$ is C^k in the strong-* topology and*

$$\frac{\partial}{\partial u_i}\langle \hat{T}_u, g \rangle = \left\langle \frac{\partial}{\partial u_i}\hat{T}_u, g \right\rangle \tag{3.5}$$

for $g \in \Sigma$.

If T_u is continuous in u and integrable $\left(\text{i.e. } \left\langle \int T_u du, \phi \right\rangle = \int \langle T_u, \phi \rangle du < \infty \text{ for all } \phi \in \mathscr{S}(\boldsymbol{R}^d)\right)$, then

$$\left\langle \int \hat{T}_u du, g \right\rangle = \int \langle \hat{T}_u, g \rangle du \tag{3.6}$$

for all $g \in \Sigma$.

Now consider the Dirac distribution $\delta_u \in \mathscr{S}^*(\boldsymbol{R}^d)$, $u \in \boldsymbol{R}^d$. Form

$$p(u) := \langle \hat{\delta}_u(f), 1 \rangle \tag{3.7}$$

for f satisfying the hypothesis of Theorem 3.3. Then, since $1 \in \Sigma$, (3.7) makes sense and $p(u)$ is C^∞. Furthermore for $\phi \in \mathscr{S}(\boldsymbol{R}^d)$

$$\begin{aligned}\int_{\boldsymbol{R}^d} \phi(u)p(u)du &= \langle \phi \circ f, 1 \rangle \\ &= \int_{\mathscr{N}^*} (\phi \circ f)(x) d\mu(x)\end{aligned}$$

i.e. $p(u)$ is the (smooth) density of f and we have proved

THEOREM 3.5 (Malliavin, Watanabe, $\cdots$). *Let f satisfying the hypothesis of Theorem* 3.3. *Then it has a C^∞-density on $\boldsymbol{R}^d$, given by* (3.7).

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