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THE INNER PRODUCT OF AN AUTOMORPHIC WAVE FORM WITH THE PULLBACK OF AN EISENSTEIN SERIES

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In this paper we shall show a relation between a special value of an automorphic wave form and the inner product of the automorphic wave form with the pullback of an Eisenstein series on the upper half space. The main theorem is Theorem 3 in the end of this paper. As is shown in P. B. Garrett [13], pullbacks of Eisenstein series on Siegel upper half spaces have interesting properties as a kernel function of an integral operator. It is natural to try to investigate pullbacks of Eisenstein series of Hilbert type. We can say that Theorem 3 clarifies a property of such pullbacks in a special case. The idea of the proof is a lifting of automorphic forms by theta functions. We discuss a lifting of automorphic wave forms in 1, 2 and 3, and obtain Theorem 2 in the end of 3 as a result. We can prove Theorem 3 without much difficulty by using Theorem 2.

1. We denote, as usual, by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ the ring of rational integers, the rational number field the real and the complex number fields. For $z \in \boldsymbol{C}$, we define $\sqrt{z} = z^{1/2}$ so that $-\pi/2 < \arg(z^{1/2}) \leq \pi/2$.

We discuss a lifting of an automorphic wave form by means of theta functions for a quadratic form with the signature $(+2, -2)$. Denote by H the upper half plane. For $g_1, g_2 \in G = G_\infty = SL(2, \boldsymbol{R})$, put

$$(1.1) \qquad \Theta(z, g_1, g_2) = v \sum_{X=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in L} \bar{\chi}(pd) e^{2\pi i(-up\det X + i2^{-1}pv\,\mathrm{tr}\,{}^t({}^tg_1Xg_2)({}^tg_1Xg_2))},$$

$$(1.2) \qquad \tilde{\Theta}(z, g_1, g_2) = v \sum_{X=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{L}} \chi(a) e^{2\pi i(-u\det X + i2^{-1}v\,\mathrm{tr}\,{}^t({}^tg_1Xg_2)({}^tg_1Xg_2))}$$

where

$$L = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,\ b,\ c,\ pd \in \boldsymbol{Z}\right\},\ \tilde{L} = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \boldsymbol{Z}\right\},\quad z = u + iv \in H$$

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and χ is a primitive character modulo a prime $p \equiv 1 \pmod 4$. Assume $\chi(-1) = 1$. We consider χ as a character on $\boldsymbol{Z}_p$ by extending χ to a character on $\boldsymbol{Z}_p^\times$ in the usual way and putting $\chi(p) = 0$. Then we have

PROPOSITION 1.

$$\Theta(-1/pz, g_1, g_2) = p^{-1}\mathfrak{g}(\bar{\chi})\tilde{\Theta}(z, g_1, g_2)\,, \tag{1.3}$$

with $\mathfrak{g}(\chi) := \sum_{l \bmod p} \chi(l)e^{2\pi i l/p}$ *and for* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma = \Gamma_0(p)$,

$$\Theta(\gamma z, g_1, g_2) = \bar{\chi}(d)\Theta(z, g_1, g_2), \tag{1.4}$$

$$\Theta(z, \gamma g_1, g_2) = \Theta(z, g_1, \gamma g_2) = \chi(d)\Theta(z, g_1, g_2)\,, \tag{1.5}$$

$$\tilde{\Theta}(\gamma z, g_1, g_2) = \chi(d)\tilde{\Theta}(z, g_1, g_2)\,, \tag{1.6}$$

$$\tilde{\Theta}(z, \gamma g_1, g_2) = \tilde{\Theta}(z, g_1, \gamma g_2) = \chi(d)\tilde{\Theta}(z, g_1, g_2)\,. \tag{1.7}$$

Proof. It is sufficient to prove (1.3), (1.4) and (1.5). Put $X = \begin{pmatrix} m & n \\ x & y \end{pmatrix} \in L$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, then ${}^t\gamma^{-1}X = \begin{pmatrix} * & * \\ * & -bn + ay \end{pmatrix} \in L$ and $\bar{\chi}(p(-bn + ay)) = \bar{\chi}(pay) = \chi(d)\bar{\chi}(py)$, which proves $\Theta(z, \gamma g_1, g_2) = \chi(d)\Theta(z, g_1, g_2)$. The proof of $\Theta(z, g_1, \gamma g_2) = \chi(d)\Theta(z, g_1, g_2)$ is the same. As for the proof of (1.3) and (1.4), we can apply transformation formulas in [7], which are summarized in [16] as Proposition 0, for example. However we describe the transformation formulas in the language of adeles to prove (1.3) and (1.4) for later use. For an archimedean or non-archimedean place v, $\boldsymbol{Q}_v$ denotes the completion at v. Let A be the adele ring of $\boldsymbol{Q}$ and denote by $\varphi(K^n)$ the Schwartz Bruhat space on K^n when $K = A$ or $\boldsymbol{Q}_v$. Define an additive character ψ_v (respectively ψ) of $\boldsymbol{Q}_v$ (respectively A) by $\psi_v(x) = e^{2\pi i\,(\text{the principal part of } x)}$ if v is non-archimedean and $\psi_v(x) = e^{2\pi i x}$ if v is archimedean (respectively $\psi((x_v)) = \prod_v \psi_v(x_v)$). Denote by $d_v x$ the Haar measure on $\boldsymbol{Q}_v$ normalized by $\int_{Z_q} d_q x = 1$ and $\int_0^1 d_\infty x = 1$. Put $d(x_v) = \prod_v d_v x$. Define a partial "Fourier" transformation $\mathscr{F}_{m,v}$ in $\mathscr{S}(\boldsymbol{Q}_v^2)$ by $\mathscr{F}_{m,v}f(x, y) = \int_{Q_v} f(x, z)\cdot \psi_v(mzy)d_v z$ with $m \in \boldsymbol{Q}$ and a transformation $\lambda(g)$ in $\varphi(\boldsymbol{Q}_v^2)$ by $\lambda(g)f(x, y) = f((x, y)g)$ for $g \in G_v = SL(2, \boldsymbol{Q}_v)$. Then $\mathscr{F}_{m,v}^{-1} = |m|_v \mathscr{F}_{-m,v}$. Define representations $r_{v,1}$, $r_{v,2}$ and r_v of G_v by $r_{v,1}(g) = \mathscr{F}_{-p,v}^{-1}\lambda(g)\mathscr{F}_{-p,v}$, $r_{v,2}(g) = \mathscr{F}_{p,v}^{-1}\lambda(g)\mathscr{F}_{p,v}$ and $r_v(g) = r_{v,1}(g) \otimes r_{r,2}(g)$. Then the Poisson summation formula

$$\sum_{x\in Q^2}(\prod_v \mathscr{F}_{m,v})f(x)=\sum_{x\in Q^2}f(x)$$

is valid for $f\in\mathscr{S}(A^2)$, and therefore

$$(1.8)\qquad \sum_{x\in Q^4}(\prod_v r_v(g))f(x)=\sum_{x\in Q^4}f(x)$$

holds for $f\in\mathscr{S}(A^4)$ and $g\in G_Q=SL(2,\boldsymbol{Q})$. Define a representation r of $G_A=SL(2,A)$ on $\mathscr{S}(A^4)$ by $r((g_v))=\prod_v r_v(g_v)$. When K denotes $\boldsymbol{Q}_v$ or A, we define a mapping α from K^4 to $M_{2,2}(K)$ by $\alpha(a,b,c,d)=\begin{pmatrix}a&c\\d&b\end{pmatrix}$. Define a representation ρ_v (respectively ρ) of $G_v\times G_v$ (respectively $G_A\times G_A$) on $\boldsymbol{Q}_v^4$ (respectively A^4) by

$$(1.9)\qquad x\rho_v(g,h)=\alpha^{-1}({}^tg\alpha(x)h)\ (\text{respectively } x\rho(g,h)=\alpha^{-1}({}^tg\alpha(x)h)).$$

For $g,h,k\in G_A$ and $f\in\mathscr{S}(A)$, put

$$(1.10)\qquad \Theta_A(g;h,k;f)=\sum_{x\in Q^4}r(g)f(x\rho(h,k)).$$

Then (1.8) shows that

$$(1.11)\qquad \Theta_A(\gamma g;h,k;f)=\Theta_A(g;h,k;f)$$

holds for $\gamma\in G_Q$. For a prime $q\neq p$, let f_q be the characteristic function of $\boldsymbol{Z}_q^4$. Define a function f_∞ (respectively f_p) on $\boldsymbol{R}^4$ (respectively $\boldsymbol{Q}_p^4$) by $f_\infty(a,b,c,d)=e^{-p\pi(a^2+b^2+c^2+d^2)}$ (respectively $f_p(a,b,c,d)=\bar{\chi}(pb)\varphi(a,pb,c,d)$ with the characteristic function φ of $\boldsymbol{Z}_p^4$) and define a function f on A^4 by $f((a_v),(b_v),(c_v),(d_v))=\prod_v f_v(a_v,b_v,c_v,d_v)$. Then it is easy to see that the restriction of the function $\Theta_A(g;h,k;f)$ of $(g,h,k)\in G_A\times G_A\times G_A$ to $G\times G\times G$ is equal to $\Theta(gi,h,k)$. For a subset A in $\boldsymbol{Q}_p$, denote the characteristic function of A by $\varphi(\ ;A)$. Then for $f^{(1)}(a,b)=\bar{\chi}(pb)\varphi(a;\boldsymbol{Z}_p)\varphi(b;1/p\boldsymbol{Z}_p^\times)$ we have $\tilde{f}^{(1)}(a,b)=\mathscr{F}_{-p,p}f^{(1)}(a,b)=pG(\bar{\chi})\varphi(a;\boldsymbol{Z}_p)\chi(-bp)\varphi(b;1/p\boldsymbol{Z}_p^\times)$ with $G(\chi)=\int_{Z_p}\chi(u)\psi_p(u/p)du$ and for $f^{(2)}(a,b)=\varphi(a;\boldsymbol{Z}_p)\varphi(b;\boldsymbol{Z}_p)$ we have $\tilde{f}^{(2)}(a,b)=\mathscr{F}_{p,p}f^{(2)}(a,b)=\varphi(a;\boldsymbol{Z}_p)\varphi(b;1/p\boldsymbol{Z}_p)$. For $\sigma=\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}$ such that $\alpha,\beta,\delta\in\boldsymbol{Z}_p$, $\gamma\in p\boldsymbol{Z}_p$ and $\alpha\beta-\gamma\delta=1$, we can easily see that $\lambda(\sigma)\tilde{f}^{(1)}=\chi(\delta)\tilde{f}^{(1)}$ and $\lambda(\sigma)\tilde{f}^{(1)}=\chi(\delta)\tilde{f}^{(1)}$. These imply (1.4) since obviously $\Theta_A(gk_q;h,k;f)=\Theta_A(g;h,k;f)$ holds for $k_q\in SL(2,\boldsymbol{Z}_q)$ with all primes $q\neq p$. It can easily be verified that

$$\begin{aligned}\mathscr{F}_{-p,p}^{-1}\tilde{f}^{(1)}(a,b)&=\int_{Q_p}\tilde{f}^{(1)}(x,-a)\psi_p(pxb)|p|_pd_px\\&=G(\bar{\chi})\chi(ap)\varphi(a;1/p\boldsymbol{Z}_p^\times)\varphi(b;1/p\boldsymbol{Z}_p)\end{aligned}$$

and that

$$\mathscr{F}_{p,p}^{-1}\tilde{f}^{(2)}(a, b) = \int_{Q_p} \tilde{f}^{(2)}(x, -a)\psi_p(-pxb)|p|_p d_p x$$
$$= p^{-1}\varphi(a; 1/pZ_p)\varphi(b; 1/pZ_p)\,.$$

Put $f'((a_v), (b_v), (c_v), (d_v)) = f'_p(a_p, b_p, c_p, d_p) \prod_{v\neq p} f_v(a_v, b_v, c_v, d_v)$ where $f'_p(a, b, c, d) = \chi(pa)\varphi(a; 1/pZ_p)\varphi(b; 1/pZ_p)\varphi(c; 1/pZ_p)\varphi(d; 1/pZ_p)$. Then for $\omega = (\omega_v)$ such that $\omega_\infty = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\omega_v = 1$ if $v \neq \infty$ and for $g = (g_v)$ such that $g_\infty = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{v} & 0 \\ 0 & \sqrt{v}^{-1} \end{pmatrix}$ with $z = u + iv$ and $g_v = 1$ if $v \neq \infty$, we have

$$(1.12)\qquad \begin{aligned} \Theta(-1/z, h, k) &= \Theta_A(\omega g; h, k; f) = \Theta_A(g\tilde{\omega}; h, k; f) \\ &= p^{-2}\mathfrak{g}(\bar{\chi})\Theta_A(g; h, k; f') = p^{-2}\mathfrak{g}(\bar{\chi})\Theta(p^{-1}z, h, k) \end{aligned}$$

with $\tilde{\omega} = (\tilde{\omega}_v)$ where $\tilde{\omega}_\infty = 1$ and $\tilde{\omega}_v = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ for $v \neq \infty$ provided that $h, k \in G$. Hence the proof of Proposition 1 is completed.

For $w = x + iy \in H$, put $g_w = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}$ and let

$$(1.13)\qquad \theta(z, w_1, w_2) = \Theta(z, g_{w_1}, g_{w_2})\,, \qquad \tilde{\theta}(z, w_1, w_2) = \tilde{\Theta}(z, g_{w_1}, g_{w_2})\,.$$

Denote by $T_w^\chi(q)$ Hecke operators acting on a space of functions $\varphi(w)$ of w such that $\varphi(\gamma w) = \chi(d)\varphi(w)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ by the rule

$$T_w^\chi(q)\varphi(w) = \sum_i \varphi\left(\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}\gamma_i w\right)\bar{\chi}(d_i)$$

where the sum is extended over all $\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ such that $\Gamma\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}\Gamma = \bigcup_i \Gamma\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}\gamma_i$ (disjoint).

Let φ be as above and ψ is a function satisfying $\psi(\gamma w) = \bar{\chi}(d)\psi(w)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Then we have, if the integrals converge,

$$\int_{\Gamma\backslash H} (T_z^\chi(q)\varphi(z))\psi(z)d_0 z = \int_{\Gamma\backslash H} \varphi(z)(T_z^\chi(q)^*\psi(z))d_0 z\,.$$

Here $T_z^\chi(q)^*$ is the operator defined by $T_z^\chi(q)^*\psi(z) = \sum_i \psi\left(\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}\sigma_i z\right)\chi(d_i)$ where the sum is extended over all $\sigma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ such that $\Gamma\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}\Gamma = \bigcup_i \Gamma\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}\sigma_i$ and $d_0 z = \dfrac{du dv}{v^2}$. It is easy to see that $T_z^\chi(q)^* = \chi(q)T_z^{\bar{\chi}}(q)$ if $q \neq p$.

PROPOSITION 2. *For any prime q*

$$T_z^{\chi}(q)^*\theta(z, w_1, w_2) = T_{w_1}^{\chi}(q)\theta(z, w_1, w_2) = T_{w_2}^{\chi}(q)\theta(z, w_1, w_2)\,,$$

whether q equals p or not.

This proposition is almost proved by H. Yoshida ([8]). However we need the exact result convenient to us, so we describe the proof for the sake of completeness and convenience, after H. Yoshida.

Proof. Let the notation be as in the proof of Proposition 1. Put $\tilde{G} = \tilde{G}_\infty = GL(2, \boldsymbol{R})$, $\tilde{G}_q = GL(2, \boldsymbol{Q}_q)$, $\tilde{G}_{\boldsymbol{Q}} = GL(2, \boldsymbol{Q})$ and $\tilde{G}_A = GL(2, A)$. For primes $q \neq p$, put $\tilde{K}_q = GL(2, \boldsymbol{Z}_q)$ and $K_q = SL(2, \boldsymbol{Z}_q)$. Put

$$\tilde{K}_p = \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \boldsymbol{Z}_p) \,|\, c \equiv 0 \pmod{p}\right\},$$

$K_p = \tilde{K}_p \cap SL(2, \boldsymbol{Z}_p)$, $K_\infty = SO(2)$, $\mathfrak{Z}_v = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \Big| a \in \boldsymbol{Q}_v^\times\right\}$, $\mathfrak{Z}_{\boldsymbol{Q}} = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \Big| a \in \boldsymbol{Q}^\times\right\}$ and $\mathfrak{Z}_A = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \Big| a \in A^\times\right\}$. As usual we can extend the character χ to the character χ_A on $\mathfrak{Z}_A/\mathfrak{Z}_{\boldsymbol{Q}}$ and a function φ on H satisfying $\varphi(\gamma z k) = \bar{\chi}(d)\varphi(z) = \bar{\chi}(d)\varphi(-\bar{z})$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p)$ to the function φ_A on $\tilde{G}_A$ satisfying

$$\varphi_A(\gamma g k_\infty k_p k \mathfrak{z}) = \mathscr{S}_A(g)\chi_A(\mathfrak{z})\chi_A(d) \tag{1.14}$$

for $\gamma \in \tilde{G}_{\boldsymbol{Q}}$, $k_\infty \in \tilde{K}_\infty = O(2)$, $\mathfrak{z} \in \mathfrak{Z}_A$, $k \in \prod_{v \neq p,\infty} \tilde{K}_v$ and $k_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{K}_p$. Let φ be as above, q is a prime other than p and put $\psi(z) = \bar{\chi}(q)T_z^{\bar{\chi}}(q)\varphi(z)$, then we see that $\psi_A(g) = (T(q)\psi_A)(g) = \sum_i \varphi_A(g\beta_i)$ where the sum is extended over the elements β_i in $\tilde{G}_q$ such that $\tilde{K}_q\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}\tilde{K}_q = \bigcup_i \beta_i \tilde{K}_q$ (disjoint). We can extend a function Ψ on G_A satisfying

$$\Psi(\gamma g k k_p) = \Psi(g)\chi_A(d_p) \tag{1.15}$$

for $\gamma \in G_{\boldsymbol{Q}}$, $k \in \prod_{v \neq p} K_v$ and $k_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_p$ to a function $\tilde{\Psi}$ on $\tilde{G}_A$ satisfying (1.14) by putting $\tilde{\Psi}(\gamma g_\infty k \tau) = \Psi(g_\infty k)\chi_A(t_p)$ for $\gamma \in \tilde{G}_{\boldsymbol{Q}}$, $g_\infty \in G_\infty$, $k \in \prod_{v \neq \infty} K_v$, $\tau = (\tau_v) \in \tilde{G}_A$ where $\tau_\infty = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$, $r > 0$ and $\tau_q = \begin{pmatrix} 1 & 0 \\ 0 & t_q \end{pmatrix}$, $t_q \in \boldsymbol{Z}_q^\times$. Θ_A defined in the proof of Proposition 1 is a function on $G_A \times G_A \times G_A$ having the same types of property as (1.15) with respect to each variable so we can extend it to a function θ_A on $\tilde{G}_A \times \tilde{G}_A \times \tilde{G}_A$ which satisfies (1.14) with respect to the first variable and (1.14) modified by replacing χ_A by $\bar{\chi}_A$ with respect to the second and third variables. Let $g=\begin{pmatrix}1&u\\0&1\end{pmatrix}\begin{pmatrix}\sqrt{v}&0\\0&\sqrt{v}^{-1}\end{pmatrix}\in G_\infty$, $h=\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}\sqrt{y}&0\\0&\sqrt{y}^{-1}\end{pmatrix}\in G_\infty$ $k\in G_\infty$, $z=u+iv$ and $w=x+iy$. Choose the elements β_i in $\tilde{G}_q$ such that $\tilde{K}_q\begin{pmatrix}1&0\\0&q\end{pmatrix}\tilde{K}_q=\bigcup_i\beta_i\tilde{K}_q$ (disjoint). We can take elements in $\Gamma\begin{pmatrix}1&0\\0&q\end{pmatrix}\Gamma$ as such β_i so that $\Gamma\begin{pmatrix}q&0\\0&1\end{pmatrix}\Gamma=\bigcup_i\Gamma\alpha_i$ with $\alpha_i=q\beta_i^{-1}$. Note that we can take $\begin{pmatrix}1&0\\0&q\end{pmatrix}$, $\begin{pmatrix}q&i\\0&1\end{pmatrix}$, $(i=1,2,\cdots,q)$ as β_0,β_i, $(i=1,2,\cdots,q)$. We denote also by $\iota_{v'}(\gamma)$ an element (g_v) in $\tilde{G}_A$ such that $g_v=1$ if $v\neq v'$; $g_{v'}=\gamma$ and also by $\delta[\gamma]$ an element (h_v) in $\tilde{G}_A$ such that $h_v=\gamma$ for all v to avoid confusing $\iota_{v'}(\gamma)$ with $\delta[\gamma]$ when $\gamma\in\tilde{G}_Q$. We denote by $d(\sigma)$ the lower right entry d of $\sigma=\begin{pmatrix}a&b\\c&d\end{pmatrix}$. Then, by using (1.15), we have

$$\begin{aligned}
&T_z^\chi(q)^*\theta(z,w,ki)\\
&=\sum_i\theta_A(\iota_\infty(\alpha_ig);\ \iota_\infty(h),\ \iota_\infty(k);\ f)\chi(d(\alpha_i))\\
&=\chi(q)\sum_i\theta_A(\iota_\infty(\beta_i^{-1}g);\ \iota_\infty(h),\ \iota_\infty(k);\ f)\chi(d(\beta_i^{-1}))\\
&=\chi(q)\sum_i\theta_A(\delta[\beta_i^{-1}]\iota_\infty(g)\iota_q(\beta_i);\ \iota_\infty(h),\ \iota_\infty(k);f)\\
&=\chi(q)\sum_i\theta_A(\iota_\infty(g)\iota_q(\beta_i);\ \iota_\infty(h),\ \iota_\infty(k);f)\\
&=\chi(q)\sum_i\theta_A\left(\delta\left[\begin{pmatrix}1&0\\0&q^{-1}\end{pmatrix}\right]\iota_\infty(g)\iota_q(\beta_i);\ \iota_\infty(h),\ \iota_\infty(k);f\right)\\
&=\sum_i\theta_A\left(\iota_\infty\left(\begin{pmatrix}1&0\\0&q^{-1}\end{pmatrix}g\right)\iota_q\left(\begin{pmatrix}1&0\\0&q^{-1}\end{pmatrix}\beta_i\right);\ \iota_\infty(h),\ \iota_\infty(k);f\right)\\
&=\sum_i\theta_A\left(\iota_\infty\left(\begin{pmatrix}q^{1/2}&0\\0&q^{-1/2}\end{pmatrix}g\right)\iota_q\left(\begin{pmatrix}1&0\\0&q^{-1}\end{pmatrix}\beta_i\right);\ \iota_\infty(h),\ \iota_\infty(k);f)\\
&=\theta_A(\iota_\infty(g);\ \iota_\infty(h),\iota_\infty(k);\tilde{f})
\end{aligned}$$

with $\tilde{f}=q\tilde{f}_q\tilde{f}_\infty\prod_{v\neq q,\infty}f_v$ where $\tilde{f}_q=\sum_i r_q\left(\begin{pmatrix}1&0\\0&q^{-1}\end{pmatrix}\beta_i\right)f_q$, $\tilde{f}_\infty(x)=f_\infty(\sqrt{q}x)$ and f, f_v are the same as in the proof of Proposition 1. We can define the representations, which we also denote by ρ_v and ρ, of $\tilde{G}_v\times\tilde{G}_v$ and $\tilde{G}_A\times\tilde{G}_A$ by (1.9). It is easy to see that

$$\begin{aligned}
&T_w^\chi(q)\theta(z,w,ki)\\
&=\sum_i\theta_A(\iota_\infty(g);\ \iota_\infty(h)\iota_q(\beta_i),\ \iota_\infty(k);f)\\
&=\chi(q)\sum_i\theta_A(\iota_\infty(g);\ \iota_\infty\left(\begin{pmatrix}q^{1/2}&0\\0&q^{-1/2}\end{pmatrix}h\right)\iota_q\left(\begin{pmatrix}1&0\\0&q^{-1}\end{pmatrix}\beta_i\right),\ \iota_\infty(k);f)
\end{aligned}$$

$$= \chi(q)\sum_i \sum_{x \ni Q^4} r(\iota_\infty(g)) f\Big(x\rho\Big(\delta\Big[\begin{pmatrix}1 & 0\\ 0 & q\end{pmatrix}\Big], 1\Big)$$

$$\times \rho\Big(\iota_\infty\Big(\begin{pmatrix}q^{1/2} & 0\\ 0 & q^{-1/2}\end{pmatrix}h\Big),\ \iota_\infty(k)\Big)\rho\Big(\iota_q\Big(\begin{pmatrix}1 & 0\\ 0 & q^{-1}\end{pmatrix}\beta_i\Big), 1\Big)\Big)$$

$$= \sum_i \sum_{x \in Q^4} r(\iota_\infty(g))$$

$$\times f\Big(x\rho\Big(\iota_\infty\Big(\begin{pmatrix}q^{1/2} & 0\\ 0 & q^{1/2}\end{pmatrix}h\Big),\ \iota_\infty(k)\Big)\rho(\iota_q(\beta_i), 1)\Big)$$

$$= \theta_A(\iota_\infty(g);\ \iota_\infty(h),\ \iota_\infty(k);\ \hat{f})$$

with $\hat{f} = \tilde{f}_\infty \hat{f}_q \prod_{v \neq q,\infty} f_v$ where $\hat{f}_q(x) = \sum_i f_q(x\rho_q(\beta_i, 1))$. Therefore it is sufficient to show that $q\tilde{f}_q = \hat{f}_q$ in order to prove the former part of Proposition 2 in case $q \neq p$. Choose the elements $\delta_i \in \Gamma$ such that $\Gamma\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}\Gamma = \bigcup_i \Gamma\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}\delta_i$ (disjoint). Then it holds $K_p\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}K_p = \bigcup_i \delta_i^{-1}\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}K_p$ and we have

$$T_z^\chi(p)^*\theta(z, w, ki)$$

$$= \sum_i \theta_A\Big(\iota_\infty\Big(\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}\delta_i g\Big);\ \iota_\infty(h),\ \iota_\infty(k); f\Big)\chi(d(\delta_i))$$

$$= \sum_i \theta_A\Big(\iota_\infty\Big(\begin{pmatrix}1 & 0\\ 0 & 1/p\end{pmatrix}\delta_i g\Big);\ \iota_\infty(h),\ \iota_\infty(k);\ f\Big)\chi(d(\delta_i))$$

$$= \sum_i \theta_A\Big(\iota_\infty(g)\,\iota_p\Big(\delta_i^{-1}\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Big);\ \iota_\infty(h),\ \iota_\infty(k); f\Big)\chi(d(\delta_i))$$

$$= \sum_i \theta_A\Big(\delta\Big[\begin{pmatrix}1 & 0\\ 0 & p^{-1}\end{pmatrix}\Big]\iota_\infty(g)\,\iota_p\Big(\delta_i^{-1}\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Big);\ \iota_\infty(h),\ \iota_\infty(k); f\Big)\chi(d(\delta_i))$$

$$= \sum_i \theta_A\Big(\iota_\infty\Big(\begin{pmatrix}1 & 0\\ 0 & p^{-1}\end{pmatrix}g\Big)\iota_p\Big(\begin{pmatrix}1 & 0\\ 0 & p^{-1}\end{pmatrix}\delta_i^{-1}\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Big);\ \iota_\infty(h),\ \iota_\infty(k); f\Big)\chi(d(\delta_i))$$

$$= \sum_i \theta_A\Big(\iota_\infty\Big(\begin{pmatrix}p^{1/2} & 0\\ 0 & p^{-1/2}\end{pmatrix}g\Big)\iota_p\Big(\begin{pmatrix}1 & 0\\ 0 & p^{-1}\end{pmatrix}\delta_i^{-1}\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Big);\ \iota_\infty(h),\ \iota_\infty(k); f\Big)\chi(d(\delta_i))$$

$$= \theta_A(\iota_\infty(g);\ \iota_\infty(h),\ \iota_\infty(k);\ \tilde{f})$$

with $\tilde{f} = p\tilde{f}_p\tilde{f}_\infty \prod_{v \neq p,\infty} f_v$ where $\tilde{f}_p = \sum_i \chi(d(\delta_i))\, r_p\Big(\begin{pmatrix}1 & 0\\ 0 & p^{-1}\end{pmatrix}\delta_i^{-1}\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Big)f_p$, $\tilde{f}_\infty(x) = f_\infty(\sqrt{p}\,x)$ and f, f_v are the same as in the proof of Proposition 1. Choose the elements $\gamma_i \in \Gamma$ such that $\Gamma\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Gamma = \bigcup_i \Gamma\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\gamma_i$ (disjoint). Then it holds $K_p\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}K_p = \bigcup_i \gamma_i^{-1}\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}K_p$ and we have

$$
\begin{aligned}
&T_w^\chi(p)\theta(z, w, ki)\\
&= \sum_i \theta_A(\iota_\infty(g);\ \iota_\infty\left(\begin{pmatrix}1&0\\0&p\end{pmatrix}\gamma_i h\right),\ \iota_\infty(k); f)\bar\chi(d(\gamma_i))\\
&= \sum_i \theta_A(\iota_\infty(g);\ \iota_\infty(h)\iota_p\left(\gamma_i^{-1}\begin{pmatrix}p&0\\0&1\end{pmatrix}\right),\ \iota_\infty(k);\ f)\bar\chi(d(\gamma_i))\\
&= \sum_i \theta_A(\iota_\infty(g);\ \iota_\infty\left(\begin{pmatrix}p^{-1/2}&0\\0&p^{1/2}\end{pmatrix}h\right)\iota_p\left(\begin{pmatrix}p^{-1}&0\\0&1\end{pmatrix}\gamma_i^{-1}\begin{pmatrix}p&0\\0&1\end{pmatrix}\right),\ \iota_\infty(k); f)\\
&\quad\times\bar\chi(d(\gamma_i))\\
&= \sum_i\sum_{x\in \boldsymbol{Q}^4} r_\infty(\iota_\infty(g))\bar\chi(d(\gamma_i))f(x\rho\left(\delta\left[\begin{pmatrix}p&0\\0&1\end{pmatrix}\right],1\right)\\
&\quad\times \rho\left(\iota_\infty\left(\begin{pmatrix}p^{-1/2}&0\\0&q^{1/2}\end{pmatrix}h\right),\ \iota_\infty(k)\right)\rho\left(\iota_p\left(\begin{pmatrix}p^{-1}&0\\0&1\end{pmatrix}\gamma_i^{-1}\begin{pmatrix}p&0\\0&1\end{pmatrix}\right),1\right))\\
&= \sum_i\sum_{x\in \boldsymbol{Q}^4} r_\infty(\iota_\infty(g))\bar\chi(d(\gamma_i))\\
&\quad\times f\left(x\rho\left(\iota_\infty\left(\begin{pmatrix}p^{1/2}&0\\0&p^{1/2}\end{pmatrix}h\right),\ \iota_\infty(k)\right)\rho\left(\iota_p\left(\gamma_i^{-1}\begin{pmatrix}p&0\\0&1\end{pmatrix}\right),1\right)\right)\\
&= \theta_A(\iota_\infty(g);\ \iota_\infty(h),\ \iota_\infty(k);\hat f)
\end{aligned}
$$

with $\hat f = \tilde f_\infty \hat f_p \prod_{v\neq p,\infty} f_v$ where $\tilde f_p(x) = \sum_i \bar\chi(d(\gamma_i))f_p\left(x\rho_p\left(\gamma_i^{-1}\begin{pmatrix}p&0\\0&1\end{pmatrix},1\right)\right)$, Therefore it is sufficient to show that $p\tilde f_p = \hat f_p$ in order to prove the former part of Proposition 2 in case $q = p$. Since we can take $\begin{pmatrix}1&0\\-pi&1\end{pmatrix}$ $(i = 0, \cdots, p-1)$ and $\begin{pmatrix}1&-i\\0&1\end{pmatrix}$, $(i = 0, \cdots, p-1)$ as δ_i and γ_i respectively, it is sufficient to show that

$$
\sum_{i=0}^{p-1} f_p\left(x\rho_p\left(\begin{pmatrix}1&i\\0&1\end{pmatrix}\begin{pmatrix}p&0\\0&1\end{pmatrix},1\right)\right) = \sum_{i=0}^{p-1} pr_p\left(\begin{pmatrix}1&0\\i&1\end{pmatrix}\right)f_p(x) \quad (=p\tilde f_p(x))
$$

for all $x\in \boldsymbol{Q}_p^4$.

Let the notation be as in the proof of Proposition 1. Then $f_p(a, b, c, d) = f^{(1)}(a, b)f^{(2)}(c, d)$ and $r_p(g)f_p = r_{p,1}(g)f^{(1)}r_{p,2}(g)f^{(2)}$. Therefore we compute $r_{p,i}(g)f^{(i)}$, $(i = 1, 2)$. Since $r_{p,1}(g) = \mathscr{F}^{-1}_{-p,p}\lambda(g)\mathscr{F}_{-p,p}$ and

$$
\lambda\begin{pmatrix}1&0\\i&1\end{pmatrix}\tilde f^{(1)}(a, b) = pG(\bar\chi)\varphi(a + ib; \boldsymbol{Z}_p)\varphi(b; 1/p\boldsymbol{Z}_p)\chi_A(-bp),
$$

we have

$$
r_{p,1}\left(\begin{pmatrix}1&0\\i&1\end{pmatrix}\right)f^{(1)}(a, b) = G(\bar\chi)\int_{1/p\boldsymbol{Z}_p^\times}\varphi(a + iu; \boldsymbol{Z}_p)\chi_A(pu)du
$$

$$= G(\bar{\chi})\varphi(a; 1/p\boldsymbol{Z}_p^\times)\varphi(b; 1/p\boldsymbol{Z}_p)\chi_A(-pi^{-1}a)\psi(-pi^{-1}ab)$$

if $i \neq 0$. Recall that

$$r_{p,2}(g) = \mathscr{F}_{p,p}^{-1}\lambda(g)\mathscr{F}_{p,p} \quad \text{and} \quad \tilde{f}^{(2)}(c, d) = \varphi(c; \boldsymbol{Z}_p)\varphi(d; 1/p\boldsymbol{Z}_p)\,,$$

then we have

$$\lambda\left(\begin{pmatrix}1 & 0\\ i & 1\end{pmatrix}\right)\tilde{f}^{(2)}(c, d) = \varphi(c + id; \boldsymbol{Z}_p)\varphi(d; 1/p\boldsymbol{Z}_p)$$

and

$$r_{p,2}\begin{pmatrix}1 & 0\\ i & 0\end{pmatrix}\tilde{f}_p^{(2)}(c, d) = p^{-1}\psi(pi^{-1}cd)\varphi(c; 1/p\boldsymbol{Z}_p)\varphi(d; 1/p\boldsymbol{Z}_p)$$

if $i \neq 0$. Hence we have

$$\begin{aligned} p\tilde{f}_p(a, b, c, d) = p(&\varphi(a; \boldsymbol{Z}_p)\varphi(b; 1/p\boldsymbol{Z}_p)\varphi(c; \boldsymbol{Z}_p)\varphi(d; \boldsymbol{Z}_p)\bar{\chi}_A(pb) \\ &+ \sum_{k=1}^{p-1} p^{-1}G(\bar{\chi})\psi(pk(cd - ab))\chi_A(-pka) \\ &\times \varphi(c; 1/p\boldsymbol{Z}_p)\varphi(d; 1/p\boldsymbol{Z}_p)\varphi(a; 1/p\boldsymbol{Z}_p^\times)\varphi(b; 1/p\boldsymbol{Z}_p)\,. \end{aligned}$$

We can easily see that $\hat{f}_p(a, b, c, d) = 0$ unless $b \in 1/p\boldsymbol{Z}_p$ and that

$$\begin{aligned} &\hat{f}_p(a, b, c, d) = p\bar{\chi}(pb) && \text{if } b \in 1/p\boldsymbol{Z}_p^\times,\ d \in \boldsymbol{Z}_p,\ a \in \boldsymbol{Z}_p,\ c \in \boldsymbol{Z}_p, \\ &\hat{f}_p(a, b, c, d) = \bar{\chi}(pb) && \text{if } b \in 1/p\boldsymbol{Z}_p^\times, d \in \boldsymbol{Z}_p, a \in 1/p\boldsymbol{Z}_p^\times, c \in 1/p\boldsymbol{Z}_p, \\ &\hat{f}_p(a, b, c, d) = \bar{\chi}(pa)\bar{\chi}(p^2(ab - cd)) && \text{if } b \in 1/p\boldsymbol{Z}_p,\ d \in 1/p\boldsymbol{Z}_p^\times,\ a \in 1/p\boldsymbol{Z}_p^\times, \\ & && \quad c \in 1/p\boldsymbol{Z}_p,\ p^2(ad - bc) \in \boldsymbol{Z}_p^\times, \\ &\hat{f}_p(a, b, c, d) = 0 && \text{otherwise.} \end{aligned}$$

Using this, it is easy to see that $p\tilde{f}_p(x) = \hat{f}_p(x)$ in each case stated above. On the other hand, since we have

$$\begin{aligned} q\tilde{f}_q(a, b, c, d) &= q\left(r_q\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right) + \sum_{i=1}^{q} r_q\left(\begin{pmatrix}q & i\\ 0 & q^{-1}\end{pmatrix}\right)\right)f_q(a, b, c, d) \\ &= qf_q(a, b, c, d) + q^{-1}\sum_{i=1}^{q}\psi_q(p(bc - ad)qi)f_q(qa, qb, qc, qd) \end{aligned}$$

for $q \neq p$, we can easily verify $q\tilde{f}_q(x) = \hat{f}_q(x)$ case by case. These complete the proof of Proposition 2.

We call a function φ satisfying following conditions on H an automorphic wave form with character χ:

1. For $z = u + iv \in H$, $\varphi(z)$ is an eigenfunction of $D_z = v^2\left(\dfrac{\partial^2}{\partial u^2} + \dfrac{\partial^2}{\partial v^2}\right)$,

i.e., $D_z\varphi = \lambda\varphi$ with $\lambda \in \boldsymbol{C}$.

2. $\varphi(\gamma z) = \chi(d)\varphi(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

3. $\varphi(u + iv) = O(v^{\kappa})$, $v \to \infty$, with a constant $\kappa \in \boldsymbol{R}$, uniformly in u.

Put $\rho = (1 + \sqrt{1 + 4\lambda})/2$, $\nu = \sqrt{1/4 + \lambda}$. Let u_j be a representative of a Γ-equivalence class of cusps and σ_j is an element of $SL(2, \boldsymbol{Z})$ such that $\sigma_j\infty = u_j$. Then φ has the following Fourier expansion:

$$\varphi(\sigma_j z) = a^{(j)}v^{\rho} + b^{(j)}v^{1-\rho} + \sum_{n \neq 0} a_n^{(j)} v^{1/2} K_\nu(2\pi|n|v/N_j)e^{2\pi n i u/N_j} \tag{1.16}$$

with the modified Bessel function K_ν and an appropriate integer N_j. Let $u_0 = \infty$ and $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and put $a_n^{(0)} = a_n$. Then $N_0 = 1$. We call φ a cusp form if $a^{(j)} = b^{(j)} = 0$ for all j. For simplicity, we assume $\varphi(-\bar{z}) = \varphi(z)$ which means $a_n = a_{-n}$ for all n. Notice that a cusp form which is a common eigenfunction for all Hecke operators is the sum of two common eigenfunctions φ_+, φ_- such that $\varphi_+(-\bar{z}) = \varphi_+(z)$, $\varphi_-(-\bar{z}) = -\varphi_-(z)$. We further assume $a_1 = a_{-1} = 1$. Define

$$F(w_1, w_2) = \int_{\Gamma \backslash H} \varphi(z)\theta(z, w_1, w_2)d_0 z \tag{1.17}$$

where $d_0 z = v^{-2}dudv$ for $z = u + iv$. The following proposition guarantees that F is an eigenfunction of D_{w_1}, D_{w_2}.

PROPOSITION 2.

$$D_z\theta(z, w_1, w_2) = D_{w_1}\theta(z, w_1, w_2) = D_{w_2}\theta(z, w_1, w_2)\,. \tag{1.18}$$

We can easily prove this in the same way as in the proof of [7], Lemma 1.5. Thus we skip the proof.

By Proposition 1, Proposition 2 and the additional assumptions, we obtain the Fourier expansion of F, and can easily see the equality

$$F(w_1, w_2) = c\varphi(w_1)\varphi(w_2) \tag{1.19}$$

with a certain constant c which can not be determined by these Propositions and assumptions. Therefore our next task is to determine the constant c. For this purpose, we consider the integral

$$\int_0^\infty \int_0^1 F(w, w)dxy^{s-1}dy \qquad (w = x + iy)\,. \tag{1.20}$$

On one hand, for $\varphi(w) = \sum_{n \neq 0} a_{|n|} y^{1/2} K_\nu(2\pi|n|y)e^{2\pi n i x}$, it holds

$$
(1.21)\quad
\begin{aligned}
\int_0^\infty \int_0^1 F(w, w)dxy^{s-1}dy &= c\int_0^\infty \int_0^1 \varphi(w)\varphi(w)dxy^{s-1}dy \\
&= 2c(\sum_{n\geq 1} a_n^2 n^{-(s+1)})(2\pi)^{-(s+1)} \int_0^\infty K_\nu(y)^2 y^{s-1}dy \\
&= 2cL(s+1, \chi)(\sum_{n\geq 1} a_n^2 n^{-(s+1)}) \\
&\quad \times (2\pi)^{-(s+1)} \int_0^\infty K_\nu(y)^2 y^{s-1}dy\,,
\end{aligned}
$$

(see [1]) with $L(s, \chi) = \sum_{n=1}^\infty \chi(n)n^{-s}$. On the other hand, substituting the right-hand side of (1.17) for $F(w, w)$ in (1.20) and exchanging the order of integrations, we obtain

$$
(1.22)\quad
\begin{aligned}
&\int_0^\infty \int_0^1 F(w, w)dxy^{s-1}dy \\
&\quad = \int_{\Gamma\backslash H} \varphi(z) \int_0^\infty \int_0^1 \theta(z, w, w)y^{s-1}dxdyd_0z\,, \qquad (w = x + iy)\,.
\end{aligned}
$$

We can easily justify this formal computation by the absolute convergence of the integrals for sufficient large $\Re s$. It is more convenient to consider, instead of $\int_0^\infty \int_0^1 \theta(z, w, w)y^{s-1}dxdy$, another integral

$$
(1.23)\quad E_I(z, s) = \int_0^\infty \int_0^1 \tilde{\theta}(z, w, w)y^{s-1}dxdy\,.
$$

Now recall the definition (1.2) and (1.13) of $\tilde{\Theta}(z, g_1, g_2)$ and $\tilde{\theta}(z, w_1, w_2)$, then we see that an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of L in (1.2) is uniquely written as the sum of two elements $\begin{pmatrix} a & e/2 \\ e/2 & d \end{pmatrix}$, $\begin{pmatrix} 0 & f/2 \\ -f/2 & 0 \end{pmatrix}$ where e and f are integers with the same parity. Since ${}^t g \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for $g \in G$, we can decompose the summation defining $\tilde{\Theta}(z, g_1, g_2)$ into two parts, only when $g_1 = g_2$, as follows.

$$
(1.24)\quad \tilde{\Theta}(z, g, g) = v^{1/2}\overline{\theta(z)}\Theta_3(z, g) + v^{1/2}\overline{\theta_0(z)}\Theta_{3,0}(z, g)
$$

where

$$
(1.25)\quad \theta(z) = \sum_{n\in \boldsymbol{Z}} e^{2\pi i n^2 z}\,, \qquad \theta_0(z) = \sum_{n\in \boldsymbol{Z}, n;\text{odd}} e^{2\pi i n^2 z/4}\,,
$$

$$
(1.26)\quad \Theta_3(z, g) = \sqrt{v} \sum_{X=\begin{pmatrix} a & b \\ b & c \end{pmatrix}\in L_3} \chi(a)e^{2\pi i(-u\det X + i2^{-1}v\,\mathrm{tr}(({}^t gXg)^2))}
$$

and

$$\Theta_{3,0}(z, g) = \sqrt{v} \sum_{X=\begin{pmatrix} a & b \\ b & c \end{pmatrix} \in L_{3,0}} \chi(a) e^{2\pi i(-u \det X + i2^{-1} v \operatorname{tr}(({}^t g X g)^2))}, \tag{1.27}$$

with $L_3 = \left\{\begin{pmatrix} a & b \\ b & c \end{pmatrix} \middle| a, b, c \in \boldsymbol{Z}\right\}$ and $L_{3,0} = \left\{\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \middle| a, b, c \in \boldsymbol{Z},\ b;\ \text{odd}\right\}$. Therefore we obtain

$$\begin{aligned} E_I(z, s) &= \sqrt{v}\,\overline{\theta(z)} \int_0^\infty \int_0^1 \Theta_3\left(z, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}\right) dx y^{s-1} dy \\ &+ \sqrt{v}\,\overline{\theta_0(z)} \int_0^\infty \int_0^1 \Theta_{3,0}\left(z, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}\right) dx y^{s-1} dy. \end{aligned} \tag{1.28}$$

We have to make explicit computation of two integrals in (1.28) to show that they are equal to Siegel's generalizations of Eisenstein-Epstein series. Put

$$E_\theta(z, s) = \int_0^\infty \int_0^1 \Theta_3\left(z, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}\right) dx y^{s-1} dy, \tag{1.29}$$

$$E_{\theta,0}(z, s) = \int_0^\infty \int_0^1 \Theta_{3,0}\left(z, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}\right) dx y^{s-1} dy. \tag{1.30}$$

We treat only the first integral here because the second integral is reduced to the first. $E_\theta(z, s)$ can be transformed to the following Fourier series in z.

$$\begin{aligned} E_\theta(z, s) &= \sqrt{v} \sum_{n \in \boldsymbol{Z}} e^{2\pi i n u} \int_0^\infty \int_0^1 y^{s-1} \sum_{X=\begin{pmatrix} a & b \\ b & c \end{pmatrix} \in L_3, -\det X = n} \chi(a) \\ &\times \exp\left(-\pi v \operatorname{tr}\left(\left(\begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} X \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}\right)^2\right)\right) dx dy. \end{aligned} \tag{1.31}$$

Since $a \neq 0$ in (1.31) and $\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & am+b \\ am+b & am^2+2bm+c \end{pmatrix}$ $\in L_3$ for $m \in \boldsymbol{Z}$ if $\begin{pmatrix} a & b \\ b & c \end{pmatrix} \in L_3$, we can extend the interval of the integration with respect to x in (1.31) by letting b run over representatives modulo a instead of letting b run over all integers in the summation of (1.31). Therefore we obtain

$$\begin{aligned} E_\theta(z, s) &= \sqrt{v} \sum_{n \in \boldsymbol{Z}} e^{2\pi i n u} \int_0^\infty \int_{-\infty}^\infty \sum_{\substack{a \neq 0 \\ a \in \boldsymbol{Z}}} \sum_{\substack{b \bmod a, \\ b^2 \equiv n \bmod a}} \chi(a) y^{s-1} \\ &\times \exp\left(-\pi v \operatorname{tr}\left(\left(\begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} a & b \\ b & (b^2-n)/a \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right.\right.\right. \\ &\qquad \left.\left.\left.\times \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}\right)^2\right)\right) dx dy. \end{aligned} \tag{1.32}$$

Change the variable x to $x - b/a$, then the summand of the sum in the above integral becomes independent of b and the signature of a. Hence we have

$$E_\theta(z, s) = 2\sqrt{v}\sum_{n\in \boldsymbol{Z}} e^{2\pi i n u} \sum_{\substack{a>0\\ a\in \boldsymbol{Z}}} \chi(a)A_a(n)\int_0^\infty\int_{-\infty}^\infty y^{s-1}$$
$$\times \exp\left(-\pi v\,\mathrm{tr}\left(\left(\begin{pmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{pmatrix}\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & -n/a\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{pmatrix}\right)^2\right)\right) \times dxdy\,, \tag{1.33}$$

where we denote by $A_a(n)$ the number of solutions modulo a for the equation $x^2 \equiv n \pmod{a}$. Next, change the variables y, x to $a^{-1}y$, $a^{-1}x$ or $|n|^{1/2}a^{-1}y$, $|n|^{1/2}a^{-1}x$ according as n is equal to 0 or not. Then we have

$$E_\theta(z, s) = 2\sqrt{v}\,\Xi(s, 0)\,W_0(s, \pi v) + 2\sqrt{v} \sum_{\substack{n\in \boldsymbol{Z}\\ n\neq 0}} e^{2\pi i n u}|n|^{(s+1)/2}\Xi(s, n)\,W_{-\mathrm{sgn}\, n}(s, \pi|n|v) \tag{1.34}$$

where

$$\Xi(s, n) = \sum_{n=1}^{\infty} \chi(a)A_a(n)a^{-(s+1)}\,, \tag{1.35}$$

$$W_\varepsilon(s,v) = \int_0^\infty\int_{-\infty}^\infty y^{s-1} \times \exp\left(-\pi v\,\mathrm{tr}\left(\left(\begin{pmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{pmatrix}\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \varepsilon\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{pmatrix}\right)^2\right)\right) dxdy \tag{1.36}$$

and $\mathrm{sgn}\, n = \ n/|n|$.

LEMMA 1.

$$W_{-\mathrm{sgn}\, n}(s, \pi|n|v) = 2^{s/2-1}\Gamma((s+1)/2)(2\pi v)^{-(s+1)/2} \rho_n(s+1, v)\begin{cases}|n|^{-(s+1)/2} & (n \neq 0)\,,\\ 1 & (n = 0)\,,\end{cases} \tag{1.37}$$

where

$$\rho_n(s, v) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i n v \xi}d\xi}{(1 - i\xi)^{(s+1)/2}(1 + i\xi)^{s/2}}\,. \tag{1.38}$$

Proof. By the definition we can express $W_\varepsilon(v)$ as an integral of the modified Bessel function, namely,

$$W_\varepsilon(s, v) = 1/2\int_0^\infty \sqrt{x}^{\,-2vx}\int_0^\infty e^{-v(y+(x+\varepsilon)^2/y)}y^{s/2-1}dydx$$

$$= \int_0^\infty |\varepsilon + x|^{s/2} e^{-2vx} K_{-s/2}(2v|x+\varepsilon|)\sqrt{x}\, dx\,.$$

Using another integral expression

$$|\varepsilon + x|^{s/2} K_{-s/2}(2v|x+\varepsilon|)$$
$$= v^{-s/2} 2^{-1} \pi^{-1/2} \Gamma(s/2 + 1/2) \int_{-\infty}^{\infty} \frac{e^{-i\xi(2vx+2v\varepsilon)}}{(\xi^2+1)^{s/2+1/2}} d\xi$$

for $K_{-s/2}$ and changing the order of the integrations, we have

$$W_\varepsilon(s, v) = v^{-s/2} 2^{-1} \pi^{-1/2} \Gamma(s/2 + 1/2)$$
$$\times \int_{-\infty}^{\infty} \frac{e^{-i2\xi v\varepsilon}}{(\xi^2+1)^{s/2+1/2}} d\xi \int_0^\infty \sqrt{x}\, e^{-2vx(1+i\xi)} dx$$

By changing the variable x to $x/2v(1+i\xi)$ and rotating the path of integration to let it go back to the real axis, we have

$$\int_0^\infty \sqrt{x}\, e^{-2vx(1+i\xi)} dx = \sqrt{2v(1+i\xi)}^{\,-1} \int_0^{2v(1+i\xi)} \sqrt{x}\, e^{-x} dx$$
$$= \sqrt{2v(1+i\xi)\pi}^{\,-1}\,.$$

Hence we obtain

$$W_\varepsilon(s, v) = v^{-s/2} 2^{-3/2} \Gamma(s/2 + 1/2) \int_{-\infty}^{\infty} \frac{e^{i2\xi v\varepsilon}}{(\xi^2+1)^{s/2+1/2}(1-i\xi)^{1/2}} d\xi$$

by changing the variable ξ to $-\xi$, which completes the proof.

Lemma 1 shows that

$$(1.39)\quad E_\theta(z, s) = 2^{s/2} \Gamma((s+1)/2)(2\pi v)^{-(s+1)/2} \sqrt{v} \sum_{n \in Z} e^{2\pi i n u} \rho_n(s+1, v) \Xi(s, n)\,.$$

2. After Siegel we define generalizations of Eisenstein-Epstein series, which we simply call Eisenstein series in this paper, and compute their Fourier expansions. For integers a, c such that $(a, c) = 1$ and $c > 0$, put

$$(2.1)\qquad \lambda_1(a/c) = 2^{-1} \sqrt{i/c} \sum_{h=1}^{2c} e^{\pi i h^2 a/c}\,,$$

then we obtain

LEMMA 2. *Let* $\left(\frac{\ }{\ }\right)$, ε_d *be as in* [2], *namely* $\left(\frac{a}{b}\right) = \left(\frac{a}{|b|}\right)_J (a, b)_\infty$ *with Jacobi symbol* $\left(\frac{\ }{\ }\right)_J$ *and Hilbert symbol* $(\ ,\ )_\infty$ *at the infinite place, and* ε_d

is 1 *or* i *according as* $d \equiv 1 \pmod 4$ *or* $d \equiv -1 \pmod 4$. *Then*

$$\lambda_1(-a/c) = i\varepsilon_a\left(\frac{-2c}{a}\right), \text{ if } c \text{ is a even positive integer and } (a, c) = 1,$$

$$\lambda_1(-a/c) = 0, \text{ if } a \text{ and } c \text{ are odd, } c > 0 \text{ and } (a, c) = 1,$$

$$\lambda_1(-a/c) = \sqrt{i}\,\varepsilon_c\left(\frac{2a}{c}\right), \text{ if } c \text{ is a odd positive integer and } a \text{ is even.}$$

Now let us define Eisenstein series by

$$\varphi_\chi(z, s) = v^{s/2} \sum_{\substack{(a,c)=1 \\ c>0}} \lambda_1(-a/c)\chi(c)(cz - a)^{-1/2}|cz - a|^{-s}, \tag{2.2}$$

$$\psi_\chi(z, s) = v^{s/2} \sum_{\substack{(a,c)=1 \\ c>0;\text{odd}}} \lambda_1(-a/c)\chi(c)(cz - a)^{-1/2}|cz - a|^{-s}, \tag{2.3}$$

$$\phi_\chi(z, s) = v^{s/2} \sum_{\substack{(a,c)=1 \\ c>0;\text{even}}} \lambda_1(-a/c)\chi(c)(cz - a)^{-1/2}|cz - a|^{-s}, \tag{2.4}$$

where the sum in the first line is extended over all integers a, c such that $(a, c) = 1$ and $c > 0$; the sum in the second line is extended over all integers a, c such that $(a, c) = 1$, c is odd and $c > 0$; and the sum in the last line is extended over all integers a, c such that $(a, c) = 1$, c is even and $c > 0$. Then, by Lemma 2, we have obviously

$$\varphi_\chi(z, s) = \psi_\chi(z, s) + \phi_\chi(z, s). \tag{2.5}$$

Put $\Theta(z) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 z}$, $j'(\gamma, z) = \Theta(\gamma z)/\Theta(z)$ and $J(\gamma, z) = cz + d$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. Then it holds as well known that $j'(\gamma, z) = \varepsilon_d^{-1}\left(\frac{2c}{d}\right)J(\gamma, z)^{1/2}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$. Put

$$E^{\chi_p}(z, s) = \sum_{\gamma \in \Gamma(2)_\infty \backslash \Gamma_0(2p, 2)} j'(\gamma, z)^{-1}\chi_p(d)|J(\gamma, z)|^{-s}, \tag{2.6}$$

where χ_p is a character defined by $\chi_p(n) = \left(\frac{n}{p}\right)\chi(n)$, $\Gamma(2)_\infty = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2) \,\middle|\, c = 0\right\}$ and $\Gamma_0(N, M) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \,\middle|\, b \equiv 0 \pmod M\right\}$. Then one can easily derive from Lemma 2 that

$$\psi_\chi(z, s) = v^{s/2}\sqrt{i}\,\sqrt{z}^{\,-1}|z|^{-s}E^{\chi_p}(-1/pz, s), \tag{2.7}$$

$$\phi_\chi(z, s) = v^{s/2}p^{(1+2s)/2}\chi(2)j'(\mu_p, pz)^{-1}|J(\mu_p, pz)|^{-s}E^{\chi_p}(\mu_p pz, s), \tag{2.8}$$

with $\mu_p = \begin{pmatrix} pm & 2n \\ 2 & p \end{pmatrix}$ where m, n are integers such that $p^2m - 4n = 1$.

Now let us compute the Fourier expansions of $\varphi_\chi(z, s)$, $\psi_\chi(z, s)$ after [3]. Observing that $\lambda_1(a + 2c/c) = \lambda_1(a/c)$ for $a, c \in \boldsymbol{Z}$ coprime to each other, we have

$$
(2.9)\quad
\begin{aligned}
\varphi_\chi(2z, s) &= (2v)^{s/2} \sum_{\substack{(a,c)=1\\ c>0}} \lambda_1(a/c)\chi(c)c^{-s-1/2}(2z + a/c)^{-(s+1)/2}(2\bar{z} + a/c)^{-s/2}\\
&= (2v)^{s/2} \sum_{\substack{a/c \bmod 2\\ c>0}} \sum_{n=-\infty}^{\infty} \lambda_1(a/c + 2n)\chi(c)c^{-s-1/2}\\
&\quad \times (2z + a/c + 2n)^{-(s+1)/2}(2\bar{z} + a/c + 2n)^{-s/2}\\
&= (2v)^{s/2} \sum_{\substack{a/c \bmod 2\\ c>0}} \lambda_1(a/c)\chi(c)(2c)^{-s-1/2}\\
&\quad \times \sum_{n=-\infty}^{\infty} (z + a/2c + n)^{-(s+1)/2}(\bar{z} + a/2c + n)^{-s/2}\,.
\end{aligned}
$$

Since

$$
\sum_{n=-\infty}^{\infty} (z + n)^{-(s+1)/2}(\bar{z} + n)^{-s/2} = \sum_{n=-\infty}^{\infty} i^{-1/2}v^{-s+1/2}\rho_n(s, v)e^{2\pi i n u}
$$

with

$$
\rho_n(s, v) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i n v \xi}\, d\xi}{(1 - i\xi)^{(s+1)/2}(1 + i\xi)^{s/2}}\,,
$$

we have

$$
(2.10)\quad
\begin{aligned}
\varphi_\chi(z, s) &= 2^{s/2}v^{-s/2+1/2} \sum_{\substack{a/c \bmod 2\\ c>0}} \lambda_1(a/c)\chi(c)(2c)^{-1/2-s} \sum_{n=-\infty}^{\infty} i^{-1/2}\rho_n(s, v)e^{2\pi i n(u+a/2c)}\\
&= 2^{s/2}v^{-(s-1)/2} \sum_{n=-\infty}^{\infty} j_1(n, s)\rho_n(s, v)e^{2\pi i n u}\,,
\end{aligned}
$$

where

$$
(2.11)\quad
\begin{aligned}
j_1(n, s) &= \sum_{\substack{a/c \bmod 2\\ c>0}} \lambda_1(a/c)\chi(c)(2c)^{-1/2-s}i^{-1/2}e^{\pi i n a/c}\\
&= 2^{-s-3/2} \sum_{c=1}^{\infty} c^{-s-1}\chi(c) \sum_{h=1}^{2c} \sum_{\substack{a \bmod 2c\\ (a,c)=1}} e^{-\pi i(h^2-n)a/c}.
\end{aligned}
$$

Put $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s}$, then

$$
(2.12)\quad L(s, \chi)j_1(n, s) = 2^{-s-1/2} \sum_{b=1}^{\infty} b^{-s}\chi(b)A_{2b}(n)\,.
$$

Put

$$
(2.13)\quad \xi_p(s, n) = \sum_{r=0}^{\infty} p^{-rs}\chi(p^r)A_{p^r}(n)\,,
$$

for all primes p and put

$$\eta_2(s, n) = \sum_{r=0}^{\infty} 2^{-rs}\chi(2^r)A_{2^{r+1}}(n)\,, \tag{2.14}$$

then $\xi_2(s, n) = 1 + 2^{-s}\chi(2)\eta_2(s, n)$ and

$$L(s, \chi)j_1(n, s) = 2^{-s-1/2}\Big(\prod_{p\neq 2}\xi_p(s, n)\Big)\eta_2(s, n)\,. \tag{2.15}$$

In the exactly same way we have

$$\psi_\chi(z, s) = 2^{s/2}v^{-(s-1)/2}\sum_{n=-\infty}^{\infty} j(n, s)\rho_n(s, v)e^{2\pi inu}\,, \tag{2.16}$$

where

$$j(n, s) = \sum_{\substack{a/c \bmod 2\\ c>0;\ \text{odd}}} \lambda_1(a/c)\chi(c)(2c)^{-1/2-s}i^{-1/2}e^{\pi ina/c}\,. \tag{2.17}$$

Put $L_2(s, \chi) = \sum_{\substack{n=1\\ n;\ \text{odd}}}^{\infty} \chi(n)n^{-s}$, then

$$L_2(s, \chi)j(n, s) = 2^{-s-1/2}\sum_{\substack{b=1\\ b;\ \text{odd}}}^{\infty} b^{-s}\chi(b)A_{2b}(n) = 2^{-s-1/2}\prod_{p\neq 2}\xi_p(s, n)\,. \tag{2.18}$$

Since $\Xi(s, n) = \prod_p \xi_p(s, n)$ and $\xi_2(s, n) = 1 + 2^{-s}\chi(2)\eta_2(s, n)$,

$$\begin{aligned} &L_2(s, \chi)\psi_\chi(2z, s) + 2^{-s}\chi(2)L(s, \chi)\varphi_\chi(2z, s)\\ &\quad = 2^{-s/2-1/2}v^{-(s-1)/2}\sum_{n=-\infty}^{\infty} e^{2\pi inu}\rho_n(s, v)\Xi(s, n)\,. \end{aligned} \tag{2.19}$$

Recalling that

$$E_\theta(z, s) = 2^{s/2}\Gamma((s+1)/2)(2\pi v)^{-(s+1)/2}\sqrt{v}\sum_{n\in \boldsymbol{Z}} e^{2\pi inu}\rho_n(s+1, v)\Xi(s+1, n)\,, \tag{2.20}$$

we obtain

THEOREM 1. *The integral $E_\theta(z, s)$, defined in* (1.29), *of the theta function for the ternary quadratic form defined in* (1.26) *is a linear combination of Eisenstein series φ_χ and ψ_χ defined in* (2.2) *and* (2.3), *that is,*

$$\begin{aligned} E_\theta(z, s) &= 2^{s+1}\Gamma((s+1)/2)(2\pi)^{-(s+1)/2}\\ &\quad\times (L_2(s, \chi)\psi_\chi(2z, s) + 2^{-s}\chi(2)L(s, \chi)\varphi_\chi(2z, s))\,. \end{aligned}$$

3. We continue to compute the right-hand side of the equality (1.22) in this section. First, by changing the variable z to $-1/pz$ and using (1.23), (1.28), (1.29), (1.30), we can transform it to the following forms:

$$
\begin{aligned}
&\sqrt{p}^{-1}\int_{\Gamma\backslash H}\varphi(-1/pz)\int_0^\infty\int_0^1\tilde{\theta}(z,w,w)y^{s-1}dxdyd_0z \\
(3.1)\qquad &= \sqrt{p}^{-1}\int_{\Gamma\backslash H}\varphi(-1/pz)E_I(z,s)d_0z \\
&= \sqrt{p}^{-1}\int_{\Gamma\backslash H}(\sqrt{v}\,\varphi(-1/pz)\overline{\theta(z)}E_\theta(z,s) \\
&\quad + \sqrt{v}\,\varphi(-1/pz)\overline{\theta_0(z)}E_{\theta,0}(z,s))d_0z\,.
\end{aligned}
$$

Note that $\theta(z)$ and $E_\theta(z,s)$ are automorphic under $\Gamma_0(4p)$; and so is $\overline{\theta_0(z)}E_{\theta,0}(z,s)$ because

$$(3.2)\qquad \tilde{\Theta}(z,g,g) = v^{1/2}\overline{\theta(z)}\Theta_3(z,g) + v^{1/2}\overline{\theta_0(z)}\Theta_{3,0}(z,g)$$

is automorphic under $\Gamma_0(p)$. Therefore we can decompose the integral in (3.1) as follows.

$$
\begin{aligned}
(3.3)\qquad &\sqrt{p}^{-1}[\Gamma_0(p):\Gamma_0(4p)]^{-1}\Big(\int_{\mathscr{D}}\sqrt{v}\,\varphi(-1/pz)\overline{\theta(z)}E_\theta(z,s)d_0z \\
&\quad + \int_{\mathscr{D}}\sqrt{v}\,\varphi(-1/pz)\overline{\theta_0(z)}\,E_{\theta,0}(z,s)\,d_0z\Big)\,,
\end{aligned}
$$

where $\mathscr{D}$ denotes the fundamental domain $\Gamma_0(4p)\backslash H$. To investigate the second integral, consider

$$(3.4)\qquad \Theta_{3,0}(z,g) = \sqrt{v}\sum_{X=\left(\begin{smallmatrix}a & b\\ b & c\end{smallmatrix}\right)\in L_{3,0}}\chi(a)e^{2\pi i(-u\det X + i2^{-1}v\,\mathrm{tr}(({}^tgXg)^2))}\,,$$

then we see that

$$(3.5)\qquad \Theta_{3,0}(z,g) = \Theta_3^*(z,g) + \Theta_3(z,g)$$

when we put

$$(3.6)\qquad \Theta_3^*(z,g) = \sqrt{v}\sum_{X=\left(\begin{smallmatrix}a & b\\ b & c\end{smallmatrix}\right)\in L_3^*}\chi(a)e^{2\pi i(-u\det X + i2^{-1}v\,\mathrm{tr}(({}^tgXg)^2))}\,,$$

with $L_3^* = \left\{\begin{pmatrix}a & b/2\\ b/2 & c\end{pmatrix}\,\middle|\, a,b,c\in\boldsymbol{Z}\right\}$. By [7],

$$(3.7)\qquad \Theta_3(\sigma z,g) = \sqrt{2i}^{-1}\sqrt{pz+4}\,\chi(4)\Theta_3^*(z,g)$$

with $\sigma = \begin{pmatrix}4m & n\\ p & 4\end{pmatrix}$ where m, n are integers such that $16m - pn = 1$. Since σ normalizes $\Gamma_0(4p,4)$, $\Theta_3^*(z,g)$ is automorphic with respect to $\Gamma_0(4p,4)$. Namely, it holds $\Theta_3^*(\gamma z,g) = \chi(d)\sqrt{cz+d}\,\Theta_3^*(z,g)$ for $\gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma_0(4p,4)$. Hence we can decompose the second integral in (3.3) into two terms, that is,

$$\begin{aligned}(3.8)\quad &\int_{\mathscr{D}} \sqrt{v}\,\varphi(-1/pz)\overline{\theta_0(z)}E_{\theta,0}(z,s)d_0z \\ &= \int_{\mathscr{D}} \sqrt{v}\,\varphi(-1/pz)\overline{\theta_0(z)}\int_0^\infty\int_0^1 \Theta_{3,0}\left(z, \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{pmatrix}\right)dxy^{s-1}dy \\ &= 1/4\int_{\mathscr{D}'} \sqrt{v}\,\varphi(-1/pz)\overline{\theta_0(z)} \\ &\quad\times \int_0^\infty\int_0^1 \Theta_3^*\left(z, \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{pmatrix}\right)dxy^{s-1}dyd_0z \\ &\quad + 1/4\int_{\mathscr{D}'} \sqrt{v}\,\varphi(-1/pz)\overline{\theta_0(z)} \\ &\quad\times \int_0^\infty\int_0^1 \Theta_3\left(z, \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{pmatrix}\right)dxy^{s-1}dyd_0z\end{aligned}$$

with $\mathscr{D}' = \Gamma_0(4p, 4)\backslash H$. By (3.7) the first term is equal to

$$(3.9)\qquad 1/4\int_{\mathscr{D}'} \sqrt{v}\,\varphi(-1/pz)\overline{\theta_0(z)}\sqrt{2i}\,(pz+4)^{-1/2}\chi(4)E_\theta(\sigma z, s)d_0z\,,$$

and since $\sigma\mathscr{D}'$ is also a fundamental domain of $\Gamma_0(4p)$ and can be taken as the domain of integration in (3.9) instead of $\mathscr{D}'$, it is transformed to

$$(3.10)\quad 1/4\int_{\mathscr{D}'} \sqrt{v}\,|pz-4m|^{-1}\varphi(\omega_p\sigma^{-1}z)\overline{\theta_0(\sigma^{-1}z)}\,\sqrt{2i}\,(p\sigma^{-1}z+4)^{-1/2}\chi(4)E_\theta(z,s)d_0z$$

with $\omega_p = \begin{pmatrix}0 & -1\\ p & 0\end{pmatrix}$. Put

$$(3.11)\quad \Theta_1(z) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 z}\,,\quad \Theta_2(z) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i n^2 z}\,,\quad \Theta_3(z) = \sum_{n=-\infty}^{\infty} e^{\pi i (n+1/2)^2 z},$$

then the well known transformation formulas

$$(3.12)\quad \begin{aligned} &\Theta_1(z+1) = \Theta_2(z)\,, && \Theta_2(z+1) = \Theta_1(z)\,,\\ &\Theta_3(z+1) = \sqrt{i}\,\Theta_3(z)\,, && \Theta_1(-1/z) = \sqrt{z/i}\,\Theta_1(z)\,,\\ &\Theta_2(-1/z) = \sqrt{z/i}\,\Theta_3(z)\,, && \Theta_3(-1/z) = \sqrt{z/i}\,\Theta_3(z)\,,\end{aligned}$$

hold. Using these and the explicit transformation formula for $\Theta_1(z) = \theta(2^{-1}z)$,

$$(3.13)\qquad \Theta_1(\gamma z) = \left(\frac{2c}{d}\right)\varepsilon_d^{-1}\sqrt{cz+d}\,\Theta_1(z) \text{ for } \gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma(2)\,,$$

we have

$$
\begin{aligned}
\theta_0(\sigma^{-1}z) &= \Theta_3(\alpha\sigma^{-1}z) = \Theta_3(\omega^{-1}\sigma'\alpha^{-1}z)\\
&= i^{-1/2}(\sigma'\alpha^{-1}z)^{1/2}\Theta_2(\sigma'\alpha^{-1}z) = i^{-1/2}(\sigma'\alpha^{-1}z)^{1/2}\Theta_1(\tilde{\sigma}\varepsilon\alpha^{-1}z)\\
&= i^{-1/2}(\sigma'\alpha^{-1}z)^{1/2}\varepsilon_{-n-8}^{-1}(8\varepsilon\alpha^{-1}z - n - 8)^{1/2}\Theta_1(\varepsilon\alpha^{-1}z)\\
&= \sqrt{2i}^{\,-1}\sqrt{pz - 4m}\,\Theta_2(2^{-1}z)
\end{aligned}
\tag{3.14}
$$

where $\sigma' = \omega\alpha\sigma^{-1}\alpha\begin{pmatrix}1/2 & 0\\ 0 & 1/2\end{pmatrix} = \begin{pmatrix}p & -2m\\ 8 & -n\end{pmatrix}$, $\varepsilon = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $\alpha = \begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$ and $\tilde{\sigma} = \varepsilon\sigma'\varepsilon^{-1} = \begin{pmatrix}p+8 & *\\ 8 & -n-8\end{pmatrix}$. Here notice that $n \equiv -1 \pmod 4$ in (3.14) since $16m - pn = 1$; therefore $\varepsilon_{-n-8} = 1$. Observing that $\varphi(\omega_p\sigma^{-1}z) = \chi(4)\varphi(\omega_p z)$, we see that the first term of (3.8) equals

$$
1/4\int_{\mathscr{D}'}\sqrt{v}\,\varphi(\omega_p z)\overline{\Theta_2(2^{-1}z)}E_\theta(z, s)d_0 z\,. \tag{3.15}
$$

Put $\gamma_i = \begin{pmatrix}1 & i\\ 0 & 1\end{pmatrix}$, $(i = 1, 2, 3, 4)$, then $\Gamma_0(4p) = \cup_{i=1}^4 \Gamma_0(4p, 4)\gamma_i$. Hence we can suppose that $\mathscr{D}' = \cup_{i=1}^4 \gamma_i\mathscr{D}$. Therefore (3.15) becomes

$$
1/4\int_{\mathscr{D}}\sqrt{v}\,\varphi(\omega_p z)\overline{\sum_{i=1}^{4}\Theta_2\left(\begin{pmatrix}1 & i\\ 0 & 2\end{pmatrix}z\right)}E_\theta(z, s)d_0 z\,. \tag{3.16}
$$

By the definition of $\Theta_2(z)$, we have $\overline{\sum_{i=1}^4 \Theta_2\left(\begin{pmatrix}1 & i\\ 0 & 2\end{pmatrix}z\right)} = 4\theta(z)$ and therefore (3.16) becomes

$$
\int_{\mathscr{D}}\sqrt{v}\,\varphi(\omega_p z)\theta(z)E_\theta(z, s)d_0 z\,, \tag{3.17}
$$

which is identical with the first integral in (3.3).

We show that the second term of (3.8)

$$
1/4\int_{\mathscr{D}'}\sqrt{v}\,\varphi(-1/pz)\overline{\theta_0(z)}E_\theta(z, s)d_0 z \tag{3.18}
$$

is 0. Since $\mathscr{D}' = \cup_{i=1}^4 \gamma_i\mathscr{D}$ and $\theta_0(z) = \Theta_3(2z)$, it equals

$$
1/4\int_{\mathscr{D}}\sqrt{v}\,\varphi(\omega_p z)\overline{\sum_{i=1}^{4}\Theta_3\left(\begin{pmatrix}2 & 2i\\ 0 & 1\end{pmatrix}z\right)}E_\theta(z, s)d_0 z \tag{3.19}
$$

which is 0 because $\overline{\sum_{i=1}^4 \Theta_3\left(\begin{pmatrix}2 & 2i\\ 0 & 1\end{pmatrix}z\right)} = 0$ by the definition (3.11). Thus we obtain

$$
\begin{aligned}
&\int_0^\infty\int_0^1 F(w,w)dxy^{s-1}dy\\
&\quad = 2p^{-1}\mathfrak{g}(\chi)[\Gamma_0(p): \Gamma_0(4p)]^{-1}\int_{\mathscr{D}}\sqrt{v}\,\varphi(-1/pz)\overline{\theta(z)}E_\theta(z, s)d_0 z.
\end{aligned}
\tag{3.20}
$$

By Theorem 1 and (2.5), we have

$$E_\theta(z, s) = \Gamma((s+1)/2)(2\pi)^{-(s+1)/2}\chi(2)L(s, \chi) \times (\bar{\chi}(2)2^{s+1}\psi_\chi(2z, s+1) + \phi_\chi(2z, s+1)). \tag{3.21}$$

Therefore by (2.7) and (2.8) we have

$$\begin{aligned}&\int_{\mathscr{D}} \sqrt{v}\,\varphi(-1/pz)\overline{\theta(z)}E_\theta(z, s)d_0z \\ &= \Gamma((s+1)/2)(2\pi)^{-(s+1)/2}\chi(2)L(s+1, \chi) \\ &\quad \times (2^{(s+1)/2}p^{s+3/2}\chi(2)P_1(s) + \bar{\chi}(2)2^{3(s+1)/2}\sqrt{i}\,P_2(s))\end{aligned} \tag{3.22}$$

where

$$\begin{aligned}P_1(s) = \int_{\mathscr{D}} &\varphi(-1/pz)\overline{\theta(z)}v^{s/2+1}j'(\mu_p, 2pz)^{-1} \\ &\times |J(\mu_p, 2pz)|^{-s-1}E^{\chi p}(\mu_p 2pz, s+1)d_0z,\end{aligned} \tag{3.23}$$

$$\begin{aligned}P_2(s) = \int_{\mathscr{D}} &\varphi(-1/pz)\overline{\theta(z)}v^{s/2+1} \\ &\times \sqrt{2z}^{-1}|2z|^{-s-1}E^{\chi p}(-1/2pz, s+1)d_0z.\end{aligned} \tag{3.24}$$

Put $\nu = \begin{pmatrix} p & -n \\ -4p & p^2m \end{pmatrix}$, then ν normalizes $\Gamma_0(4p)$ and we can take $\nu^{-1}\mathscr{D}$ as a domain of integration instead of $\mathscr{D}$. Hence, changing the variable z to νz, we have

$$\begin{aligned}P_1(s) = \int_{\mathscr{D}} &\varphi\left(\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}\nu z\right)\overline{\theta(\nu z)}v^{s/2+1}(\sqrt{p}\,|4z - pm|)^{s+2}j'\left(\mu_p, \begin{pmatrix} 2p & 0 \\ 0 & 1 \end{pmatrix}\nu z\right)^{-1} \\ &\times \left|J\left(\mu_p, \begin{pmatrix} 2p & 0 \\ 0 & 1 \end{pmatrix}\nu z\right)\right|^{-s-1}E^{\chi p}\left(\mu_p\begin{pmatrix} 2p & 0 \\ 0 & 1 \end{pmatrix}\nu z, s+1\right)d_0z.\end{aligned} \tag{3.25}$$

Since $\mu_p\begin{pmatrix} 2p & 0 \\ 0 & 1 \end{pmatrix}\nu = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}\nu\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}^{-1} = \begin{pmatrix} 4 & -pm \\ p & -n \end{pmatrix} \in \Gamma_0(p)$ and $\nu\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -n \\ -4 & p^2m \end{pmatrix} \in \Gamma_0(4)$, we have $E^{\chi p}\left(\mu_p\begin{pmatrix} 2p & 0 \\ 0 & 1 \end{pmatrix}\nu z, s+1\right) = E^{\chi p}(2z, s+1)$, $\varphi\left(\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}\nu z\right) = \chi(-n)\varphi(z) = \bar{\chi}(4)\varphi(z)$ and $\theta(\nu z) = (-4pz + p^2m)^{1/2}\theta(pz)$. Therefore we obtain

$$P_1(s) = p^{-(s+1)/2}\bar{\chi}(4)\int_{\mathscr{D}} \varphi(z)\overline{\theta(pz)}v^{s/2+1}E^{\chi p}(2z, s+1)d_0z. \tag{3.26}$$

As for $P_2(s)$ in (3.24), changing the variable z to $-1/4pz$, we have

$$P_2(s) = p^{(s+2)/2}\sqrt{i}^{-1}\int_{\mathscr{D}} \varphi(4z)\overline{\theta(pz)}v^{s/2+1}E^{\chi p}(2z, s+1)d_0z. \tag{3.27}$$

Hence it follows from (2.22) that

$$\begin{aligned}
&\int_{\mathscr{D}} \sqrt{v}\,\varphi(-1/pz)\overline{\theta(z)}E_\theta(z,s)d_0z \\
&\quad = \Gamma((s+1)/2)(2\pi)^{-(s+1)/2}L(s+1,\chi)p^{(s+2)/2} \\
(3.28)\qquad &\quad \times \left(2^{(s+1)/2}\int_{\mathscr{D}} \varphi(z)\overline{\theta(pz)}v^{s/2+1}E^{\chi_p}(2z,s+1)d_0z\right. \\
&\quad \left. + 2^{3(s+1)/2}\int_{\mathscr{D}} \varphi(4z)\overline{\theta(pz)}v^{s/2+1}E^{\chi_p}(2z,s+1)d_0z\right).
\end{aligned}$$

Denote the first (respectively the second) integral in (3.28) by $Q_1(s)$ (respectively $Q_2(s)$). Then the usual computation for convolutions gives that for $\varphi(z) = \sum_{n\neq 0} a_{|n|}v^{1/2}K_\nu(2\pi|n|v)e^{2\pi niu}$,

$$\begin{aligned}
Q_1(s) &= \int_{\mathscr{D}} \varphi(z)\overline{\theta(pz)}v^{s/2+1}E^{\chi_p}(2z,s+1)d_0z \\
&= \sum_{\gamma\in(\Gamma_\infty\backslash\Gamma_0)(4p)} \int_{\mathscr{D}} v\langle\gamma z\rangle^{(s+2)/2}\varphi(\gamma z)\overline{\theta(p\gamma z)}d_0z \\
(3.29)\qquad &= \int_0^\infty\int_0^1 v^{(s+2)/2}\varphi(z)\overline{\theta(pz)}d_0z \\
&= \int_0^\infty 2\sum_{n=1}^\infty a_{pn^2}K_\nu(2\pi pn^2v)e^{-2\pi pn^2v}v^{s/2-1/2}dv \\
&= 2(2\pi)^{-(s+1)/2}\sum_{n=1}^\infty a_{pn^2}(pn^2)^{-(s+1)/2}\int_0^\infty v^{(s-1)/2}K_\nu(v)e^{-v}dv
\end{aligned}$$

where $\Gamma_\infty = \left\{\pm\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}\middle| n\in \mathbf{Z}\right\}$. By the same reasoning,

$$\begin{aligned}
Q_1(s) &= \int_0^\infty\int_0^1 v^{(s+2)/2}\varphi(4z)\overline{\theta(pz)}d_0z \\
(3.30)\qquad &= \int_0^\infty 4\sum_{n=1}^\infty a_{pn^2}K_\nu(8\pi pn^2v)e^{-8\pi pn^2v}v^{s/2-1/2}dv \\
&= 4(8\pi)^{-(s+1)/2}\sum_{n=1}^\infty a_{pn^2}(pn^2)^{-(s+1)/2}\int_0^\infty v^{(s-1)/2}K_\nu(v)e^{-v}dv\,,
\end{aligned}$$

hence we have

$$\begin{aligned}
&\int_{\mathscr{D}} \sqrt{v}\,\varphi(-1/pz)\overline{\theta(z)}E_\theta(z,s)d_0z \\
&\quad = \Gamma((s+1)/2)(2\pi)^{-(s+1)/2}L(s+1,\chi)p^{(s+2)/2} \\
(3.31)\qquad &\quad \times 2(2\pi)^{-(s+1)/2}\sum_{n=1}^\infty a_{pn^2}(pn^2)^{-(s+1)/2}\int_0^\infty v^{(s-1)/2}K_\nu(v)e^{-v}dv \\
&\quad \times (2^{(s+1)/2} + 2\cdot 4^{-(s+1)/2}2^{3(s+1)/2}) \\
&\quad = 6\Gamma((s+1)/2)(2\pi)^{-(s+1)}L(s+1,\chi)p^{1/2}a_p2^{(s+1)/2}
\end{aligned}$$

$$\times \left(\sum_{n=1}^{\infty} a_{n^2} n^{-(s+1)}\right) \int_0^\infty v^{(s-1)/2} K_\nu(v) e^{-v} dv .$$

Therefore we finally obtain

$$(3.32)\quad \begin{aligned} &\int_0^\infty \int_0^1 F(w, w) dx y^{s-1} dy \\ &= \Gamma((s+1)/2)(2\pi)^{-(s+1)} L(s+1, \chi) a_p 2^{(s+3)/2} p^{-1/2} \mathfrak{g}(\bar{\chi}) \\ &\quad \times \left(\sum_{n=1}^{\infty} a_{n^2} n^{-(s+1)}\right) \int_0^\infty v^{(s-1)/2} K_\nu(v) e^{-v} dv . \end{aligned}$$

Comparing this with (1.21), we have

$$(3.33)\quad \begin{aligned} &c \int_0^\infty K_\nu(y)^2 y^{s-1} dy \\ &= \Gamma((s+1)/2) a_p 2^{(s+3)/2} p^{-1/2} \mathfrak{g}(\bar{\chi}) \int_0^\infty v^{(s-1)/2} K_\nu(v) e^{-v} dv . \end{aligned}$$

From the well known formulas (see [5], p. 101 and p. 37)

$$(3.34)\quad \begin{aligned} &\int_0^\infty v^{(s-1)/2} K_\nu(v) e^{-v} dv \\ &= \sqrt{\pi}\, 2^{-(s+1)/2} \Gamma(s/2 + 1/2 + \nu) \Gamma(s/2 + 1/2 - \nu) \Gamma(s/2 + 1)^{-1} , \end{aligned}$$

$$(3.35)\quad \begin{aligned} &\int_0^\infty K_\nu(y)^2 y^{s-1} dy \\ &= 2^{s-2} \Gamma(s/2 + 1/2 + \nu) \Gamma(s/2 + 1/2 - \nu) \Gamma(s/2 + 1/2)^2 \Gamma(s+1)^{-1} \end{aligned}$$

and $\Gamma(2s) = 2^{2s-1} \pi^{-1/2} \Gamma(s) \Gamma(s + 1/2)$,

$$(3.36)\quad c = 2^3 a_p p^{-1/2} \mathfrak{g}(\bar{\chi})$$

follows.

We can easily verify the following lemma:

LEMMA 3. *Let* $\varphi(z) = \sum_{n\neq 0} a_{|n|} y^{1/2} K_\nu(2\pi|n|y) e^{2\pi n i x}$ *be a cusp form with the character* χ, *which is a common eigenfunction for all Hecke operators, and assume* $a_1 = 1$. *Define the cusp form* ϕ *with the character* $\bar{\chi}$ *by* $\phi(z) = \varphi(-1/pz)$. *Then* ϕ *is a common eigenfunction for all Hecke operators and its first Fourier coefficient* λ_φ *is given by*

$$\lambda_\varphi = \mathfrak{g}(\bar{\chi}) a_p^{-1} .$$

Let ϕ be as in Lemma 3 and put

$$(3.37)\quad G(w_1, w_2) = \int_{\Gamma\backslash H} \phi(z) \tilde{\theta}(z, w_1, w_2) d_0 z$$

with $\Gamma=\Gamma_0(p)$. Then by (1.3), (1.17), (1.19) and (3.36) we have

$$G(w_1, w_2)=\mathfrak{g}(\bar{\chi})F(w_1, w_2)=2^3 a_p\sqrt{p}\,\varphi(w_1)\varphi(w_2)\,. \tag{3.38}$$

Put $\psi(z)=\lambda_\varphi^{-1}\phi(z)$, then ψ is a common eigenfunction for all Hecke operators whose first Fourier coefficient is unity and therefore

$$\int_{\Gamma\backslash H}\psi(z)\tilde{\theta}(z,\ -1/pw_1,\ -1/pw_2)d_0z=\lambda_\varphi^{-1}2^3 a_p\sqrt{p}\,\varphi(-1/pw_1)\varphi(-1/pw_2)\,. \tag{3.39}$$

Hence we obtain

THEOREM 2. *Let ψ be a cusp form with the character $\bar{\chi}$, which is a common eigenfunction for all Hecke operators. Assume that the first Fourier coefficient of ψ is unity. Then it holds*

$$\int_{\Gamma\backslash H}\psi(z)\tilde{\theta}(z,\ -1/pw_1,\ -1/pw_2)d_0z=2^3\sqrt{p}\,\mathfrak{g}(\bar{\chi})\psi(w_1)\psi(w_2)\,.$$

4. Let $\mathfrak{o}$ be the ring of integers in an imaginary quadratic field $\boldsymbol{Q}(\sqrt{-p})$ with a rational prime $p\equiv 1 \pmod 4$. Then $\mathfrak{o}=\boldsymbol{Z}+\sqrt{-p}\boldsymbol{Z}$. Put $\mathfrak{P}=\sqrt{-p}\,\mathfrak{o}$ and denote the upper half space $\{(z, v)\,|\,z\in\boldsymbol{C},\ v>0\}$ by $\boldsymbol{H}$. Let us define the action of $G_C=SL(2, \boldsymbol{C})$ on $\boldsymbol{H}$ in the usual way. For $a\in\boldsymbol{C}$, put $\tilde{a}=\begin{pmatrix}a & 0\\ 0 & \bar{a}\end{pmatrix}$. For $w=(z, v)\in\boldsymbol{H}$ and $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in G_C$, put $\tilde{w}$ $\begin{pmatrix}z & -v\\ v & \bar{z}\end{pmatrix}$, and define $w'=\gamma w=(z', v')\in\boldsymbol{H}$ by the relation

$$\tilde{w}'=(\tilde{a}\tilde{w}+\tilde{b})(\tilde{c}\tilde{w}+\tilde{d})^{-1}\,.$$

Put

$$E_0(w, s)=\sum_{n\in\mathfrak{o},m\in\mathfrak{P}} v^s(|mz+n|^2+|m|^2v^2)^{-s}\bar{\chi}(n) \tag{4.1}$$

where χ is the character on $\mathfrak{o}/\mathfrak{P}$ defined by $\chi(n+\sqrt{-p}\,l)=\chi(n)$ for $n, l\in\boldsymbol{Z}$. Then E_0 is a non-holomorphic Eisenstein series on $\boldsymbol{H}$ satisfying $E_0(\gamma w, s)=\chi(d)E_0(w, s)$ for $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma_{\mathfrak{P}}=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL(2, \mathfrak{o})\,|\,c\equiv 0 \pmod{\mathfrak{P}}\right\}$.

Put

$$F_0(w, s)=\sum_{n\in\mathfrak{P},m\in\mathfrak{o}} v^s(|mz+n|^2+|m|^2v^2)^{-s}\bar{\chi}(m) \tag{4.2}$$

then $E_0(w, s)=F_0(\omega w, s)$ with $\omega=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. F_0 has an usual integral expression, that is,

$$(4.3)\quad \begin{aligned} F_0(w, s) &= \pi^s \Gamma(s)^{-1} \\ &\times \int_0^\infty \eta^{s-1} \sum_{n\in\mathfrak{P}, m\in\mathfrak{o}} \bar{\chi}(m) \exp(-\pi\eta v^{-1}(|mz+n|^2+|m|^2v^2))d\eta\,. \end{aligned}$$

The series in the integrand can be expressed as a theta function for a quadratic form over $K = \boldsymbol{Q}(\sqrt{-p})$. Put $\mathfrak{P}^* = \sqrt{-p}^{-1}(\boldsymbol{Z} + \sqrt{-p}^{-1}\boldsymbol{Z})$, then, using the Poisson summation formula, we obtain

$$(4.5)\quad \begin{aligned} &\sum_{n\in\mathfrak{P}} \exp(\pi\eta v^{-1}(|mz+n|^2+|m|^2v^2)) \\ &= \sum_{n\in\mathfrak{P}^*} p^{-3/2}\eta^{-1}v \exp(-2\pi i\Re e(mnz) - \pi(|m|^2\eta + |n|^2\eta^{-1})v) \end{aligned}$$

so that

$$(4.6)\quad \begin{aligned} &\sum_{n\in\mathfrak{P}, m\in\mathfrak{o}} \bar{\chi}(m) \exp(-\pi\eta v^{-1}(|mz+n|^2+|m|^2v^2)) \\ &= p^{-3/2}\eta^{-1}v \sum_{m\in\mathfrak{o}, n\in\mathfrak{P}^*} \bar{\chi}(m) \\ &\quad\times \exp(-2\pi i\Re e(mnz) - \pi(|m|^2\eta + |n|^2\eta^{-1})v)\,. \end{aligned}$$

Hence we obtain

$$(4.7)\quad \begin{aligned} F_0(w, s) &= \pi^s\Gamma(s)^{-1}p^{-3/2}\int_0^\infty \eta^{s-2} \\ &\times v \sum_{m\in\mathfrak{o}, n\in\mathfrak{P}^*} \bar{\chi}(m) \exp(-2\pi i\Re e(mnz) - \pi(|m|^2\eta + |n|^2\eta^{-1})v)d\eta\,. \end{aligned}$$

Now we consider the pullbacks of E_0, F_0. It is easy to see that for $z = u + iv \in H$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{R})$, $\gamma(u, v) = (u', v')$ holds denoting $\gamma z = \dfrac{az+b}{cz+d} = u' + iv'$. Since $E_0(\gamma w, s) = \chi(d)E_0(w, s)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\mathfrak{P}}$ and $\Gamma_{\mathfrak{P}} \cap SL(2, \boldsymbol{R}) = \Gamma_0(p)$, we see that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p)$, $E(\gamma(u+iv)) = \chi(d)E(u+iv)$ holds and for $\sigma \in \Gamma^0(p)$, $F(\sigma(u+iv)) = \bar{\chi}(d)F(u+iv)$ holds when we put $E(u+iv) = E_0((u, v), s)$, $F(u+iv) = F_0((u, v), s)$ with $u, v \in \boldsymbol{R}$. If $z = u \in \boldsymbol{R}$, the series in the right-hand side of (4.7) can be viewed as a theta function for a quaternary quadratic form over $\boldsymbol{Q}$. In fact, put $n = (-p)^{-1/2}(-b - a(-p)^{-1/2})$ and $m = d + (-p)^{1/2}c$ with $a, b, c, d \in \boldsymbol{Z}$, then

$$\begin{aligned} \Re e(mn) &= \det\begin{pmatrix} a & b \\ c & d/p \end{pmatrix}, \\ |m|^2\eta + |n|^2\eta^{-1} &= (d^2 + pc^2)\eta + p^{-1}(b^2 + p^{-1}a^2)\eta^{-1} \\ &= \mathrm{tr}\left({}^t\!\left(h_\eta\begin{pmatrix} a & b \\ c & d/p \end{pmatrix}l_p\right)\left(h_\eta\begin{pmatrix} a & b \\ c & d/p \end{pmatrix}l_p\right)\right), \end{aligned}$$

with

$$h_\eta = \begin{pmatrix} (\eta p^{3/2})^{-1/2} & 0 \\ 0 & (\eta p^{3/2})^{1/2} \end{pmatrix}, \qquad l_p = \begin{pmatrix} p^{-1/4} & 0 \\ 0 & p^{1/4} \end{pmatrix}.$$

Thus we obtain

$$F_0((u, v), s) = \pi^s \Gamma(s)^{-1} p^{-1/2} \int_0^\infty \eta^{s-2} \Theta((u+iv)/p, h_\eta, l_p) d\eta \tag{4.8}$$

where $\Theta(z, g, h)$ is the theta function defined in (1.1). It follows from (1.3) that

$$F_0(\omega(u, v), s) = \pi^s \Gamma(s)^{-1} p^{-3/2} \mathfrak{g}(\bar{\chi}) \int_0^\infty \eta^{s-2} \tilde{\Theta}(u+iv, h_\eta, l_p) d\eta . \tag{4.9}$$

Since $\omega \tilde{h}_\eta {}^t\omega_p = h_\eta$ and $\omega_p l_p \omega^{-1} = l_p$ with $\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\omega_p = \begin{pmatrix} 0 & -p^{-1/2} \\ p^{1/2} & 0 \end{pmatrix}$ and $\tilde{h}_\eta = \begin{pmatrix} \eta^{1/2} p^{1/4} & 0 \\ 0 & \eta^{-1/2} p^{-1/4} \end{pmatrix}$, it holds

$$\tilde{\Theta}(z, h_\eta, l_p) = \tilde{\Theta}(z, \omega_p \tilde{h}_\eta, \omega_p l_p) .$$

Therefore, for a cusp form ψ, putting $\mathscr{D}_1 = \Gamma_0(p)\backslash H$ and $z = u + iv$, we have

$$\begin{aligned} &\int_{\mathscr{D}_1} E_0((u, v), s) \psi(z) d_0 z \\ &\quad = p^{-3/2} \mathfrak{g}(\bar{\chi}) \pi^s \Gamma(s)^{-1} \int_0^\infty \eta^{s-2} \int_{\mathscr{D}_1} \tilde{\Theta}(z, \omega_p \tilde{h}_\eta, \omega_p l_p) \psi(z) d_0 z d\eta . \end{aligned} \tag{4.10}$$

On the other hand, Theorem 2 asserts

$$\int_{\mathscr{D}_1} \tilde{\Theta}(z, \omega_p \tilde{h}_\eta, \omega_p l_p) \psi(z) d_0 z d\eta = 2^3 \sqrt{p}\, \mathfrak{g}(\chi) \psi(p^{1/2} \eta i) \psi(p^{-1/2} i) . \tag{4.11}$$

Thus we obtain

$$\begin{aligned} &\int_{\mathscr{D}_1} E_0((u, v), s) \psi(z) d_0 z \\ &\quad = \pi^s \Gamma(s)^{-1} 2^3 \psi(p^{-1/2} i) \int_0^\infty \eta^{s-2} \psi(p^{1/2} \eta i) d\eta . \end{aligned} \tag{4.12}$$

Since $\int_0^\infty K_\nu(\eta) \eta^{s-1} d\eta = 2^{s-2} \Gamma((s-\nu)/2) \Gamma((s+\nu)/2)$, we finally have

THEOREM 3. *Let χ be a primitive Dirichlet character modulo a prime $p \equiv 1 \pmod 4$. Assume $\chi(-1) = 1$. Let $\psi(z) = \sum_{n \neq 0} a_{|n|} v^{1/2} K_\nu(2\pi |n| v) e^{2\pi n i u}$ be a cusp form with the character $\bar{\chi}$, which is a common eigenfunction for all Hecke operators, and assume $a_1 = 1$. Let $E_0(w, s)$ be the Eisenstein series defined by* (4.1) *on the upper half space. Then it holds*

$$\int_{\Gamma_0(p)\backslash H} E_0((u, v), s)\psi(u + iv)v^{-2}dudv$$
$$= \pi^{1/2}\Gamma(s)^{-1}\Gamma\Big(\frac{s - 1/2 - \nu}{2}\Big)\Gamma\Big(\frac{s - 1/2 + \nu}{2}\Big)$$
$$\times 2^2\psi(p^{-1/2}i)p^{-(s-1)/2}L(s - 1/2, \psi)$$

where $L(s, \psi) = \sum_{n=1}^{\infty} a_n n^{-s}$.

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