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A DECOMPOSITION FORMULA FOR REPRESENTATIONS*

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Let H be the Levi subgroup of a parabolic subgroup of a split reductive group G. In characteristic zero, an irreducible representation V of G decomposes when restricted to H into a sum $V = \oplus m_\alpha W_\alpha$ where the W_α's are distinct irreducible representations of H. We will give a formula for the multiplicities m_α. When H is the maximal torus, this formula is Weyl's character formula. In theory one may deduce the general formula from Weyl's result but I do not know how to do this.

My formula will also be valid in a Grothendieck group in positive characteristic. The proof uses a modification of Demazure's character formula [1] but I think that my formulation is more useful for calculations.

§1. The fundamentals

Let $T \subset B \subset G$ be a maximal torus contained in a Borel subgroup of G. The characters (or weights) of T are identified with characters of B. The Grothendieck group of finite dimensional B-modules is the free abelian group generated by the weights, which we will call the group ring.

We have G-linearized coherent sheaves on the homogeneous space G/B [5, 3]. The G-linearized invertible sheaves correspond to characters of B. For each weight ψ, we have an invertible sheaf $\mathcal{O}_{G/B}(\psi)$. If ψ is dominant, then $\mathcal{O}_{G/B}(\psi)$ has non-zero sections. A general G-linearized coherent sheaf $\mathscr{W}$ has a composition series with invertible factors $\mathcal{O}_{G/B}(\psi_i)$ for $0 \leq i \leq \operatorname{rank} \mathscr{W} = n$. Then we write

$$[\mathscr{W}] = \sum_{1 \leq i \leq n} \psi_i .$$

Thus the class $[\mathscr{W}]$ determines the image of $\mathscr{W}$ in the Grothendieck group of G-linearized coherent sheaves. This symbol is contained in the group ring of the characters.

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We will need some linear operators on the group ring. Let α be a root. We will define a linear operator L_α by the rules:

$$L_\alpha(\psi) = \begin{cases} \sum_{0 \leq i \leq \langle \psi, \alpha^v \rangle} \psi\alpha^{-i} & \text{if } \langle \psi, \alpha^v \rangle \geq 0 \\ 0 & \text{if } \langle \psi, \alpha^v \rangle = -1 \\ -\sum_{1 \leq n \leq -\langle \psi, \alpha^v \rangle - 1} \psi\alpha^n & \text{if } \langle \psi, \alpha^v \rangle \leq -2 \end{cases}$$

Let α be a basic root. Let $P = P(\alpha)$ be the parabolic subgroup containing B with exactly one negative root $-\alpha$. Consider the projection $\pi\colon G/B \to G/P$. If $\mathscr{W}$ is a G-linearized coherent sheaf on G/B, then $\pi^*\pi_*\mathscr{W}$ and $\pi^*R^1\pi_*\mathscr{W}$ are G-linearized coherent sheaves on G/B. The difference $[\pi^*\pi_*\mathscr{W}] - [\pi^*R^1\pi_*\mathscr{W}]$ is additive in $\mathscr{W}$ because $R^in_*\mathscr{W} = 0$ for $i > 1$ and π is flat. Thus we have a linear operation $\pi^*\pi_*$ on the group ring such that $\pi^*\pi_*(\psi) \equiv [\pi^*\pi_*\mathscr{O}_{G/B}(\psi)] - [\pi^*R^1\pi_*\mathscr{O}_{G/B}(\psi)]$. The principal result is

THEOREM 1. $L_\alpha(\psi) = \pi^*\pi_*(\psi)$.

Proof. Now π is a $P/B \approx \boldsymbol{P}^1$-bundle and $\langle \psi, \alpha^v \rangle$ is the fiber degree of $\mathscr{O}_{G/B}(\psi)$. By Serre's theorem, $\pi_*\mathscr{O}_{G/B}(\psi) = 0$ if $\langle \psi, \alpha^v \rangle < 0$ and $R^1\pi_*\mathscr{O}_{G/B}(\psi) = 0$ if $\langle \psi, \alpha^v \rangle > -2$. Thus if $\langle \psi, \alpha^v \rangle = -1$, $\pi^*\pi_*(\psi) = 0$ and the formula is true. If $\langle \psi, \alpha^v \rangle \geq 0$, then $\pi_*\mathscr{O}_{G/B}(\psi)$ is locally free of rank $1 + \langle \psi, \alpha^v \rangle$. Then $\pi^*\pi_*\mathscr{O}_{G/B}(\psi)$ a G-equivariant filtration with factors

$$\psi, \psi\alpha^{-1}, \cdots, \psi\alpha^{-\langle \psi, \alpha v \rangle}.$$

This can be checked on a fiber where it is rather trivial property of $\boldsymbol{P}^1$ and rank 1 groups. Hence the formula is true. For the case $\langle \psi, \alpha^v \rangle \leq -2$, note that $\mathscr{O}_{G/B}(\alpha^{-1})$ is the relative dualizing sheaf for π. Hence $R^1\pi_*\mathscr{O}_{G/B}(\alpha^{-1})$ is trivial as a G-sheaf. By duality we have a G-equivariant perfect pairing $R^1\pi_*\mathscr{O}_{G/B}(\psi) \otimes \pi_*\mathscr{O}_{G/B}(\psi^{-1}\alpha^{-1}) \to \mathscr{O}_{G/P}$. It follows that $\pi^*R^1\pi_*\mathscr{O}_{G/B}(\psi)$ has composition factors $\psi_1, \cdots, \psi_r$ where $\psi_1^{-1}, \cdots, \psi_r^{-1}$ are composition factors of $\pi^*\pi_*\mathscr{O}_{G/B}(\psi^{-1}\alpha^{-1})$ but $\langle \psi^{-1}\alpha^{-1}, \alpha^v \rangle \geq 2 - 2 = 0$. Hence the last set of characters is $\psi^{-1}\alpha^{-1}, \cdots, \psi\alpha^{-(1+\langle \psi^{-1}, \alpha v \rangle - 2)}$. Thus $\{\psi_1, \cdots, \psi_r\}$ is $\{\psi\alpha, \cdots, \psi\alpha^{(-1-\langle \psi, \alpha^v \rangle)\alpha}\}$. In other words the formula is true in this case.

Q.E.D.

The above duality gives a symmetry in the formula for L. In fact $L_\alpha(\psi) = -L_\alpha(\psi\alpha^{-(\langle \omega, \alpha^v \rangle + 1)\alpha})$. Recall the twisted action $s^*\psi = s(\psi\rho)^{-1}$ of the Weyl group on weights where ρ is the square root of the product of the positive roots. Here $s_\alpha^*\psi = \psi\alpha^{-(\langle \psi, \alpha^v \rangle + 1)}$ where s_α is the symmetry about α.

Thus $L_\alpha(\psi) = -L_\alpha(s_\alpha^*\psi)$.

Given a G-linearized sheaf $\mathscr{W}$ on G/B, the cohomology groups $H^i(G/B, \mathscr{W})$ are G-modules. Thus we may regard the Euler charactreistic $\chi(\mathscr{W}) = \sum(-1)^i H^i(G/B, \mathscr{W})$ as an element of the Grothendieck group of G-modules. When $\mathscr{W} = \mathcal{O}_{G/B}(\psi)$ we will denote its Euler characteristic by $\chi_{G/B}(\psi)$. Also we extend $\chi_{G/B}$ to all of the group ring additively.

A useful identity due to Hirzebruch and Borel is

THEOREM 2. *For any s in the Weyl group*

$$\chi_{G/B}(\psi) = (-1)^{\text{length}(s)}\chi_{G/B}(s^*\psi)\,.$$

Proof. As s is the product of symmetries s_α about basic roots, we may assume that $s = s_\alpha$. This theorem will follow from the symmetry of L if we prove

LEMMA 3. $\chi_{G/B}(\psi) = \chi_{G/B}(L_\alpha(\psi))$.

Proof. By the Leray spectral sequence for π and the additivity of Euler characteristics we have

$$\chi_{G/B}(\psi) = \chi(\pi_*\mathcal{O}_{G/B}(\psi)) - \chi(R^1\pi_*\mathcal{O}_{G/B}(\psi))\,.$$

The point is that last quantity equals $\chi_{G/B}(\pi^*\pi_*\psi)$ which equals $\chi[L_\alpha(\psi)]$ by Theorem 1. The point is a direct consequence of Lemma 4 where $f = \pi$ and $\mathscr{W} = R^i\pi_*\mathcal{O}_{G/B}(\psi)$.

LEMMA 4. *Let f: $X \to Y$ be a morphism such that $f_*\mathcal{O}_X \approx \mathcal{O}_Y$ and $R^if_*\mathcal{O}_X = 0$ if $i>0$. For any locally free sheaf $\mathscr{W}$ on Y, we have natural isomorphisms*

$$H^i(X, f^*\mathscr{W}) \approx H^i(Y, \mathscr{W})\,.$$

Proof. By the projection formula, $R^if_*f^*\mathscr{W} \approx R^if_*\mathcal{O}_X \otimes \mathscr{W}$. Thus $\mathscr{W} = \mathcal{O}_Y \otimes \mathscr{W}$ is the only non-zero direct image of $f^*\mathscr{W}$. The isomorphism follows by a degenerate Leray spectral sequence. Q.E.D.

To use Theorem 2 one should note that $s(\psi\rho) = s^*(\psi)\rho$. We may always find an element of the Weyl group such that $(s^*\psi)\rho$ is contained in the positive Weyl chamber. Here are two possibilities. If ψ is singular; i.e. $\langle\psi\rho, \beta^v\rangle = 0$ for some root β, then $\langle(s^*\psi)\rho, \alpha^v\rangle = 0$ for some basic root α, i.e., $\langle s^*\psi, \alpha^v\rangle = -1$. Thus by Lemma 3, $\chi_{G/B}(s^*\psi) = 0$ and hence by Theorem 2, $\chi_{G/B}(\psi) = 0$. If $\chi\rho$ is non-singular, $\chi_{G/B}(\psi) = (-1)^{\text{lengths}}[V_G(s^*\psi)]$

where $V_G(\sigma)$ is the induced G-module $\Gamma(G/B, \mathcal{O}_{G/B}(\sigma))$ for a dominant weight σ. This equality follows from the Borel-Weil vanishing theorem; $H^i(G/B, \mathcal{O}_{G/B}(\sigma)) = 0$ for $i > 0$ [2, 4].

§ 2. A variation

Let Q be a parabolic subgroup of G which contains B. We want to decompose as a Q-module the induced representation $V_G(\psi)$ for a positive weight ψ. As we have just seen $\chi_{G/B}(\psi) = [V_G(\psi)]$. Thus we will decompose Euler characteristic for arbitrary $\bar{\omega}$. For any G-module M we have the restricted Q-module $M = \operatorname{res}_Q M$. The operation res_Q extends to an operator res_Q from the Grothendieck group of G to that of Q.

Recall that Schubert variety in G/B is the closure of a B-orbits. We will be working with two Q-invariant Schubert varieties $X \subsetneqq Y$ such that there is a basic root α such that X and Y have the same image in $G/P(\alpha)$ under the projection π. In [2] X is called a moving divisor in Y. The geometry of this situation is very simple. Let σ_Y and σ_X be π restricted to Y and X. Then $\sigma_Y\colon Y \to \pi Y$ is a $\boldsymbol{P}^1$-fibration and $\sigma_X\colon X \to \pi Y$ is birational.

Let $\mathscr{W}$ be Q-linearized coherent sheaf on Y which is induced by a G-linearized sheaf on G/B. The Grothendieck group of such sheaves is the group ring again. We will also consider the analogous sheaves on X. Consider $\sigma_X^*\sigma_{Y*}\mathscr{W} \equiv [\sigma_X^*\sigma_{Y*}\mathscr{W}] - [\sigma_X^*R^1\sigma_{Y*}\mathscr{W}]$ in the Grothendieck group for X. The operation $\sigma_X^*\sigma_{Y*}$ is additive because the direct images $R^i\sigma_{Y*}\mathscr{W}$ commute with base extension by σ_X.

Thus we may regard $\sigma_X^*\sigma_{Y*}$ as a transformation of the group ring into itself. Let $\sigma_X^*\sigma_{Y*}\mathcal{O}_Y(\psi) \equiv \sigma_X^*\sigma_{Y*}(\psi)$.

THEOREM 5. $\sigma_X^*\sigma_{Y*}(\psi) = L_\alpha(\psi)$.

Proof. This theorem follows from Theorem 1. Explicitly by base extension $R^i\pi_*\mathcal{O}_{G/B}(\psi)|_{\pi Y} \approx R^i\sigma_{Y*}\mathcal{O}_Y(\psi)$. Hence $\sigma_X^*R^i\sigma_{Y*}\mathcal{O}_Y(\psi) = \pi^*R^i\pi_*\mathcal{O}_{G/B}(\psi)|_X$. Thus $\sigma_X^*\sigma_{Y*}(\psi) = \pi^*\pi_*(\psi)|_X$ which equals $L_\alpha(\psi)$ by Theorem 1. Q.E.D.

We may regard the Euler characteristics $\chi_Y(\mathscr{W}) = \sum(-1)^i H^i(Y, \mathscr{W})$ and $\chi_X(\mathscr{W}) = \sum(-1)^i H^i(X, \mathscr{W})$ in the Grothendieck group of Q-modules for any Q-linearized coherent sheaf $\mathscr{W}$ on Y or X. These operations extend additively to the corresponding Grothendieck groups. For any weight ψ, let $\chi_X(\psi) = \chi_X(\mathcal{O}_X(\psi))$ and similarly for Y.

THEOREM 6. $\chi_Y(\psi) = \chi_X(L_\alpha(\psi))$.

Proof. This is a variation of Lemma 3. By the Leray spectral sequence for σ_Y, $\chi_Y(\psi) = \chi(\sigma_{Y*}\mathcal{O}_Y(\psi)) - \chi(R^1\sigma_{Y*}\mathcal{O}_Y(\psi))$. Now the point is that the last difference is $\chi(\sigma_{X*}\sigma_Y(\psi))$ as σ_X satisfies the hypothesis for Lemma 4 by [6]. Thus we get $\chi_Y(\psi) = \chi_X(L_\alpha(\psi))$ by Theorem 5. Q.E.D.

Next we start with a chain $G/B = Y_0 \supset Y_1 \supset \cdots \supset Y_n = Q/Q \cap B$ of Q-invariant Schubert varieties such that Y_i is a moving divisor in Y_{i-1} with the root α_i. For the most interesting case where Q approximates G most closely the geometry of the Q-invariant Schubert varieties is worked out in detail in [2]. In this case we get by induction

COROLLARY 7.

a) $\chi_{Q/Q\cap B}(L_{\alpha_n}\cdots L_{\alpha_i}\psi) = \chi_{Y_{i-1}}(\psi)$ *and*
b) $\chi_{G/B}(\psi) = \chi_{Q/Q\cap B}(L_{\alpha_n}\cdots L_{\alpha_1}\psi)$.

By the vanishing theorems in [4, 6], if ψ is dominant, $H^i(Y_j, \mathcal{O}_{Y_j}(\psi)) = 0$ for $i > 0$. Thus $\chi_{Y_j}(\psi) = [\Gamma(Y_j, \mathcal{O}_{Y_j}(\psi))]$ and we get

THEOREM 8. *If ψ is dominant,*

a) $[\Gamma(Y_i, \mathcal{O}_{Y_i}(\psi))] = \chi_{Q/Q\cap B}(L_{\alpha_n}\cdots L_{\alpha_i}\psi)$ *and*
b) $[\mathrm{res}_Q V_G(\psi)] = \chi_{Q/Q\cap B}(L_{\alpha_n}\cdots L_{\alpha_1}\psi)$.

The only thing remaining is to replace Q by its Levi subgroup H. Let $B' = B \cap H$. Then we have

$$[\mathrm{res}_H V_G(\psi)] = \chi_{H/B'}(L_{\alpha_n}\cdots L_{\alpha_1}\psi)$$

where the last Euler characteristics can be expressed in terms of the induced representations $V_H(\psi)$. This gives the decomposition formula.

In case $Q = B$, $\chi_{Q/Q\cap B}$ is the identity and one gets formulas analogous to Demazure's character formula. Also in characteristic zero it should be recalled that the induced representation $V_G(\psi)$ are irreducible.

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