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THE RING OF INVARIANTS OF MATRICES

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§ 1. Introduction

We denote by $M(n)$ the space of all $n \times n$-matrices with their coefficients in the complex number field C and by G the group of invertible matrices $GL(n, C)$. Let $W = M(n)^l$ be the vector space of l-tuples of $n \times n$-matrices. We denote by $\rho: G \to GL(W)$ a rational representation of G defined as follows:

$$\rho(S)(A(1), A(2), \cdots, A(l)) = (SA(1)S^{-1}, SA(2)S^{-1}, \cdots, SA(l)S^{-1})$$

if $S \in G$, $A(i) \in M(n)$ $(i = 1, 2, \cdots, l)$.

This action of G defines an action of G on an algebra $C[W] = C[x_{ij}(1), \cdots, x_{ij}(l)]$ of all polynomial functions on W. We denote by $C[W]^G$ the subalgebra of G invariant polynomials. This is a finitely generated subalgebra of $C[W]$.

If $l = 1$ it is a classical result that this ring of invariants is a polynomial ring in n variables. In fact the coefficients of characteristic polynomial of the matrix $X(1) = (x_{ij}(1))$ are algebraically independent invariants and the ring of invariants is generated by them. By the Newton's formula all coefficients of characteristic polynomial of $X(1)$ are expressed by n traces

$$\mathrm{Tr}\,(X(1)),\ \mathrm{Tr}\,(X^2(1)),\ \cdots,\ \mathrm{Tr}\,(X(1)^n)\,,$$

and hence $C[x_{ij}(1)]^G$ is the polynomial ring generated by these traces.

Procesi [5] has shown the following important

THEOREM 1.1. *The ring of invariants $C[W]^G$ is generated by all traces $\mathrm{Tr}\,(X(i_1) \cdots X(i_j))$ $(j = 1, 2, \cdots)$, where $X(i_1) \cdots X(i_j)$ runs all possible noncommutative monomials.*

The object of this paper is to determine the Poincaré series of $C[W]^G$ and to determine generators of $C[W]^G$ for some cases.

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The following notations are fixed throughout:

$\boldsymbol{C}$ the field of complex numbers

$\boldsymbol{N}$ additive semigroup of nonnegative integers

$\boldsymbol{Q}$ the field of rational numbers

For a complex number z, we denote by $\bar{z}$ its complex conjugate and set $e(z) = \exp 2\pi\sqrt{-1}\, z$.

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§ 2. Poincaré series

We give $\boldsymbol{C}[W]$ the structure of $\boldsymbol{N}^l$-graded algebra by defining $\deg x_{ij}(k)$ to be the k-th unit coordinate vector ε_k in $\boldsymbol{N}^l$. Let

$$\boldsymbol{C}[W] = \bigoplus_{d \in \boldsymbol{N}^l} \boldsymbol{C}[W]_d ,$$

where $\boldsymbol{C}[W]_d$ is a vector space spanned over $\boldsymbol{C}$ by the monomials in $\boldsymbol{C}[W]$ of degree $d \in \boldsymbol{N}^l$. Then $\boldsymbol{C}[W]^G$ has the structure

$$\boldsymbol{C}[W]^G = \bigoplus_{d \in \boldsymbol{N}^l} \boldsymbol{C}[W]^G_d ,$$

of an $\boldsymbol{N}^l$-graded algebra given by

$$\boldsymbol{C}[W]^G_d = \boldsymbol{C}[W]^G \cap \boldsymbol{C}[W]_d .$$

The Poincaré series of $\boldsymbol{C}[W]^G$ is the formal power series $P(z_1, \cdots, z_l)$ in l-variables $z_1, \cdots, z_l$ defined by

$$P(z_1, \cdots, z_l) = \sum_{d \in \boldsymbol{N}^l} \dim_{\boldsymbol{C}} \boldsymbol{C}[W]^G_d z^d$$

where $z^d = z_1^{d_1} \cdots z_l^{d_l}$ with $d = (d_1, \cdots, d_l)$.

A theorem of Hilbert-Serre implies that $P(z_1, \cdots, z_l)$ is a rational function in l variables $z_1, \cdots, z_l$. By using a classical method of Molien-Weyl, we shall calculate this rational function.

For each diagonal unitary matrix ε with diagonal entries

$$\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n ,$$

since $|\varepsilon_i| = 1$ $(i = 1, 2, \cdots, n)$, we can put $\varepsilon_i = e(\varphi_i)$ $(0 \leqq \varphi_i \leqq 1)$. We set

$$\Delta = \prod_{i<j} (e(\varphi_i) - e(\varphi_j)) .$$

Then the normalized volume element on the group consisting of diagonal unitary matrices is given by

$$\frac{1}{n!}\Delta\bar{\Delta}d\varphi_1\cdots d\varphi_n\,,\ [8]\,.$$

We define polynomials in one variable z by

$$\Delta(z)=\prod_{i<j}(e(\varphi_i)-ze(\varphi_j))$$

and

$$\bar{\Delta}(z)=\prod_{i<j}(\overline{e(\varphi_i)}-z\overline{e(\varphi_j)})\,.$$

THEOREM 2.1. *The Poincaré series* $P(z_1,\cdots,z_l)$ *is*

$$\frac{1}{n!\prod_{i=1}^{l}(1-z_i)^n}\int_0^1\cdots\int_0^1\frac{\Delta\bar{\Delta}}{\prod_{i=1}^{l}\Delta(z_i)\bar{\Delta}(z_i)}d\varphi_1\cdots d\varphi_n\,,$$
$$|z_1|<1,\cdots,|z_l|<1\,.$$

Proof. Let $f(z)$ be a polynomial in one variable z defined as

$$\begin{aligned}f(z)&=\det(I_n-\rho(\varepsilon)z),\quad I_n=\text{the } n\times n\text{-identity matrix},\\&=\prod_{1\leqq i<j\leqq n}(1-z\varepsilon_i\varepsilon_j^{-1})\\&=(1-z)^n\Delta(z)\bar{\Delta}(z)\,.\end{aligned}$$

Then by the Molien-Weyl formula [8], the Poincaré series $P(z_1,\cdots,z_l)$ equals

$$\frac{1}{n!}\int_0^1\cdots\int_0^1\frac{\Delta\bar{\Delta}}{f(z_1)\cdots f(z_l)}d\varphi_1\cdots d\varphi_n\,,\quad |z_i|<1\,.$$

By changing variables from $\varphi_1,\cdots,\varphi_n$ to $\varepsilon_1,\cdots,\varepsilon_n$, we have

$$\begin{aligned}&P(z_1,\cdots,z_l)\\&=\left(\frac{1}{2\pi\sqrt{-1}}\right)^n\frac{1}{n!\prod_{i=1}^{l}(1-z_i)^n}\int_{c_1}\cdots\int_{c_n}\frac{\Delta\bar{\Delta}}{\prod_{i=1}^{l}\Delta(z_i)\bar{\Delta}(z_i)}d\varepsilon_1\cdots d\varepsilon_n\,,\end{aligned}$$

where C_k denotes the unit circle $|\varepsilon_k|=1$ in the complex ε_k-plane. Thus the Poincaré series $P(z_1,\cdots,z_l)$ can be calculated in principle by means of residues. Since

$$\Delta(z)\bar{\Delta}(z)=(-z)^{(n(n-1))/2}(\varepsilon_1\cdots\varepsilon_n)^{1-n}\prod_{i<j}(\varepsilon_i-z\varepsilon_j)\left(\varepsilon_i-\frac{1}{z}\varepsilon_j\right),$$

we have

$$\begin{aligned}\frac{\Delta\bar{\Delta}}{\prod_{i=1}^{l}\Delta(z_i)\bar{\Delta}(z_i)}&=(-1)^{(n(n-1)(l-1))/2}(z_1\cdots z_l)^{(n(1-n))/2}(\varepsilon_1\cdots\varepsilon_n)^{(n-1)(l-1)}\\&\times\frac{D(\varepsilon_1,\cdots,\varepsilon_n)}{\prod_{p=1}^{l}\prod_{i<j}(\varepsilon_i-z_p\varepsilon_j)(\varepsilon_i-(1/z_p)\varepsilon_j)}\,,\end{aligned}$$

where $D(\varepsilon_1, \cdots, \varepsilon_n) = \prod_{i<j} (\varepsilon_i - \varepsilon_j)^2$. And so we can rewrite Theorem 2.1 as

$$\text{(2.2)} \quad \begin{aligned} &P(z_1, \cdots, z_l) \\ &= (-1)^{(n(n-1)(l-1))/2} \frac{1}{n! \prod_{i=1}^{l} (1 - z_i)^n (z_1 \cdots z_l)^{(n(n-1))/2}} \left(\frac{1}{2\pi\sqrt{-1}} \right)^n \\ &\times \int \cdots \int \frac{(\varepsilon_1 \cdots \varepsilon_n)^{(n-1)(l-1)-1} D(\varepsilon_1, \cdots, \varepsilon_n)}{\prod_{p=1}^{l} \prod_{i<j} (\varepsilon_i - z_p \varepsilon_j)(\varepsilon_i - (1/z_p \varepsilon_j)} d\varepsilon_1 \cdots d\varepsilon_n . \end{aligned}$$

PROPOSITION 2.3. *The Poincaré series $P(z_1, \cdots, z_l)$ $(l \geqq 2)$ satisfies the following functional equation*

$$P(z_1^{-1}, \cdots, z_l^{-1}) = (-1)^{n(l-1)+1} (z_1, \cdots, z_l)^{n^2} P(z_1, \cdots, z_l).$$

Proof. Consider a rational function $I(z_1, \cdots, z_l)$ defined in $|z_1| < 1, \cdots, |z_l| < 1$ as follows

$$I(z_1, \cdots, z_l) = \int_{c_1} \cdots \int_{c_n} F_{z_1, \cdots, z_l}(\varepsilon_1, \cdots, \varepsilon_n) d\varepsilon_1 \cdots d\varepsilon_n ,$$

where

$$F_{z_1, \cdots, z_l}(\varepsilon_1, \cdots, \varepsilon_n) = \frac{(\varepsilon_1 \cdots \varepsilon_n)^{(n-1)(l-1)-1} D(\varepsilon_1, \cdots, \varepsilon_n)}{\prod_{p=1}^{l} \prod_{i<j} (\varepsilon_i - z_p \varepsilon_j)(\varepsilon_i - (1/z_p)\varepsilon_j)} .$$

Set inductively

$$\begin{aligned} I_1(\varepsilon_1, \cdots, \varepsilon_n) &= F_{z_1, \cdots, z_l}(\varepsilon_1, \cdots, \varepsilon_n) , \\ I_{i+1}(\varepsilon_{i+1}, \cdots, \varepsilon_n) &= \int_{|\varepsilon_i|=1} I_i(\varepsilon_i, \varepsilon_{i+1}, \cdots, \varepsilon_n) d\varepsilon_i , \\ &(i = 1, \cdots, n-1) . \end{aligned}$$

Then we find that $I_i(\varepsilon_i, \cdots, \varepsilon_n)$ is, as a function of ε_i, holomorphic at $\varepsilon_i = \infty$. If $|z_1| > 1, \cdots, |z_l| > 1$, we have

$$\begin{aligned} I(z_1^{-1}, \cdots, z_l^{-1}) &= \int_{c_1} \cdots \int_{c_n} F_{z_1, \cdots, z_l}(\varepsilon_1, \cdots, \varepsilon_n) d\varepsilon_1 \cdots d\varepsilon_n \\ &= (-1)^{n-1} \int_{c_1^{-1}} \cdots \int_{c_{n-1}^{-1}} \int_{c_n} F_{z_1, \cdots, z_l}(\varepsilon_1, \cdots, \varepsilon_n) d\varepsilon_1 \cdots d\varepsilon_n . \end{aligned}$$

By the Cauchy integral formula we have

$$I(z_1^{-1}, \cdots, z_l^{-1}) = (-1)^{n-1} I(z_1, \cdots, z_l) ,$$

and hence we obtain the result by 2.2.

We consider $\boldsymbol{C}[W]$ as a $\boldsymbol{N}$-graded algebra

$$\boldsymbol{C}[W] = \bigoplus_{d \in \boldsymbol{N}} \boldsymbol{C}[W]_d$$

by defining $\deg x_{ij}(k) = 1$ and define the Poincaré series $P(z)$ in one variable z by

$$P(z) = P(z, \cdots, z) = \sum_{d \in N} \dim_C C[W]_d^G z^d .$$

Then it follows from (2.2) that the Poincaré series $P(z)$ equals

$$(2.4) \qquad (-1)^{(n(n-1)(l-1))/2} \frac{1}{n!(1-z)^{nl} z^{(n(n-1)l)/2}} \left(\frac{1}{2\pi\sqrt{-1}} \right)^n \times \int \cdots \int \frac{(\varepsilon_1 \cdots \varepsilon_n)^{(n-1)(l-1)-1} D(\varepsilon_1, \cdots, \varepsilon_n)}{(\prod_{i<j} (\varepsilon_i - z\varepsilon_j)(\varepsilon_i - (1/z)\varepsilon_j))^l} d\varepsilon_1 \cdots d\varepsilon_n .$$

Let $f_1, \cdots, f_m$ be a homogeneous system of parameters of the N-graded algebra $C[W]^G$. By a theorem of Hochster and Roberts [4], $C[W]^G$ is a free module over the polynomial ring $C[f_1, \cdots, f_m]$. Let $\varphi_1, \cdots, \varphi_r$ be a homogeneous system of generators of this module,

$$C[W]^G = \bigoplus_{i=1}^{r} \varphi_i C[f_1, \cdots, f_m] .$$

We claim that $m = (l-1)n^2+1$. For $w \in W$, we denote by G_w the isotropy subgroup of $GL(n, C)$ at w. If $l \geqq 2$, there exists a dense open subset U of w such that $G_w = \{e\}$. Then it follows from a theorem of Rosenlicht [6] that the transcendence degree of $C[W]^G$ is equals $\dim W - \dim G + 1$. This shows that $m = (l-1)n^2 + 1$. Formanek [1] has shown that the field of rational invariants $C(W)^G$ is unirational of transcendence degree $(l-1)n^2 + 1$.

We set

$$\deg f_i = d_i , \qquad d_1 \leqq \cdots \leqq d_m$$
$$\deg \varphi_j = e_j , \qquad 0 = e_1 \leqq \cdots \leqq e_r .$$

By Proposition 2.3, $P(z)$ satisfies the following functional equation

$$P(z^{-1}) = (-1)^{(l-1)n^2+1} z^{n^2 l} P(z) .$$

This equation is equivalent to

$$d_1 + \cdots + d_m - e_{r-i+1} = n^2 l + e_i , \qquad i = 1, \cdots, r .$$

In particular we have

$$e_i + e_{r-i+1} = e_r , \qquad i = 1, \cdots, l ,$$
$$e_r = d_1 + \cdots + d_m - n^2 l$$

and

(2.5) $$n^2 l = \sum_{j=1}^{m} d_j - \frac{2}{r} \sum_{i=1}^{r} e_i .$$

Let α and β be the first and second Laurant coefficients of $P(z)$ respectively. Then the Laurant expansion of the Poincaré series $P(z)$ begins with

$$P(z) = \frac{\alpha}{(1-z)^m} + \frac{\beta}{(1-z)^{m-1}} + \cdots .$$

By 2.5.9 Lemma (7), it follows that

(2.6) $$\alpha = \frac{r}{d_1 \cdots d_m}$$

and

$$\beta = \frac{r \sum_{i=1}^{m} (d_j - 1) - 2 \sum_{i=1}^{r} e_i}{2 d_1 \cdots d_m} .$$

Then it follows from (2.5) that

(2.7) $$\frac{\beta}{\alpha} = \frac{n^2 - 1}{2} .$$

We shall need the following important theorem due to Hilbert [3].

THEOREM 2.8. *Assume that some invariants $I_1, \cdots, I_\mu$ have a property that their vanishing implies the vanishing of all invariants. Then the ring of invariants is integral over the subring generated by $I_1, \cdots, I_\mu$.*

§ 3. The ring of invariants of 2 × 2 matrices

In this section we shall be concerned with the ring of invariants of 2×2 matrices. Throughout this section we assume that $l \geqq 2$.

PROPOSITION 3.1. (1) *The Poincaré series $P_2(z)$ is given by*

$$P_2(z) = (-1)^{l-1} \frac{1}{2(l-1)!(1-z)^{2l}} \left(\frac{d}{d\varepsilon}\right)^{l-1} \frac{\varepsilon^{l-2}(\varepsilon-1)^2}{(z\varepsilon-1)^l}\bigg|_{\varepsilon=z} .$$

(2) *The Laurant expansion of $P_2(z)$ at $a = 1$ begins with*

$$P_2(z) = \frac{[l-1]_{l-2}}{(l-1)!\,2^{2l-1}(1-z)^{4l-3}} + \frac{3[l-1]_{l-2}}{(l-1)!\,2^{2l}(1-z)^{4l-4}} + \cdots ,$$

where $[l-1]_{l-2} = (l-1)l(l+1) \cdots (2l-4)$.

(3) *If* $C[X(1), \cdots, X(l)]^{GL(2)} = \bigoplus_{i=1}^{r} \varphi_i C[f_1, \cdots, f_{4l-3}]$, *where* $f_1, \cdots, f_{4l-3}$ *is a system of parameters of* $C[X(1), \cdots, X(l)]^{GL(2)}$, *we have*

$$r = \frac{[l-1]_{l-2}}{(l-1)!} \prod_{i=1}^{4l-3} \frac{\deg(f_i)}{2^{2l-1}}.$$

Proof. (1) follows from (2.4). By a direct computation, we see that the first Laurant coefficient at $z = 1$ equals

$$\frac{[l-1]_{l-2}}{(l-1)!\, 2^{2l-1}}.$$

Then (2) follows from (2.7). (3) is an immediate consequence from (2) and (2.6).

We denote by C_l a subring of $C[X(1), \cdots, X(l)]^{GL(2)}$ generated by traces $\mathrm{Tr}(X(i)X(j))$, $1 \leqq i, j \leqq l$, $\mathrm{Tr}(X(i))$, $1 \leqq i \leqq l$.

PROPOSITION 3.2. *The ring of invariants* $C[X(1), \cdots, X(l)]^{GL(2)}$ *is integral over* C_l.

Proof. By Theorem 1.1, it is enough to show

$$(*) \qquad \begin{array}{l} \text{if } \mathrm{Tr}(A_iA_j) = \mathrm{Tr}(A_i) = 0 \ (A_i, A_j \in M(2, C),\ 1 \leqq i, j \leqq l), \\ \mathrm{Tr}(A_{i_1}A_{i_2} \cdots A_{i_k}) = 0 \qquad \text{for any } k,\ 1 \leqq i_1, \cdots, i_k \leqq l. \end{array}$$

We shall prove $(*)$ by induction on l. By making the substitution $A_i \to BA_iB^{-1}$ $(B \in GL(2, C))$, we can assume $A_1 = 0$ or $A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

If $A_1 = 0$, by the inductive hypothesis $(*)$ is true. If $A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we have $A_i = \begin{pmatrix} 0 & \alpha_i \\ 0 & 0 \end{pmatrix}$, $a_i \in C$ $(1 \leqq i \leqq l)$. Because $\mathrm{Tr}(A_1A_i) = 0$ and $A_i^2 = 0$. $1 \leqq i \leqq l$. This shows that $\mathrm{Tr}(A_{i_1}A_{i_2} \cdots A_{i_k}) = 0$. This completes the proof.

If $l = 2$ or 3, $\mathrm{Tr}(X(i)X(j))$ $(1 \leqq i, j \leqq l)$, $\mathrm{Tr}(X(i))$ $(1 \leqq i \leqq l)$ is a homogeneous system of parameters of $C[X(1), \cdots, X(l)]^{GL(2)}$.

PROPOSITION 3.3. (E. Formanek, P. Halpin and W.C.W. Li [2])

$$\begin{aligned} &C[X(1), X(2)]^{GL(2)} \\ &\quad = C[\mathrm{Tr}(X(1)), \mathrm{Tr}(X(2)), \mathrm{Tr}(X(1)^2), \mathrm{Tr}(X(2)^2), \mathrm{Tr}(X(1)X(2))] \end{aligned}$$

Proof. By (3) Proposition 3.1, we have $r = 1$ and we obtain the result.

§ 4. The ring of invariants $\boldsymbol{C}[X(1), X(2)]^{GL(3)}$

In this section we treat the case: $n = 3$ and $l = 2$. Set

$$f_1 = \mathrm{Tr}\,(X(1))\,, \quad f_2 = \mathrm{Tr}\,(X(1)^2)\,, \quad f_3 = \mathrm{Tr}\,(X(1)^3)\,,$$
$$f_4 = \mathrm{Tr}\,(X(2))\,, \quad f_5 = \mathrm{Tr}\,(X(2)^2)\,, \quad f_6 = \mathrm{Tr}\,(X(2)^3)\,,$$
$$f_7 = \mathrm{Tr}\,(X(1)X(2))\,, \quad f_8 = \mathrm{Tr}\,(X(1)X(2)^2)\,, \quad f_9 = \mathrm{Tr}\,(X(1)^2X(2))\,,$$
$$f_{10} = \mathrm{Tr}\,(X(1)^2X(2)^2)\,, \quad f_{11} = \mathrm{Tr}\,(X(1)X(2)X(1)^2X(2)^2)\,.$$

We denote by C the subring of $\boldsymbol{C}[X(1), X(2)]^{GL(3)}$ generated by ten invariants $f_1, \cdots, f_{10}$ which are algebraically independent.

THEOREM 4.1. $f_1, \cdots, f_{10}$ is a system of parameters of the ring $\boldsymbol{C}[X(1), X(2)]^{GL(3)}$ and

$$\boldsymbol{C}[X(1), X(2)]^{GL(3)} = C \oplus f_{11}C\,.$$

Proof. Let A_1 and A_2 be 3×3-matrices which satisfy the following condition: $f_1(A_1, A_2) = \cdots = f_{10}(A_1, A_2) = 0$.

Since $\mathrm{Tr}\,(A_i) = \mathrm{Tr}\,(A_i^2) = \mathrm{Tr}\,(A_i^3) = 0$, $i = 1, 2$, we have $A_1^3 = A_2^3 = 0$. If $A_1^2 = A_2^2 = 0$, it follows from the Cayley-Hamilton theorem that $A_1A_2A_1 = A_2A_1A_2 = 0$ and hence we have, for any k, $\mathrm{Tr}\,(A_{i_1}A_{i_2} \cdots A_{i_k}) = 0$, $1 \leqq i_1, \cdots, i_k \leqq 2$. Assume now that $A_1^2 \neq 0$. Then, by making the substitution

$$A_i \longrightarrow BA_iB^{-1}\,, \qquad i = 1, 2\,,$$

we can assume that A_1 and A_2 are of the form

$$A_1 = \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

The equations $\mathrm{Tr}\,(A_1A_2) = \mathrm{Tr}\,(A_1^2A_2) = \mathrm{Tr}\,(A_2) = 0$ imply $a_{11} + a_{22} + a_{33} = a_{21} + a_{32} = a_{31} = 0$ and $\mathrm{Tr}\,(A_1^2A_2^2) = 0$ implies $a_{21}a_{32} = 0$. Hence we have $a_{31} = a_{21} = a_{32} = 0$. This shows that A_2 is an upper triangular matrix with zero diagonal entries. Consequently $\mathrm{Tr}\,(A_{i1}A_{i2} \cdots A_{ik}) = 0$, $i_1, i_2, \cdots, i_k = 1, 2$ for any k.

If A_1 or A_2 is the zero matrix, all traces are zero by our assumption. Therefore $\boldsymbol{C}[X(1), X(2)]^{GL(3)}$ is integral over C. Since the transcendence degree of the ring $\boldsymbol{C}[X(1), X(2)]^{GL(3)}$ is ten, $f_1, \cdots, f_{10}$ is a homogeneous system of parameters.

Consider the Poincaré series $P(z_1, z_2)$. By the theorem of Hochster and Roberts $\boldsymbol{C}[X(1), X(2)]^{GL(3)}$ is a free module over the subring C. Therefore there is a polynomial $F(z_1, z_2)$ in two variables such that

$$P(z_1, z_2) = \frac{F(z_1, z_2)}{(1-z_1)(1-z_1^2)(1-z_1^3)(1-z_2)(1-z_2^2)(1-z_2^3)(1-z_1z_2)(1-z_1^2z_2)(1-z_1z_2^2)(1-z_1^2z_2^2)} .$$

It follows from the functional equation of $P(z_1, z_2)$ that $F(z_1, z_2)$ satisfies the following relation

$$F(z_1, z_2) = (z_1z_2)^3F(z_1^{-1}, z_2^{-1}) .$$

And it is easily shown that $F(z_1, z_2) = 1 + z_1^3z_2^3$. Therefore $C[X(1), X(2)]^{GL(3)}$ is generated by $f_1, \cdots, f_{10}$ and an invariant φ of degree (3, 3).

Invariants $\mathrm{Tr}(X(1)X(2)X(1)^2X(2)^2)$, $\mathrm{Tr}(X(2)X(1)X(2)^2X(1)^2)$ and $\mathrm{Tr}(X(1)\cdot X(2)X(1)X(2)X(1)X(2))$ span the vector space $\boldsymbol{C}[X(1), X(2)]^{GL(3)}_{(3,3)}$ consisting of invariants of degree (3, 3). By the Cayley-Hamilton theorem, we find that $\mathrm{Tr}(X(1)X(2)X(1)X(2)X(1)X(2)) \in C$ and $\mathrm{Tr}(X(1)X(2)X(1)X(2)^2) + \mathrm{Tr}(X(2)\cdot X(1)X(2)^2X(1)^2) \in C$. Therefore the ring of invariants $\boldsymbol{C}[X(1), X(2)]^{GL(3)}$ is generated by $f_1, \cdots, f_{11}$ and $C[X(1), X(2)]^{GL(3)} = C \oplus f_{11}C$. This completes the proof.

§5. The ring of invariants $\boldsymbol{C}[X(1), X(2)]^{GL(4)}$

We denote by $\mathrm{Sym}(n)$ the symmetric group of n letters and recall the multi-linearlized Cayley-Hamilton theorem for $n \times n$-matrices $Y_1, \cdots, Y_n$:

$$\sum_{\pi \in \mathrm{Sym}(n)} Y_{\pi(1)} \cdots Y_{\pi(n)} + \sum_{k=1}^{n} \sum_{u} \sum_{\pi \in \mathrm{Sym}(n)} q_u \mathrm{Tr}(Y_{\pi(1)} \cdots Y_{\pi(u_1)}) \cdots Y_{\pi(n-k+1)} Y_{\pi(n-k+2)} \cdots Y_{\pi(n)} = 0 ,$$

for suitable $q_u \in \boldsymbol{Q}$ and suitable j-tuples $u = (u_1, \cdots, u_j)$ such that $1 \leqq u_1 \leqq u_2 \leqq \cdots \leqq u_j$ and $u_1 + \cdots + u_i = k$.

PROPOSITION 5.1. *The ring of invariants $\boldsymbol{C}[X(1), X(2)]^{GL(4)}$ is generated by invariants of the form*

$$\mathrm{Tr}(X(1)^{\alpha_1}X(2)^{\alpha_2}X(1)^{\alpha_3}X(2)^{\alpha_4}) , \quad 0 \leqq \alpha_1, \alpha_2, \alpha_3 \leqq 3 ,$$
$$\mathrm{Tr}(X(1)X(2)X(1)^2X(2)^2X(1)^3X(2)^3) , \quad \mathrm{Tr}(X(1)X(2)X(1)X(2)^2X(1)X(2)^3) ,$$
$$\mathrm{Tr}(X(2)X(1)X(2)X(1)^2X(2)X(1)^3) .$$

Proof. We claim that any invariant $\mathrm{Tr}(X(1)^{\alpha_1}X(2)^{\alpha_2} \cdots X(1)^{\alpha_{2r-1}}X(2)^{\alpha_{2r}})$, $0 \leqq \alpha_1, \cdots, \alpha_{2r} \leqq 3$ $(r > 6)$, can be written as a polynomial in $T(X(1)^{\beta_1}X(2)^{\beta_2} \cdots X(1)^{\beta_5}X(2)^{\beta_6}$, $0 \leqq \beta_1, \cdots, \beta_6 \leqq 3$. We work by induction on r. We assume that, for any $r' < r$, this assertion is true. Apply the multi-linearlized Cayley-Hamilton theorem for 4×4-matrices X_1, X_2, X_3, X_4 to the case $X_1 = X(1)^{\alpha_1}$, $X_2 = X(2)^{\alpha_2}$, $X(1)^{\alpha_3}$, $X_3 = X(2)^{\alpha_4}$, $X_4 = X(1)^{\alpha_5}X(2)^{\alpha_6}$. Then by the inductive hypothesis we conclude the assertion. A similar argument shows that any invariant of the form

$$\mathrm{Tr}\,(X(1)^{\alpha_1}X(2)^{\alpha_2}X(1)^{\alpha_3}X(2)^{\alpha_4}X(1)^{\alpha_5}X(2)^{\alpha_6})\,, \quad 1 \leqq \alpha_1, \alpha_2, \cdots, \alpha_6 \leqq 3\,,$$

is written as a polynomial in $T_r(X(1)^{\alpha_1}X(2)^{\alpha_2}X(1)^{\alpha_3}X(2)^{\alpha_4})$, $0 \leqq \alpha_1, \cdots, \alpha_4 \leqq 3$, $\mathrm{Tr}\,(X(1)X(2)X(1)^2X(2)^2X(1)^3X(2)^3)$, $\mathrm{Tr}\,(X(1)X(2)X(1)X(2)^2X(1)X(2)^3)$, $\mathrm{Tr}\,(X(2)\cdot X(1)X(2)X(1)^2X(2)X(1)^3)$. The proposition is proved.

Set

$$\begin{aligned}
&f_1 = \mathrm{Tr}\,(X(1))\,, \quad f_2 = \mathrm{Tr}\,(X(1)^2)\,, \quad f_3 = \mathrm{Tr}\,(X(1)^3)\,, \quad f_4 = \mathrm{Tr}\,(X(1)^4)\,,\\
&f_5 = \mathrm{Tr}\,(X(2))\,, \quad f_6 = \mathrm{Tr}\,(X(2)^2)\,, \quad f_7 = \mathrm{Tr}\,(X(2)^3)\,, \quad f_8 = \mathrm{Tr}\,(X(2)^4)\,,\\
&f_9 = \mathrm{Tr}\,(X(1)X(2))\,, \quad f_{10} = \mathrm{Tr}\,(X(1)^2X(2)^2)\,, \quad f_{11} = \mathrm{Tr}\,(X(1)X(2)^2)\,,\\
&f_{12} = \mathrm{Tr}\,(X(1)^2X(2))\,, \quad f_{13} = \mathrm{Tr}\,(X(1)X(2)^3)\,, \quad f_{14} = \mathrm{Tr}\,(X(1)^3X(2))\,,\\
&f_{15} = T_r\,(X(1)X(2)X(1)X(2))\,, \quad f_{16} = \mathrm{Tr}\,(X(1)X(2)^2X(1)X(2)^2)\,,\\
&f_{17} = \mathrm{Tr}\,(X(2)X(1)^2X(2)X(1)^2)\,.
\end{aligned}$$

We denote by C a subring of $\boldsymbol{C}[X(1), X(2)]^{GL(4)}$ generated by $f_1, \cdots, f_{17}$.

PROPOSITION 5.2. *$f_1, \cdots, f_{17}$ is a homogeneous system of parameters oj the ring of invariants $\boldsymbol{C}[X(1), X(2)]^{GL(4)}$.*

Proof. Since the transcendence degree of the ring $\boldsymbol{C}[X(1), X(2)]^{GL(4)}$ is 17, it is enough to show that, for 4×4-matrices A_1 and A_2, $f_1(A_1, A_2) = \cdots = f_{17}(A_1, A_2) = 0$ imply $\mathrm{Tr}\,(A_{i_1}, A_{i_2} \cdots A_{i_k}) = 0$, $i_1, \cdots, i_k = 1, 2$ for any k.

Notice that $A_1^4 = A_2^4 = 0$, since $f_1(A_1, A_2) = \cdots = f_8(A_1, A_2) = 0$. Assume that $A_1^3 \neq 0$. Then, by the substitution $A_i \to BA_iB^{-1}$, $B \in GL(4)$ and $i = 1, 2$, we can assume that

$$A_1 = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

It follows from the equations $\mathrm{Tr}\,(A_1^2A_2) = \mathrm{Tr}\,(A_1^3A_2) = 0$ that $a_{41} = a_{31} + a_{42} = 0$ and the Cayley-Hamilton theorem shows that the equation $\mathrm{Tr}\,(A_1^2A_2A_1^2A_2) = 0$ implies $\mathrm{Tr}\,(A_1^2A_1A_2A_1A_2) = 0$.

Since

$$A_1A_2 = \begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

it follows from the equation $\mathrm{Tr}\,(A_1^2A_1A_2A_2A_1A_2) = 0$ that $a_{31}a_{42} = 0$ and hence we have $a_{31} = a_{42} = 0$. Then it follows from the relation $\mathrm{Tr}\,(A_1A_2) = a_{21} + a_{32} + a_{43} = 0$ that $\mathrm{Tr}\,(A_1^2A_2^2) = a_{21}a_{32} + a_{32}a_{43} = -a_{32}^2$ and we obtain $a_{32} = 0$.

Since

$$\mathrm{Tr}\,(A_1A_2A_1A_2) = \mathrm{Tr}\left[\begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix}^2\right]$$

$$= a_{21}^2 + a_{43}^2\,,$$

$a_{21} = a_{43} = a_{32} = 0$ and hence A_2 is a 4×4 upper triangular matrix with zero diagonal entries. Consequently we can conclude that $\mathrm{Tr}\,(A_{i_1}, A_{i_2} \cdots A_{i_k}) = 0$, $1 \leqq i_1, i_2, \cdots, i_k \leqq 2$ for any k. By the same argument, we obtain the same conclution if $A_2^3 \neq 0$.

We next assume that $A_1^3 = A_2^3 = 0$ and either A_1^2 or A_2^2 is not zero. Then we can take A_1 as

$$A_1 = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix},$$

and divide into two cases:

Case 1.

$$A_1 = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

In this case, it follows from the equations $\mathrm{Tr}\,(A_1^2A_2) = 0$, $\mathrm{Tr}\,(A_1A_2A_1A_2) = 0$ and $\mathrm{Tr}\,(A_1A_2) = 0$ that $a_{21} = a_{31} = a_{32} = 0$.

Therefore A_1A_2 and $A_1^2A_2$ are upper triangular matrices with zero diagonal entries. Similarly, replacing A_2 by A_2^2, we see that $A_1A_2^2$ and $A_1^2A_2^2$ are also upper triangular matrices with zero diagonal entries. This shows that $\mathrm{Tr}(A_{i_1}A_{i_2}\cdots A_{i_k})=0$, $1\leqq i_1, i_2, \cdots, i_k\leqq 2$ for any k.

Case 2.

$$A_1=\begin{bmatrix}0&&&\\&0&1&\\&&0&1\\&&&0\end{bmatrix}\qquad A_2=\begin{bmatrix}a_{11}&a_{12}&a_{13}&a_{14}\\a_{21}&a_{22}&a_{23}&a_{24}\\a_{31}&a_{32}&a_{33}&a_{34}\\a_{41}&a_{42}&a_{43}&a_{44}\end{bmatrix}.$$

In this case, by the equation $\mathrm{Tr}(A_1^2A_2)=0$, we have $a_{42}=0$.

Since

$$A_1A_2=\begin{bmatrix}0&0&0&0\\a_{31}&a_{32}&a_{33}&a_{34}\\a_{41}&a_{42}&a_{43}&a_{44}\\0&0&0&0\end{bmatrix}$$

and $\mathrm{Tr}(A_1A_2A_1A_2)=0$, we have $a_{32}=a_{43}=0$. Then we find that $A_1A_2A_1=a_{33}A_1^2$ and, replacing A_2 by A_2^2, $A_1A_2^2A_1=bA_1^2$. Here b denotes the (3, 3)-entry of the matrix A_2^2.

Notice that, for any 4×4-matrix $X=(x_{ij})$,

$$A_1^2X=\begin{bmatrix}0&x_{32}&x_{33}&x_{34}\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}.$$

Therefore we can conclude that $\mathrm{Tr}(A_{i1}A_{i2}\cdots A_{ik})=0$ for any k.

If $A_1^2=A_2^2=0$, we have evidently $\mathrm{Tr}(A_{i1}A_{i2}\cdots A_{ik})=0$. This completes the proof.

Proposition 5.2 shows that C is a polynomial ring in 17 variables and $\boldsymbol{C}[X(1), X(2)]^{GL(4)}$ is a free module over C.

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