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Nagoya Math. J.
Vol. 102 (1986), 135-154

ON THE GORENSTEINNESS OF REES ALGEBRAS OVER LOCAL RINGS

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Introduction

Let (A, m, k) be a Noetherian local ring and I an ideal of A. We set $R(I) = \bigoplus_{n \geq 0} I^n$ and call this graded A-algebra the Rees algebra of I. The importance of the Rees algebra $R(I)$ is in the fact that $\operatorname{Proj} R(I)$ is the blowing up of $\operatorname{Spec} A$ with center in $V(I)$. The Cohen-Macaulayness of Rees algebras was studied by many mathematicians. In [GS] S. Goto any Y. Shimoda gave a criterion for $R(m)$ to be Cohen-Macaulay under the assumption that A is Cohen-Macaulay. Their results have been generalized to $R(I)$ in [HI].

Let grade $(I) \geq 2$. The purpose of this paper is to characterize the Gorensteinness of $R(I)$ in terms of canonical modules of A and the associated graded ring $G(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$. The notion of canonical modules of local rings plays an important role in the homological theory of local rings, cf. [HK]. The canonical modules of graded rings defined over a field were introduced and studied extensively in [GW]. In Section 1 we introduce the notion of canonical modules of graded rings defined over a local ring. Our definition of canonical modules coincides with that of [GW] if the local ring is a field. In Section 2 we collect several facts about the behaviour of the local cohomology modules of Rees algebras. Section 3 will be devoted to the proof of our criterion of the Gorensteinness of $R(I)$ and to the construction of an example of a local ring (A, m, k) such that $R(m)$ is Gorenstein but A is not Cohen-Macaulay.

§ 1. Local cohomology of graded rings

In this section we give a brief summary of the theory of local cohomology and duality of graded rings.

Let $R = \bigoplus_{n \in Z} R_n$ be a Noetherian graded ring and let M, N be graded

Received January 29, 1985.

R-modules. Let us denote the category of graded R-modules by $M_H(R)$. A morphism in $M_H(R)$ $f\colon M \to N$ is an R-linear map such that $f(M_n) \subset N_n$ for all $n \in \boldsymbol{Z}$. Let $n \in \boldsymbol{Z}$. We denote by $M(n)$ the graded R-module whose grading is defined by $M(n)_m = M_{n+m}$ for all $m \in \boldsymbol{Z}$. Let $\mathscr{H}om_R(M, N)_n$ be the abelian group of all homomorphisms from M into $N(n)$ in $M_H(R)$. Let $\mathscr{H}om_R(M, N) = \bigoplus_{n \in \boldsymbol{Z}} \mathscr{H}om_R(M, N)_n$. Then $\mathscr{H}om_R(M, N)$ is a graded R-module whose homogeneous component of degree n is $\mathscr{H}om_R(M, N)_n$. A graded R-module E is injective (resp. projective) in $M_H(R)$ if the functor $\mathscr{H}om_R(\ \ , E)$ (resp. $\mathscr{H}om_R(E, \ \ )$) from $M_H(R)$ into itself is an exact functor.

The tensor product $M \otimes_R N$ is a graded R-module whose n-th homogeneous component is the abelian group generated by the elements of the form $x \otimes y$ with $x \in M_i$, $y \in N_j$ and $i + j = n$.

The category $M_H(R)$ is an abelian category with enough injectives (cf. [Gr$_1$], (1, 10)). A homomorphism $f\colon M \to N$ in $M_H(R)$ is called essential if f is an injection and for any non-trivial graded R-submodule L of N we have $f(M) \cap L \neq 0$. The injective envelope of a graded R-module M is an injective object $\mathscr{E}_R(M)$ of $M_H(R)$ with an essential homomorphism $M \to \mathscr{E}_R(M)$ in $M_H(R)$.

The following proposition describes the structure of injective objects in $M_H(R)$.

Proposition (1.1). (1) *Let M be a graded R-module. Then*

$$\mathrm{Ass}_R(\mathscr{E}_R(M)) = \mathrm{Ass}_R(M)\,.$$

(2) *Let E be an injective object of $M_H(R)$. Then E is indecomposable if and only if $E = \mathscr{E}_R(R/p)(n)$ for some homogeneous prime ideal of R and for some $n \in \boldsymbol{Z}$.*

(3) *Every injective object of $M_H(R)$ can be decomposed into a direct sum of indecomposable injective objects of $M_H(R)$. This decomposition is unique up to isomorphism.*

Proof. This is [GW], (1.2.1).

For $i \geq 0$ the functor $\mathscr{E}xt_R^i(\ \ ,\ \ )$ is defined to be the i-th derived functor of the functor $\mathscr{H}om_R(\ \ ,\ \ )$. Suppose that M is a finitely generated graded R-module. Then $\mathscr{H}om_R(M, N) = \mathrm{Hom}_R(M, N)$ as underlying R-modules. Hence $\mathscr{E}xt_R^i(M, N) = \mathrm{Ext}_R^i(M, N)$ for all $i \geq 0$. For any $p \in \mathrm{Spec}\,(R)$ and for any R-module L we define

$$\mu^i(p, L) = \dim_{k(p)} \mathrm{Ext}_{R_p}^i(k(p), L_p)\,,$$

where $k(p) = R_p/pR_p$, and call this number the i-th Bass number of M at p (cf. [B]).

PROPOSITION (1.2). *Let M be a graded R-module and let*

$$0 \to M \to I^0 \to I^1 \to \cdots \to I^n \to I^{n+1} \to \cdots$$

be a minimal injective resolution of M in $M_H(R)$. Then for any homogeneous prime ideal p and for any integer $i \geq 0$, $\mu^i(p, M)$ is equal to the number of the graded R-modules of the form $\mathscr{E}_R(R/p)(n)$ which appear in I^i as direct summands.

Proof. This is [GW], (1.2.4).

In this paper a Noetherian graded R is called defined over a local ring if R_0 is a Noetherian local ring and $R_n = 0$ for $n < 0$. If R is defined over a local ring we denote the graded ring $R \otimes_{R_0} \hat{R}_0$ by $\hat{R}$, where $\hat{R}_0$ is the completion of R_0. In the rest of this section R denotes a graded ring defined over a local ring (R_0, m_0, k) and M denotes the maximal homogeneous ideal of R. R can be regarded as a graded R_0-module in a natural way. Let E_{R_0} be the injective envelope of k as an R_0-module. We denote by $\mathscr{E}_{R_0}$ the graded R-module whose underlying R_0-module is E_{R_0} and whose grading is given by $[\mathscr{E}_{R_0}]_0 = E_{R_0}$ and $[\mathscr{E}_{R_0}]_n = 0$ for $n \neq 0$.

DEFINITION (1.3). $\mathscr{E}_R(k) = \mathscr{H}om_{R_0}(R, \mathscr{E}_{R_0})$.

PROPOSITION (1.4). (1) *$\mathscr{E}_R(k)$ is the injective envelope of R/M in $M_H(R)$.*

(2) *$\mathscr{H}om_R(\mathscr{E}_R(k), \mathscr{E}_R(k)) = R \otimes_{R_0} \hat{R}_0$, where $\hat{R}_0$ is the completion of R_0.*

Proof. (1) As in the non-graded case, in order to show that $\mathscr{E}_R(k)$ is injective in $M_H(R)$ it is enough to show that for any homogeneous ideal of R and for any integer n every homomorphism $f\colon I(n) \to \mathscr{E}_R(k)$ can be extended to a homomorphism $f'\colon R(n) \to \mathscr{E}_R(k)$. Since

$$\mathscr{H}om_{R_0}(R, \mathscr{E}_{R_0}) \subset \operatorname{Hom}_{R_0}(R, E_{R_0}) = \prod_{i \in \mathbf{Z}} \operatorname{Hom}_{R_0}(R_i, E_{R_0}),$$

and since $\operatorname{Hom}_{R_0}(R, E_{R_0})$ is an injective R-module f can be extended to an R-homomorphism $f''\colon R \to \operatorname{Hom}_{R_0}(R, E_{R_0})$.

Let $f''(1) = (g_i)_{i \in \mathbf{Z}}$, where $g_i \in \operatorname{Hom}_{R_0}(R_{-i}, E_{R_0})$. Since f is homogeneous for any homogeneous element $x \in I$ we have $xg_j = 0$ for $j \neq -n$. This shows that the homomorphism f' in $M_H(R)$ defined by $f'(1) = g_{-n} \in$

$\mathrm{Hom}_{R_0}(R_n, E_{R_0})$ extends f. It is not difficult to show that $\mathrm{Supp}(\mathscr{E}_R(k)) = M$. Moreover we have

$$\begin{aligned}\mathscr{H}om_R(R/M, \mathscr{E}_R(k)) &= \mathscr{H}om_R(R/M, \mathscr{H}om_{R_0}(R, \mathscr{E}_{R_0}))\\ &= \mathscr{H}om_{R_0}(R/M, \mathscr{E}_{R_0})\\ &= k.\end{aligned}$$

This shows that $\mathscr{E}_R(k)$ is the injective envelope of R/M in $M_H(R)$.

(2)
$$\begin{aligned}\mathscr{H}om_R(\mathscr{E}_R(k), \mathscr{E}_R(k)) &= \mathscr{H}om_R(\mathscr{E}_R(k), \mathscr{H}om_{R_0}(R, \mathscr{E}_{R_0}))\\ &= \mathscr{H}om_{R_0}(\mathscr{E}_R(k), \mathscr{E}_{R_0})\\ &= \mathscr{H}om_{R_0}(\mathscr{H}om_{R_0}(R, \mathscr{E}_{R_0}), \mathscr{E}_{R_0})\\ &= \bigoplus_{n\in \boldsymbol{Z}} \mathrm{Hom}_{R_0}(\mathrm{Hom}_{R_0}(R_n, E_{R_0}), E_{R_0})\\ &= \bigoplus_{n\in \boldsymbol{Z}} R_n \otimes_{R_0} \hat{R}_0\\ &= R \otimes_{R_0} \hat{R}_0 .\end{aligned}$$

PROPOSITION (1.5). *Let R be a graded ring defined over a complete local ring R_0 and N a graded R-module. Then, we have:*

(1) *If N is Noetherian (resp. Artinian) $\mathscr{H}om_R(N, \mathscr{E}_R(k))$ is Artinian (resp. Noetherian).*

(2) *If N is Noetherian or Artinian*

$$\mathscr{H}om_R(\mathscr{H}om_R(N, \mathscr{E}_R(k)), \mathscr{E}_R(k)) = N .$$

Proof. Using Proposition (1.4) this can be proved as in [M].

For every integer $i \geq 0$ we put

$$\mathscr{H}^i_M(\ \ ) = \varinjlim_n \mathscr{E}xt^i_R(R/M^n,\ \ )$$

and call it the i-th local cohomology functor, where R is a graded ring defined over a local ring and M is the maximal homogeneous ideal of R. $\mathscr{H}^i_M(\ \ )$ is the i-th derived functor of $\mathscr{H}^0_M(\ \ )$ (cf. [Gr$_2$] and [HK]).

DEFINITION (1.6). Suppose that R_0 is complete. We put

$$\mathscr{K}_R = \mathscr{H}om_R(\mathscr{H}^d_M(R), \mathscr{E}_R(k)) ,$$

where $d = \dim R$, and call this graded R-module the canonical module of R.

If R_0 is not complete a graded R-module $\mathscr{K}_R$ is a canonical module of R if there is an isomorphism in $M_H(\hat{R})$ $\mathscr{K}_{\hat{R}} = \mathscr{K}_R \otimes_R \hat{R}$.

PROPOSITION (1.7). *If there is a canonical module of R it is a finitely generated R-module and unique up to isomorphisms.*

Proof. Since $\hat{R}$ is faithfully flat over R it is sufficient to show that $\mathscr{K}_{\hat{R}}$ is finitely generated. But this follows from Proposition (1.5). For the proof of the uniqueness it is enough to show that if K and L are finitely generated graded R-modules such that $K \otimes_R \hat{R} = L \otimes_R \hat{R}$ then $K = L$. Let $f \in \mathscr{H}om_{\hat{R}}(K \otimes_R \hat{R}, L \otimes_R \hat{R})_0$ be an isomorphism. Since $\hat{R}$ is flat over R and K is finitely generated over R one gets

$$\begin{aligned}\mathscr{H}om_{\hat{R}}(K \otimes_R \hat{R}, L \otimes_R \hat{R}) &= \mathscr{H}om_R(K, L) \otimes_R \hat{R} \\ &= \mathscr{H}om_R(K, L) \otimes_{R_0} \hat{R}_0\end{aligned}$$

which implies that $\mathscr{H}om_{\hat{R}}(K \otimes_R \hat{R}, L \otimes_R \hat{R})_0$ is the completion of $\mathscr{H}om_R(K, L)_0$ since $\mathscr{H}om_R(K, L)_0$ is a finitely generated R_0-module. Let $\mathscr{H}om_R(K, L)_0^{\wedge}$ be the m_0-adic completion of $\mathscr{H}om_R(K, L)_0$. For any integer $n > 0$ there is a homomorphism $f_n \in \mathscr{H}om_R(K, L)_0$ such that $f - f_n \in m_0^n \mathscr{H}om_R(K, L)_0^{\wedge}$. By assumption f_n induces an isomorphism $\bar{f}_n\colon K/m_0^n K \to L/m_0^n L$. Hence f_n is a surjective homomorphism. Since K/MK and L/ML are isomorphic there exist finitely generated graded free R-modules F and G of the same rank $\dim_k K/MK$ such that there are surjective homomorphisms in $M_H(R)$ $g\colon F \to K$ and $h\colon G \to L$. Let $S = \operatorname{Ker}(g)$ and $T = \operatorname{Ker}(h)$. We get a commutative diagram with exact rows

$$\text{(I)}\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & F & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle b_n} & & \downarrow{\scriptstyle a_n} & & \downarrow{\scriptstyle f_n} & & \\ 0 & \longrightarrow & T & \longrightarrow & G & \longrightarrow & L & \longrightarrow & 0\,. \end{array}$$

a_n is an isomorphism since F and G are free R-modules of the same rank. Since $\bar{f}_n$ is an isomorphism from (I) we get

$$T \subset b_n(S) + m_0^n G \cap T\,.$$

By Artin-Rees lemma there is an integer $r > 0$ such that

$$m_0^n G \cap T = m_0^{n-r}(m_0^r G \cap T) \qquad \text{for } n > r\,.$$

Therefore we get $T \subset b_n(S) + m_0 T$ for $n > r$. By Nakayama's lemma $T = b_n(S)$. From (I) one knows that f_n is an isomorphism.

Let us recall that R is Cohen-Macaulay (resp. Gorenstein) if and only if R_M is Cohen-Macaulay (resp. Gorenstein), see [AG], [MR] and [GW].

PROPOSITION (1.8). *Let $d = \dim R$ and assume that R_0 is complete. Then R is Cohen-Macaulay if and only if for any finitely generate graded R-module N and for all $i \geq 0$ we have*

$$\mathscr{H}om_R(\mathscr{H}^i_M(N), \mathscr{E}_R(k)) = \mathscr{E}xt^{d-i}_R(N, \mathscr{K}_R).$$

Proof. Suppose that R is Cohen-Macaulay. We will show that the functor $T^i(\ ) = \mathscr{H}om_R(\mathscr{H}^{d-i}_M(\ ), \mathscr{E}_R(k))$ is the i-th derived functor of $\mathscr{H}om_R(\ , \mathscr{K}_R)$. We must show that

(1) from the short exact sequence $0 \to N' \to N \to N'' \to 0$ we obtain the long exact sequence

$$0 \to T^0(N'') \to T^0(N) \to T^0(N') \to T^1(N'') \to T^1(N) \to T^1(N') \to \cdots$$

(2) $T^i(R) = 0$ for $i > 0$.

Since $\mathscr{E}_R(k)$ is an injective object in $M_H(R)$ (1) follows from the long exact sequence of the local cohomology. (2) follows from the fact that for any graded R-module N $\mathscr{H}om_R(N, \mathscr{E}_R(k)) = 0$ if and only if $N = 0$. The converse is immediate.

PROPOSITION (1.9). *Suppose that R is Cohen-Macaulay. Then R is Gorenstein if and only if R has a canonical module $\mathscr{K}_R$ and $\mathscr{K}_R = R(n)$ for some $n \in \boldsymbol{Z}$.*

Proof. Recall that R is Gorenstein if and only if

$$\mathscr{E}xt^i_R(R/M, R) = \begin{cases} R/M & \text{for } i = \dim R \\ 0 & \text{for } i \neq \dim R\,. \end{cases}$$

If R is Gorenstein we have $\mathscr{H}^d_M(R) = \mathscr{E}_R(k)(n)$ for some $n \in \boldsymbol{Z}$. Hence $\mathscr{K}_{\hat{R}} = \hat{R}(-n)$ by Proposition (1.3). By the uniqueness of canonical modules we have $\mathscr{K}_R = R(-n)$. Conversely assume that $\mathscr{K}_R = R(-n)$ for some $n \in \boldsymbol{Z}$. By Proposition (1.8) we get

$$\mathscr{E}xt^i_{\hat{R}}(\hat{R}/\hat{M}, \hat{R}) = \mathscr{H}om_{\hat{R}}(\mathscr{H}^{d-i}_{\hat{M}}(\hat{R}/\hat{M}, \mathscr{E}_{\hat{R}}(k))(n)$$

for all $i \geq 0$, where $\hat{M}$ is the maximal homogeneous ideal of $\hat{R}$. Hence $\hat{R}$ is Gorenstein since $\mathscr{H}^0_{\hat{M}}(\hat{R}/\hat{M}) = \hat{R}/\hat{M}$ and $\mathscr{H}^i_{\hat{M}}(\hat{R}/\hat{M}) = 0$ for $i > 0$. Since $\hat{R}$ is faithfully flat over R it follows that R is Gorenstein.

Remark. Let $a = \max\{n \mid \mathscr{H}^d_M(R)_n \neq 0\}$. If R is Gorenstein we have $\mathscr{K}_R = R(a)$. In the sequel we denote this number by $a(R)$ and call it the a-invariant of R.

PROPOSITION (1.10). *Let $R \to S$ be a finite homomorphism of graded rings defined over local rings. Assume that R is Cohen-Macaulay and has a canonical module. Then*

$$\mathscr{K}_S = \mathscr{E}xt_R^r(S, \mathscr{K}_R),$$

where $r = \dim R - \dim S$.

Proof. Let n_0 be the maximal ideal of S_0 and $\hat{S}_0$ be the n_0-adic completion of S_0. Since S_0 is finite over R_0 we have $\hat{S}_0 = S_0 \otimes_{R_0} \hat{R}_0$. Let N be the maximal homogeneous ideal of S and $\hat{N} = N \otimes_{R_0} \hat{R}_0$. Let $\hat{S} = S \otimes_{R_0} \hat{R}_0$. Note that $\mathscr{H}om_{\hat{R}}(\hat{S}, \mathscr{E}_{\hat{R}}(k))$ is the injective envelope of $\hat{S}/\hat{N}$ in $M_H(\hat{S})$.

$$\begin{aligned}
\mathscr{K}_{\hat{S}} &= \mathscr{H}om_{\hat{S}}(\mathscr{H}^s_{\hat{N}}(\hat{S}), \mathscr{E}_{\hat{S}}(\hat{S}/\hat{N})) \qquad (s = \dim S) \\
&= \mathscr{H}om_{\hat{S}}(\mathscr{H}^s_{\hat{M}}(\hat{S}), \mathscr{H}om_{\hat{R}}(\hat{S}, \mathscr{E}_{\hat{R}}(k))) \\
&= \mathscr{H}om_{\hat{R}}(\mathscr{H}^s_{\hat{M}}(\hat{S}), \mathscr{E}_{\hat{R}}(k)) \\
&= \mathscr{E}xt^r_{\hat{R}}(\hat{S}, \mathscr{K}_{\hat{R}}) \\
&= \mathscr{E}xt^r_R(S, \mathscr{K}_R) \otimes_R \hat{R}.
\end{aligned}$$

Since S is finite over R it follows that $\mathscr{K}_S = \mathscr{E}xt_R^r(S, \mathscr{K}_R)$.

COROLLARY (1.11). *If moreover R is Gorenstein in Proposition* (1.10) *we get $\mathscr{K}_S = \mathscr{E}xt_R^r(S, R)(n)$ for some $n \in \mathbf{Z}$.*

From Corollary (1.11) one knows that for any $p \in \operatorname{Supp}(\mathscr{K}_S)$ $(\mathscr{K}_S)_p$ is a canonical module of the local ring S_p in the sense of [HK].

§ 2. Preliminaries

In this section we collect fundamental facts about the local cohomology of Rees algebras over Noetherian local rings.

Let (A, m, k) be a local ring and I an ideal of A. We put $R(I) = \bigoplus_{n \geq 0} I^n$ and call this graded A-algebra the Rees algebra of I. Let $I = (a_1, \cdots, a_n)$. Then $R(I)$ can be identified with the subalgebra $A[a_1X, \cdots, a_nX]$ of the polynomial ring $A[X]$ in one variable. Throughout this paper we use this identification without mentioning. Let $M = mR(I) + (a_1X, \cdots, a_nX)R(I)$ be the maximal homogeneous ideal of $R(I)$. Let $G(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ be the associated graded ring of I. Note that

$$G(I) = R(I)/IR(I) \quad \text{and} \quad A = R(I)/R(I)_+,$$

where $R(I)_+ = \bigoplus_{n>0} I^n$. Let $\ell(I) = \dim R(I)/mR(I)$; we call this number the analytic spread of I. The analytic spread $\ell(I)$ of I is equal to the minimum number of generators of a minimal reduction of I if the residue field k is infinite (cf. [NR]).

PROPOSITION (2.1). *Let (A, m, k) be a local ring and I an ideal of A with* $\mathrm{ht}(I) > 0$. *If $R(I)$ is Cohen-Macaulay then*

a) $a(G(I)) < 0$ *and*

b) *for $i < \dim A$ we have*

$$\mathscr{H}^i_M(G(I))_n = \begin{cases} H^i_m(A) & \text{for } n = -1 \\ 0 & \text{for } n \neq -1. \end{cases}$$

Proof. For b) see the proof of [HI], Proposition (1.5). Let $J = R(I)_+$. From the exact sequences

$$0 \longrightarrow J \longrightarrow R(I) \longrightarrow A \longrightarrow 0$$

and

$$0 \longrightarrow J(1) \longrightarrow R(I) \longrightarrow G(I) \longrightarrow 0$$

we obtain the exact sequences of local cohomology

$$0 \longrightarrow H^d_m(A) \longrightarrow \mathscr{H}^{d+1}_M(J) \xrightarrow{f} \mathscr{H}^{d+1}_M(R(I)) \longrightarrow 0$$

and

$$0 \longrightarrow \mathscr{H}^d_M(G(I)) \longrightarrow \mathscr{H}^{d+1}_M(J)(1) \xrightarrow{g} \mathscr{H}^{d+1}_M(R(I)) \longrightarrow 0,$$

where $d = \dim A$. From this one gets the isomorphisms

$$f_n\colon \mathscr{H}^{d+1}_M(J)_n \longrightarrow \mathscr{H}^{d+1}_M(R(I))_n \qquad \text{for } n \neq 0$$

and surjective homomorphisms

$$g_n\colon \mathscr{H}^{d+1}_M(J)_n \longrightarrow \mathscr{H}^{d+1}_M(R(I))_{n-1} \qquad \text{for all } n.$$

Since $\mathscr{H}^{d+1}_M(J)$ and $\mathscr{H}^{d+1}_M(R(I))$ are Artinian $R(I)$-modules their homogeneous components of sufficiently large degree are zero. By an easy diagram chase we know that $\mathscr{H}^{d+1}_M(J)_n = 0$ for $n \geq 1$ and $\mathscr{H}^{d+1}_M(R(I))_n = 0$ for $n \geq 0$. Now it is easy to see that $a(G(I)) < 0$.

COROLLARY (2.2). *Let A and I be the same as in Proposition* (2.2). *Then $\mathscr{H}^{d+1}_M(R(I))_n = 0$ for $n \geq 0$.*

Proof. This follows from the proof of Proposition (2.1).

If, inparticular, $I = m$ we have the following result.

PROPOSITION (2.3). *If $d = \dim A > 0$ the following conditions are* *equivalent.*

(1) $R(m)$ *is Cohen-Macaulay.*

(2) a) $a(G(m)) < 0$ *and*

b) *for* $i < d$ *we have*

$$\mathscr{H}^i_M(G(m))_n = \begin{cases} H^i_m(A) & \text{for } n = -1 \\ 0 & \text{for } n \neq -1 . \end{cases}$$

In this case A *and* $G(m)$ *are Buchsbaum.*

Proof. See [I_1].

For the technical simplicity in the rest of this paper we assume that every local ring has an infinite residue field.

LEMMA (2.4). *Let* A *and* I *be the same as above and let* q *be a minimal reduction of* I. *We put* $r(q) = \min\{r \in Z \mid I^{r+1} = qI^r\}$. *If* $\mathrm{ht}(I) = \ell(I)$ *we have* $r(q) \geq a(G(I)) + \mathrm{ht}(I)$.

Proof. See [HI], Lemma (2.3).

LEMMA (2.5). *Let* A *and* I *be the same as above. We put*

$$n_i = \max\{n \in Z \mid \mathscr{H}^i_M(G(I))_n \neq 0\} \qquad \text{for } 0 \leq i \leq d = \dim A .$$

If I *is* m*-primary we have* $r(q) \leq \max_i\{n_i + i\}$ *for any minimal reduction* q *of* I.

Proof. Let $x \in A$. We denote by x^* the initial form of x with respect to I. Let $q = (a_1, \cdots, a_d)$ and $q^* = (a_1^*, \cdots, a_d^*)$. Then

$$r(q) = \max\{r \in Z \mid (G(I)/q^*)_r \neq 0\} .$$

If $\dim G(I) = 0$ the assertion is clear. Let $\dim G(I) > 0$. Since the residue field is infinite we may assume that $l_{G(I)}((0 : a_1^*)) < \infty$. From the exact sequences

$$0 \longrightarrow G(I)/(0 : a_1^*)(-1) \longrightarrow G(I) \longrightarrow G(I)/a_1^*G(I) \longrightarrow 0$$

and

$$0 \longrightarrow (0 : a_1^*) \longrightarrow G(I) \longrightarrow G(I)/(0 : a_1^*) \longrightarrow 0$$

we get the exact sequence

$$\mathscr{H}^i_M(G(I)) \longrightarrow \mathscr{H}^i_M(G(I)/a_1^*G(I)) \longrightarrow \mathscr{H}^{i+1}_M(G(I))(-1)$$

for $0 \leq i \leq d$. Let $n_i' = \max\{n \in Z \mid \mathscr{H}^i_M(G(I)/a_1^*G(I))_n \neq 0\}$. Then $n_i' \leq \max\{n_i, n_{i+1} + 1\}$. By induction we have $r(q) \leq \max\{n_i + i\}$.

§ 3. The Gorensteinness of Rees algebras

This section is devoted to the proof of the following theorem.

THEOREM (3.1). *Let (A, m, k) be a local ring and I an ideal of A. Suppose that $R(I)$ is Cohen-Macaulay and* $\operatorname{grade}(I) \geq 2$. *Then the following conditions are equivalent.*

(1) *$R(I)$ is Gorenstein.*

(2) *$K_A = A$ and $\mathscr{K}_{G(I)} = G(I)(-2)$.*

Remark. Since A and $G(I)$ are homomorphic images of $R(I)$, A and $G(I)$ have canonical modules if $R(I)$ is Gorenstein.

We need several preliminaries to prove this theorem.

LEMMA (3.2). *Let A be a local ring which has a canonical module K_A. Then the following conditions are equivalent.*

(1) *A satisfies (S_2).*

(2) *$\hat{A}$ satisfies (S_2).*

(3) $\operatorname{Hom}_A(K_A, K_A) = A$.

Proof. See [A], (4.4) and (4.5).

LEMMA (3.3). *Let A and I be the same as in Theorem* (3.1). *Let $a \in I - I^2$ be an element whose initial form in $G(I)$ is a non zero-divisor. We put $\bar{R} = R(I)/(a, aX)$. If $R(I)$ is Gorenstein and* $\operatorname{grade}(I) \geq 2$ *we have $\mathscr{H}_M^{d-1}(\bar{R}) = 0$, where* $\dim A = d$.

Proof. Let $R = R(I)$ and $G = G(I)$. Since a is a non zero divisor, by Propositions (1.8) and (1.9), it is enough to show that $\mathscr{E}xt^1_{R/aR}(\bar{R}, R/aR) = 0$. Let $I = (a_1, \cdots, a_n)$. Then we have the exact sequence

$$(R/aR)^n(-1) \xrightarrow[\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}]{} R/aR(-1) \xrightarrow[aX]{} R/aR \longrightarrow \bar{R} \longrightarrow 0 .$$

Applying the functor $\mathscr{H}om_{R/aR}(\ , R/aR)$ to this sequence, we see

$$\mathscr{E}xt^1_{R/aR}(\bar{R}, R/aR) = (aR : IR)/(a, aX) .$$

Let $f^m \in (aR : IR)$, where $f \in I^m$ and $m \geq 0$. Then we have

$$f \in (aA : I) \cap (I^{m+1} : I) \subset (aA : I) \cap (I^{m+1} : a) .$$

Since $\operatorname{grade}(I) \geq 2$ we have $(aA : I) = aA$. That a^* is a non zero-divisor of $G(I)$ is equivalent to that $(I^m : a) = I^{m-1}$ for all $m > 0$. Hence we have $(I^{m+1} : a) = I^m$. Therefore $f \in I^m \cap aA = aI^{m-1}$. This means $(aR : IR) = (a, aX)$, which completes the proof.

LEMMA (3.4). *Let A and I be the same as in Theorem (3.1). Assume that $R(I)$ is Cohen-Macaulay and $\mathscr{K}_{G(I)} = G(I)(-2)$. Then*

$$\mathscr{H}om_{R(I)}(k, \mathscr{H}_M^{d+1}(R(I)))_n = 0 \qquad \textit{for } n \neq -1,$$

where $d = \dim A$ and $k = R(I)/M$.

Proof. Let R and G be as in the proof of Lemma (3.3). Put $J = \bigoplus_{n>0} R_n$. Then we get the exact sequence (cf. the proof of Proposition (2.1))

$$\text{(I)}\quad 0 \longrightarrow \mathscr{H}om_R(k, H_m^d(A)) \longrightarrow \mathscr{H}om_R(k, \mathscr{H}_M^{d+1}(J)) \xrightarrow{f} \mathscr{H}om_R(k, \mathscr{H}_M^{d+1}(R)) \longrightarrow \mathscr{E}xt_R^1(k, H_m^d(A))$$

and

$$\text{(II)}\quad 0 \longrightarrow \mathscr{H}om_R(k, \mathscr{H}_M(G)) \longrightarrow \mathscr{H}om_R(k, \mathscr{H}_M^{d+1}(J))(1) \xrightarrow{g} \mathscr{H}om_R(k, \mathscr{H}_M^{d+1}(R)) \longrightarrow \mathscr{E}xt_R^1(k, \mathscr{H}_M^d(G)).$$

Since $\mathscr{H}om_R(k, H_m^d(A))$ is concentrated in degree 0 and since $\mathscr{E}xt_R^1(k, H_m^d(A))_n = 0$ for $n \leq -2$ from (I) we get isomorphisms

$$f_n\colon \mathscr{H}om_R(k, \mathscr{H}_M^{d+1}(J))_n \longrightarrow \mathscr{H}om_R(k, \mathscr{H}_M^{d+1}(R))_n$$

for $n \leq -2$. By assumption $\mathscr{H}om_R(k, \mathscr{H}_M^d(G))_n = 0$ for $n \neq -2$. (II) yields injective homomorphisms

$$g_n\colon \mathscr{H}om_R(k, \mathscr{H}_M^{d+1}(J))_n \longrightarrow \mathscr{H}om_R(k, \mathscr{H}_M^{d+1}(R))_{n-1}$$

for $n \leq -2$. Since $\mathscr{H}_M^{d+1}(J)$ and $\mathscr{H}_M^{d+1}(R)$ are Artinian

$$\mathscr{H}om_R(k, \mathscr{H}_M^{d+1}(J))_n = \mathscr{H}om_R(k, \mathscr{H}_M^{d+1}(R))_n = 0 \qquad \text{for } n \ll 0.$$

Now, it is easy to see that

$$\mathscr{H}om_R(k, \mathscr{H}_M^{d+1}(R))_n = 0$$

for $n \leq -2$. On the other hand, by Corollary (2.2) we have $\mathscr{H}_M^{d+1}(R)_n = 0$ for $n \geq 0$. This completes the proof.

LEMMA (3.5). *Let (A, m, k) be a local ring and I an ideal of A such that $R(I)$ is Cohen-Macaulay. Suppose that* $\operatorname{grade}(I) \geq n > 0$, *Then A and $G(I)$ satisfy (S_n).*

Proof. We may assume that A is complete. Let $G = G(I)$. Let B be a Gorenstein local ring such that A is a homomorphic image of B and $d = \dim A = \dim B$. Let n be the maximal ideal of B. By the local duality we have

$$\mathrm{Ext}_B^i(A, B) = \mathrm{Hom}_B(H_n^{d-i}(A), E_B(B/n)) \qquad \text{for } i \geq 0\,,$$

where $E_B(B/n)$ is the injective envelope of B/n as B-module. By Proposition (2.1) we see that $\mathrm{Ext}_B^i(A, B)$ is annihilated by I for $i > 0$. Let $p \in \mathrm{Spec}(A)$ and P be the inverse image of p in B. Then if $p \not\supset I$ we have

$$\mathrm{Ext}_{B_P}^i(A_p, B_P) = 0 \qquad \text{for } i > 0\,.$$

Hence A_p is Cohen-Macaulay. If $p \supset I$ we have depth $A_p \geq n$ by assumption. Therefore A satisfies (S_n).

To prove the assertion on G we use induction on $\dim A/I$. Let $\dim A/I = 0$. By Proposition (2.1) we know that $l_G(\mathscr{H}_M^i(G)) < \infty$ for $i < d$ and depth $G_N \geq n$, where N is the maximal homogeneous ideal of G. Hence G satisfies (S_n) because G_Q is Cohen-Macaulay for $Q \in \mathrm{Spec}(G) - \{N\}$. Let $\dim A/I > 0$. Note that G can be written as a homomorphic image of a Gorenstein graded ring of the same dimension. By Proposition (1.8) we see that G_p is Cohen-Macaulay if $p \not\supset G_+$, where $G_+ = \bigoplus_{n>0} G_n$. Assume that $p \supset G_+$ and $p \neq N$. Then $p \cap A/I = P/I$ for some $P \in \mathrm{Spec}(A) - \{m\}$. Since $R(I)_P$ is Cohen-Macaulay and $\dim A/I > \dim A_P/IA_P$ one knows that G_P satisfies (S_n) by induction on $\dim A/I$.

Proof of Theorem (3.1). First we show that if $\mathrm{ht}(I) > 0$ and $R(I)$ is Cohen-Macaulay then there is an element $a \in I - I^2$ whose initial form in $G(I)$ is a non zero-divisor. Since $\mathrm{ht}(IR(I)) > 0$ one can choose an element $b \in I$ which is a non zero-divisor on $R(I)$. Noting that $R(I)/IR(I) + IXR(I) = A/I$, we have $\mathrm{ht}(IR(I) + IXR(I)) = \dim R(I) - \dim A/I = d + 1 - \dim A/I \geq 2$. Since the residue field of A is infinite we can choose an element $c + aX$ of $IR(I) + IXR(I)$ such that $b, c + aX$ is an $R(I)$-sequence and $a \in I - I^2$. Since b is also a non zero-divisor on A one can easily verify that $(bR(I) : bX) = IR(I)$. This implies that there exists an exact sequence

$$0 \longrightarrow G(I)(-1) \longrightarrow R(I)/bR(I) \longrightarrow R(I)/(b, bX)R(I) \longrightarrow 0\,.$$

By the choice of $c + aX$ we see that $c + aX$ is a non zero-divisor on $G(I)$. The canonical image of $c + aX$ in $G(I) = R(I)/IR(I)$ is nothing but the initial form of a because $c \in I$. Therefore the initial form a^* of a in $G(I)$ is a non zero-divisor on $G(I)$.

(1) $\Rightarrow$ (2): Let $R = R(I)$ and $G = G(I)$. Let $a \in I - I^2$ be as above. Since a is a non zero-divisor on A there are two exact sequences

$$(\sharp) \qquad 0 \longrightarrow A \longrightarrow R/aXR \longrightarrow R/(a, ax) \longrightarrow 0$$

and

$$(\sharp\sharp) \qquad 0 \longrightarrow G(-1) \longrightarrow R/aR \longrightarrow R/(a, aX) \longrightarrow 0\,.$$

These exact sequences induce the exact sequences by Lemma (3.3)

$$(+) \qquad 0 \longrightarrow \mathscr{H}^d_M(A) \longrightarrow \mathscr{H}^d_M(R/aXR) \longrightarrow \mathscr{H}^d_M(R/(a, aX)) \longrightarrow 0$$

and

$$(++) \qquad 0 \longrightarrow \mathscr{H}^d_M(G)(-1) \longrightarrow \mathscr{H}^d_M(R/aR) \longrightarrow \mathscr{H}^d_M(R/(a, aX)) \longrightarrow 0\,,$$

where $d = \dim A$ and M is the maximal homogeneous ideal of R as before. Since R is Gorenstein $\mathscr{K}_R = R(n)$ for some $n \in Z$. Since aX is a non zero divisor of degree 1 we have $\mathscr{K}_{R/aXR} = R/aXR(n+1)$. From the exact sequence (+) we know that $n = -1$ and $K_A = A/J$ for some ideal J of A. From (++) we have $\mathscr{K}_G = G/L(-2)$ for some homogeneous ideal L of G. By Lemma (3.5) A and G satisfy (S_2), hence by Lemma (3.2) we have $J = 0$ and $L = 0$.

(2) $\Rightarrow$ (1): From the exact sequences ($\sharp$) and ($\sharp\sharp$) we get two injections $\mathscr{H}^{d-1}_M(R/(a, aX)) \to \mathscr{H}^d_M(A)$ and $\mathscr{H}^{d-1}_M(R/(a, aX)) \to \mathscr{H}^d_M(G)(-1)$ since R is Cohen-Macaulay. From the first one we know that $\mathscr{H}^{d-1}_M(R/(a, aX))$ is concentrated in degree 0. The assumption $\mathscr{K}_G = G(-2)$ shows that $\mathscr{H}^d_M(G)_n = 0$ for $n \geq -1$. From the second injection we see that $\mathscr{H}^{d-1}_M(R/(a, aX)) = 0$. Hence we have the exact sequences (+) and (++). By Lemma (3.4) we know that $\mathscr{H}om_R(k, \mathscr{H}^d_M(R/aXR))$ is concentrated in degree 0. By (+) we get

$$\mathscr{H}om_R(k, \mathscr{H}^d_M(R/aXR)) = \operatorname{Hom}_A(k, H^d_m(A))$$

since $\mathscr{H}^d_M(R/aR)_n = \mathscr{H}^d_M(R/(a, aX))_n = 0$ for $n \geq 0$ by Corollary (2.2). By the assumption $K_A = A$ we have $\operatorname{Hom}_A(k, H^d_m(A)) = k$. This shows that R is Gorenstein.

Let I be an ideal of a local ring and q a minimal reduction of I. We put $r(q) = \min\{r \mid I^{r+1} = qI^r\}$. We call $r(q)$ the reduction exponent of q.

COROLLARY (3.6) *Let A be a local ring and I an ideal of A such that* $\mathrm{ht}(I) = \ell(I) > 0$ *and $R(I)$ is Cohen-Macaulay. Then we have*:

(1) *Suppose that $a(G(I)) \geq -2$. Then we have $r(q) = \mathrm{ht}(I) - 1$ or* $\mathrm{ht}(I) - 2$ *for any minimal reduction q of I.*

(2) *Suppose moreover that* $\mathrm{grade}(I) \geq 2$ *and $R(I)$ is Gorenstein. Then for any minimal reduction q of I we have $r(q) = \mathrm{ht}(I) - 2$ if and only if* $\mathrm{depth}\, A \geq \dim A/I + 2$.

Proof. (1) By induction on $\dim A/I$. If $\dim A/I = 0$ this follows from Lemmas (2.4) and (2.5). Let $\dim A/I > 0$. Choose an element $b \in A$ whose image in A/I is a part of system of parameters of A/I. By Proposition (1.5) b is a non zero-divisor on $G(I)$ and $R(I)$ and we have $R(I)/bR(I) = R(I(A/bA))$ and $G(I) = G(I(A/bA))$. It is easy to see that the ideal $I(A/bA)$ in A/bA satisfies the same assumption on I. By induction hypothesis we have $r(q(A/bA)) = \mathrm{ht}(I) - 1$ or $\mathrm{ht}(I) - 2$. By Nakayama's lemma we have $r(q(A/bA)) = r(q)$.

(2) First we assume that $\mathrm{depth}\, A \geq \dim A/I + 2$. If $\dim A/I = 0$ we have $r(q) = \mathrm{ht}(I) - 2$ by Proposition (2.1), Lemmas (2.4) and (2.5). We proceed by induction on $\dim A/I$. Let $\dim A/I > 0$. Then by assumption $\mathrm{depth}\, A \geq 3$. Let a_1, a_2 be a regular sequence in I. One can choose an element $b \in m$ so that a_1, a_2, b is a regular sequence and the image of b in A/I is a part of system of parameters of A/I. Then $\mathrm{grade}(I(A/bA)) \geq 2$ and $R(I(A/bA))$ is Gorenstein. Since $\mathrm{depth}\, A/bA \geq \dim A/(b, I) + 2$ we have $r(q) = \mathrm{ht}(I) - 2$ by induction hypothesis.

Conversely assume that $r(q) = \mathrm{ht}(I) - 2$. Let $b_1, \cdots, b_s \in m$ be a system of parameters of A/I. We set $\bar{A} = A/(b_1, \cdots, b_s)$. Since $b_1, \cdots, b_s$ is a regular sequence we have only to show that $\mathrm{depth}\, \bar{A} \geq 2$. Let $\bar{I} = I\bar{A}$ and $\bar{q} = q\bar{A}$. Since $r(\bar{q}) = \mathrm{ht}(\bar{I}) - 2$ we see that $\mathscr{H}^h_M(G(\bar{I}))_n = 0$ for $n \geq -1$ by Lemma (2.4), where $h = \mathrm{ht}(I)$. By [HI], Proposition (1.5) we know that $b_1, \cdots, b_s$ is a $G(I)$-sequence. Let $q_i = (b_1, \cdots, b_i)$ for $1 \leq i \leq s$. Then we see that $G(I)/q_iG(I) = G(I(A/q_i))$. We set $G_i = G(I)/q_iG(I)$ for $1 \leq i \leq s$. Then we have an exact sequence

$$\mathscr{H}^{d-i}_M(G_{i-1}) \xrightarrow{b_i} \mathscr{H}^{d-i}_M(G_{i-1}) \longrightarrow \mathscr{H}^{d-i}_M(G_i) \longrightarrow \mathscr{H}^{d-i+1}_M(G_{i-1}).$$

Since $r(q(A/q_i)) = \mathrm{ht}(I) - 2$ we know that $\mathscr{H}^{d-i}_M(G_i)_n = 0$ for $n \geq -1$ by Lemma (2.4). By Proposition (2.1) we see that $H^{d-i}_m(A/q_{i-1}) = b_iH^{d-i}_m(A/q_{i-1})$ for $1 \leq i \leq s$.

This implies that $K_{A/q_i} = A/q_i$ for $0 \leq i \leq s$, where $q_0 = 0$. In particular, $K_{\bar{A}} = \bar{A}$. One sees that depth $\bar{A} \geq 2$ by [A]. The following is a generalization of a result in [GS].

COROLLARY (3.7). *Let A be a Cohen-Macaulay local ring and I an ideal of A with* $\mathrm{ht}(I) = \ell(I) \geq 2$. *Then the following conditions are equivalent.*

(1) *$R(I)$ is Gorenstein.*

(2) *$G(I)$ is Gorenstein and $a(G(I)) = -2$.*

(3) *$G(I)$ is Gorenstein and there exists a minimal reduction q of I such that $r(q) = \mathrm{ht}(I) - 2$.*

In this case A is Gorenstein.

Proof. This follows from Theorem (3.1), Corollary (3.6) and the fact that the Gorensteinness of $G(I)$ implies that of A.

COROLLARY (3.8). *Let A and I be the same as in Corollary* (3.6). *Suppose that*

(1) *$R(I)$ is Gorenstein,*

(2) *$l_A(H^i_m(A)) < \infty$ for $i < d = \dim A$ and*

(3) $2\,\mathrm{ht}(I) \leq \dim A$.

Then A is Gorenstein.

Proof. By Corollary (3.7) it is sufficient to prove that A is Cohen-Macaulay. Let $b_1, \cdots, b_s$ be a system of parameters of A/I. We put $q_i = (b_1, \cdots, b_i)$ and $G_i = G(I)/q_iG(I)$ for $1 \leq i \leq s$. Let $a_1, \cdots, a_h$, $h = \mathrm{ht}(I)$, be a minimal generators of a minimal reduction of I. Then $a_1, \cdots, a_h, b_1, \cdots, b_s$ is a system of parameters of A. Since $l_A(H^i_m(A)) < \infty$ for $i < d$ we know that if $t \leq \operatorname{depth} A$ then any t elements of a system of parameters of A form a regular sequence by [CST], (3.3). Hence we have grade$(I(A/q_i)) \geq 2$ for $1 \leq i < s$ by [HI], Proposition (1.5). We are going to show that $H^{d-s+i}_m(A/q_{s-j}) = 0$ for $2 \leq j \leq s-2$ and $1 \leq i \leq j-1$ by induction on j. Let $j = 2$. From the exact sequence

$$\mathscr{H}^{d-s+1}_M(G_{s-2}) \xrightarrow{b_{s-1}} \mathscr{H}^{d-s+1}_M(G_{s-2}) \longrightarrow \mathscr{H}^{d-s+1}_M(G_{s-1})$$

we get $H^{d-s+1}_m(A/q_{s-2}) = b_{s-1}H^{d-s+1}_m(A/q_{s-2})$ by Theorem (3.1) and Proposition (2.1) since $R(I(A/q_{s-1}))$ is Gorenstein. By the assumption (2) we get $H^{d-s+1}_m(A/q_{s-2}) = 0$. Let us assume that our assertion is true for $j < s-2$ and we will prove that the assertion is true for $j+1$. Since b_{s-j} is a non zero-divisor on A/q_{s-j-i} we obtain the exact sequence

$$H_m^{d-s+i}(A/q_{s-j-1}) \xrightarrow{b_{s-j}} H_m^{d-s+i}(A/q_{s-j-1}) \longrightarrow H_m^{d-s+i}(A/q_{s-j})$$

for $i>0$. By the induction hypothesis $H_m^{d-s+i}(A/q_{s-j})=0$ for $1\leq i\leq j-1$. Therefore $H_m^{d-s+i}(A/q_{s-j-1})=0$ for $1\leq i\leq j-1$ by assumption (2). It remains to prove that $H_m^{d-s+j}(A/q_{s-j-1})=0$. But this can be proved by the same method used for $j=2$. Hence, in particular, we get $H_m^i(A)=0$ for $h+1\leq i\leq d$. By assumption (3) we get depth $A\geq \dim A/I+1\geq h+1$ cf. [HI]. Therefore A is Cohen-Macaulay.

§ 4. Example

In this section we construct a local ring (A, m, k) such that $R(m)$ is Gorenstein but A is not Cohen-Macaulay. For a local ring A we denote the multiplicity of A by $e(A)$.

LEMMA (4.1). *Let* (A, m, k) *be a local ring with* $\dim A=3$ *and* $q=(a_1, a_2, a_3)$ *be a minimal reduction of* m. *Let*

$$I=((a_1, a_2):a_3)+((a_2, a_3):a_1)+((a_1, a_3):a_2)+m^2.$$

Then $R(m)$ *is Cohen-Macaulay if and only if* $m^3=qm^2$ *and* $l_A(I/m^2)=3(l_A(A/q)-e(A))+3$.

Proof. See [I_2], Theorem 5.

LEMMA (4.2). *Let* A *be the same as in Lemma* (4.1). *Suppose that* $R(m)$ *is Cohen-Macaulay and* A *is not Cohen-Macaulay. If* $l_A(m/m^2)=6$ *we have*

(1) A *is a Buchsbaum ring with* depth $A=2$ *and* $l_A(H_m^2(A))=1$,

(2) $m^2=qm$ *for any minimal reduction* q *of* m *and*

(3) $e(A)=3$.

Proof. See [I_2], Corollary 11.

EXAMPLE (1). Let k be a field and X_i, Y_i $(1\leq i\leq 3)$ be indeterminates over k. We put

$$A=k[[X_1, X_2, X_3, Y_1, Y_2, Y_3]]/(X_1Y_1+X_2Y_2+X_3Y_3, (Y_1, Y_2, Y_3)^2).$$

Then A is not Cohen-Macaulay but $R(m)$ is Cohen-Macaulay. By Lemma (4.2) $e(A)=3$ (cf. [I_1] and [I_2]).

EXAMPLE (2). Let k be a field of $\mathrm{ch}(k) = 2$ and $X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4$ indeterminates over k. Let

$$\begin{aligned} A &= k[[X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4]]/J \\ &= k[[x_1, x_2, x_3, y_1, \cdots, y_4]] \end{aligned}$$

where J is the ideal generated by $X_1Y_1 + X_2Y_2 + X_3Y_3$, Y_1^2, Y_2^2, Y_3^2, Y_4^2, Y_1Y_4, Y_2Y_4, Y_3Y_4, $Y_1Y_2 - X_3Y_4$, $Y_2Y_3 - X_1Y_4$ and $Y_1Y_3 - X_2Y_4$.

Then A is not Cohen-Macaulay but $R(m)$ is Gorenstein.

To prove this we need the following lemma.

LEMMA (4.3). $(0 : y_4) = (y_1, \cdots, y_4)$

Proof. Let $R = k[[X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4]]$ and let $f \in (J \colon Y_4)$. Since

$$\begin{aligned} (J \colon Y_4) = (&X_1Y_1 + X_2Y_2 + X_3Y_3,\ Y_1Y_2 - X_3Y_4,\ Y_2Y_3 - X_1Y_4,\ Y_1Y_3 - X_2Y_4) \colon \\ &Y_4 + (Y_1, \cdots, Y_4) \end{aligned}$$

we may assume that f belongs to the first ideal on the right side. Let us write

$$\begin{aligned} fY_4 = &(g_1 + g_1'Y_4)(X_1Y_1 + X_2Y_2 + X_3Y_3) + (g_2 + g_2'Y_4)(Y_1Y_2 - X_3Y_4) \\ &+ (g_3 + g_3'Y_4)(Y_2Y_3 - X_1Y_4) + (g_4 + g_4'Y_4)(Y_1Y_3 - X_2Y_4), \end{aligned}$$

where $g_i \in k[[X_1, X_2, X_3, Y_1, Y_2, Y_3]]$. From this we see that

$$\text{(I)} \qquad g_1(X_1Y_1 + X_2Y_2 + X_3Y_3) + g_2Y_1Y_2 + g_3Y_2Y_3 + g_4Y_1Y_3 = 0$$

and

$$\text{(II)} \qquad f \equiv -g_2X_3 - g_3X_1 - g_4X_4 \mod (Y_1, \cdots, Y_4).$$

From (I) we have

$$Y_1(g_1X_1 + g_2Y_2 + g_4Y_3) + Y_2(g_1X_2 + g_3Y_3) = 0.$$

Since Y_1, Y_2 is a regular sequence in R we have

$$\begin{aligned} g_1X_1 + g_2Y_2 + g_4Y_3 &= hY_2 \\ g_1X_2 + g_3Y_3 &= -hY_1 \end{aligned}$$

for some $h \in R$. Since X_1, Y_2, Y_3 and X_2, Y_1, Y_3 are regular sequences in R there are elements $a_1, a_2, a_3, b_1, b_2, b_3$ of R such that

$$\text{(III)} \qquad (g_1, g_2 - h, g_4) = (X_1, Y_2, Y_3)\begin{pmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{pmatrix}$$

$$\text{(IV)} \qquad (g_1, h, g_3) = (X_2, Y_1, Y_3)\begin{pmatrix} 0 & b_1 & b_2 \\ -b_1 & 0 & b_3 \\ -b_2 & -b_3 & 0 \end{pmatrix}.$$

Hence

$$g_1 = -a_1 Y_2 - a_2 Y_3 = -b_1 Y_1 - b_2 Y_3 .$$

Since $\mathrm{ch}(k) = 2$ we have

$$\left.\begin{aligned} a_2 + b_2 &\equiv 0 \\ a_1 &\equiv 0 \\ b_1 &\equiv 0 \end{aligned}\right\} \mod (Y_1, \cdots, Y_4).$$

By (II), (III) and (IV) we obtain

$$\begin{aligned} f &\equiv a_1 X_1 X_3 + b_1 X_2 X_3 + (a_2 + b_2) X_1 X_2 \\ &\equiv 0 \mod (Y_1, \cdots, Y_4). \end{aligned}$$

Proof of Example (2). By Lemma (4.3) we have $(0 : y_4) = (y_1, \cdots, y_4)$. From the exact sequence

$$0 \longrightarrow A/(0 : y_4) \longrightarrow A \longrightarrow A/y_4 A \longrightarrow 0$$

we get $e(A) = e(A/(0 : y_4)) + e(A/y_4 A)$. Since $A/y_4 A$ is isomorphic to the local ring in Example (1) and since $A/(0 : y_4)$ is a regular local ring we have $e(A) = 4$. It is easy to see that x_1, x_2, x_3 is a system of parameters of A and that $((x_1, x_2) : x_3) = (x_1, x_2, y_3)$, $((x_2, x_3) : x_1) = (x_2, x_3, y_1)$ and $((x_1, x_3) : x_2) = (x_1, x_3, y_2)$. This shows that A is not Cohen-Macaulay. It is easy to verify that $m^2 = (x_1, x_2, x_3)m$. By Lemma (4.1) we see that $R(m)$ is Cohen-Macaulay. By Theorem (3.1) it is enough to show that $\mathscr{K}_{G(m)} = G(m)(-2)$. Since A is defined by homogeneous polynomials

$$G(m) = k[X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4]/J^* = k[x_1, x_2, \cdots, y_4],$$

where J^* is generated by the polynomials generating J. Let $S = k[x_1, x_2, x_3]$. Then S is a polynomial ring with $\dim S = 3$ and $G(m)$ is generated by 1, y_1, y_2, y_3, y_4 as an S-module. Since $G(m)$ has rank 4 as an S-module and depth $G(m) = 2$ we get a finite free resolution of $G(m)$ as S-module

$$0 \longrightarrow S(-2) \xrightarrow[{[0,\, x_1,\, x_2,\, x_3,\, 0]}]{} S \oplus S^4(-1) \xrightarrow[d]{} G(m) \longrightarrow 0,$$

where d is given by $d(e_0) = 1$ and $d(e_i) = y_i$ for $1 \leq i \leq 4$, with suitable free basis $e_0, e_1, \cdots, e_4$ of $S \oplus S^4(-1)$ with $\deg(e_0) = 0$ and $\deg(e_i) = 1$ for

$1 \leq i \leq 4$. By Corollary (1.11) $\mathscr{K}_{G(m)} = \mathscr{H}om_S(G(m), S(-3))$.

The $G(m)$-structure of $\mathscr{K}_{G(m)}$ is given by

$$(xf)(y) = f(xy) \qquad \text{for } f \in \mathscr{H}om_S(G(m), S(-3)) \text{ and } x, y \in G(m).$$

$\mathscr{K}_{G(m)}$ is generated by e_0^*, $x_2e_3^* - x_3e_2^*$, $x_3e_1^* - x_1e_3^*$, $x_1e_2^* - x_2e_1^*$ and e_4^* as an S-module, where e_i^* is the dual base of e_i with $\deg(e_0^*) = 3$ and $\deg(e_i^*) = 2$ for $1 \leq i \leq 4$. Using the fact that $\mathrm{ch}(k) = 2$ we can easily verify the following relations as $G(m)$-module.

$$\begin{aligned} y_4e_4^* &= e_0^* \\ y_1e_4^* &= x_2e_3^* - x_3e_2^* \\ y_2e_4^* &= x_3e_1^* - x_1e_3^* \\ y_3e_4^* &= x_1e_2^* - x_2e_1^* \end{aligned}$$

Hence $\mathscr{K}_{G(m)} = G(m)(-2)$ and hence $R(m)$ is Gorenstein by Theorem (3.1).

Example (3). Let A be same as in Example (2). We put $B = A[[T_1, \cdots, T_n]]$, where $T_1, \cdots, T_n$ are indeterminates over A. Let $I = mB$. Then $R(I) = R(m) \otimes_A B$ is Gorenstein since B is faithfully flat over A. If $n \geq 3$ we have $2\,\mathrm{ht}(I) = 6 \leq \dim B$. But B is not Gorenstein.

Remark. a) If in Example (2) $\mathrm{ch}(k) \neq 2$ A is not Buchsbaum. This can be seen as follows. If A is Buchsbaum we have

$$\begin{aligned} e(A) &= l_A(A/(x_1, x_2, x_3)) - l_A((x_1, x_2) : x_3/(x_1, x_2)) \\ &= 5 - 1 = 4. \end{aligned}$$

On the other hand one can easily see $(0 : y_4) \supset (y_1, \cdots, y_4, x_1x_2)$ and $\dim A/(0 : y_4) < 3$. This implies $e(A) = e(A/y_4A) = 3$, a contradiction.

b) Example (3) shows that Corollary (3.8) is false without any restriction on the local cohomology modules of A.

References

[A] Y. Aoyama, Some basic results on canonical modules, J. Math. Kyoto Univ., **23** (1983), 85–94.

[AG] Y. Aoyama and S. Goto, On the type of graded Cohen-Macaulay rings, J. Math. Kyoto Univ., **15** (1975), 275–284.

[B] H. Bass, On the ubiquity of Gorenstein rings, Math. Z., **82** (1963), 8–28.

[Gr$_1$] A. Grothendieck, Sur quelque points d'algebre homologique, Tohoku Math. J., **IX** (1957), 119–221.

[Gr$_2$] ——, Local cohomology, Lect. Notes in Math., **41**, Berlin-Heidelberg-New York, 1967.

[GS] S. Goto and Y. Shimoda, On the Rees algebras of Cohen-Macaulay local rings, Commutative algebra (analytic methods), Lecture Notes in Pure and Applied Mathematics, **68** (1982), 201–231.

[GW] S. Goto and K. Watanabe, On graded rings I, J. Math. Soc. Japan, **30** (1978), 179–213.

[HI] M. Herrmann and S. Ikeda, Remarks on lifting of Cohen-Macaulay property, Nagoya Math. J., **92** (1983), 121–132.

[HK] J. Herzog and E. Kunz, Der kanonische Modul eines Cohen-Macaulay Rings, Springer Lect. Notes in Math., **238** (1971).

[I_1] S. Ikeda, The Cohen-Macaulayness of the Rees algebras of local rings, Nagoya Math. J., **89** (1983), 47–63.

[I_2] ——, Remarks on Rees algebras and graded rings with multiplicity 3, Preprint.

[M] E. Matlis, Injective modules over Noetherian rings, Pacific J. Math., **8** (1958), 511–528.

[MR] J. Matijevic and P. Roberts, A conjecture of Nagata on graded Cohen-Macaulay rings, J. Math. Kyoto Univ., **14** (1974) 125–128.

[NR] D. G. Northcott and D. Rees, Reductions of ideals in local rings, Proc. Camb. Phil. Soc., **50** (1954), 145–158.

[CST] N. T. Cuong, P. Schenzel and N. V. Trung, Verallgemeinerte Cohen-Macaulay-Moduln, Math. Nachr., **85** (1978), 57–73.