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THE MIXED HODGE STRUCTURE ON THE FUNDAMENTAL GROUP OF THE FIBER TYPE 2-ARRANGEMENT

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Abstract. The complement of an arrangement of hyperplanes is a good example of the mixed Hodge structure on the fundamental group of an algebraic variety. We compute its isomorphic class using iterated integrals in the fiber type case and then get the combinatorial and projective invariant.

Introduction

The mixed Hodge structure on the homotopy group of the algebraic variety was constructed in two different ways. One way is Morgan's construction based on Sullivan's theory of minimal models [M]. The other way is Hain's method based on the bar construction [H5]. We shall deal with Hain's method as this approach is very natural from the topological viewpoint and it directly gives precise results on the fundamental group.

Due to Hain [H1], we can construct a mixed Hodge structure on the fundamental group of an algebraic variety using iterated integrals defined by K-T. Chen as follows. Let V be an algebraic variety over $\mathbb{C}$. We fix a point x of V and consider the truncation

$$\mathbb{Z}\pi_1(V,x)/J^{s+1}$$

of the group algebra of the fundamental group $\pi_1(V,x)$ over $\mathbb{Z}$ by some power of its augmentation ideal J. An iterated integral is a function on the space of paths in V. Let us denote the space of iterated integrals with length $\leqq s$ that are *homotopy functionals* on the space of loops based at x (i.e. its value depends only on the homotopy class of the loop.), by

$$H^0(B_s(V),x).$$

Then the integral map

$$H^0(B_s(V),x) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\pi_1(V,x)/J^{s+1},\mathbb{C})$$

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is an isomorphism (K-T. Chen). As an iterated integral $\int \omega_1 \cdots \omega_r$ is in F^p if the total number of d's in ω_j's is $\geqq p$, a Hodge filtration on $H^0(B_s(V), x)$ can be defined. The weight filtration is given by the length filtration when V is smooth and projective:

$$W_l(\mathbb{Z}\pi_1(V,x)/J^{s+1})^* = (\mathbb{Z}\pi_1(V,x)/J^{l+1})^* \cong H^0(B_l(V), x)$$

If $H^1(V)$ is a pure Hodge structure of weight 2, then the weight filtration on $\mathbb{Z}\pi_1(V,x)/J^{s+1}$ is defined by

$$W_{2l+1} = W_{2l} = (\mathbb{Z}\pi_1(V,x)/J^{l+1})^* \cong H^0(B_l(V), x).$$

Thus the Hodge and weight filtration induced by those define a mixed Hodge structure on $\mathbb{Z}\pi_1(V,x)/J^{s+1}$.

In particular, when $s = 1$, there is an isomorphism

$$\mathbb{C}\pi_1(V,x)/J^2 \cong \mathbb{C} \oplus H_1(V,\mathbb{C})$$

of mixed Hodge structures. And, since the mixed Hodge structure on $H_1(V)$ is independent of the base point, so is the same on $\mathbb{C}\pi_1(V,x)/J^2$. Thus, the interesting case is when $s = 2$; the mixed Hodge structure on $\mathbb{C}\pi_1(V,x)/J^3$ will vary with the base point. In fact,

THEOREM. *If $V = \mathbb{P}^1 - \{a_1, \cdots, a_n\}$, then the polarized mixed Hodge structure on $J(V,t)/J^3$ determines (V,t) up to biholomorphism.*

And if V is a smooth projective algebraic curve, the similar theorem is obtained by Hain and Pulte ([H1]).

On the other hand $\mathbb{P}^1 - \{a_1, \cdots, a_n\}$ can be seen as the complement of an arrangement of hyperplanes in $\mathbb{C}$. We consider the same arguments in the case of the complement of some 2-arrangement. An arrangement $\mathcal{A}$ of hyperplanes in $\mathbb{C}^2$ is called *fiber type* if

$$\mathcal{A} = \{H_1, \cdots, H_n, G_1, \cdots, G_m\}$$

satisfies the following conditions.

(1) Each G_j is parallel to G_1, and each H_i is not parallel to G_1. (i.e. For each $i \neq j$, $G_i \cap G_j = \emptyset$ and for each k, l, $H_k \cap G_l \neq \emptyset$.)

(2) If $H_i \cap H_j \neq \emptyset$, then there exists unique G_k such that

$$H_i \cap H_j \subset G_k.$$

In general, if $H_1(V,\mathbb{Z})$ is torsionfree, there is an exact sequence

$$0 \longrightarrow H^1(V) \longrightarrow \mathrm{Hom}(J/J^3,\mathbb{Z}) \longrightarrow K \longrightarrow 0,$$

where K is the kernel of the cup product $H^1(V)\otimes H^1(V) \to H^2(V)$. When $H^1(V)$ has a pure Hodge structure, the mixed Hodge structure on the dual of J/J^3 is a separated extension of Hodge structures. The set of suitable classes of extensions of K by H^1 forms an abelian group $\mathrm{Ext}(K,H^1)$, and there is an abelian group isomorphism (see [Ca])

$$\psi : \mathrm{Ext}(K,H^1) \to \frac{\mathrm{Hom}(K,H^1)_{\mathbb{C}}}{F^0\mathrm{Hom}(K,H^1)_{\mathbb{C}} + \mathrm{Hom}(K,H^1)_{\mathbb{Z}}}.$$

For a pointed fiber type 2-arrangement $(\mathcal{A},b)$, with suitable each basis of H^1 and K, the concrete description of $\psi((J(M(\mathcal{A},b)/J^3)^*)$ gives that it depends only on cross ratios

$$\lambda_{ij} \quad (1 \leqq i < j \leqq n) \qquad \lambda'_{ij} \quad (1 \leqq i < j \leqq m)$$

arising from $(\mathcal{A},b)$. Two pointed fiber type 2-arrangements are called *cross ratio equivalent* if these respective cross ratios coincide. Then we obtain the following result.

MAIN THEOREM. *Let $(\mathcal{A},b)$ and $(\mathcal{A}',b')$ be pointed fiber type 2-arrangements. If there is a ring isomorphism*

$$\varphi : \mathbb{Z}\pi_1(M(\mathcal{A}),b)/J^3 \to \mathbb{Z}\pi_1(M(\mathcal{A}'),b')/J^3$$

that induces an isomorphism of mixed Hodge structures, then $(\mathcal{A},b)$ and $(\mathcal{A}',b')$ are cross ratio equivalent.

§1. The mixed Hodge structure on π_1

In this section, we review some results on the mixed Hodge structure on π_1 using Hain's method [H1].

The mixed Hodge structure on π_1

Let $k = \mathbb{R}$ or $\mathbb{C}$, and M a smooth manifold. Denote the set of piecewise smooth path $\gamma : [0,1] \to M$ by PM and the subset of loops based at x by $P_x M$. $E_k^{\cdot}(M)$ denotes the de Rham complex of C^∞ k-valued forms on M. First we denote the iterated integral as follows. For $\omega_1, \ldots, \omega_r \in E_k^1(M)$ and $\gamma \in PM$, define

$$\int_\gamma \omega_1 \cdots \omega_r = \int \cdots \int_{0 \leqq t_1 \leqq \cdots \leqq t_r \leqq 1} f_1(t_1) \cdots f_r(t_r) dt_1 \cdots dt_r,$$

where $f_j(t)dt = \gamma^* \omega_j$. $\int \omega_1 \cdots \omega_r$ denotes the function $PM \to k$, $\gamma \to \int_\gamma \omega_1 \cdots \omega_r$. If $r = 0$, it is the constant function. A linear combination of such functions and the constant function is called an *iterated path integral*, and a linear combination of a constant function and iterated integrals $\int \omega_1 \cdots \omega_r$ with $r \leqq s$ is called an iterated integral *of length* $\leqq s$. Sometime we shall denote the integration value of an iterated integral I on a path γ by $< I, \gamma >$.

A function F on PM is *homotopy functional*, if $F(\gamma)$ depends only on the homotopy class of γ relative to its endpoints.

We define subspaces of iterated integrals;

$B_s(M) = \{$iterated integrals on M of length $\leqq s$ $\}$,
$H^0(B_s(M), x) = \{I \in B_s(M) \mid I$ is homotopy functional on $P_x M$ $\}$,
$\overline{B}_s(M) = \{I \in B_s(M)$ with zero constant term$\}$,
$H^0(\overline{B}_s(M), x) = H^0(B_s(M), x) \cap \overline{B}_s(M)$.

Note that

$$B_s(M) = k \oplus \overline{B}_s(M).$$

Suppose that $A^{\cdot}$ is a subdifferential graded algebra of $E_k^{\cdot}(M)$ such that the inclusion $A^{\cdot} \to E_k^{\cdot}(M)$ is a quasi isomorphism. $B_s(A^{\cdot})$ is a set of iterated integrals on M spanned by $\int \omega_1 \cdots \omega_r$ where each $\omega_j \in A^1$ and $0 \leqq r \leqq s$. We define subspaces $H^0(B_s(A^{\cdot}), x)$, $\overline{B}_s(A^{\cdot})$, and $H^0(\overline{B}_s(A^{\cdot}), x)$ in the same way.

Let G be a group and R a commutative ring. We denote by RG the group algebra of G over R and by J its augmentation ideal.

THEOREM 1.1. (Chen) *For each $x \in M$ and $s > 0$, the integration map*

$$H^0(B_s(A^{\cdot}), x) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\pi_1(M, x)/J^{s+1}, k)$$

is an isomorphism.

Proof. The proof is given in [H1,2,3]. cf. [C1,2,3].

COROLLARY 1.2.

$$H^0(\overline{B}_s(A^{\cdot}),x) \to \mathrm{Hom}_{\mathbb{Z}}(J(M,x)/J^{s+1},k)$$

is an isomorphism.

Using this, we can give the mixed Hodge structure on $\mathbb{Z}\pi_1(M,x)/J^{s+1}$.

THEOREM 1.3. (Morgan, Hain) *If M is an algebraic variety over $\mathbb{C}$ and $x \in V$, then there is a mixed Hodge structure on*

$$\mathbb{Z}\pi_1(M,x)/J^{s+1}$$

that is natural with respect to morphism of pointed varieties. Moreover, if $s \geqq t$, then the quotient map

$$\mathbb{Z}\pi_1(M,x)/J^{s+1} \to \mathbb{Z}\pi_1(M,x)/J^{t+1}$$

induces a morphism of mixed Hodge structures.

Proof. The theorem is proved by induction s, using the following proposition. For detail, see [H1].

PROPOSITION 1.4. (Hain) *There is a natural isomorphism*

$$\mathbb{C} \oplus H^1(M,\mathbb{C}) \to H^0(B_1(M),x)$$

$$\lambda \oplus \omega \to \lambda + \int \omega$$

and for all s, the sequence

$$0 \longrightarrow H^0(B_{s-1}(M),x) \longrightarrow H^0(B_s(M),x) \xrightarrow{p} \otimes^s H^1(M,\mathbb{C})$$

is exact, where p takes the iterated integral I to the function

$$p(I) : \otimes^s H_1(M,\mathbb{C}) \longrightarrow \mathbb{C}$$

$$[\alpha_1] \otimes \cdots \otimes [\alpha_s] \longrightarrow < I, \prod_{j=1}^{s}(\alpha_j - 1) >$$

for each loop α_j based at x.

Proof. The proof is given in [H1].

Extensions of mixed Hodge structures

We also review the extension of mixed Hodge structures, (cf. [Ca] and [H1]).

A *separated extension of Hodge structures* is an exact sequence

$$0 \to A \to E \to B \to 0$$

of mixed Hodge structures, where A is a pure Hodge structure of weight m, B is a pure Hodge structure of weight n, and $n > m$. Two extensions

$$0 \to A \to E_j \to B \to 0, \qquad j = 1, 2,$$

are *congruent* if there is an isomorphism of mixed Hodge structures $\Phi : E_1 \to E_2$ such that

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}"] & E_1 \arrow[r] \arrow[d, "\Phi"] & B \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & A \arrow[r] & E_2 \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

commutes. The set of congruence classes of extensions of B by A forms an abelian group that we shall denote by $\mathrm{Ext}(B, A)$. There is an abelian group isomorphism

$$\psi : \mathrm{Ext}(B, A) \to \frac{\mathrm{Hom}(B, A)_{\mathbb{C}}}{F^0\mathrm{Hom}(B, A)_{\mathbb{C}} + \mathrm{Hom}(B, A)_{\mathbb{Z}}}$$

that is given as follows. If

$$0 \longrightarrow A \xrightarrow{\ i\ } E \xrightarrow{\ p\ } B \longrightarrow 0$$

is an extension, choose a Hodge filtration preserving section $s_F : B \to E$ of p and a retraction $r_{\mathbb{Z}} : E \to A$ of i that is defined over $\mathbb{Z}$. Composing these gives an element $\psi(E) = r_{\mathbb{Z}} \circ s_F$ of $\mathrm{Hom}(B, A)_{\mathbb{C}}$. It can be checked that $\psi(E) = r_{\mathbb{Z}} \circ s_F$ is well-defined modulo $F^0\mathrm{Hom}(B, A)_{\mathbb{C}} + \mathrm{Hom}(B, A)_{\mathbb{Z}}$. It is easy to construct an inverse of ξ. For detail see [Ca].

When B is of weight $2p$, then $r_{\mathbb{Z}} \circ s_F$ also induces a homomorphism

$$\mu : B_{\mathbb{Z}}^{p,p} \to A_{\mathbb{C}}/F^pA + A_{\mathbb{Z}}$$

called the *motif* of the extension (cf. [Ca]) where $B_{\mathbb{Z}}^{p,p} = B_{\mathbb{Z}} \cap B^{p,p}$.

We can express the mixed Hodge structure on $(J/J^3)^*$ as an extension.

LEMMA 1.5. *Suppose that (X, x) is a path connected, pointed topological space. If $H_1(X, \mathbb{Z})$ is torsion-free, then there is an exact sequence*

$$0 \longrightarrow H_{\mathbb{Z}}^1(X) \xrightarrow{i} \mathrm{Hom}_{\mathbb{Z}}(J(X,x)/J^3, \mathbb{Z}) \xrightarrow{p} H_{\mathbb{Z}}^1(X) \otimes H_{\mathbb{Z}}^1(X) \longrightarrow H_{\mathbb{Z}}^2(X).$$

Here $i(z)(g-1) =< z, g >$, where $g \in \pi_1(X,x)$ and $z \in H^1(X)$. If $\phi \in (J/J^3)^*$ and α, β are loops based at x, then

$$p(\phi)([\alpha] \otimes [\beta]) =< \phi, (\{\alpha\} - 1)(\{\beta\} - 1) > .$$

Proof. See [H1].

§2. Fiber type arrangements

DEFINITION 2.1. Let $\mathcal{A}$ be 2-arrangement i.e. a finite set of hyperplanes in $\mathbb{C}^2$ and $M(\mathcal{A})$ a complement of $\mathcal{A}$ in $\mathbb{C}^2$. We call $\mathcal{A}$ *affine fiber type* if $\mathcal{A}$ is a set $\{H_1, \cdots, H_n, G_1, \cdots, G_m\}$ of hyperplanes in $\mathbb{C}^2$ satisfying the following conditions

(1) Each G_j is parallel to G_1 and each H_i is not parallel to G_1. (i.e. For each $i \neq j$, $G_i \cap G_j = \emptyset$ and for each k, l, $H_k \cap G_l \neq \emptyset$.)

(2) If $H_i \cap H_j \neq \emptyset$, then there exists unique G_k such that

$$H_i \cap H_j \subset G_k.$$

We assume that all H_i's are not parallel each other. (In this paper arrangements mean affine arrangements.), cf. see [FR] and [J].

Now we reconsider the extension of $(J/J^3)^*$. For a fiber type 2-arrangement $\mathcal{A}$ and a base point b of the complement $M = M(\mathcal{A})$ of $\mathcal{A}$, there is the extension

$$0 \longrightarrow H^1(M) \longrightarrow (J/J^3)^* \longrightarrow K \longrightarrow 0$$

of $(J/J^3)^*$ where J is the augmentation ideal of the group algebra of $\pi_1(M(\mathcal{A}), b)$ over $\mathbb{Z}$ and K is the kernel of the cup-product $H^1(M) \otimes H^1(M) \to H^2(M)$. Since the first cohomology is pure of weight 2 and the kernel K of its cup product is pure of weight 4, there is the extension isomorphism

$$\psi : \mathrm{Ext}(K, H^1) \to \mathrm{Hom}(K, H^1)_{\mathbb{C}}/\mathrm{Hom}(K, H^1)_{\mathbb{Z}}.$$

We shall give the description of $\psi((J/J^3)^*)$ for $(\mathcal{A}, b)$. First each basis of $H^1(M)$, $H_1(M)$, K and K^* can be given as follows.

As we take a coordinate (x, y) in $\mathbb{C}^2$, we can assume that

$$\mathcal{A} = \{H_1, \cdots, H_n, G_1, \cdots, G_m\},$$

where each H_i is defined by the equation $y = h_i(x)$ and each G_j is defined by the equation $g_j(x) = 0$ where h_i, g_j are linear polynomials in x. Set

$$\omega_i = \frac{1}{2\pi\sqrt{-1}} d\log(y - h_i(x)) \qquad 1 \leqq i \leqq n$$

and

$$\eta_j = \frac{1}{2\pi\sqrt{-1}} d\log(g_j(x)) \qquad 1 \leqq j \leqq m.$$

Brieskorn showed that the cohomology of the complement of hyperplanes $H_k = \{l_k = 0\}$ is generated by forms

$$w_k = \frac{1}{2\pi\sqrt{-1}} d\log(l_k),$$

in general [B](for detail, see [OT: 5.4]). Then ω_i's and η_j's generate the cohomology $H^*(M(\mathcal{A}), \mathbb{Z})$. Set

$$BH^1 = \{\omega_1, \cdots, \omega_n, \eta_1, \cdots, \eta_m\}.$$

Let $\mathcal{B}$ be the subarrangement $\{G_1 \cdots, G_m\}$ of $\mathcal{A}$. Fix a base point $b = (x_0, y_0)$ of $M(\mathcal{A})$. We put

$$F = \{(x_0, y) \in M(\mathcal{A})\} = \mathbb{C} - \{y_1, \cdots, y_n\}$$

and

$$B = \{(x, y_0) \in M(\mathcal{B})\} = \mathbb{C} - \{x_1, \cdots, x_m\},$$

where

$$y_i = h_i(x_0) \qquad 1 \leqq i \leqq n$$

and

$$x_j = \operatorname{Ker} g_j \qquad 1 \leqq j \leqq m.$$

Choose loops α_i, $1 \leqq i \leqq n$, based at y_0 in F such that, for each i, α_i is anti-clockwise around y_i and nullhomotopic in $\mathbb{C} - \{y_1, \cdots, \check{y}_i, \cdots, y_n\}$. In the same way we choose loops β_j, $1 \leqq j \leqq m$ based at x_0 in B. It is clear that

$$\int_{\alpha_i} \omega_j = \delta_{ij} \quad \int_{\beta_i} \eta_j = \delta_{ij} \quad \int_{\alpha_i} \eta_j = \int_{\beta_i} \omega_j = 0,$$

where δ_{ij} is Kronecker's delta. Consequently, $[\alpha_1], \cdots, [\alpha_n], [\beta_1], \cdots, [\beta_m]$ is the dual basis of $H_1(M(\mathcal{A}), \mathbb{Z})$. Set

$$BH_1 = \{\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_m\}.$$

Remark 2.2. In general, let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{C}^N$ and $M = M(\mathcal{A})$ the complement of $\mathcal{A}$. The mixed Hodge structure on the cohomology $H^i(M)$ is pure. Moreover any element of $H^i(M)$ has the Hodge type (i,i) (see [Sh]).

We hope to find the basis of K. In general, if a vector space H of dimension n has a basis $\tau_1, \cdots, \tau_n$, then we can choose the basis of $H \otimes H$ as

$$\frac{1}{2}[\tau_i, \tau_j] = \frac{1}{2}(\tau_i\tau_j - \tau_j\tau_i) \qquad 1 \leqq i < j \leqq n$$

$$\frac{1}{2}\{\tau_i, \tau_j\} = \frac{1}{2}(\tau_i\tau_j + \tau_j\tau_i) \qquad 1 \leqq i \leqq j \leqq n.$$

We put

$$[\omega]_{ij} = \begin{cases} [\omega_i, \omega_j] + [\omega_j, \eta_k] + [\eta_k, \omega_i] & \text{if } \emptyset \neq H_i \cap H_j \subset G_k \\ [\omega_i, \omega_j] & \text{if } H_i \cap H_j = \emptyset \end{cases}$$

$$[\eta]_{ij} = [\eta_i, \eta_j]$$

$$\{\omega\}_{ij} = \{\omega_i, \omega_j\}$$

$$\{\eta\}_{ij} = \{\eta_i, \eta_j\}.$$

We can obtain the following proposition.

Proposition 2.3.

$$BK_{\mathbb{Q}} = \left\{ \begin{array}{ll} \frac{1}{2}[\omega]_{ij} & 1 \leqq i < j \leqq n \\ \frac{1}{2}[\eta]_{ij} & 1 \leqq i < j \leqq m \\ \frac{1}{2}\{\omega\}_{ij} & 1 \leqq i \leqq j \leqq n \\ \frac{1}{2}\{\eta\}_{ij} & 1 \leqq i \leqq j \leqq m \\ \frac{1}{2}\{\omega_i, \eta_j\} & 1 \leqq i \leqq n, 1 \leqq j \leqq m \end{array} \right\}$$

is a basis of $K_{\mathbb{Q}}$.

Let $\mathcal{H} = \{F_1, \cdots, F_l\}$ be an arrangement with $F_i = ker\varphi_i$. We define a basis $\tau_1, \cdots, \tau_l$ of $H^1(M(\mathcal{H}))$ by $\tau_i = \frac{1}{2\pi\sqrt{-1}} d\log\varphi_i$. Let $Z_1, \cdots, Z_l$ be the dual basis of $H_1(M(\mathcal{H}))$. Denote the free associative algebra they generate by $\mathbb{C} < Z_1, \cdots, Z_l >$. We shall denote its augmentation ideal by I. The geometric lattice $L(\mathcal{H})$ consists of the subspaces of the form

$$F_{i_1} \cap \cdots \cap F_{i_p} \quad \text{where } \{i_1, \cdots, i_p\} \subset \{1, \cdots, l\}.$$

Let $L^2(\mathcal{H})$ be a set of codimension-two elements of $L(\mathcal{H})$. For $K \in L^2(\mathcal{H})$, we set $\mathcal{H}_K = \{F_i \in \mathcal{H} | K \subset F_i\}$ and define the relation ideal R_K of $\mathbb{C} < Z_1, \cdots, Z_l >$ by

$$[Z_{i_\nu}, Z_{i_1} + \cdots + Z_{i_p}] = 0 \qquad 1 \leqq \nu \leqq p,$$

where $\mathcal{H}_K = \{F_{i_1}, \cdots, F_{i_p}\}$. Let $R = (R_K)_{K \in L^2(\mathcal{H})}$ and define

$$A_s = \mathbb{C} < Z_1, \cdots, Z_l > /R + I^{s+1}.$$

Set

$$\omega = \omega_1 Z_1 + \cdots + \omega_l Z_l \qquad \in H^1(M(\mathcal{H})) \otimes A_s.$$

The relations guarantee that $\omega \wedge \omega = 0$. For detail, see [K1,2,3].

Proof of Proposition 2.3. Let $\mathcal{A}$ be the fiber type 2-arrangement and $M = M(\mathcal{A})$ its complement. We take the basis $\omega_1, \cdots, \omega_n, \eta_1, \cdots, \eta_m$ of $H^1(M)$ and let $X_1, \cdots, X_n, Y_1, \cdots, Y_m$ be a dual basis of $H_1(M)$. Set

$$A_s = \mathbb{C} < X_i, Y_j > /R + I^{s+1}$$

and

$$\tau = \sum \omega_i X_i + \sum \eta_j Y_j,$$

where R is the relation ideal for $\mathcal{A}$. Since $\mathcal{A}$ is fiber type, for $K \in L^2(\mathcal{A})$, we can write

$$\mathcal{A}_K = \{H_{i_1}, \cdots, H_{i_p}, G_{j_K}\}$$

and its relation is

$$[X_{i_\nu}, X_{i_1} + \cdots + X_{i_p} + Y_{j_K}] = 0 \qquad 1 \leqq \nu \leqq p$$

$\Longleftrightarrow$

$$[X_{i_\nu}, Y_{j_K}] = \sum_{\mu=1}^{p} [X_{i_\mu}, X_{i_\nu}] \qquad 1 \leqq \nu \leqq p.$$

Hence, any $[X_i, Y_j]$ is generated by $[X_i, X_j]$'s. For $\emptyset \neq H_i \cap H_j \subset G_k$, there exists uniquely $K_{ij} \in L^2(\mathcal{A})$ such that $H_i, H_j, G_k \in \mathcal{A}_{K_{ij}}$, and we set

$$\mathcal{A}_{K_{ij}} = \{H_i, H_j, G_k, H_{i_1}, \cdots, H_{i_p}\},$$

where $i, j, i_1, \cdots, i_p$ are distinct each other. We can find the relations including terms $[X_i, X_j]$;

$$\begin{aligned} [X_i, Y_k] &= [X_j, X_i] + \sum_{\mu=1}^{p} [X_{i_\mu}, X_i] \\ &= -[X_i, X_j] + \sum_{\mu=1}^{p} [X_{i_\mu}, X_i] \end{aligned}$$

and

$$[X_j, Y_k] = [X_i, X_j] + \sum_{\mu=1}^{p} [X_{i_\mu}, X_j].$$

Any other relation has no terms of $[X_i, X_j]$. Therefore, the coefficient of $[X_i, X_j]$ in $\tau \cdot \tau$ is

$$\frac{1}{2}([\omega_i, \omega_j] - [\eta_k, \omega_j] + [\eta_k, \omega_i]) = \frac{1}{2}[\omega]_{ij}.$$

If $H_i \cap H_j = \emptyset$, then there is no such relation. The coefficient of $[X_i, X_j]$ is only $\frac{1}{2}[X_i, X_j]$. Consequently we can write

$$\begin{aligned} \tau \cdot \tau = & \sum_{i<j} \frac{1}{2}[\omega]_{ij}[X_i, X_j] + \sum_{i<j} \frac{1}{2}[\eta]_{ij}[Y_i, Y_j] \\ & + \sum_{i \leqq j} \frac{1}{2}\{\omega\}_{ij}\{X_i, X_j\} + \sum_{i \leqq j} \frac{1}{2}\{\eta\}_{ij}\{Y_i, Y_j\} + \sum_{i,k} \frac{1}{2}\{\omega_i, \eta_k\}\{X_i, Y_k\}. \end{aligned}$$

Thus $\frac{1}{2}[\omega]_{ij}, \frac{1}{2}[\eta]_{ij}, \frac{1}{2}\{\ ,\ \}$ is independent in $H^1 \otimes H^1$. Since $\dim(H^2(M)) = mn$, it is a basis of K whose dimension is $(m+n)^2 - mn$.

□

Then $\{(\gamma - 1) | \gamma \in BH_1\}$ is a basis of $J/J^2 \cong H_1$ and BH^1 is its dual basis of $(J/J^2)^* \cong H^1$. Moreover $BK_{\mathbb{Q}}$ is a basis of $K \cong (J^2/J^3)^*$. Set

$$BK_{\mathbb{Q}}^* = \left\{ \begin{array}{ll} [\alpha_i - 1, \alpha_j - 1] & 1 \leqq i < j \leqq n \\ [\beta_i - 1, \beta_j - 1] & 1 \leqq i < j \leqq m \\ \{\alpha_i - 1, \alpha_j - 1\} & 1 \leqq i \leqq j \leqq n \\ \{\alpha_i - 1, \beta_j - 1\} & 1 \leqq i \leqq n, 1 \leqq j \leqq m \end{array} \right\}.$$

LEMMA 2.4. *$BK_{\mathbb{Q}}^*$ is a dual basis of $J^2/J^3 \cong K_{\mathbb{Q}}^*$.*

Proof. Since α_i, β_j are dual of ω_i, η_j respectively, it is enough to prove the following lemma.

LEMMA 2.5. *Let M be a smooth manifold, τ_1, τ_2 smooth 1-forms on M and γ_1, γ_2 loops based at $x \in M$. Then*

(1) $< \int \tau_1 \tau_2\ ,\ (\gamma_1 - 1)(\gamma_2 - 1) >= \int_{\gamma_1} \tau_1 \cdot \int_{\gamma_2} \tau_2$.

(2) $< \int \frac{1}{2}[\tau_1, \tau_2]\ ,\ [\gamma_1 - 1, \gamma_2 - 1] >= \int_{\gamma_1} \tau_1 \cdot \int_{\gamma_2} \tau_2 - \int_{\gamma_2} \tau_1 \cdot \int_{\gamma_1} \tau_2$.

(3) $< \int \frac{1}{2}\{\tau_1, \tau_2\}\ ,\ \{\gamma_1 - 1, \gamma_2 - 1\} >= \int_{\gamma_1} \tau_1 \cdot \int_{\gamma_2} \tau_2 + \int_{\gamma_2} \tau_1 \cdot \int_{\gamma_1} \tau_2$.

(4) $< \int \frac{1}{2}[\tau_1, \tau_2]\ ,\ \{\gamma_1 - 1, \gamma_2 - 1\} >=< \int \frac{1}{2}\{\tau_1, \tau_2\}\ ,\ [\gamma_1 - 1, \gamma_2 - 1] >= 0.$

Proof. (1) See [H1; (2.13) (b)].

(2) Using (1), we get

$$\begin{aligned} < \int \frac{1}{2}[\tau_1, \tau_2], [\gamma_1 - 1, \gamma_2 - 1] > &= \frac{1}{2}\{\int_{\gamma_1} \tau_1 \cdot \int_{\gamma_2} \tau_2 - \int_{\gamma_1} \tau_2 \cdot \int_{\gamma_2} \tau_1\} \\ &\quad - \frac{1}{2}\{\int_{\gamma_2} \tau_1 \cdot \int_{\gamma_1} \tau_2 - \int_{\gamma_2} \tau_1 \cdot \int_{\gamma_1} \tau_2\} \\ &= \int_{\gamma_1} \tau_1 \cdot \int_{\gamma_2} \tau_2 - \int_{\gamma_2} \tau_1 \cdot \int_{\gamma_1} \tau_2. \end{aligned}$$

Also we obtain (3) and (4) in the similar way. □

It is note that $BK_{\mathbb{Q}}$ is not a basis of $K_{\mathbb{Z}}$. Though we can choose the following basis of $K_{\mathbb{Z}}$. Set

$$\omega_{ij} = \begin{cases} \omega_i\omega_j - \eta_k\omega_j + \eta_k\omega_i & \text{if } \emptyset \neq H_i \cap H_j \subset G_k \\ \omega_i\omega_j & \text{if } H_i \cap H_j = \emptyset \end{cases}$$

$$\eta_{ij} = \eta_i\eta_j.$$

COROLLARY 2.6.

$$BK_{\mathbb{Z}} = \left\{ \begin{array}{ll} \omega_{ij} & 1 \leqq i < j \leqq n \\ \eta_{ij} & 1 \leqq i < j \leqq m \\ \{\omega\}_{ij} & 1 \leqq i < j \leqq n \\ \{\eta\}_{ij} & 1 \leqq i < j \leqq m \\ \{\omega_i, \eta_j\} & 1 \leqq i \leqq n, 1 \leqq j \leqq m \\ \omega_{ii} = \omega_i\omega_i & 1 \leqq i \leqq n \\ \eta_{ii} = \eta_i\eta_i & 1 \leqq i \leqq m \end{array} \right\}$$

is a basis of $K_{\mathbb{Z}}$. And also

$$BK_{\mathbb{Z}}^* = \left\{ \begin{array}{ll} [\alpha_i - 1, \alpha_j - 1] & 1 \leqq i < j \leqq n \\ [\beta_i - 1, \beta_j - 1] & 1 \leqq i < j \leqq m \\ (\alpha_j - 1)(\alpha_i - 1) & 1 \leqq i < j \leqq n \\ (\beta_j - 1)(\beta_i - 1) & 1 \leqq i < j \leqq m \\ (\alpha_i - 1)(\beta_j - 1) & 1 \leqq i \leqq n, 1 \leqq j \leqq m \\ (\alpha_i - 1)(\alpha_i - 1) & 1 \leqq i \leqq n \\ (\beta_i - 1)(\beta_i - 1) & 1 \leqq i \leqq m \end{array} \right\}$$

is its dual basis of $K_{\mathbb{Z}}^*$.

Proof. From Proposition 2.3 it is clear that $BK_{\mathbb{Z}}$ is a basis of $K_{\mathbb{Z}}$. Using lemma 2.5, we can check its duality. For example,

$$< \int \omega_{ij}, [\alpha_i - 1, \alpha_j - 1] >= \int_{\alpha_i} \omega_i \int_{\alpha_j} \omega_j - \int_{\alpha_j} \omega_i \int_{\alpha_i} \omega_j = 1$$

$$< \int \omega_{ij}, (\alpha_j - 1)(\alpha_i - 1) >= \int_{\alpha_j} \omega_i \int_{\alpha_i} \omega_j = 0$$

$$< \int \omega_{ij}, (\alpha_s - 1)(\beta_t - 1) >= \int_{\alpha_s} \omega_i \int_{\beta_t} \omega_j - \int_{\alpha_s} \eta_k \int_{\beta_t} \omega_j + \int_{\alpha_s} \eta_k \int_{\beta_t} \omega_i = 0.$$

$\square$

Consequently, for $(\mathcal{A}, b)$, we give the description of $\psi((J/J^3)^*)$ by

$$\psi((J/J^3)^*) = \sum_{\chi \in BK_{\mathbb{Z}}^*, \gamma \in BH_1} I(\chi : \gamma) \ \chi \otimes (\gamma - 1)^* \mathrm{mod}(K^* \otimes H_1^*)_{\mathbb{Z}}$$

$$\in (K^* \otimes H_1^*)_{\mathbb{C}} / (K^* \otimes H_1^*)_{\mathbb{Z}} \cong \mathrm{Hom}(K, H^1)_{\mathbb{C}} / \mathrm{Hom}(K, H^1)_{\mathbb{Z}}$$

where, since a dual of χ is $\chi^* \in BK_{\mathbb{Z}}$, we define

$$I(\chi : \gamma) = \int_\gamma \chi^*.$$

§3. The cross ratio equivalence

We compute the description of $\psi((J/J^3)^*)$ for $(\mathcal{A}, b)$. First we prepare the following lemma.

LEMMA 3.1. *Set*

$$\tau_1 = \frac{1}{2\pi\sqrt{-1}} \frac{dz}{z - z_1}, \quad \tau_2 = \frac{1}{2\pi\sqrt{-1}} \frac{dz}{z - z_2}.$$

Suppose that, for $i = 1, 2$, γ_i is a loop based at z_0 in $\mathbb{C}$ anti-clockwisely around z_i which is nullhomotopic in $\mathbb{C} - \{z_j\}$, $j \neq i$. Then

$$\int_{\gamma_1} \frac{1}{2}[\tau_1, \tau_2] = \frac{1}{2\pi\sqrt{-1}} \log(\lambda)$$

$$\int_{\gamma_2} \frac{1}{2}[\tau_1, \tau_2] = \frac{1}{2\pi\sqrt{-1}} \log((1 - \lambda)^{-1}).$$

Here λ is the cross ratio

$$\lambda = [z_0, z_1, z_2, \infty] = \frac{z_0 - z_2}{z_1 - z_2},$$

and also

$$(1 - \lambda)^{-1} = [z_2, z_0, z_1, \infty] = \frac{z_2 - z_1}{z_0 - z_1}.$$

Proof. Since

$$\int_{\gamma_1} \frac{dz}{z - z_1} \frac{dz}{z - z_2} = 2\pi\sqrt{-1} \log(\frac{z_0 - z_2}{z_1 - z_2})$$

and

$$\int_{\gamma_2} \frac{dz}{z - z_1} \frac{dz}{z - z_2} = 2\pi\sqrt{-1} \log(\frac{z_1 - z_2}{z_0 - z_2}),$$

then

$$\begin{aligned}\int_{\gamma_1}[\tau_1,\tau_2] &= \frac{1}{2\pi\sqrt{-1}}(\log(\frac{z_0-z_2}{z_1-z_2})-\log(\frac{z_1-z_2}{z_0-z_2}))\\ &= \frac{1}{2\pi\sqrt{-1}}(\log(\frac{z_0-z_2}{z_1-z_2})^2)\\ &= 2\cdot\frac{1}{2\pi\sqrt{-1}}\log(\frac{z_0-z_2}{z_1-z_2}).\end{aligned}$$

From $[\tau_1,\tau_2]=-[\tau_2,\tau_1]$, we get

$$\begin{aligned}\int_{\gamma_2}[\tau_1,\tau_2] &= -\int_{\gamma_2}[\tau_2,\tau_1]\\ &= -2\cdot\frac{1}{2\pi\sqrt{-1}}\log(\frac{z_0-z_1}{z_2-z_1})\\ &= 2\cdot\frac{1}{2\pi\sqrt{-1}}\log(\frac{z_2-z_1}{z_0-z_1}).\end{aligned}$$

□

COROLLARY 3.2. *Let* $a_1,a_2\neq 0$ *and*

$$\tau_1=\frac{1}{2\pi\sqrt{-1}}\frac{a_1dz}{a_1z+b_1},\quad \tau_2=\frac{1}{2\pi\sqrt{-1}}\frac{a_1dz}{a_2z+b_2}.$$

And γ_1 *is a loop based at* z_0 *anti-clockwisely around* $(-\frac{b_1}{a_1})$ *which is nullhomotopic in* $\mathbb{C}-\{-\frac{b_2}{a_2}\}$. *Then*

$$\int_{\gamma_1}\frac{1}{2}[\tau_1,\tau_2]=\frac{1}{2\pi\sqrt{-1}}\log(\frac{a_1(a_2z_0+b_2)}{a_1b_2-b_1a_2}).$$

Proof. Set $z_1=-b_1/a_1$, $z_2=-b_2/a_2$. Applying to Lemma 5.1, we obtain

$$\exp(2\pi\sqrt{-1}\int_{\gamma_1}\frac{1}{2}[\tau_1,\tau_2])=\frac{z_0-z_2}{z_1-z_2}=\frac{z_0+b_2/a_2}{-b_1/a_1+b_2/a_2}=\frac{a_1(a_2z_0+b_2)}{a_1b_2-b_1a_2}.$$

□

Using this lemma, we can get the following proposition.

PROPOSITION 3.3.

(1) *For each* $1 \leqq i < j \leqq n$ *and* $\gamma \in BH_1$,

$$\int_\gamma \frac{1}{2}[\omega]_{ij} = \begin{cases} l(\lambda_{ij}) & \text{if } \gamma = \alpha_i \\ l((1-\lambda_{ij})^{-1}) & \text{if } \gamma = \alpha_j \\ l(\lambda_{ij}^{-1}(\lambda_{ij}-1)) & \text{if } \emptyset \neq H_i \cap H_j \subset G_k \text{ and } \gamma = \beta_k \\ 0 & \text{otherwise }. \end{cases}$$

(2) *For each* $1 \leqq i < j \leqq m$ *and* $\gamma \in BH_1$,

$$\int_\gamma \frac{1}{2}[\eta]_{ij} = \begin{cases} l(\lambda'_{ij}) & \text{if } \gamma = \beta_i \\ l((1-\lambda'_{ij})^{-1}) & \text{if } \gamma = \beta_j \\ 0 & \text{otherwise }. \end{cases}$$

(3) *For each* $1 \leqq i \leqq j \leqq n$, $1 \leqq k \leqq l \leqq m$ *and* $\gamma \in BH_1$,

$$\int_\gamma \frac{1}{2}\{\omega\}_{ij} = \begin{cases} \frac{1}{2} & \text{if } i = j \text{ and } \gamma = \alpha_i \\ 0 & \text{otherwise }, \end{cases}$$

$$\int_\gamma \frac{1}{2}\{\eta\}_{kl} = \begin{cases} \frac{1}{2} & \text{if } k = l \text{ and } \gamma = \beta_k \\ 0 & \text{otherwise }. \end{cases}$$

For each $1 \leqq i \leqq n$, $1 \leqq j \leqq m$ *and* $\gamma \in BH_1$,

$$\int_\gamma \frac{1}{2}\{\omega_i, \eta_j\} = 0.$$

Here

$$l(z) = \frac{1}{2\pi\sqrt{-1}} \log(z),$$

$$\lambda_{ij} = [y_0, y_i, y_j, \infty] = \frac{y_0 - y_j}{y_i - y_j},$$

and

$$\lambda'_{ij} = [x_0, x_i, x_j, \infty] = \frac{x_0 - x_j}{x_i - x_j}.$$

Proof. **(1)** $\frac{1}{2}[\omega]_{ij}$ **with** $H_i \cap H_j = \emptyset$**.** For each $1 \leqq s \leqq n$,

$$\int_{\alpha_s} \omega_i \omega_j = \int_{\alpha_s} \frac{dy - dh_i(x)}{y - h_i(x)} \frac{dy - dh_j(x)}{y - h_j(x)} = \int_{\alpha_s} \frac{dy}{y - y_i} \frac{dy}{y - y_j}.$$

Applying to Lemma 3.1, we get

$$\int_{\alpha_i} \frac{1}{2}[\omega]_{ij} = l([y_0, y_i, y_j, \infty]) = l(\lambda_{ij})$$

and

$$\int_{\alpha_j} \frac{1}{2}[\omega]_{ij} = l([y_j, y_0, y_i, \infty]) = l((1 - \lambda_{ij})^{-1}).$$

(1) $\frac{1}{2}[\omega]_{ij}$ **with** $\emptyset \neq H_i \cap H_j \subset G_k$**.** For each $1 \leqq s, t \leqq n$ and each $1 \leqq u \leqq m$, since

$$\int_{\alpha_s} [\omega_t, \eta_u] = \int_{\alpha_s} \omega_t \eta_u - \int_{\alpha_s} \eta_u \omega_t = \int_{\alpha_s} \omega_t - \int_{\alpha_s} \omega_t = 0,$$

we get

$$\int_{\alpha_s} [\omega]_{ij} = \int_{\alpha_s} [\omega_i, \omega_j].$$

Applying to Lemma 3.1, we obtain

$$\int_{\alpha_i} \frac{1}{2}[\omega]_{ij} = l([y_0, y_i, y_j, \infty]) = l(\lambda_{ij})$$

and

$$\int_{\alpha_j} \frac{1}{2}[\omega]_{ij} = l([y_j, y_0, y_i, \infty]) = l((1 - \lambda_{ij})^{-1}).$$

For each $1 \leqq l \leqq m$,

$$\int_{\beta_l} [\omega]_{ij} = \int_{\beta_l} [\omega_i, \omega_j] + \int_{\beta_l} ([\omega_j, \eta_k] + [\eta_k, \omega_i]) = \int_{\beta_l} [\omega_j, \eta_k] + \int_{\beta_l} [\eta_k, \omega_i]$$

and for $l \neq k$

$$\int_{\beta_l} [\omega]_{ij} = 0.$$

Then it is enough to compute

$$\int_{\beta_k} [\eta_k, \omega_i].$$

Set

$$\begin{aligned} H_i: &\quad y = a_i x + b_i \\ H_j: &\quad y = a_j x + b_j \\ G_k: &\quad (a_i - a_j)x + (b_i - b_j) = 0. \end{aligned}$$

By means of Corollary 3.2 we get

$$\int_{\beta_k} \frac{1}{2}[\eta_k, \omega_i] = \int_{\beta_k} \frac{1}{2}[\frac{1}{2\pi\sqrt{-1}} \frac{(a_i - a_j)dx}{(a_i - a_j)x + (b_i - b_j)}, \frac{1}{2\pi\sqrt{-1}} \frac{a_i dx}{a_i x + b_i - y_0}]$$

$$= \frac{1}{2\pi\sqrt{-1}} \log(\frac{(a_i - a_j)(a_i x_0 + b_i - y_0)}{(a_i - a_j)(b_i - y_0) - a_i(b_i - b_j)})$$

$$= \frac{1}{2\pi\sqrt{-1}} \log(\frac{(a_i - a_j)(y_i - y_0)}{a_i b_j - (a_i - a_j)y_0}),$$

and, in the same way,

$$\int_{\beta_k} \frac{1}{2}[\eta_k, \omega_j] = \frac{1}{2\pi\sqrt{-1}} \log(\frac{(a_i - a_j)(y_j - y_0)}{a_i b_j - (a_i - a_j)y_0}).$$

Consequently we obtain

$$\int_{\beta_k} \frac{1}{2}\omega_{ij} = l(\frac{(a_i - a_j)(y_i - y_0)}{a_i b_j - (a_i - a_j)y_0} \cdot \frac{a_i b_j - (a_i - a_j)y_0}{(a_i - a_j)(y_i - y_0)})$$

$$= l(\frac{y_i - y_0}{y_j - y_0}) = l([y_i, y_0, y_j, \infty]) = l(\lambda_{ij}^{-1}(\lambda_{ij} - 1)).$$

(2) $\frac{1}{2}[\eta]_{ij}$**.** We apply $\eta_i, \eta_j, \beta_i, \beta_j$ to Lemma 3.1 and then get

$$\int_{\beta_i} \frac{1}{2}\eta_{ij} = l([x_0, x_i, x_j, \infty]) = l(\lambda'_{ij})$$

and

$$\int_{\beta_j} \frac{1}{2}\eta_{ij} = l([x_j, x_0, x_i, \infty]) = l((1 - \lambda'_{ij})^{-1}).$$

(3) $\frac{1}{2}\{\ ,\ \}$**.** In general, the shuffle formula of iterated integrals (see [H1; (2.11)]) gives

$$\int_\gamma \{\omega, \eta\} = \int_\gamma \omega\eta + \int_\gamma \eta\omega = \int_\gamma \omega \int_\gamma \eta.$$

By the duality between ω_i, η_k, and $[\alpha_i]$, $[\beta_k]$, for any braces basis $\frac{1}{2}\{\omega,\eta\}$ of K with $\omega\neq\eta$ and $[\gamma]\in\{[\alpha_i],[\beta_k]\}$, we get

$$\int_\gamma \frac{1}{2}\{\omega,\eta\}=0.$$

For any braces base $\frac{1}{2}\{\omega,\omega\}$ of K and a dual base $[\gamma]$ of ω, we get

$$\int_\gamma \frac{1}{2}\{\omega,\omega\}=\frac{1}{2}(\int_\gamma \omega)^2=\frac{1}{2}.$$

And also for $[\gamma']\neq[\gamma]$,

$$\int_{\gamma'} \frac{1}{2}\{\omega,\omega\}=0.$$

□

COROLLARY 3.4. *The extension isomorphism ψ associated with the mixed Hodge structure on $(J/J^3)^*$ is*

$$\begin{aligned}\psi((J/J^3)^*) \;=\; & \sum_{(i,j,k)\in C} l(\lambda_{ij})[\alpha_i-1,\alpha_j-1]\otimes(\alpha_i-1)^*\\ & \quad + l((1-\lambda_{ij})^{-1})[\alpha_i-1,\alpha_j-1]\otimes(\alpha_j-1)^*\\ & \quad + l(\lambda_{ij}^{-1}(\lambda_{ij}-1))[\alpha_i-1,\alpha_j-1]\otimes(\beta_k-1)^*\\ + & \sum_{(i,j)\in P} l(\lambda_{ij})[\alpha_i-1,\alpha_j-1]\otimes(\alpha_i-1)^*\\ & \quad + l((1-\lambda_{ij})^{-1})[\alpha_i-1,\alpha_j-1]\otimes(\alpha_j-1)^*\\ + & \sum_{1\leqq i<j\leqq m} l(\lambda'_{ij})[\beta_i-1,\beta_j-1]\otimes(\beta_i-1)^*\\ & \quad + l((1-\lambda'_{ij})^{-1})[\beta_i-1,\beta_j-1]\otimes(\beta_j-1)^*\\ & \qquad \bmod (K^*\otimes H_1^*)_{\mathbb{Z}}\end{aligned}$$

where

$$C=C(\mathcal{A})=\{(i,j,k)|1\leqq i<j\leqq n,\emptyset\neq H_i\cap H_j\subset G_k\}$$

and

$$P=P(\mathcal{A})=\{(i,j)|1\leqq i<j\leqq n, H_i\cap H_j=\emptyset\}.$$

Proof. In order to prove this we compute its motif. According to §2, there is an extension

$$0 \longrightarrow H^1(M) \longrightarrow (J(M,b)/J^3)^* \longrightarrow K \longrightarrow 0$$

of $(J(M,b)/J^3)^*$. Since $H^1(M)$ is a pure Hodge structure of weight 2 its motif is

$$\mu : K_{\mathbb{Z}} \to H^1_{\mathbb{C}}(M)/H^1_{\mathbb{Z}}(M).$$

The exponential map gives a canonical identification

$$H^1_{\mathbb{C}}(M)/H^1_{\mathbb{Z}}(M) \to (\mathbb{C}^*)^n \times (\mathbb{C}^*)^m$$

$$\sum a_i\omega_i + \sum b_j\eta_j \to (\exp 2\pi\sqrt{-1}a_i, \exp 2\pi\sqrt{-1}b_j).$$

Thus, for $z_1 \otimes z_2 \in K$ and $[\gamma] \in H_1(M)$,

$$\mu(z_1 \otimes z_2)[\gamma] = \exp 2\pi\sqrt{-1}\int_\gamma z_1z_2.$$

$$\omega_{ij} = \begin{cases} \frac{1}{2}[\omega]_{ij} + \frac{1}{2}\{\omega\}_{ij} + \frac{1}{2}\{\omega_i, \eta_k\} + \frac{1}{2}\{\omega_j, \eta_k\} & \text{if } \emptyset \neq H_i \cap H_j \subset G_k \\ \frac{1}{2}[\omega]_{ij} + \frac{1}{2}\{\omega\}_{ij} & \text{if } H_i \cap H_j = \emptyset \end{cases}$$

and

$$\eta_{ij} = \frac{1}{2}[\eta]_{ij} + \frac{1}{2}\{\eta\}_{ij}.$$

Then, using the proposition 3.3, we can compute values of a basis $BK_{\mathbb{Z}}$ of $K_{\mathbb{Z}}$ obtained in §2. ∎

Remark 3.5. Using

$$\int_\gamma \omega\eta + \int_\gamma \eta\omega = \int_\gamma \omega \int_\gamma \eta,$$

we get

$$\int_\gamma \frac{1}{2}[\omega, \eta] = \int_\gamma \omega\eta - \frac{1}{2}\int_\gamma \omega \int_\gamma eta.$$

It leads

$$\int_{\alpha_i} \omega_i\omega_j = \int_{\alpha_i} \frac{1}{2}[\omega_i, \omega_j] = -\int_{\alpha_i} \omega_j\omega_i.$$

From this fact and the above proposition we also obtain the corollary.

This corollary leads the following definition and theorem.

DEFINITION 3.6. Let $\mathcal{A}$ be a fiber type 2-arrangement and b a base point of $M(\mathcal{A})$. A pair $(\mathcal{A}, b)$ is called a *pointed fiber type 2-arrangement*. Two pointed fiber type 2-arrangements $(\mathcal{A}, b)$ and $(\mathcal{A}', b')$ are *cross ratio equivalent* if there is a one-to-one correspondence between $\mathcal{A}$ and $\mathcal{A}'$ satisfying following conditions (1), (2) and (3); suppose that

$$\mathcal{A} = \{H_1, \cdots, H_n, G_1, \cdots, G_m\}$$

and

$$\mathcal{A}' = \{H_1', \cdots, H_n', G_1', \cdots, G_m'\},$$

where H_i', G_j' are corresponded to H_i, G_j, respectively.

(1) $H_i \cap H_j = \emptyset \Longleftrightarrow H_i' \cap H_j' = \emptyset$

(2) $\emptyset \neq H_i \cap H_j \subset G_k \Longleftrightarrow \emptyset \neq H_i' \cap H_j' \subset G_k'$

(3)
$$\lambda_{ij}(\mathcal{A}, b) = \lambda_{ij}(\mathcal{A}', b')$$
and
$$\lambda_{ij}'(\mathcal{A}, b) = \lambda_{ij}'(\mathcal{A}', b')$$

Consequently we obtain the following theorem.

THEOREM 3.7. *Let $(\mathcal{A}, b)$ and $(\mathcal{A}', b')$ be pointed fiber type 2-arrangements. If there is a ring isomorphism*

$$\varphi : \mathbb{Z}\pi_1(M(\mathcal{A}), b)/J^3 \to \mathbb{Z}\pi_1(M(\mathcal{A}'), b')/J^3$$

which induces an isomorphism of mixed Hodge structures, then $(\mathcal{A}, b)$ and $(\mathcal{A}', b')$ are cross ratio equivalent.

Proof. The isomorphism φ of mixed Hodge structures induces an isomorphism of Hodge structures on $W_{-1}/W_{-2} = J/J^2 \cong H_1$, on H^1 and on $K = \mathrm{Ker}(H^1 \otimes H^1 \to H^2)$. Thus there is a one-to-one correspondence between $\mathcal{A}$ and $\mathcal{A}'$ satisfying conditions (1), (2). And also it induces a congruent class of extensions

$$0 \longrightarrow H^1(M) \longrightarrow (J/J^3)^* \longrightarrow K \longrightarrow 0$$

of K by H^1. Hence, according to Corollary 3.4, we get $\lambda_{ij}(\mathcal{A}, b) = \lambda_{ij}(\mathcal{A}', b')$ and $\lambda_{ij}'(\mathcal{A}, b) = \lambda_{ij}'(\mathcal{A}', b')$. ∎

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