W. Chen, G. Hu and S. Lu
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CRITERION OF (L^p, L^r) BOUNDEDNESS FOR A CLASS OF MULTILINEAR OSCILLATORY SINGULAR INTEGRALS

WENGU CHEN, GUOEN HU AND SHANZHEN LU

Abstract. In this paper, we consider a kind of multilinear operators related to oscillatory singular integrals with rough kernels and give a criterion of certain boundedness for this kind of operators.

§1. Introduction

During the last decade, there has been significant progress in the study of oscillatory singular integral operators with polynomial phases. A prototypical work in this area is Ricci and Stein's paper [8]. Suppose that $K(x)$ is a function defined on $\mathbf{R}^n \backslash \{0\}$ such that

(i) $K(x)$ is homogeneous of $-n$,

(ii) $\displaystyle\int_{R_1<|x|<R_2} K(x)dx = 0,\ 0 < R_1 < R_2 < \infty.$

Ricci and Stein showed that for real-valued polynomial $P(x,y)$ defined on $\mathbf{R}^n \times \mathbf{R}^n$, if $K(x) \in C^1(\mathbf{R}^n \backslash \{0\})$, then the operator

$$Tf(x) = p.v. \int_{\mathbf{R}^n} e^{iP(x,y)} K(x-y) f(y) dy, \tag{1.1}$$

is bounded on $L^p(\mathbf{R}^n), 1 < p < \infty$, with bound depending only on the total degree of $P(x,y)$, not on the coefficients of $P(x,y)$. Subsequently, Chanillo and Christ [1] showed that $K(x) \in C^1(\mathbf{R}^n \backslash \{0\})$ is also a sufficient condition such that T is of weak type $(1,1)$. Lu and Zhang [7] found out a simple criterion on L^p-boundedness for oscillatory singular integrals with polynomial phases when the kernels satisfy only a size conditions.

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This paper is a continuation of our previous work [2], [3]. We shall extend above result of [7] to the case of multilinear oscillatory singular integral operators. Let us consider the following multilinear operators

$$(1.2)\quad T_{A_1,A_2}f(x) = \int_{\mathbf{R}^n} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+M-1}} \prod_{j=1}^{2} R_{m_j}(A_j;x,y)f(y)dy,\ n \geq 2,$$

where $M = m_1 + m_2$, Ω is homogeneous of degree zero, $R_m(A;x,y)$ denotes the m-th order Taylor series remainder of A at x expanded about y, more precisely

$$R_m(A;x,y) = A(x) - \sum_{|\alpha|<m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha.$$

For functions A_1 and A_2, one has derivatives of order $m_1 - 1$ in $\mathrm{BMO}(\mathbf{R}^n)$, another has derivatives of order m_2 in $L^{r_0}, 1 < r_0 \leq \infty$. We will give a criterion of (L^p, L^r) boundedness for T_{A_1,A_2}.

To begin with, let us introduce two concepts (see [7]).

DEFINITION 1. A real valued polynomial $P(x,y)$ is called non–trivial if $P(x,y)$ does not take the form of $P_0(x) + P_1(y)$, where P_0 and P_1 are polynomials defined on $\mathbf{R}^n$.

DEFINITION 2. We will say that the non–trivial polynomial $P(x,y)$ has property $\mathcal{P}$, if P satisfies

$$P(x+h, y+h) = P(x,y) + R_0(x,h) + R_1(y,h),$$

where R_0 and R_1 are real polynomials.

DEFINITION 3. We say that a non–trivial polynomial $P(x,y)$ is non–degenerate if

$$P(x,y) = \sum_{|\alpha|\leq k, |\beta|\leq l} a_{\alpha\beta} x^\alpha y^\beta,\ k,\ l \quad \text{are two positive integers}$$

$$\text{and} \quad \sum_{|\alpha|=k, |\beta|=l} |a_{\alpha\beta}| > 0.$$

Now we formulate our main result.

THEOREM 1. *Let* Ω *be homogeneous of degree zero and belong to* $L^q(S^{n-1})$ *for some* $q > 1$. *If* A_1 *has derivatives of order* m_1-1 *in* $\mathrm{BMO}(\mathbf{R}^n)$, A_2 *has derivatives of order* m_2 *in* $L^{r_0}, 1 < r_0 \leq \infty$, *then for* $1/r = 1/p + 1/r_0, 1 < p, r < \infty$, *the following two facts are equivalent:*

(i) *If* $P(x, y)$ *is a non-degenerate real-valued polynomial, then* T_{A_1,A_2} *is bounded from* L^p *to* L^r *with bound*

$$C(\mathrm{deg}P, n)(\sum_{|\alpha|=m_1-1} \|D^\alpha A_1\|_{\mathrm{BMO}})(\sum_{|\beta|=m_2} \|D^\beta A_2\|_{r_0});$$

(ii) *The truncated operator*

$$S_{A_1,A_2}f(x) = \int_{|x-y|<1} \frac{\Omega(x-y)}{|x-y|^{n+M-1}} \prod_{j=1}^{2} R_{m_j}(A_j; x, y) f(y) dy$$

is bounded from L^p *to* L^r *with bound*

$$C(\sum_{|\alpha|=m_1-1} \|D^\alpha A_1\|_{\mathrm{BMO}})(\sum_{|\beta|=m_2} \|D^\beta A_2\|_{r_0});$$

where degP denotes the total degree of the polynomial $P(x, y)$.

§2. Proof of Theorem 1

To prove Theorem 1, we will use some lemmas.

LEMMA 1. (see [4]) *Let* $b(x)$ *be a function on* $\mathbf{R}^n$ *with* m*-th order derivatives in* $L^s(\mathbf{R}^n)$ *for some* s, $n < s \leq \infty$. *Then*

$$|R_m(b; x, y)| \leq C_{m,n}|x-y|^m \sum_{|\alpha|=m} \Big(\frac{1}{|I_x^y|} \int_{I_x^y} |D^\alpha b(z)|^s dz\Big)^{1/s},$$

where I_x^y *is the cube centered at* x, *with sides parallel to the axes and whose diameter is* $2\sqrt{n}|x-y|$.

LEMMA 2. *Let* Ω_0 *be homogeneous of degree zero and integrable on* S^{n-1}. *For* k *a positive integer and* $j = 1, 2, \cdots, k$, $A_j(x)$ *have derivatives of order* m_j *in* L^{r_j}, $1 < r_j \leq \infty, 1 \leq s < \infty$, *let*

$$\begin{aligned}&\tilde{M}^{\Omega_0}_{A_1,A_2,\cdots,A_k} f(x)\\ &\quad= \sup_{r>0} r^{-(n+M)} \int_{|x-y|<r} |\prod_{j=1}^{k} [R_{m_j}(A_j; x, y)]^s \Omega_0(x-y) f(y)| dy,\end{aligned}$$

where $M/s = m_1 + m_2 + \cdots + m_k$. *If* $1 < p < \infty, 1 < r < \infty, 1/r = 1/p + \sum_{j=1}^{k} s/r_j$, *then*

$$\|\tilde{M}^{\Omega_0}_{A_1,A_2,\cdots,A_k} f\|_r \le C\|\Omega_0\|_{L^1(S^{n-1})} \prod_{j=1}^{k} \Big(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|^s_{r_j} \Big) \|f\|_p.$$

For the special case $s = 1$, Lemma 2 was proved by Cohen and Gosselin [5]. If $1 < s < \infty$, the lemma can be proved by repeating the argument used in [5].

LEMMA 3. *Let* Ω , A_1, A_2 *be the same as that in Theorem* 1. *Denote*

$$M^{\Omega}_{A_1,A_2} f(x) = \sup_{r>0} r^{-(n+M-1)} \int_{|x-y|<r} |\Omega(x-y) \prod_{j=1}^{2} R_{m_j}(A_j; x, y) f(y)| dy.$$

If $1 < p,\ r < \infty,\ 1/r = 1/p + 1/r_0$, *then*

$$\|M^{\Omega}_{A_1,A_2} f\|_r \le C\|\Omega\|_q \Big(\sum_{|\alpha|=m_1-1} \|D^\alpha A_1\|_{\mathrm{BMO}} \Big) \Big(\sum_{|\beta|=m_2} \|D^\beta A_2\|_{r_0} \Big) \|f\|_p.$$

Proof. It suffices to prove the lemma for $\overline{M}^{\Omega}_{A_1,A_2}$, a variant of $M^{\Omega}_{A_1,A_2}$:

$$\begin{aligned} &\overline{M}^{\Omega}_{A_1,A_2} f(x) \\ &= \sup_{r>0} r^{-(n+M-1)} \int_{r/2<|x-y|<r} |\Omega(x-y) \prod_{j=1}^{2} R_{m_j}(A_j; x, y) f(y)| dy. \end{aligned}$$

For fixed $x \in \mathbf{R}^n, r > 0$, let $Q(x, r)$ be the cube centered at x and having sidelength $2\sqrt{n}r$,
set

$$A_1^Q(y) = A_1(y) - \sum_{|\alpha|=m_1-1} \frac{1}{\alpha!} m_{Q(x,r)}(D^\alpha A_1) y^\alpha,$$

where $m_{Q(x,r)}(D^\alpha A_1)$ denotes the mean value of $D^\alpha A_1$ on $Q(x, r)$. By the observation
of Cohen and Gosselin [4], we have

$$R_{m_1}(A_1; x, y) = R_{m_1}(A_1^Q; x, y).$$

Hölder's inequality then gives

$$\begin{aligned}\overline{M}^{\Omega}_{A_1,A_2}f(x) &\le \sup_{r>0}\Big(r^{-n-m_2q}\int_{|x-y|<r}|\Omega(x-y)R_{m_2}(A_2;x,y)|^q|f(y)|dy\Big)^{1/q}\\ &\quad\times\sup_{r>0}\Big(r^{-n-(m_1-1)q'}\int_{r/2<|x-y|<r}|R_{m_1}(A_1^Q;x,y)|^{q'}|f(y)|dy\Big)^{1/q'}\\ &= \mathrm{I}(f)(x)^{1/q}\mathrm{II}(f)(x)^{1/q'}.\end{aligned}$$

It follows from Lemma 2 that

$$\|\mathrm{I}(f)\|_{r_1}\le C\|\Omega\|^q_{L^q(S^{n-1})}\Big[\sum_{|\beta|=m_2}\|D^\beta A_2\|_{r_0}\Big]^q\|f\|_p,$$

where $1/r_1 = 1/p + q/r_0$.

For the estimate of $\mathrm{II}(f)$, we consider two cases:

(i) $m_1 = 1$, in this case, $A_1 \in \mathrm{BMO}$ and

$$\begin{aligned}\mathrm{II}(f)(x) &= \sup_{r>0} r^{-n}\int_{r/2<|x-y|<r}|A_1(x)-A_1(y)|^{q'}|f(y)|dy\\ &\le C^{q'}_{A_1}(f)(x),\end{aligned}$$

where the notation $C^{q'}_{A_1}(f)$ comes from [6]. By Theorem 2.4 in [6], we have

$$\|\mathrm{II}(f)\|_p \le C\|f\|_p.$$

(ii) $m_1 > 1$, in this case, we observe that if $r/2 < |x-y| < r$, then for $s > n$,

$$\begin{aligned}&|R_{m_1-1}(A_1^Q;x,y)|\\ &\le C_{m_1,n}|x-y|^{m_1-1}\sum_{|\alpha|=m_1-1}\Big(\frac{1}{|I_x^y|}\int_{I_x^y}|D^\alpha A_1(z)-m_{Q(x,r)}(D^\alpha A_1)|^s dz\Big)^{1/s}\\ &\le C_{m_1,n}\sum_{|\alpha|=m_1-1}\|D^\alpha A_1\|_{\mathrm{BMO}}|x-y|^{m_1-1}.\end{aligned}$$

So,

$$\begin{aligned}|R_{m_1}(A_1^Q;x,y)| &\le |R_{m_1-1}(A_1^Q;x,y)| + \Big|\sum_{|\alpha|=m_1-1}\frac{1}{\alpha!}D^\alpha A_1^Q(y)(x-y)^\alpha\Big|\\ &\le C\Big(\sum_{|\alpha|=m_1-1}\|D^\alpha A_1\|_{\mathrm{BMO}} + \sum_{|\alpha|=m_1-1}|D^\alpha A_1(y)-m_{Q(x,r)}(D^\alpha A_1)|\Big)\\ &\qquad\times|x-y|^{m_1-1}.\end{aligned}$$

Thus for any $t > 1$,

$$\begin{aligned}\mathrm{II}(f)(x) &\le C_{m_1,n} \sum_{|\alpha|=m_1-1} \|D^\alpha A_1\|_{\mathrm{BMO}}^{q'} Mf(x) \\ &\quad + C \sum_{|\alpha|=m_1-1} \sup_{r>0} r^{-n} \int_{|x-y|<r} |D^\alpha A_1(y) - m_{Q(x,r)}(D^\alpha A_1)|^{q'} |f(y)| dy \\ &\le C \sum_{|\alpha|=m_1-1} \|D^\alpha A_1\|_{\mathrm{BMO}}^{q'} M_t f(x),\end{aligned}$$

where Mf denotes the Hardy-Littlewood maximal function of f and $M_t f(x) = [M(|f|^t)(x)]^{1/t}$. Hölder's inequality and the above estimate yield that

$$\begin{aligned}\|\overline{M}^{\Omega}_{A_1,A_2} f\|_r &\le \|\mathrm{I}(f)\|_{r_1}^{1/q} \|\mathrm{II}(f)\|_p^{1/q'} \\ &\le C\Big(\sum_{|\alpha|=m_1-1} \|D^\alpha A_1\|_{\mathrm{BMO}}\Big)\Big(\sum_{|\beta|=m_2} \|D^\beta A_2\|_{r_0}\Big) \|f\|_p.\end{aligned}$$

LEMMA 4. *Let Ω; A_1, A_2 be the same as the assumption in Theorem 1, $b(x,y) \in L^\infty(\mathbf{R}^n \times \mathbf{R}^n)$. Let $1 < p$, $r < \infty$ and $1/r = 1/p + 1/r_0$. Suppose that the operator*

$$Tf(x) = \int_{\mathbf{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+M-1}} \prod_{j=1}^{2} R_{m_j}(A_j; x, y) b(x,y) f(y) dy$$

is bounded from L^p to L^r with bound

$$A\Big(\sum_{|\alpha|=m_1-1} \|D^\alpha A_1\|_{\mathrm{BMO}}\Big)\Big(\sum_{|\beta|=m_2} \|D^\beta A_2\|_{r_0}\Big).$$

Then the truncated operator

$$T_1 f(x) = \int_{|x-y|<1} \frac{\Omega(x-y)}{|x-y|^{n+M-1}} \prod_{j=1}^{2} R_{m_j}(A_j; x, y) b(x,y) f(y) dy$$

is bounded from L^p to L^r with bound

$$C(A + \|b\|_\infty)\Big(\sum_{|\alpha|=m_1-1} \|D^\alpha A_1\|_{\mathrm{BMO}}\Big)\Big(\sum_{|\beta|=m_2} \|D^\beta A_2\|_{r_0}\Big).$$

Proof. Without loss of generality, we may assume that

$$\sum_{|\alpha|=m_1-1} \|D^\alpha A_1\|_{\mathrm{BMO}} = \sum_{|\beta|=m_2} \|D^\beta A_2\|_{r_0} = 1.$$

For each fixed $h \in \mathbf{R}^n$, we split f into three parts as

$$f = f_1 + f_2 + f_3$$

where

$$f_1(y) = f(y)\chi_{\{|y-h|<1/2\}}(y),$$

and

$$f_2(y) = f(y)\chi_{\{1/2\le|y-h|<5/4\}}(y).$$

Let $\phi_h \in C_0^\infty(\mathbf{R}^n)$ such that $\operatorname{supp}\phi_h \subset \{y : |y-h| < 4\}$, $\phi_h(y) = 1$ if $|y-h| < 2$, and $\|D^\nu\phi_h\|_\infty \le c$ for all multi-index ν. Set

$$A_2^{\bar h}(y) = R_{m_2}(A_2; y, \bar h)\phi_{\bar h}(y)\text{with}|\bar h - h| < 3/4.$$

It is easy to verify that if $|x-h| < 1/4$, then

$$T_1 f_1(x) = \int_{\mathbf{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+M-1}} R_{m_1}(A_1; x, y) R_{m_2}(A_2^{\bar h}; x, y) b(x,y) f_1(y)\,dy.$$

Thus

$$\int_{|x-h|<1/4} |T_1 f_1(x)|^r dx \le A^r\Big(\sum_{|\beta|=m_2} \|D^\beta A_2^{\bar h}\|_{r_0}\|f_1\|_p\Big)^r.$$

For each fixed multi-index β, $|\beta| = m_2$, write

$$D^\beta A_2^{\bar h}(y) = \sum_{\beta=\mu+\nu} C_{\mu,\nu} R_{m_2-|\mu|}(D^\mu A_2; y, \bar h) D^\nu \phi_{\bar h}(y).$$

Using the formula

$$R_m(F; y, \bar h) = \sum_{|\alpha|=m} \frac{m}{\alpha!} \int_0^1 (1-t)^{m-1} D^\alpha F(\bar h + t(y-\bar h))(y-\bar h)^\alpha dt,$$

we have

$$\left(\int_{|y-\bar h|<4} |R_m(F; y, \bar h)|^r dy\right)^{1/r}$$

$$\leq C \sum_{|\alpha|=m} \int_0^1 \left(\int_{|y-\bar{h}|<4} |D^\alpha F(\bar{h} + t(y-\bar{h}))|^r dy \right)^{1/r} dt$$

$$= C \sum_{|\alpha|=m} \int_0^1 \left(\int_{|z-\bar{h}|<4t} |D^\alpha F(z)|^r t^{-n} dz \right)^{1/r} dt$$

$$\leq CM_r \left(\sum_{|\alpha|=m} |D^\alpha F| \chi_{B(h,5)} \right) (\bar{h}).$$

Using this inequality, we obtain

$$(*) \qquad \sum_{|\beta|=m_2} \|D^\beta A_2^{\bar{h}}\|_{r_0} \leq CM_{r_0} \Big(\sum_{|\alpha|=m_2} |D^\alpha A_2| \chi_{B(h,5)} \Big)(\bar{h}).$$

We can choose $\bar{h}$, $|\bar{h}-h| < 3/4$, so that the right hand side of $(*)$ is majorized by

$$C \Big\| \sum_{|\alpha|=m} |D^\alpha A_2| \chi_{B(h,5)} \Big\|_{r_0}.$$

This shows that

$$(2.1) \qquad \int_{|x-h|<1/4} |T_1 f_1(x)|^r dx \\ \leq CA^r \sum_{|\beta|=m_2} \bigg(\Big(\int_{|y-h|<8} |D^\beta A_2(y)|^{r_0} dy \Big)^{1/r_0} \|f_1\|_p \bigg)^r.$$

If $|x-h| < 1/4$ and $1/2 \leq |y-h| < 5/4$, then $1/4 < |x-y| < 3/2$. So we see that for $|x-h| < 1/4$,

$$|T_1 f_2(x)| \leq \|b\|_\infty \int_{1/4<|x-y|<3/2} \left| \frac{\Omega(x-y)}{|x-y|^{n+M-1}} \prod_{j=1}^2 R_{m_j}(A_j; x, y) f_2(y) \right| dy \\ \leq C\|b\|_\infty M^\Omega_{A_1, A_2^{\bar{h}}} f_2(x).$$

Lemma 3 now tells us that

$$\int_{|x-h|<1/4} |T_1 f_2(x)|^r dx \leq C\|b\|_\infty^r \Big(\sum_{|\beta|=m_2} \|D^\beta A_2^{\bar{h}}\|_{r_0} \|f_2\|_p \Big)^r$$

$$\leq C\|b\|_\infty^r \Big(\sum_{|\beta|=m_2} \Big(\int_{|y-h|<8} |D^\beta A_2(y)|^{r_0} dy \Big)^{1/r_0} \|f_2\|_p \Big)^r. \tag{2.2}$$

Obviously, we have $T_1 f_3(x) = 0$ for $|x-h| < 1/4$. Combining inequalities (2.1) and (2.2) leads to

$$\begin{aligned}
&\int_{|x-h|<1/4} |T_1 f(x)|^r dx \\
&\leq C(A^r + \|b\|_\infty^r) \sum_{|\beta|=m_2} \Big(\int_{|y-h|<8} |D^\beta A_2(y)|^{r_0} dy \Big)^{r/r_0} \Big(\int_{|y-h|<2} |f(y)|^p dy \Big)^{r/p}.
\end{aligned}$$

Integrating the last inequality with respect to h gives that

$$\begin{aligned}
\|T_1 f\|_r^r &\leq C(A^r + \|b\|_\infty^r) \sum_{|\beta|=m_2} \Big(\int_{\mathbf{R}^n} \int_{|y-h|<8} |D^\beta A_2(y)|^{r_0} dy dh \Big)^{r/r_0} \\
&\qquad \times \Big(\int_{\mathbf{R}^n} \int_{|y-h|<2} |f(y)|^p dy dh \Big)^{r/p} \\
&\leq C(A^r + \|b\|_\infty^r) \sum_{|\beta|=m_2} \|D^\beta A_2\|_{r_0}^r \|f\|_p^r.
\end{aligned}$$

This completes the proof of Lemma 4.

Proof of Theorem 1. We only treat the case that

$$\sum_{|\alpha|=m_1-1} \|D^\alpha A_1\|_{\mathrm{BMO}} = \sum_{|\beta|=m_2} \|D^\beta A_2\|_{r_0} = 1.$$

First we show that (ii) implies (i). Let k and l be two positive integers, $P(x,y)$ be a non-degenerate real-valued polynomial with degree k in x and l in y. Write

$$P(x,y) = \sum_{|\alpha|\leq k, |\beta|\leq l} a_{\alpha\beta} x^\alpha y^\beta.$$

By dilation invariance, we may assume that $\sum_{|\alpha|=k,|\beta|=l} |a_{\alpha\beta}| = 1$. Decompose T_{A_1,A_2} as

$$\begin{aligned}
&T_{A_1,A_2} f(x) \\
&= \int_{|x-y|<1} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+M-1}} \prod_{j=1}^{2} R_{m_j}(A_j; x, y) f(y) dy
\end{aligned}$$

$$+\sum_{d=1}^{\infty}\int_{2^{d-1}\le|x-y|<2^d} e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^{n+M-1}}\prod_{j=1}^{2}R_{m_j}(A_j;x,y)f(y)dy$$

$$= T^0_{A_1,A_2}f(x)+\sum_{d=1}^{\infty}T^d_{A_1,A_2}f(x).$$

We first consider the operator $T^d_{A_1,A_2}$, $d\ge 1$. We claim that if $D^\beta A_2$ is in L^∞ for all $|\beta|=m_2$, then

$$\|T^d_{A_1,A_2}f\|_2\le C2^{-\varepsilon_1 d}\sum_{|\beta|=m_2}\|D^\beta A_2\|_\infty\|f\|_2,\ d\ge 1, \tag{2.3}$$

where ε_1 is independent of d and f. If this is done, then by interpolating between inequality (2.3) and the crude estimate

$$\|T^d_{A_1,A_2}f\|_p\le C\sum_{|\beta|=m_2}\|D^\beta A_2\|_\infty\|f\|_p,\ 1<p<\infty,$$

we can get

$$\|T^d_{A_1,A_2}f\|_p\le C2^{-\varepsilon_2 d}\sum_{|\beta|=m_2}\|D^\beta A_2\|_\infty\|f\|_p,\ 1<p<\infty. \tag{2.4}$$

For each fixed p and r_0, we choose $\tilde{r_0}$, $\tilde{r}$ such that $1<\tilde{r_0}<r_0$, $1/p+1/\tilde{r_0}=1/\tilde{r}<1$. Lemma 3 then tells us that

$$\|T^d_{A_1,A_2}f\|_{\tilde{r}}\le C\sum_{|\beta|=m_2}\|D^\beta A_2\|_{\tilde{r_0}}\|f\|_p,\ 1<p<\infty.$$

We regard $T^d_{A_1,A_2}$ as a linear operator of A_2. Thus the inequality (2.4) together with the last inequality states that

$$\|T^d_{A_1,A_2}f\|_r\le C2^{-\varepsilon d}\sum_{|\beta|=m_2}\|D^\beta A_2\|_{r_0}\|f\|_p,\ 1<p<\infty, \tag{2.5}$$

where ε is a positive constant. Summing over all $d\ge 1$, we obtain

$$\|\sum_{d=1}^{\infty}T^d_{A_1,A_2}f\|_r\le C\|f\|_p.$$

To prove (2.3), we may assume $\sum_{|\beta|=m_2} \|D^\beta A_2\|_\infty = 1$. Define

$$\tilde{T}^d_{A_1,A_2} f(x) = \int_{1<|x-y|\le 2} e^{iP(2^{d-1}x, 2^{d-1}y)} \frac{\Omega(x-y)}{|x-y|^{n+M-1}} \prod_{j=1}^{2} R_{m_j}(A_j; x, y) f(y)dy.$$

By dilation invariance, it is enough to prove that

$$\|\tilde{T}^d_{A_1,A_2} f\|_2 \le C 2^{-\varepsilon d} \|f\|_2. \tag{2.6}$$

Decompose $\mathbf{R}^n$ into $\mathbf{R}^n = \bigcup I_i$, where I_i is a cube with side length 1, and the cubes have disjoint interiors. Set $f_i = f\chi_{I_i}$. Since the support of $\tilde{T}^d_{A_1,A_2} f_i$ is contained in a fixed multiple of I_i, so that the supports of the various terms $\tilde{T}^d_{A_1,A_2} f_i$ have bounded overlaps. Thus we have the "almost orthogonality" property

$$\|\tilde{T}^d_{A_1,A_2} f\|_2^2 \le C \sum_i \|\tilde{T}^d_{A_1,A_2} f_i\|_2^2,$$

and therefore it suffices to show

$$\|\tilde{T}^d_{A_1,A_2} f_i\|_2^2 \le C 2^{-\varepsilon d} \|f_i\|_2^2. \tag{2.7}$$

For fixed i, denote $\tilde{I}_i = 100nI_i$. Let $\phi_i(x) \in C_0^\infty(\mathbf{R}^n)$ such that $0 \le \phi_i \le 1$, ϕ_i is identically one on $10\sqrt{n}I_i$ and vanishes outside of $50\sqrt{n}I_i$, $\|D^\gamma \phi_i\|_\infty \le C_\gamma$ for all multi–index γ. Let x_0 be a point on the boundary of $80\sqrt{n}I_i$. Denote

$$A_1^{\phi_i}(y) = R_{m_1-1}(A_1(\cdot) - \sum_{|\alpha|=m_1-1} \frac{1}{\alpha!} m_{\tilde{I}_i}(D^\alpha A_1)(\cdot)^\alpha; y, x_0)\phi_i(y)$$

and for multi–index α, define

$$\tilde{T}^{d,\alpha}_{A_1,A_2} f(x) = \int_{1<|x-y|\le 2} e^{iP(2^{d-1}x, 2^{d-1}y)} \frac{\Omega(x-y) R_{m_2}(A_2; x, y)}{|x-y|^{n+M-1}} (x-y)^\alpha f(y)dy.$$

It is easy to see that

$$\tilde{T}^d_{A_1,A_2} f_i(x)$$

$$= \int_{1<|x-y|\le 2} e^{iP(2^{d-1}x,2^{d-1}y)} \frac{\Omega(x-y)R_{m_2}(A_2;x,y)}{|x-y|^{n+M-1}} R_{m_1}(A_1^{\phi_i};x,y)f_i(y)dy$$
$$= A_1^{\phi_i}(x)\tilde{T}^{d,0}_{A_1,A_2}f_i(x) - \sum_{|\alpha|<m_1-1} \frac{1}{\alpha!}\tilde{T}^{d,\alpha}_{A_1,A_2}(D^\alpha A_1^{\phi_i}f_i)(x)$$
$$- \sum_{|\alpha|=m_1-1} \frac{1}{\alpha!}\tilde{T}^{d,\alpha}_{A_1,A_2}(D^\alpha A_1^{\phi_i}f_i)(x)$$
$$= \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

Before we estimate these terms, let us state a lemma.

LEMMA 5. *There exists a positive constant $\delta = \delta(n, \mathrm{deg}P)$ such that for any $d \ge 1$ and multi-index α,*

$$\|\int_{2^{d-1}\le|x-y|<2^d} e^{iP(x,y)} \frac{\Omega(x-y)R_{m_2}(A_2;x,y)}{|x-y|^{n+M-1}}(x-y)^\alpha f(y)dy\|_p$$
$$\le C2^{-(\delta+m_1-1-|\alpha|)d}\|f\|_p,\ 1<p<\infty,$$

where constant C is independent of d, f and coefficients of $P(x,y)$.

Recall that $P(x,y) = \sum_{|\alpha|\le k,|\beta|\le l} a_{\alpha\beta}x^\alpha y^\beta$ and $\sum_{|\alpha|=k,|\beta|=l}|a_{\alpha\beta}| = 1$. Lemma 5 can be proved by an argument used in [2]. We omit the details here.

We return to the estimates of I, II and III. Note that for multi-index β, $|\beta| < m_1 - 1$,

$$D^\beta A_1^{\phi_i}(y) = \sum_{\beta=\mu+\nu} C_{\mu,\nu}R_{m_1-|\mu|-1}(D^\mu(A_1(\cdot)$$
$$- \sum_{|\alpha|=m_1-1} \frac{1}{\alpha!} m_{\tilde{I}_i}(D^\alpha A_1)(\cdot)^\alpha); y, x_0)D^\nu\phi_i(y).$$

Since that supp $\phi_i \subset 50\sqrt{n}I_i$, by Lemma 1, we have

$$|D^\beta A_1^{\phi_i}(y)| \le C \sum_{|\alpha|=m_1-1} (\frac{1}{|I^y_{x_0}|}\int_{I^y_{x_0}} |D^\alpha A_1(z) - m_{\tilde{I}_i}(D^\alpha A_1)|^t dz)^{1/t} \le C,$$

where $t > n$. Thus, it follows from Lemma 5 that

$$\|\mathrm{I}\|_2 \le \|A_1^{\phi_i}\|_\infty \|\tilde{T}^{d,0}_{A_1,A_2}f_i\|_2 \le C2^{-\delta d}\|f_i\|_2.$$

Similarly, we have

$$\|\mathrm{II}\|_2 \le C2^{-\delta d}\|f_i\|_2.$$

It remains to estimate the third term III. Note that for any $0<\gamma<n$,

$$\begin{aligned}|\tilde{T}^{d,\alpha}_{A_1,A_2}f(x)| &\le C\int_{1<|x-y|\le 2}|\Omega(x-y)f(y)|dy\\ &\le C_\gamma\|\Omega\|_{L^q(S^{n-1})}\Big(\int_{1<|x-y|\le 2}\frac{|f(y)|^{q'}}{|x-y|^{n-\gamma}}dy\Big)^{1/q'}\\ &\le C_\gamma\|\Omega\|_{L^q(S^{n-1})}[I_\gamma(|f|^{q'})(x)]^{1/q'},\end{aligned}$$

where I_γ denotes the usual fractional integral of order γ. If $p>q'$ and $\sigma>0$, we take a γ such that $0<\gamma<nq'/p$, and $1/(p+\sigma)=1/p-\gamma/nq'$. By the Hardy–Littlewood–Sobolev theorem [9], we get

$$\|\tilde{T}^{d,\alpha}_{A_1,A_2}f\|_{p+\sigma}\le C\|\Omega\|_{L^q(S^{n-1})}\|f\|_p,\ p>q',\ \sigma>0.$$

By the last inequality and Lemma 5, an interpolation will give

$$\|\tilde{T}^{d,\alpha}_{A_1,A_2}f\|_p\le C2^{-\tilde{\sigma}d}\|f\|_{p-\sigma},\ \text{for } 1<p<\infty \text{ and } 0<\sigma<\sigma_p, \tag{2.8}$$

where $\tilde{\sigma}$ is a positive constant. On the other hand, if $|\beta|=m_1-1$, then,

$$\begin{aligned}D^\beta A_1^{\phi_i}(y)=&\sum_{\beta=\mu+\nu,|\mu|<m_1-1}C_{\mu,\nu}R_{m_1-1-|\mu|}(D^\mu(A_1(\cdot)\\ &-\sum_{|\alpha|=m_1-1}\frac{1}{\alpha!}m_{\tilde{I}_i}(D^\alpha A_1)(\cdot)^\alpha);y,x_0)D^\nu\phi_i(y)\\ &+(D^\beta A_1(y)-m_{\tilde{I}_i}(D^\beta A_1))\phi_i(y).\end{aligned}$$

Thus, it follows that

$$|D^\beta A_1^{\phi_i}(y)|\le C(1+|D^\beta A_1(y)-m_{\tilde{I}_i}(D^\beta A_1)|),$$

and this shows that for any $t>1$,

$$\|D^\beta A_1^{\phi_i}\|_t\le C_t.$$

Combining the above inequality and (2.8), we obtain

$$\begin{aligned}\|\mathrm{III}\|_2 &\le C2^{-\tilde{\delta}d}\sum_{|\alpha|=m_1-1}\|D^\alpha A_1^{\phi_i}f_i\|_{2-\sigma}\le C2^{-\tilde{\delta}d}\sum_{|\alpha|=m_1-1}\|D^\alpha A_1^{\phi_i}\|_t\|f_i\|_2\\ &\le C2^{-\tilde{\delta}d}\|f_i\|_2,\end{aligned}$$

where we choose $\sigma > 0$ and $1 < t < \infty$ such that $1/2 + 1/t = 1/(2-\sigma)$.

All above estimates imply that (2.3) is true.

We turn our attention to the operator $T^0_{A_1,A_2}$. The estimate for this operator follows from the following lemma.

LEMMA 6. *Suppose that the condition* (ii) *in Theorem* 1 *holds. Then for any real-valued polynomial* $\tilde{P}(x,y)$, *the operator*

$$U_{A_1,A_2}f(x) = \int_{|x-y|<1} e^{i\tilde{P}(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+M-1}} \prod_{j=1}^{2} R_{m_j}(A_j; x, y) f(y)dy$$

satisfies

$$\|U_{A_1,A_2}f\|_r \leq C(\deg\tilde{P}, n)\|f\|_p. \tag{2.9}$$

Proof. We shall carry out our argument by a double induction on the degree in x and y of the polynomial. If the polynomial $\tilde{P}(x,y)$ depends only on x or only on y, it is obvious that the condition (ii) implies (2.9). Let u and v be two positive integers and the polynomial has degree u in x and v in y. We assume that (2.9) holds for all polynomial which are sums of monomials of degree less than u in x times monomials of any degree in y, together with monomials which are of degree u in x times monomials which are of degree less than v in y. Write $\tilde{P}(x,y)$ as

$$\tilde{P}(x,y) = \sum_{|\alpha|=u,|\beta|=v} b_{\alpha\beta} x^\alpha y^\beta + P_0(x,y),$$

where $P_0(x,y)$ satisfies the inductive assumption. We consider the following two cases.

Case I. $\sum_{|\alpha|=u,|\beta|=v} |b_{\alpha\beta}| \leq 1$. Rewrite

$$\tilde{P}(x,y) = \sum_{|\alpha|=u,|\beta|=v} b_{\alpha\beta}(x^\alpha y^\beta - y^{\alpha+\beta}) + \tilde{P}_0(x,y),$$

where $\tilde{P}_0(x,y)$ satisfies the induction assumption. It follows that

$$\begin{aligned} &U_{A_1,A_2}f(x) \\ &= \int_{|x-y|<1} e^{i\tilde{P}_0(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+M-1}} \prod_{j=1}^{2} R_{m_j}(A_j; x, y) f(y)dy \\ &\quad + \int_{|x-y|<1} (e^{i\tilde{P}(x,y)} - e^{i\tilde{P}_0(x,y)}) \frac{\Omega(x-y)}{|x-y|^{n+M-1}} \prod_{j=1}^{2} R_{m_j}(A_j; x, y) f(y)dy \\ &= U^1_{A_1,A_2}f(x) + U^2_{A_1,A_2}f(x). \end{aligned}$$

Our induction assumption now states that

$$\|U^1_{A_1,A_2}f\|_r \le C\|f\|_p.$$

Denote $\tilde{f}(y) = f(y)\chi_{\{|y|\le 2\}}$. It is easy to see $U^2_{A_1,A_2}f(x) = U^2_{A_1,A_2}\tilde{f}(x)$ when $|x| \le 1$. Thus,

$$|U^2_{A_1,A_2}f(x)| \le C\int_{|x-y|<1}\left|\frac{\Omega(x-y)}{|x-y|^{n+M-2}}\prod_{j=1}^{2}R_{m_j}(A_j;x,y)\tilde{f}(y)\right|dy, \ |x|\le 1.$$

Let $\Phi_h \in C_0^\infty(\mathbf{R}^n)$ such that $\mathrm{supp}\Phi_h \subset \{x : |x-h| < 8\}$, $\Phi_h(x) = 1$ if $|x-h| \le 4$ and $\|D^\gamma\Phi_h\|_\infty \le c$ for all multi-index γ. We have that if $|x| \le 1$, then

$$|U^2_{A_1,A_2}f(x)| \le CM^\Omega_{A_1,A_2^{\bar h}}\tilde{f}(x),$$

where $A_2^{\bar h}(y) = R_{m_2}(A_2;y,\bar h)\Phi_{\bar h}(y)$ with $|\bar h| < 3$. By the same argument used in the proof of Lemma 4, we can get that

$$\int_{|x|\le 1}|U^2_{A_1,A_2}f|^r dx \le C\sum_{|\beta|=m_2}\Big(\int_{|y|\le 8}|D^\beta A_2(y)|^{r_0}dy\Big)^{r/r_0}\Big(\int_{|y|\le 2}|f(y)|^p dy\Big)^{r/p},$$

from which the same argument as that in [8, p. 189] show that the inequality

$$\int_{|x-h|\le 1}|U_{A_1,A_2}f|^r dx \le C\sum_{|\beta|=m_2}\Big(\int_{|y-h|\le 8}|D^\beta A_2(y)|^{r_0}dy\Big)^{r/r_0}\Big(\int_{|y-h|\le 2}|f(y)|^p dy\Big)^{r/p}$$

holds for all $h \in \mathbf{R}^n$ and C is independent of h. Integrating the last inequality with respect to h and using Hölder's inequality, we finally obtain that

$$\|U_{A_1,A_2}f\|_r \le C\|f\|_p.$$

Case II. $\sum_{|\alpha|=u,|\beta|=v}|b_{\alpha\beta}| > 1$. Denote $B = (\sum_{|\alpha|=u,|\beta|=v}|b_{\alpha\beta}|)^{1/u+v}$. Write

$$\tilde{P}(x,y) = \sum_{|\alpha|=u,|\beta|=v}\frac{b_{\alpha\beta}}{B^{u+v}}(Bx)^\alpha(By)^\beta + P_0\Big(\frac{Bx}{B},\frac{By}{B}\Big) = Q(Bx,By),$$

and denote

$$\tilde{U}_{A_1,A_2}f(x) = \int_{|x-y|<B} e^{iQ(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+M-1}} \prod_{j=1}^{2} R_{m_j}(A_j;x,y)f(y)dy.$$

It is not difficult to find that (2.9) is equivalent to the estimate

$$\|\tilde{U}_{A_1,A_2}f\|_r \le C\|f\|_p. \tag{2.10}$$

we split $\tilde{U}_{A_1,A_2}f(x)$ as

$$\begin{aligned}
&\tilde{U}_{A_1,A_2}f(x)\\
&= \int_{|x-y|<1} e^{iQ(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+M-1}} \prod_{j=1}^{2} R_{m_j}(A_j;x,y)f(y)dy\\
&\quad + \sum_{d=1}^{d_0} \int_{2^{d-1}\le|x-y|<2^d} e^{iQ(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+M-1}} \prod_{j=1}^{2} R_{m_j}(A_j;x,y)f(y)dy\\
&= \tilde{U}^0_{A_1,A_2}f(x) + \tilde{U}^\infty_{A_1,A_2}f(x),
\end{aligned}$$

where $2^{d_0} = B$. The estimate for $\tilde{U}^0_{A_1,A_2}$ follows along the same line as in case I. On the other hand, by Lemma 5 and the argument used in the treatment for $T^d_{A_1,A_2}$, we have

$$\|\tilde{U}^\infty_{A_1,A_2}f\|_r \le C\|f\|_p.$$

This leads to the estimate (2.10).

Now we show that (i) implies (ii). To do this, we need to use Definition 2. We choose $Q(x,y)$ such that $Q(x,y)$ has $\mathcal{P}$ and decompose

$$\begin{aligned}
T_{A_1,A_2}f(x) &= \int_{|x-y|<1} e^{iQ(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+M-1}} \prod_{j=1}^{2} R_{m_j}(A_j;x,y)f(y)dy\\
&\quad + \int_{|x-y|\ge1} e^{iQ(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+M-1}} \prod_{j=1}^{2} R_{m_j}(A_j;x,y)f(y)dy\\
&= T^0_{A_1,A_2}f(x) + T^\infty_{A_1,A_2}f(x).
\end{aligned}$$

By Lemma 4, $T^0_{A_1,A_2}$ is bounded from L^p to L^r. The same argument as in the proof of Lemma 4 tells us that

$$\Big(\int_{|x-h|<1} |T^o_{A_1,A_2}f(x)|^r dx\Big)^{1/r}$$

$$\leq C\Big(\sum_{|\beta|=m_2}\int_{|y-h|<8}|D^\beta A_2(y)|^{r_0}dy\Big)^{1/r_0}\Big(\int_{|y-h|<2}|f(y)|^p dy\Big)^{1/p},$$

where C is independent of h. Since $Q(x,y)$ has $\mathcal{P}$, we have

$$Q(x,y)=Q(x-h,y-h)+R_0(x,h)+R_1(y,h),$$

where R_0, R_1 are real polynomials. When $|x-h|<1$, it follows that

$$\begin{aligned}S_{A_1,A_2}f(x)&=\int_{|x-y|<1}\frac{\Omega(x-y)}{|x-y|^{n+M-1}}\prod_{j=1}^{2}R_{m_j}(A_j;x,y)f(y)\chi_{B(h,2)}(y)dy\\&=e^{-iR_0(x,h)}\int_{|x-y|<1}e^{iQ(x,y)}\frac{\Omega(x-y)}{|x-y|^{n+M-1}}\prod_{j=1}^{2}R_{m_j}(A_j;x,y)\\&\qquad\times e^{-iQ(x-h,y-h)}e^{-iR_1(y,h)}f(y)\chi_{B(h,2)}(y)dy.\end{aligned}$$

Observe that the Taylor's expression of $e^{-iQ(x-h,y-h)}$ is

$$\begin{aligned}e^{-iQ(x-h,y-h)}&=\sum_{m=0}^{\infty}\frac{i^m}{m!}\Big(\sum_{\alpha,\beta}a_{\alpha\beta}(x-h)^\alpha(y-h)^\beta\Big)^m\\&=\sum_{u,v}a_{m,u,v}(x-h)^u(y-h)^v.\end{aligned}$$

For $|x-h|<1$ and $|y-h|<2$, we have

$$\begin{aligned}&\Big(\int_{|x-h|<1}|S_{A_1,A_2}f(x)|^r dx\Big)^{1/r}\\&\leq\sum_{u,v}|a_{m,u,v}|\Big[\int_{|x-h|<1}|(x-h)^u|^r|T^0_{A_1,A_2}[e^{-iR_1(\cdot,h)}f(\cdot)\chi_{B(h,2)}(\cdot)(\cdot-h)^v]|^r dx\Big]^{1/r}\\&\leq C_{A_2,h}\sum_{u,v}|a_{m,u,v}|a^u\Big[\int_{|y-h|<2}|f(y)|^p|(y-h)^v|^p dy)\Big]^{1/p}\\&\leq C_{A_2,h}\sum_{u,v}|a_{m,u,v}|a^u b^v\Big[\int_{|y-h|<2}|f(y)|^p dy\Big]^{1/p}\\&\leq C_{A_2,h}\sum_{m=0}^{\infty}\frac{1}{m!}\Big(\sum_{\alpha,\beta}|a_{\alpha\beta}|a^\alpha b^\beta\Big)^m\Big[\int_{|y-h|<2}|f(y)|^p dy\Big]^{1/p}\\&\leq C_{A_2,h}\exp\Big\{\sum_{\alpha,\beta}|a_{\alpha\beta}|a^\alpha b^\beta\Big\}\Big[\int_{|y-h|<2}|f(y)|^p dy\Big]^{1/p},\end{aligned}$$

where $C_{A_2,h} = C(\sum_{|\beta|=m_2} \int_{|y-h|<8} |D^\beta A_2(y)|^{r_0} dy)^{1/r_0}$, and $a = (1, 1, \cdots, 1)$, $b = (2, 2, \cdots, 2)$. Hence,

$$\|S_{A_1,A_2} f\|_r \leq C\|f\|_p.$$

This completes the proof of Theorem 1.

Remark 1. Consider the operator defined by:

$$T_{A_1,A_2} f(x) = \int_{\mathbf{R}^n} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+M-2}} \prod_{j=1}^{2} R_{m_j}(A_j; x, y) f(y) dy, \ n \geq 2.$$

Repeating the arguments of Theorem 1, we can obtain that

THEOREM 2. *Let $1 < p < \infty$, Ω, M be the same as that in Theorem 1. Suppose A_i have derivatives of order $m_i - 1$ in* BMO($\mathbf{R}^n$) *respectively, $i = 1, 2$. Then the following two facts are equivalent:*

(i) *If $P(x, y)$ is a non-degenerate real-valued polynomial, then T_{A_1,A_2} is bounded on $L^p(\mathbf{R}^n)$ with bound*

$$C(\deg P, n)\Big(\sum_{|\alpha|=m_1-1} \|D^\alpha A_1\|_{\mathrm{BMO}}\Big)\Big(\sum_{|\beta|=m_2-1} \|D^\beta A_2\|_{\mathrm{BMO}}\Big);$$

(ii) *The truncated operator*

$$S_{A_1,A_2} f(x) = \int_{|x-y|<1} \frac{\Omega(x-y)}{|x-y|^{n+M-2}} \prod_{j=1}^{2} R_{m_j}(A_j; x, y) f(y) dy$$

is bounded on $L^p(\mathbf{R}^n)$ with bound

$$C\Big(\sum_{|\alpha|=m_1-1} \|D^\alpha A_1\|_{\mathrm{BMO}}\Big)\Big(\sum_{|\beta|=m_2-1} \|D^\beta A_2\|_{\mathrm{BMO}}\Big).$$

Remark 2. Here we give an example which satisfies the condition (ii) of Theorem 1. The example of Theorem 2 is analogous.

In R^1, suppose $A_1(x) = \log(1 + |x|)$, $A_2(x) = x$, and $\Omega(x) = \mathrm{sgn}(x)$, then $A_1 \in \mathrm{BMO}(R)$, A_2 has derivatives of order 1 in $L^\infty(R)$.

$$\begin{aligned} S_{A_1,A_2} f(x) &= \int_{|x-y|<1} \frac{\Omega(x-y)}{|x-y|^2} (A_1(x) - A_1(y))(A_2(x) - A_2(y)) f(y) dy \\ &= \int_{|x-y|<1} \frac{1}{|x-y|} \log\Big(\frac{1+|x|}{1+|y|}\Big) f(y) dy. \end{aligned}$$

Therefore $|S_{A_1,A_2} f(x)| \leq M f(x)$. So S_{A_1,A_2} is bounded on $L^p(R)$.

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Wengu Chen
Department of Mathematics
Capital Normal University
Beijing 100037
People's Republic of China

Guoen Hu
Department of Mathematics
Beijing Normal University
Beijing 100875
People's Republic of China

Shanzhen Lu
Department of Mathematics
Beijing Normal University
Beijing 100875
People's Republic of China
lusz@bnu.edu.cn