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ON THE DETERMINISM OF THE DISTRIBUTIONS OF MULTIPLE MARKOV NON-GAUSSIAN SYMMETRIC STABLE PROCESSES

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Abstract. Consider a non-Gaussian $S\alpha S$ process $X = \{X(t); t \in T\}$ which is expressed as a canonical representation $X(t) = \int_{u \leq t, u \in T} F(t,u)\, dZ(u)$, $t \in T$, and is continuous in probability. If X is n-ple Markov, then X has determinism of dimension $n+1$. That is, any $S\alpha S$ process $\tilde{X} = \{\tilde{X}(t); t \in T\}$ having the same $(n+1)$-dimensional distributions with X is identical in law with X.

§1. Introdution

In this paper we consider the determinism of the distribution of an $S\alpha S$ (= symmetric α-stable) random field ($0 < \alpha \leq 2$) in the following sense.

DEFINITION. We say that an $S\alpha S$ random field $X = \{X(s); s \in S\}$ has *determinism of dimension* n if any $S\alpha S$ random field $\tilde{X} = \{\tilde{X}(s); s \in S\}$ having the same n-dimensional distributions with X is identical in law with X.

In this definition, "X and $\tilde{X}$ have the same n-dimensional distributions" means that $(X(s_1), X(s_2), \cdots, X(s_n))$ and $(\tilde{X}(s_1), \tilde{X}(s_2), \cdots, \tilde{X}(s_n))$ have a common distribution for any choice of distinct $s_1, s_2, \cdots, s_n \in S$. "$X$ and $\tilde{X}$ are identical in law" means that they have the same finite-dimensional distributions of all dimensions. Obviously, if X has determinism of dimension n, then X has determinism of dimension m for $m > n$. A centered Gaussian random field is symmetric stable with the index $\alpha = 2$ and has determinism of dimension 2 because any finite-dimensional distribution is expressed by its covariance function. On the other hand, it is not easy to answer a question whether a particular non-Gaussian $S\alpha S$ random field ($0 < \alpha < 2$) has determinism of a given dimension or not. However, the determinism of some non-Gaussian $S\alpha S$ random fields has been studied.

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Y. Sato ([5], Theorem 1) proved that any finite-dimensional distribution of a non-Gaussian $S\alpha S$ random field on $\boldsymbol{R}^d$ of Chentsov type is determined by its own $(d+1)$-dimensional marginal distributions. Her proof tells us that this random field has determinism of dimension $d+2$. Y. Sato and S. Takenaka ([6], Propositions 3.1 and 3.3) gave two concrete examples of self-similar $S\alpha S$ random fields of exponent H $(0<\alpha<2,\ 0<H<\alpha^{-1})$ of generalized Chentsov type, which have the same 2-dimensional distributions and have different 3-dimensional distributions. This fact means that these two random fields do not have determinism of dimension 2 (in fact, it can be proved that one of these has determinism of dimension 3 and the other has determinism of dimension 5). T. Mori ([4], Theorem 6.1) showed that there exists a one-to-one correspondence between stochastically continuous, linearly additive non-Gaussian $S\alpha S$ random fields on $\boldsymbol{R}^d$ and locally finite, bundleless measures on the space of all $(d-1)$-hyperplanes in $\boldsymbol{R}^d$. This fact implies that each of these random fields has determinism of dimension $d+1$.

Inspired by these results, in this paper we discuss the determinism of a multiple Markov non-Gaussian $S\alpha S$ process with a canonical representation. We obtain the fact that this process has determinism of dimension $n+1$ where n is the multiplicity of multiple Markov property of the process (Theorems 3.1 and 3.2). However, it is a much harder problem to find the smallest number d (≥ 2) such that this process has determinism of dimension d. We further investigate the special case where the process is stationary and its canonical representation is in the form of moving average. In this case the representation kernel is a solution of a linear differential equation with constant coefficients (Proposition 4.1). We obtain a sharper result on the determinism of some of these processes (Proposition 4.3).

§2. Preliminaries

For fixed α $(0<\alpha\leq 2)$, an $\boldsymbol{R}$-valued random variable X is called *symmetric α-stable* (in short, $S\alpha S$) if X satisfies that $E[\exp(izX)]=\exp(-c|z|^\alpha)$, $z\in\boldsymbol{R}$, for some $c\geq 0$. Especially, if $\alpha=2$, X is centered Gaussian. For $0<\alpha<2$, an $\boldsymbol{R}^n$-valued random variable X is called $S\alpha S$ if there exists a symmetric finite measure Γ on the $(n-1)$-dimensional unit sphere S^{n-1} such that $E[\exp(izX)]=\exp(-\int_{\xi=(\xi_1,\xi_2,\cdots,\xi_n)\in S^{n-1}}|\sum_{j=1}^n z_j\xi_j|^\alpha\Gamma(d\xi))$, $z=(z_1,z_2,\cdots,z_n)\in\boldsymbol{R}^n$. This Γ is uniquely determined by the distribution of X and is called the *spectral measure* of X.

From now on, the time parameter space T is an interval in $\boldsymbol{R}$. A stochastic process $X = \{X(t); t \in T\}$ is called $S\alpha S$ if any finite-dimensional distribution of X is $S\alpha S$. We assume that any process presented in this paper is separable. The notation $\sigma(X(s); s \le t)$ denotes the σ-field generated by $X(s)$, $s \le t$. If an $S\alpha S$ process $Z = \{Z(t); t \in T\}$ has independent increments, then there exists a unique measure μ such that $E[\exp(iz(Z(t) - Z(s)))] = \exp(-\mu((s,t])|z|^\alpha)$, $z \in \boldsymbol{R}$, for any s, $t \in T$ $(s < t)$. This μ is called the *control measure* of Z.

DEFINITION. Let $X = \{X(t); t \in T\}$ be an $S\alpha S$ process $(0 < \alpha \le 2)$. Let us assume that X is expressed as

$$(1) \qquad X(t) = \int_{u \le t, u \in T} F(t,u)\, dZ(u), \qquad t \in T,$$

where

(i) $Z = \{Z(t); t \in T\}$ is an $S\alpha S$ process with independent increments (with control measure μ) and

(ii) $F(t, \cdot)$ is a Borel measurable function on $\{u \in T; u \le t\}$ and satisfies that $\int_{(-\infty,t] \cap T} |F(t,\cdot)|^\alpha d\mu < \infty$ for any $t \in T$.

Then the formula (1) is called a *causal representation* of X.

Especially, the representation (1) is called *canonical* if $\sigma(X(s); s \le t) = \sigma(Z(s_2) - Z(s_1); s_1 < s_2 \le t)$ for any $t \in T$.

DEFINITION. Assume that a causal representation of an $S\alpha S$ stationary process $X = \{X(t); t \in \boldsymbol{R}\}$ is expressed as follows:

$$(2) \qquad X(t) = \int_{-\infty}^{t} F(t-u)\, dZ_0(u), \qquad t \in \boldsymbol{R},$$

where $Z_0 = \{Z_0(t); t \in \boldsymbol{R}\}$ is an $S\alpha S$ process with independent stationary increments and F is a Borel measurable function such that $\int_0^\infty |F(x)|^\alpha\, dx < \infty$. Then the formula (2) is called a *causal moving average representation* of X.

T. Hida [1] gave a notion of multiple Markov property for Gaussian processes with continuous time parameter. The author (K. Kojo [3]) extended the notion to general stochastic processes with continuous time parameter as follows.

DEFINITION. A stochastic process $X = \{X(t); t \in T\}$ is called *n-ple Markov of linear combination type* (in short, *LC n-ple Markov*) if X satisfies the following three conditions:

(i) For any fixed $t_0, t_1, \cdots, t_n \in T$ $(\inf(T) < t_0 < t_1 < \cdots < t_n)$, there exists an n-tuple of coefficients $(a_1, a_2, \cdots, a_n) \in \boldsymbol{R}^n \backslash \{\mathbf{0}\}$ such that

$$P\Big[\sum_{j=1}^{n} a_j X(t_j) \in B \Big| X(s), s \le t_0\Big] = P\Big[\sum_{j=1}^{n} a_j X(t_j) \in B \Big| X(t_0)\Big]$$

for any Borel set B in $\boldsymbol{R}$.

(ii) For any fixed $t_0, t_1, \cdots, t_{n-1} \in T$ $(\inf(T) < t_0 < t_1 < \cdots < t_{n-1})$, there exist no $(n-1)$-tuples $(a_1, a_2, \cdots, a_{n-1}) \in \boldsymbol{R}^{n-1} \backslash \{\mathbf{0}\}$ such that

$$P\Big[\sum_{j=1}^{n-1} a_j X(t_j) \in B \Big| X(s), s \le t_0\Big] = P\Big[\sum_{j=1}^{n-1} a_j X(t_j) \in B \Big| X(t_0)\Big]$$

for any Borel set B in $\boldsymbol{R}$.

(iii) For any fixed $t_0, t_1, \cdots, t_n \in T$ $(\inf(T) < t_0 < t_1 < \cdots < t_n)$, there exist no n-tuples $(a_1, a_2, \cdots, a_n) \in \boldsymbol{R}^n \backslash \{\mathbf{0}\}$ such that $\sum_{j=1}^{n} a_j X(t_j)$ is independent of $\sigma(X(s); s \le t_0)$.

§3. Multiple Markov property and determinism

For an $S\alpha S$ process $(0 < \alpha \le 2)$ with a canonical representation, we have the following theorem concerning the multiple Markov property.

THEOREM 3.1. (Kojo [3]) *Assume that an $S\alpha S$ process $X = \{X(t); t \in T\}$ $(0 < \alpha \le 2)$ is expressed as a canonical representation*

$$X(t) = \int_{u \le t, u \in T} F(t,u)\, dZ(u), \qquad t \in T,$$

and is continuous in probability. Then X is LC n-ple Markov if and only if F is expressed as

$$F(t,u) = \sum_{j=1}^{n} f_j(t) g_j(u) \qquad \mu\text{-a.e.}, \qquad u \le t, \tag{3}$$

where μ is the control measure of Z and g_j, f_j, $1 \le j \le n$, satisfy the following conditions:

(i) g_j, $1 \leq j \leq n$, *satisfy* $\int_{(-\infty,t]\cap T} |g_j|^\alpha d\mu < \infty$ *and are linearly independent on* $(-\infty, t] \cap T$ *for any fixed* $t \in T$ $(t \neq \inf(T))$.

(ii) $\det(f_i(t_j))_{1\leq i,j\leq n} \neq 0$ *for any* $t_1, t_2, \cdots, t_n \in T$ $(\inf(T) < t_1 < t_2 < \cdots < t_n)$.

A kernel $F(t,u)$, which is expressed as the formula (3) where g_j, f_j, $1 \leq j \leq n$, satisfy the conditions (i) and (ii), is called a *Goursat kernel of order* n. In non-Gaussian case, we obtain that this process has the following determinism.

THEOREM 3.2. *Let* $X = \{X(t); t \in T\}$ *be a non-Gaussian* $S\alpha S$ *process* $(0 < \alpha < 2)$ *which is expressed as a causal representation*

$$X(t) = \int_{u\leq t, u\in T} F(t,u)\, dZ(u), \qquad t \in T,$$

where $F(t,u)$ *is a Goursat kernel of order* n *and* $\mu(\{u \in T; u \leq t, F(t,u) = 0\}) = 0$ *for any* $t \in T$. *Then any* $S\alpha S$ *process* $\tilde{X} = \{\tilde{X}(t); t \in T\}$ *having the same* $(n+1)$*-dimensional distributions with* X *is identical in law with* X.

Before we start to prove this theorem, we state two important properties on non-Gaussian $S\alpha S$ random variables.

(i) The consistency condition between a multi-dimensional non-Gaussian $S\alpha S$ distribution and its marginal distribution is translated into the consistency condition between their spectral measures as follows: Let Γ' be the spectral measure on S^n of $(n+1)$-dimensional non-Gaussian $S\alpha S$ random variable $Y' = (Y_1, Y_2, \cdots, Y_{n+1})$ and Γ be the spectral measure on S^{n-1} of its n-dimensional marginal $Y = (Y_1, Y_2, \cdots, Y_n)$. Then Γ' and Γ satisfy the formula

$$\Gamma(B) = \int_{\xi=(\xi_1,\xi_2,\cdots,\xi_{n+1})\in\rho_n^{-1}(B)} (1 - {\xi_{n+1}}^2)^{\alpha/2}\Gamma'(d\xi) \qquad \text{for any Borel set } B \text{ in } S^{n-1},$$

where the function $\rho_n: S^n \backslash \{(0, \cdots, 0, \pm 1)\} \to S^{n-1}$ is defined as

$$\rho_n((\xi_1, \xi_2, \cdots, \xi_{n+1})) = (\xi_1/(1 - {\xi_{n+1}}^2)^{1/2}, \xi_2/(1 - {\xi_{n+1}}^2)^{1/2}, \cdots, \xi_n/(1 - {\xi_{n+1}}^2)^{1/2}).$$

(ii) The spectral measure Γ on S^{n-1} of an n-dimensional random variable $(\int_T \varphi_1\, dZ,\ \int_T \varphi_2\, dZ, \cdots, \int_T \varphi_n\, dZ)$ is concentrated on the symmetric set

$$\begin{aligned}&\left\{\sigma_n(\varphi_1(u), \varphi_2(u), \cdots, \varphi_n(u)); u \in T, \sum_{j=1}^{n} \varphi_j(u)^2 \neq 0\right\}\\ &\quad = \{(\xi_1, \xi_2, \cdots, \xi_n) \in S^{n-1}; \xi_1 : \xi_2 : \cdots : \xi_n\\ &\quad = \varphi_1(u) : \varphi_2(u) : \cdots : \varphi_n(u) \text{ for some } u \in T\},\end{aligned}$$

where the correspondence $\sigma_n\colon \boldsymbol{R}^n \backslash \{(0, \cdots, 0)\} \to S^{n-1}$ is defined as

$$\sigma_n(x_1, x_2, \cdots, x_n) = \pm \left(\frac{x_1}{\left(\sum_{j=1}^{n} x_j^2\right)^{1/2}}, \frac{x_2}{\left(\sum_{j=1}^{n} x_j^2\right)^{1/2}}, \cdots, \frac{x_n}{\left(\sum_{j=1}^{n} x_j^2\right)^{1/2}} \right).$$

This is because

$$E\left[\exp\left(i \sum_{j=1}^{n} z_j \int_T \varphi_j\, dZ\right)\right] = \exp\left(-\int_{u \in T} \left|\sum_{j=1}^{n} z_j \varphi_j(u)\right|^{\alpha} \mu(du)\right),$$
$$z = (z_1, z_2, \cdots, z_n) \in \boldsymbol{R}^n,$$

and thus Γ is expressed as

$$\Gamma(B) = \int_{C(B)} \left(\sum_{j=1}^{n} \varphi_j(x)^2\right)^{\alpha/2} \mu(dx)$$
for any symmetric Borel set B in S^{n-1},

where $C(B) = \{x \in T; \sigma_n(\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x)) \in B\}$.

Proof of Theorem 3.2. We will show that any $(n+2)$-dimensional distribution of $\tilde{X}$ coincides with the corresponding $(n+2)$-dimensional distribution of X using the consistency conditions for specific two of their common $(n+1)$-dimensional marginal distributions. In a similar way we can show that X and $\tilde{X}$ have the same higher-dimensional distributions. This proves the theorem.

Firstly let us calculate the spectral measure on S^n of an $(n+1)$-dimensional distribution of X (thus, of $\tilde{X}$). Let Γ be the spectral measure of $(X(t_1), X(t_2), \cdots, X(t_{n+1}))$ for $t_1, t_2, \cdots, t_{n+1} \in T(t_1 < t_2 < \cdots < t_{n+1})$. Define symmetric sets A_j $(\subset S^n)$, $1 \leq j \leq n+1$, as

$$\begin{aligned} A_1 &= \{\sigma_{n+1}(F(t_1,u), F(t_2,u), \cdots, F(t_{n+1},u));\\ &\qquad\qquad u \leq t_1, u \in T, \sum_{k=1}^{n+1} F(t_k,u)^2 \neq 0\}\\ &= \{(\xi_1, \xi_2, \cdots, \xi_{n+1}) \in S^n;\\ &\qquad \xi_1 : \xi_2 : \cdots : \xi_{n+1} = F(t_1,u) : F(t_2,u) : \cdots : F(t_{n+1},u)\\ &\qquad\qquad \text{for some } u (u \leq t_1, u \in T)\},\\ A_j &= \{\sigma_{n+1}(0, \cdots, 0, F(t_j,u), F(t_{j+1},u), \cdots, F(t_{n+1},u));\\ &\qquad\qquad t_{j-1} < u \leq t_j, \sum_{k=j}^{n+1} F(t_k,u)^2 \neq 0\}\\ &= \{(0, \cdots, 0, \xi_j, \xi_{j+1} \cdots, \xi_{n+1}) \in S^n;\\ &\qquad \xi_j : \xi_{j+1} : \cdots : \xi_{n+1} = F(t_j,u) : F(t_{j+1},u) : \cdots : F(t_{n+1},u)\\ &\qquad\qquad \text{for some } u(t_{j-1} < u \leq t_j)\} \end{aligned}$$

for $2 \leq j \leq n$ and

$$A_{n+1} = \{(0, \cdots, 0, \pm 1) \in S^n\}.$$

Then Γ is concentrated on $\cup_{j=1}^{n+1} A_j$. Since $\mu(\{u \in T; u \leq t, F(t,u) = 0\}) = 0$ for any $t \in T$, A_j, $1 \leq j \leq n$, are disjoint except Γ-null sets and we have

$$\Gamma(B_j) = \int_{C_j(B_j)} \left(\sum_{k=j}^{n+1} F(t_k,u)^2\right)^{\alpha/2} \mu(du) \qquad \text{for any symmetric Borel set } B_j \subset A_j$$

for $1 \leq j \leq n$ and

$$\Gamma(A_{n+1}) = \int_{t_n < u \leq t_{n+1}} |F(t_{n+1},u)|^\alpha \mu(du),$$

where

$$\begin{aligned} C_1(B_1) &= \{u \leq t_1, u \in T; \sigma_{n+1}(F(t_1,u), F(t_2,u), \cdots, F(t_{n+1},u)) \in B_1\},\\ C_j(B_j) &= \{t_{j-1} < u \leq t_j;\\ &\qquad \sigma_{n+1}(0, \cdots, 0, F(t_j,u), F(t_{j+1},u), \cdots, F(t_{n+1},u)) \in B_j\} \end{aligned}$$

for $2 \leq j \leq n$.

Now we investigate where the spectral measure on S^{n+1} of an $(n+2)$-dimensional distribution of $\tilde{X}$ is concentrated. Let $\tilde{\Gamma}'$ be the spectral measure of $(\tilde{X}(t_1), \tilde{X}(t_2), \cdots, \tilde{X}(t_{n+2}))$ for $t_1, t_2, \cdots, t_{n+2} \in T(t_1 < t_2 < \cdots < t_{n+2})$. The distribution of $(\tilde{X}(t_1), \tilde{X}(t_2), \cdots, \tilde{X}(t_{n+2}))$ satisfies the consistency condition with its $(n+1)$-dimensional marginal, the distribution of $(\tilde{X}(t_1), \tilde{X}(t_2), \cdots, \tilde{X}(t_{n+1}))$. Therefore $\tilde{\Gamma}'$ is concentrated on the disjoint union of the following $n+2$ symmetric subsets of S^{n+1}:

$$\begin{aligned} B_{1,1} &= \rho_{n+1}^{-1}(A_1) \\ &= \{(\xi_1, \xi_2, \cdots, \xi_{n+2}) \in S^{n+1} \backslash \{(0, \cdots, 0, \pm 1)\}; \\ &\qquad \xi_1 : \xi_2 : \cdots : \xi_{n+1} = F(t_1, u) : F(t_2, u) : \cdots : F(t_{n+1}, u) \\ &\qquad\qquad \text{for some } u (u \leq t_1, u \in T)\}, \end{aligned}$$

$$\begin{aligned} B_{1,j} &= \rho_{n+1}^{-1}(A_j) \\ &= \{(0, \cdots, 0, \xi_j, \xi_{j+1}, \cdots, \xi_{n+2}) \in S^{n+1} \backslash \{(0, \cdots, 0, \pm 1)\}; \\ &\qquad \xi_j : \xi_{j+1} : \cdots : \xi_{n+1} = F(t_j, u) : F(t_{j+1}, u) : \cdots : F(t_{n+1}, u) \\ &\qquad\qquad \text{for some } u (t_{j-1} < u \leq t_j)\} \end{aligned}$$

for $2 \leq j \leq n$,

$$B_{1,n+1} = \rho_{n+1}^{-1}(A_{n+1}) = \{(0, \cdots, 0, \xi_{n+1}, \xi_{n+2}) \in S^{n+1} \backslash \{(0, \cdots, 0, \pm 1)\}\}$$

and

$$B_{1,n+2} = \{(0, \cdots, 0, \pm 1) \in S^{n+1}\}.$$

On the other hand, by the consistency condition with the distribution of $(\tilde{X}(t_2), \tilde{X}(t_3), \cdots, \tilde{X}(t_{n+2}))$, $\tilde{\Gamma}'$ is concentrated on the disjoint union of the following $n+2$ symmetric subsets of S^{n+1}:

$$\begin{aligned} B_{2,1} = \{&(\xi_1, \xi_2, \cdots, \xi_{n+2}) \in S^{n+1} \backslash \{(\pm 1, 0, \cdots, 0)\}; \\ &\xi_2 : \xi_3 : \cdots : \xi_{n+2} = F(t_2, u) : F(t_3, u) : \cdots : F(t_{n+2}, u) \\ &\qquad \text{for some } u (u \leq t_2, u \in T)\}, \end{aligned}$$

$$\begin{aligned} B_{2,j} = \{&(\xi_1, 0, \cdots, 0, \xi_{j+1}, \xi_{j+2}, \cdots, \xi_{n+2}) \in S^{n+1} \backslash \{(\pm 1, 0, \cdots, 0)\}; \\ &\xi_{j+1} : \xi_{j+2} : \cdots : \xi_{n+2} = F(t_{j+1}, u) : F(t_{j+2}, u) : \cdots : F(t_{n+2}, u) \\ &\qquad \text{for some } u (t_j < u \leq t_{j+1})\} \end{aligned}$$

for $2 \leq j \leq n$,

$$B_{2,n+1} = \{(\xi_1, 0, \cdots, 0, \xi_{n+2}) \in S^{n+1} \backslash \{(\pm 1, 0, \cdots, 0)\}\}$$

and

$$B_{2,n+2} = \{(\pm 1, 0, \cdots, 0)\}.$$

Therefore $\tilde{\Gamma}'$ is concentrated on $(n+2)^2$ subsets of S^{n+1}, $B_{1,j} \cap B_{2,k}$, $1 \leq j,\, k \leq n+2$. Since $\mu(\{u \in T; u \leq t, F(t,u) = 0\}) = 0$ for any $t \in T$, $\tilde{\Gamma}'$ is concentrated on $B_{1,1} \cap B_{2,1}$, $B_{1,j} \cap B_{2,j-1}$, $2 \leq j \leq n+2$.

Define a symmetric set A_1' as

$$\begin{aligned} A_1' = \{(\xi_1, \xi_2, \cdots, \xi_{n+2}) \in S^{n+1};& \\ \xi_1 : \xi_2 : \cdots : \xi_{n+2} = F(t_1, u) &: F(t_2, u) : \cdots : F(t_{n+2}, u) \\ &\text{for some } u (u \leq t_1, u \in T)\}. \end{aligned}$$

Let us prove $B_{1,1} \cap B_{2,1} = A_1'$ except a $\tilde{\Gamma}'$-null set. We can easily see $B_{1,1} \cap B_{2,1} \supset A_1'$. Let us show $B_{1,1} \cap B_{2,1} \subset A_1'$. Suppose that $\xi = (\xi_1, \xi_2, \cdots, \xi_{n+2}) \in B_{1,1} \cap B_{2,1}$. Then there exist u_1 $(\leq t_1)$ and u_2 $(\leq t_2)$ such that

$$\xi_1 : \xi_2 : \cdots : \xi_{n+1} = F(t_1, u_1) : F(t_2, u_1) : \cdots : F(t_{n+1}, u_1)$$

and

$$\xi_2 : \xi_3 : \cdots : \xi_{n+2} = F(t_2, u_2) : F(t_3, u_2) : \cdots : F(t_{n+2}, u_2).$$

Thus we have

$$\begin{aligned} F(t_2, u_1) : F(t_3, u_1) : \cdots : F(t_{n+1}, u_1)& \\ = F(t_2, u_2) &: F(t_3, u_2) : \cdots : F(t_{n+1}, u_2). \end{aligned}$$

Here, let us recall that F is a Goursat kernel of order n. Then $F(t_{n+2}, u)$ can be expressed as

$$\begin{aligned} F(t_{n+2}, u) &= \sum_{j=1}^{n} f_j(t_{n+2}) g_j(u) \\ &= (f_1(t_{n+2}), \cdots, f_n(t_{n+2}))\, ((f_j(t_{i+1}))_{1 \leq i,j \leq n})^{-1} \\ &\qquad {}^{\mathbf{t}}(F(t_2, u), \cdots, F(t_{n+1}, u)), \quad u \leq t_2, \end{aligned}$$

where ${}^{\mathbf{t}}\boldsymbol{v}$ denotes the transposed of a row-vector $\boldsymbol{v}$. Thus we have

$$\begin{aligned} F(t_2, u_1) : F(t_3, u_1) : \cdots : F(t_{n+1}, u_1) : F(t_{n+2}, u_1)& \\ = F(t_2, u_2) : F(t_3, u_2) : \cdots &: F(t_{n+1}, u_2) : F(t_{n+2}, u_2) \end{aligned}$$

for almost all such u_1 and u_2. This implies that almost all ξ satisfies

$$\xi_1 : \xi_2 : \cdots : \xi_{n+1} : \xi_{n+2} \\ = F(t_1, u_1) : F(t_2, u_1) : \cdots : F(t_{n+1}, u_1) : F(t_{n+2}, u_1)$$

and thus $\xi \in A'_1$. Now we have $B_{1,1} \cap B_{2,1} = A'_1$ except a $\tilde{\Gamma}'$-null set.

Define a symmetric set A'_2 as

$$A'_2 = \{(0, \xi_2, \xi_3, \cdots, \xi_{n+2}) \in S^{n+1}; \\ \xi_2 : \xi_3 : \cdots : \xi_{n+2} = F(t_2, u) : F(t_3, u) : \cdots : F(t_{n+2}, u) \\ \text{for some } u(t_1 < u \leq t_2)\}.$$

Then we have $B_{1,2} \cap B_{2,1} = A'_2$ except a $\tilde{\Gamma}'$-null set by a similar argument. Define A'_j, $3 \leq j \leq n+2$ as

$$A'_j = \{(0, \cdots, 0, \xi_j, \xi_{j+1}, \cdots, \xi_{n+2}) \in S^{n+1}; \\ \xi_j : \xi_{j+1} : \cdots : \xi_{n+2} = F(t_j, u) : F(t_{j+1}, u) : \cdots : F(t_{n+2}, u) \\ \text{for some } u(t_{j-1} < u \leq t_j)\} \\ \text{for } 3 \leq j \leq n+1$$

and

$$A'_{n+2} = \{(0, \cdots, 0, \pm 1) \in S^{n+1}\}.$$

We easily obtain that $B_{1,j} \cap B_{2,j-1} = A'_j$, $3 \leq j \leq n+2$. Thus we find that $\tilde{\Gamma}'$ is concentrated on the disjoint union $\cup_{j=1}^{n+2} A'_j$.

Now let us consider what measure lies on A'_1. We recall again that, if $u_1, u_2 \leq t_2$ satisfy

$$F(t_2, u_1) : F(t_3, u_1) : \cdots : F(t_{n+1}, u_1) \\ = F(t_2, u_2) : F(t_3, u_2) : \cdots : F(t_{n+1}, u_2),$$

then

$$F(t_2, u_1) : F(t_3, u_1) : \cdots : F(t_{n+1}, u_1) : F(t_{n+2}, u_1) \\ = F(t_2, u_2) : F(t_3, u_2) : \cdots : F(t_{n+1}, u_2) : F(t_{n+2}, u_2).$$

Consider the following correspondence $\psi : A_1 \to A'_1$:

$$\psi : \xi = (\xi_1, \xi_2, \cdots, \xi_{n+1}) \text{ which satisfies } \xi_1 : \xi_2 : \cdots : \xi_{n+1} \\ = F(t_1, u) : F(t_2, u) : \cdots : F(t_{n+1}, u) \text{ for some } u(u \leq t_1, u \in T) \\ \mapsto \xi' = (\xi'_1, \xi'_2, \cdots, \xi'_{n+2}) \text{ which satisfies } \xi'_1 : \xi'_2 : \cdots : \xi'_{n+1} : \xi'_{n+2} \\ = F(t_1, u) : F(t_2, u) : \cdots : F(t_{n+1}, u) : F(t_{n+2}, u).$$

Let $\tilde{\psi} : A_1/\sim \to A'_1/\sim$ be the correspondence induced by ψ where the equivalence relation $\eta \sim \eta'$ denotes $\eta' = \eta$ or $\eta' = -\eta$. Then $\tilde{\psi}$ is one-to-one except a $\tilde{\Gamma}'$-null set. For any symmetric Borel set $B'_1(\subset A'_1)$, let $B_1 = \rho_{n+1}(B'_1)$. By the consistency condition between the distributions of $(\tilde{X}(t_1), \tilde{X}(t_2), \cdots, \tilde{X}(t_{n+1}))$ and $(\tilde{X}(t_1), \tilde{X}(t_2), \cdots, \tilde{X}(t_{n+2}))$,

$$\Gamma(B_1) = \int_{\xi \in \rho_{n+1}^{-1}(B_1)} (1 - {\xi_{n+2}}^2)^{\alpha/2} \tilde{\Gamma}'(d\xi).$$

Since $\mu(\{u \in T; u \leq t, F(t,u) = 0\}) = 0$ and $\tilde{\psi}$ is one-to-one, we have $\rho_{n+1}^{-1}(B_1) \cap (\cup_{j=1}^{n+2} A'_j) = B'_1$. Therefore

$$\Gamma(B_1) = \int_{\xi \in B'_1} (1 - {\xi_{n+2}}^2)^{\alpha/2} \tilde{\Gamma}'(d\xi).$$

Since $\tilde{\psi}$ is one-to-one, we have

$$\begin{aligned}\tilde{\Gamma}'(B'_1) &= \int_{C_1(B_1)} \left(1 - \frac{F(t_{n+2}, u)^2}{\sum_{j=1}^{n+2} F(t_j, u)^2}\right)^{-\frac{\alpha}{2}} \left(\sum_{j=1}^{n+1} F(t_j, u)^2\right)^{\frac{\alpha}{2}} \mu(du) \\ &= \int_{C_1(B_1)} \left(\sum_{j=1}^{n+2} F(t_j, u)^2\right)^{\frac{\alpha}{2}} \mu(du).\end{aligned}$$

Let

$$C'_1(B'_1) = \Big\{ u \leq t_1, u \in T; \sigma_{n+2}(F(t_1,u), F(t_2,u), \cdots, F(t_{n+2},u)) \in B'_1, \sum_{k=1}^{n+2} F(t_k,u)^2 \neq 0 \Big\}.$$

We can easily see that $C'_1(B'_1) = C_1(B_1)$ except a μ-null set. Thus we have

$$\tilde{\Gamma}'(B'_1) = \int_{C'_1(B'_1)} \left(\sum_{j=1}^{n+2} F(t_j, u)^2\right)^{\frac{\alpha}{2}} \mu(du)$$

Now we conclude that the measure on A'_1 is uniquely determined by the consistency conditions.

It is easy to see that the measures on A'_j, $2 \leq j \leq n+2$, are uniquely determined by the consistency condition between the distributions of $(\tilde{X}(t_2), \tilde{X}(t_3), \cdots, \tilde{X}(t_{n+2}))$ and $(\tilde{X}(t_1), \tilde{X}(t_2), \cdots, \tilde{X}(t_{n+2}))$. Hence we obtain that $\tilde{\Gamma}'$ is uniquely determined by the consistency conditions, that is, $\tilde{\Gamma}'$ coincides with the spectral measure of $(X(t_1), X(t_2), \cdots, X(t_{n+2}))$. Now we conclude that X and $\tilde{X}$ have the same $(n+2)$-dimensional distributions. $\square$

§4. The case of causal moving average processes

In this section we confine our arguments to $S\alpha S$ stationary processes represented by causal moving averages.

PROPOSITION 4.1. *Assume that an $S\alpha S$ process $X = \{X(t); t \in \boldsymbol{R}\}$ $(0 < \alpha \leq 2)$ is represented by a canonical moving average*

$$X(t) = \int_{-\infty}^{t} F(t-u)\, dZ_0(u), \qquad t \in \boldsymbol{R},$$

and is continuous in probability. Then X is LC n-ple Markov if and only if F is expressed as

$$\text{(4)}\quad F(x) = \sum_{j=1}^{r} (b_{j,m_j-1}x^{m_j-1} + b_{j,m_j-2}x^{m_j-2} + \cdots + b_{j,0})e^{-\lambda_j x}, \quad x \geq 0,$$

where $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_r$, $\sum_{j=1}^{r} m_j = n$, $b_{j,m_j-1} \neq 0$ $(1 \leq j \leq r)$.

In Gaussian case ($\alpha = 2$), this proposition is included in T. Hida's paper ([1], Theorem II.3).

Proof. By Theorem 3.1, we have only to show that $F(t-u)$ is a Goursat kernel of order n if and only if $F(x)$ is expressed as the formula (4).

'*only if*' *part*: By the assumption, for any $t \in \boldsymbol{R}$, $F(t-u)$ can be expressed as $F(t-u) = \sum_{j=1}^{n} f_j(t)g_j(u)$ for almost all $u \leq t$. Firstly we prove that f_j, $1 \leq j \leq n$, are continuous. Suppose that there exist j_0 $(1 \leq j_0 \leq n)$ and $t_0 \in \boldsymbol{R}$ such that f_{j_0} is discontinuous at t_0. This means that there exist ε (> 0) and a sequence $\{t_k\}_{k=1,2,\cdots}$ such that $t_k \to t_0$ as $k \to \infty$ and $|f_{j_0}(t_k) - f_{j_0}(t_0)| > \varepsilon$ for any k. Since X is continuous in probability,

$$\int_{-\infty}^{t_0-1} |F(t_k-u) - F(t_0-u)|^{\alpha}\, du \to 0 \qquad \text{as } k \to \infty.$$

On the other hand we have

$$\begin{aligned} &F(t_k-u) - F(t_0-u) \\ &\quad = \sum_{j=1}^{n} (f_j(t_k) - f_j(t_0))g_j(u) \end{aligned}$$

$$= \left(\sum_{l=1}^{n}(f_l(t_k)-f_l(t_0))^2\right)^{\frac{1}{2}} \sum_{j=1}^{n} \frac{f_j(t_k)-f_j(t_0)}{\left(\sum_{l=1}^{n}(f_l(t_k)-f_l(t_0))^2\right)^{1/2}} g_j(u),$$

$$u \le t_0 \wedge t_k.$$

Therefore we have

$$\int_{-\infty}^{t_0-1} \left| \sum_{j=1}^{n} \frac{f_j(t_k)-f_j(t_0)}{\left(\sum_{l=1}^{n}(f_l(t_k)-f_l(t_0))^2\right)^{1/2}} g_j(u) \right|^{\alpha} du \to 0 \qquad \text{as } k \to \infty.$$

Since the sequence $\{((f_1(t_k)-f_1(t_0))/(\sum_{l=1}^{n}(f_l(t_k)-f_l(t_0))^2)^{1/2}, \cdots, (f_n(t_k) - f_n(t_0))/(\sum_{l=1}^{n}(f_l(t_k)-f_l(t_0))^2)^{1/2})\}_{k=1,2,\cdots}$ is in S^{n-1}, there exist a subsequence $\{t_{k_m}\}_{m=1,2,\cdots}$ and $(c_1, c_2, \cdots, c_n) \in S^{n-1}$ such that $(f_j(t_{k_m}) - f_j(t_0))/(\sum_{l=1}^{n}(f_l(t_{k_m})-f_l(t_0))^2)^{1/2} \to c_j$, $1 \le j \le n$, as $m \to \infty$. By Lebesgue's convergence theorem, we have $\int_{-\infty}^{t_0-1} |\sum_{j=1}^{n} c_j g_j(u)|^{\alpha}\, du = 0$. This is contradictory to the linear independence of g_j, $1 \le j \le n$. Now we conclude that f_j, $1 \le j \le n$, are continuous.

Next we prove that F is continuous on $[0,\infty)$. Since f_j, $1 \le j \le n$, are continuous, the set $\{(u,t) \in \boldsymbol{R}^2; u \le t, F(t-u) \ne \sum_{j=1}^{n} f_j(t)g_j(u)\}$ is Borel measurable and by Fubini's theorem, this set is null. Applying Fubini's theorem again, there is a subset A of $[0,\infty)$ such that $[0,\infty)\backslash A$ is null and the set $\{y; y \in \boldsymbol{R}, F(x) \ne \sum_{j=1}^{n} f_j(x+y)g_j(y)\}$ is null for any $x \in A$. Now, for any $x_0 \in A$ and any sequence $\{x_k \in A\}_{k=1,2,\cdots}$ which tends to x_0 as $k \to \infty$, we can choose $y \in \boldsymbol{R}$ such that $F(x_k) = \sum_{j=1}^{n} f_j(x_k+y)g_j(y)$ for any $k = 1, 2, \cdots$. Since f_j, $1 \le j \le n$, are continuous, F is continuous on A. Since A is dense in $[0,\infty)$, F is continuous on $[0,\infty)$. We can easily see that g_j, $1 \le j \le n$, are also continuous on $(-\infty,\infty)$, using that F is continuous and $\det(f_i(t_j))_{1\le i,j\le n} \ne 0$ for any distinct t_j, $1 \le j \le n$.

Let us prove that f_j, $1 \le j \le n$, are differentiable on $(-\infty,\infty)$. We first show that, for any fixed $t_0 \in T$, we can choose $s_1, s_2, \cdots, s_n$ $(< t_0)$ such that $\det(\int_{s_j}^{t_0} g_i du)_{1\le i,j\le n} \ne 0$. We can choose s_1 $(< t_0)$ such that $\int_{s_1}^{t_0} g_1 du \ne 0$, if otherwise, $g_1 \equiv 0$ on $(-\infty, t_0]$ and this is a contradiction. Suppose that we choose $s_1, s_2, \cdots, s_k$ $(< t_0)$ $(1 \le k < n)$ such that $\det(\int_{s_j}^{t_0} g_i du)_{1\le i,j\le k} \ne 0$. If $\det(\int_{s_j}^{t_0} g_i du)_{1\le i,j\le k+1} = 0$ for any s_{k+1} $(< t_0)$, then there exist functions

$p_j\colon (-\infty, t_0) \cap T \to \boldsymbol{R}$, $1 \le j \le k$ such that

$$
{}^{\mathbf{t}}\left(\int_{s_{k+1}}^{t_0} g_1\, du, \cdots, \int_{s_{k+1}}^{t_0} g_{k+1}\, du\right) = \left(\int_{s_j}^{t_0} g_i\, du\right)_{1\le i\le k+1, 1\le j\le k} {}^{\mathbf{t}}(p_1(s_{k+1}), \cdots, p_k(s_{k+1})).
$$

Since $\det(\int_{s_j}^{t_0} g_i du)_{1\le i,j\le k} \ne 0$, we have

$$
{}^{\mathbf{t}}(p_1(s_{k+1}), \cdots, p_k(s_{k+1})) = \left(\left(\int_{s_j}^{t_0} g_i\, du\right)_{1\le i,j\le k}\right)^{-1} {}^{\mathbf{t}}\left(\int_{s_{k+1}}^{t_0} g_1\, du, \cdots, \int_{s_{k+1}}^{t_0} g_k\, du\right).
$$

Therefore

$$
\begin{aligned}
\int_{s_{k+1}}^{t_0} g_{k+1}\, du &= \left(\int_{s_1}^{t_0} g_{k+1}\, du, \cdots, \int_{s_k}^{t_0} g_{k+1}\, du\right) {}^{\mathbf{t}}(p_1(s_{k+1}), \cdots, p_k(s_{k+1})) \\
&= \left(\int_{s_1}^{t_0} g_{k+1}\, du, \cdots, \int_{s_k}^{t_0} g_{k+1}\, du\right) \\
&\quad \left(\left(\int_{s_j}^{t_0} g_i\, du\right)_{1\le i,j\le k}\right)^{-1} {}^{\mathbf{t}}\left(\int_{s_{k+1}}^{t_0} g_1\, du, \cdots, \int_{s_{k+1}}^{t_0} g_k\, du\right)
\end{aligned}
$$

for any s_{k+1} $(< t_0)$. This is contradictory to the linear independence of g_j, $1 \le j \le k+1$, and thus for any fixed $t_0 \in T$, we can choose $s_1, s_2, \cdots, s_n$ $(< t_0)$ such that $\det(\int_{s_j}^{t_0} g_i du)_{1\le i,j\le n} \ne 0$. Therefore the formulas $\int_{s_i}^{t} F(t-u)du = \sum_{j=1}^{n} f_j(t) \int_{s_i}^{t} g_j(u)du$, $1 \le i \le n$, imply that

$$
{}^{\mathbf{t}}(f_1(t), \cdots, f_n(t)) = \left(\left(\int_{s_i}^{t} g_j\, du\right)_{1\le i,j\le n}\right)^{-1} {}^{\mathbf{t}}\left(\int_{0}^{t-s_1} F(x)\, dx, \cdots, \int_{0}^{t-s_n} F(x)\, dx\right)
$$

holds in a neighborhood of $t = t_0$. Since the right hand side is differentiable at $t = t_0$, f_j, $1 \le j \le n$, are differentiable at $t = t_0$. Now we conclude that f_j, $1 \le j \le n$, are differentiable.

It is easy to see that F and f_j, g_j, $1 \le j \le n$, are infinitely differentiable. Moreover the formulas

$$
\begin{aligned}
\frac{d^k}{dt^k} F(t) &= \frac{\partial^k}{\partial t^k} F(t-u)\Big|_{u=0} = (-1)^k \frac{\partial^k}{\partial u^k} F(t-u)\Big|_{u=0} \\
&= (-1)^k \sum_{j=1}^{n} f_j(t) \frac{d^k}{du^k} g_j(u)\Big|_{u=0}, \qquad t > 0, \quad 0 \le k \le n,
\end{aligned}
$$

imply that F satisfies a differential equation $\sum_{k=0}^{n} q_k (d^k/dt^k)F(t) = 0$, $t \geq 0$, for some constants q_k, $0 \leq k \leq n$.

Suppose that F is also a solution of $\sum_{k=0}^{n-1} q'_k (d^k/dt^k)F(t) = 0$, $t \geq 0$, for some q'_k, $0 \leq k \leq n-1$. Then F is a Goursat kernel of order $n-1$ and this is a contradiction. Suppose that the characteristic equation $\sum_{k=0}^{n} q_k x^k = 0$ has an imaginary solution $\lambda + i\eta$. Then we have $f_{j_0}(t) = e^{\lambda t}\cos\eta t$ and $f_{j'_0}(t) = e^{\lambda t}\sin\eta t$ for some j_0, j'_0 $(1 \leq j_0, j'_0 \leq n)$. It follows that $\det(f_i(t_j))_{1\leq i,j\leq n} = 0$ for $t_j = 2\pi j/\eta$, $1 \leq j \leq n$, and this is a contradiction. Suppose that $\sum_{k=0}^{n} q_k x^k = 0$ has a non-negative solution λ. Then we have $g_{j_0}(u) = e^{-\lambda u}$ for some j_0 $(1 \leq j_0 \leq n)$. It follows that $\int_{-\infty}^{0} |g_{j_0}|^\alpha \, du = \infty$ and this is a contradiction. Hence we find that all the solutions of $\sum_{k=0}^{n} q_k x^k = 0$ are negative, so that F can be expressed as the formula (4). We finish the proof of 'only if' part.

It is easy to prove 'if' part and so we omit the proof. □

Remark 4.2. Let $X = \{X(t); t \in \boldsymbol{R}\}$ be an $S\alpha S$ process $(0 < \alpha \leq 2)$ defined as a causal moving average (2) where F is expressed as (4). In non-Gaussian case $(0 < \alpha < 2)$, the representation (2) of X is always canonical (Kojo [2]). Therefore X is LC n-ple Markov by Proposition 4.1 and has determinism of dimension $n+1$ by Theorem 3.2.

On the other hand in Gaussian case $(\alpha = 2)$, this representation (2) is not always canonical. For example, let $X = \{X(t); t \in \boldsymbol{R}\}$ be a centered Gaussian process defined as $X(t) = \int_{-\infty}^{t} (3e^{-(t-u)} - 4e^{-2(t-u)}) dB_0(u), t \in \boldsymbol{R}$, where B_0 is a Brownian motion on $\boldsymbol{R}$. This representation of X is not canonical because $\int_{-\infty}^{t} e^{2u} dB_0(u)$ is independent of $\sigma(X(s); s \leq t)$. In fact, X has a canonical representation $X(t) = \int_{-\infty}^{t} e^{-(t-u)} d\tilde{B}_0(u)$, $t \in \boldsymbol{R}$, where $\tilde{B}_0$ is another Brownian motion.

We know that a Gaussian process has determinism of dimension 2. Obviously, 2 is the smallest such dimension for the process. This fact is independent of whether the process is multiple Markov or not. However in this paper, even if a non-Gaussian $S\alpha S$ process $X = \{X(t); t \in T\}$ is multiple Markov, we cannot yet find the smallest number d (≥ 2) such that X has determinism of dimension d. But, for some restricted classes, we can show that the smallest number is less than or equal to 3. For these processes, this is a better result than what we have obtained in Theorem 3.2.

PROPOSITION 4.3. *Let F be a function on $[0, \infty)$ which satisfies the condition* (i) *or* (ii):

(i) *$F(x) \neq 0$ for $x > 0$ and $F(x+h)/F(x)$ is strictly monotone in x for any fixed $h > 0$.*

(ii) *F is expressed as $F(x) = e^{-\lambda_0 x}(b_0 + b_1 e^{-\lambda x} + b_2 e^{-2\lambda x})$ for some λ_0, $\lambda > 0$, b_0, b_1, $b_2 \neq 0$.*

Let us define a non-Gaussian $S\alpha S$ process $X = \{X(t); t \in \boldsymbol{R}\}$ $(0 < \alpha < 2)$ as a causal moving average representation $X(t) = \int_{-\infty}^{t} F(t-u)\, dZ_0(u)$, $t \in \boldsymbol{R}$. Then any $S\alpha S$ process $\tilde{X} = \{\tilde{X}(t); t \in \boldsymbol{R}\}$ having the same 3-dimensional distributions with X is identical in law with X.

For example, let $F(x) = \sum_{j=1}^{n} b_j e^{-\lambda_j x}$ where $\lambda_j > 0$, $b_j > 0$, $1 \leq j \leq n$. Then $F(x) \neq 0$ for $x > 0$ and

$$\begin{aligned}
&(F(x+h)/F(x))' \\
&= F(x)^{-2}(F'(x+h)F(x) - F(x+h)F'(x)) \\
&= F(x)^{-2}\Bigg(-\sum_{j=1}^{n} b_j \lambda_j e^{-\lambda_j(x+h)} \sum_{k=1}^{n} b_k e^{-\lambda_k x} \\
&\qquad\qquad + \sum_{j=1}^{n} b_j e^{-\lambda_j(x+h)} \sum_{k=1}^{n} b_k \lambda_k e^{-\lambda_k x}\Bigg) \\
&= F(x)^{-2} \sum_{1 \leq j < k \leq n} b_j b_k (\lambda_k - \lambda_j) e^{-(\lambda_j + \lambda_k)x}(e^{-\lambda_j h} - e^{-\lambda_k h}) > 0.
\end{aligned}$$

Thus F satisfies the condition (i). In the case (ii) we already know that X has determinism of dimension 4 by Theorem 3.2.

Proof. Firstly let us calculate the spectral measure Γ on S^2 of 3-dimensional random variable $(\tilde{X}(t), \tilde{X}(t+h_1), \tilde{X}(t+h_1+h_2))$ $(t \in \boldsymbol{R}$, h_1, $h_2 > 0)$. Define symmetric sets A_1, A_2, A_3 $(\subset S^2)$ as

$$\begin{aligned}
A_1 &= \{\sigma_3(F(x), F(x+h_1), F(x+h_1+h_2)); 0 \leq x < \infty\} \\
&= \{(\xi_1, \xi_2, \xi_3) \in S^2; \xi_1 : \xi_2 : \xi_3 = F(x) : F(x+h_1) : F(x+h_1+h_2) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for some } x (0 \leq x < \infty)\}, \\
A_2 &= \{\sigma_3(0, F(x), F(x+h_2)); 0 \leq x < h_1\} \\
&= \{(0, \xi_2, \xi_3) \in S^2; \xi_2 : \xi_3 = F(x) : F(x+h_2) \text{ for some } x (0 \leq x < h_1)\}
\end{aligned}$$

and

$$A_3 = \{(0, 0, \pm 1)\}.$$

Then Γ is concentrated on $A_1 \cup A_2 \cup A_3$. The set $\{x \geq 0; F(x) = 0\}$ is null under the condition (i) or (ii) and thus we have

$$\Gamma(B_1) = \int_{C_1(B_1)} (F(x)^2 + F(x+h_1)^2 + F(x+h_1+h_2)^2)^{\alpha/2}\, dx$$
$$\text{for any symmetric Borel set } B_1 \subset A_1,$$
$$\Gamma(B_2) = \int_{C_2(B_2)} (F(x)^2 + F(x+h_2)^2)^{\alpha/2}\, dx$$
$$\text{for any symmetric Borel set } B_2 \subset A_2$$

and

$$\Gamma(A_3) = \int_0^{h_2} |F(x)|^\alpha dx,$$

where

$$C_1(B_1) = \{0 \leq x < \infty; \sigma_3(F(x), F(x+h_1), F(x+h_1+h_2)) \in B_1\},$$
$$C_2(B_2) = \{0 \leq x < h_1; \sigma_3(0, F(x), F(x+h_2)) \in B_2\}.$$

Let $\tilde{\Gamma}'$ be the spectral measure on S^3 of 4-dimensional random variable $(\tilde{X}(t), \tilde{X}(t+h_1), \tilde{X}(t+h_1+h_2), \tilde{X}(t+h_1+h_2+h_3))$ ($t \in \boldsymbol{R}$, h_1, h_2, $h_3 > 0$). The consistency conditions with its four 3-dimensional marginal distributions imply that $\tilde{\Gamma}'$ is concentrated on symmetric sets

$$\begin{aligned} G_1 = \{(\xi_1, \xi_2, \xi_3, \xi_4) \in S^3;\ & \\ & \xi_2 : \xi_3 : \xi_4 = F(x_1) : F(x_1+h_2) : F(x_1+h_2+h_3) \\ & \qquad \text{for some } x_1 (0 \leq x_1 < \infty), \\ & \xi_1 : \xi_3 : \xi_4 = F(x_2) : F(x_2+h_1+h_2) : F(x_2+h_1+h_2+h_3) \\ & \qquad \text{for some } x_2 (0 \leq x_2 < \infty), \\ & \xi_1 : \xi_2 : \xi_4 = F(x_3) : F(x_3+h_1) : F(x_3+h_1+h_2+h_3) \\ & \qquad \text{for some } x_3 (0 \leq x_3 < \infty), \\ & \xi_1 : \xi_2 : \xi_3 = F(x_4) : F(x_4+h_1) : F(x_4+h_1+h_2) \\ & \qquad \text{for some } x_4 (0 \leq x_4 < \infty)\}, \end{aligned}$$

$$\begin{aligned} G_2 = \{(0, \xi_2, \xi_3, \xi_4) \in S^3;\ & \\ & \xi_2 : \xi_3 : \xi_4 = F(x_1) : F(x_1+h_2) : F(x_1+h_2+h_3) \\ & \qquad \text{for some } x_1 (0 \leq x_1 < \infty), \end{aligned}$$

$$\xi_3 : \xi_4 = F(x_2 + h_2) : F(x_2 + h_2 + h_3) \quad \text{for some } x_2 (0 \leq x_2 < h_1),$$
$$\xi_2 : \xi_4 = F(x_3) : F(x_3 + h_2 + h_3) \quad \text{for some } x_3 (0 \leq x_3 < h_1),$$
$$\xi_2 : \xi_3 = F(x_4) : F(x_4 + h_2) \quad \text{for some } x_4 (0 \leq x_4 < h_1)\},$$

$$A_3' = \{(0, 0, \xi_2, \xi_3) \in S^3; \\ \xi_2 : \xi_3 = F(x) : F(x + h_3) \text{ for some } x (0 \leq x < h_2)\},$$

$$A_4' = \{(0, 0, 0, \pm 1)\},$$

and that the measures on A_3' and A_4' are uniquely determined.

Here we note the following fact: If F is expressed as (i), there are no solutions $(y_1, y_2, y_3) \in [0, \infty)^3 \setminus \{(y, y, y); 0 \leq y < \infty\}$ of the equations

$$\begin{cases} F(y_2)F(y_3 + h_1) = F(y_3)F(y_2 + h_1) \\ F(y_3)F(y_1 + h_1 + h_2) = F(y_1)F(y_3 + h_1 + h_2) \\ F(y_1)F(y_2 + h_1 + h_2 + h_3) = F(y_2)F(y_1 + h_1 + h_2 + h_3) \end{cases}$$

for any fixed h_1, h_2, $h_3 > 0$. If F is expressed as (ii), the above equations have at most six solutions $(y_1, y_2, y_3) \in [0, \infty)^3 \setminus \{(y, y, y); 0 \leq y < \infty\}$. Therefore the measure on G_1 is concentrated on symmetric set

$$A_1' = \{(\xi_1, \xi_2, \xi_3, \xi_4) \in S^3; \xi_1 : \xi_2 : \xi_3 : \xi_4 \\ = F(x) : F(x + h_1) : F(x + h_1 + h_2) : F(x + h_1 + h_2 + h_3) \\ \text{for some } x (0 \leq x < \infty)\}.$$

Define the correspondence $\psi: A_1 \to A_1'$ as

$$\psi : \xi = (\xi_1, \xi_2, \xi_3) \\ \text{which satisfies } \xi_1 : \xi_2 : \xi_3 = F(x) : F(x + h_1) : F(x + h_1 + h_2) \\ \text{for some } x (0 \leq x < \infty) \\ \mapsto \xi' = (\xi_1', \xi_2', \xi_3', \xi_4') \text{ which satisfies } \xi_1' : \xi_2' : \xi_3' : \xi_4' \\ = F(x) : F(x + h_1) : F(x + h_1 + h_2) : F(x + h_1 + h_2 + h_3).$$

Then the induced correspondence $\tilde{\psi} : A_1/\sim \to A_1'/\sim$ is one-to-one except finite points of $A_1'/\sim$.

Let us consider what measure lies on A_1'. Let B_1' be a symmetric Borel set in A_1' and set $B_1 = \rho_4(B_1')$. Since $\tilde{\psi}$ is one-to-one, the measure on A_1' is uniquely determined by the consistency condition between the distributions of $(\tilde{X}(t), \tilde{X}(t+h_1), \tilde{X}(t+h_1+h_2))$ and $(\tilde{X}(t), \tilde{X}(t+h_1), \tilde{X}(t+h_1+h_2), \tilde{X}(t+h_1+h_2+h_3))$ as follows:

$$\begin{aligned}
&\tilde{\Gamma}'(B_1') \\
&= \int_{C_1(B_1)} \left(1 - \frac{F(x+h_1+h_2+h_3)^2}{F(x)^2+F(x+h_1)^2+F(x+h_1+h_2)^2+F(x+h_1+h_2+h_3)^2}\right)^{-\frac{\alpha}{2}} \\
&\qquad \times (F(x)^2 + F(x+h_1)^2 + F(x+h_1+h_2)^2)^{\alpha/2} dx, \\
&= \int_{C_1(B_1)} (F(x)^2+F(x+h_1)^2+F(x+h_1+h_2)^2+F(x+h_1+h_2+h_3)^2)^{\alpha/2} dx.
\end{aligned}$$

Let

$$\begin{aligned}
C_1'(B_1') = \{&0 \le x < \infty; \\
&\sigma_4(F(x), F(x+h_1), F(x+h_1+h_2), F(x+h_1+h_2+h_3)) \in B_1'\}.
\end{aligned}$$

Then we have $C_1'(B_1') = C_1(B_1)$ except a null set and thus

$$\begin{aligned}
\tilde{\Gamma}'(B_1') = \int_{C_1'(B_1')} &(F(x)^2 + F(x+h_1)^2 + F(x+h_1+h_2)^2 \\
&+ F(x+h_1+h_2+h_3)^2)^{\alpha/2}\, dx.
\end{aligned}$$

This implies that the measure on G_2 is uniquely determined by the consistency condition between the distributions of $(\tilde{X}(t+h_1), \tilde{X}(t+h_1+h_2), \tilde{X}(t+h_1+h_2+h_3))$ and $(\tilde{X}(t), \tilde{X}(t+h_1), \tilde{X}(t+h_1+h_2), \tilde{X}(t+h_1+h_2+h_3))$. In fact, this measure is concentrated on

$$\begin{aligned}
A_2' = \{(0, \xi_1, \xi_2, \xi_3) \in S^2; \xi_1 : \xi_2 : \xi_3 = F(x) : F(x+h_2) : F(x+h_2+h_3) \\
\text{for some } x (0 \le x < h_1)\}.
\end{aligned}$$

Hence we find that $\tilde{\Gamma}'$ is uniquely determined by the consistency conditions. Now we conclude that X and $\tilde{X}$ have the same 4-dimensional distributions. We apply the similar arguments as above or Theorem 3.2 in the case (i) or (ii) respectively and obtain that X and $\tilde{X}$ have the same finite-dimensional distributions. $\square$

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