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SYMPLECTIC STRUCTURES AND SYMMETRIES OF SOLUTIONS OF THE COMPLEX MONGE-AMPÈRE EQUATION

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Abstract. The graphs that arise from the gradients of solutions u of the homogeneous complex Monge-Ampère equation are characterized in terms of the natural symplectic structure on the cotangent bundle. This characterization is invariant under symplectic biholomorphisms. Using the symplectic structures we construct symmetries (to be called Lempert transformations) for real valued functions u which are absolutely continuous on lines. We then use these symmetries to generate interesting solutions to the homogeneous complex Monge-Ampère equation and to transform the Poincaré–Lelong equation and the ∂-equation. An example of Lempert transform is given and the main theorem is applied to prove regularity results for exterior nonlinear Dirichlet problem for the homogeneous complex Monge-Ampère equation.

§1. Introduction and Notation

The homogeneous complex Monge-Ampère equation, (in short HC-MAE), takes the form

$$\det\left[\frac{\partial^2}{\partial z_j \partial \overline{z}_k} u\right] = 0 \tag{1.1}$$

in local coordinates on a complex manifold. It is well-known that this equation is invariant under biholomorphic changes of variables and hence makes sense globally on a complex manifold. The equation takes globally the form

$$(\partial\overline{\partial}u)^n = 0, \text{ where } \partial\overline{\partial}u = \sum_{j,k=1}^{n} \left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right) dz_j \wedge d\overline{z}_k, \tag{1.2}$$

and the power is exterior power. In (1.1) and (1.2) u is a real-valued function defined on a complex manifold X of complex dimension n. If u is

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a plurisubharmonic function, then $dd^c u = 2i\partial\overline{\partial}u$ is a nonnegative $(1,1)$-current, and by a theorem of Lelong [Le.1], the coefficients $\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}$ are Borel measures for, $1 \le j,\ k \le n$.

Symplectic structures are intimately connected with HCMAE. Lempert [Lem.2] and Semmes [Se.1] have used these structures to formulate precise transformations of solutions of HCMAE. Recall that a symplectic structure on an even-dimensional smooth manifold M^{2n} is a closed nonsingular exterior differential 2-form ω, where, nonsingular means the nonvanishing of the top exterior power of ω, $\omega^n \neq 0$, on M [see An.1, We.1]. Let T^*X denote the holomorphic $(1,0)$-cotangent bundle and $\Pi_X\colon T^*X \to X$ the natural projection. On T^*X we have 1-forms and 2-forms

$$\alpha_X = \sum_j \xi_j \, dz_j, \qquad \omega_X = d\alpha = \sum_j d\xi_j \wedge dz_j, \tag{1.3}$$

where ξ_j are the holomorphic fibre coordinates relative to the holomorphic coordinate system $(z_1, \ldots, z_n)$ on a coordinate patch $\Omega \subset X$. Let $\eta = \Re\alpha_X$, $\sigma = \Im\alpha_X$, $\mu = \Re\omega_X$, $\nu = \Im\omega_X$. These correspond to canonical real forms of real symplectic geometry. It is well-known, (see [Se.1]), that the image of ∂u in T^*X which we write as

$$M = \{\xi_j - \partial_j u = 0\} \tag{1.4}$$

where $\partial_j = \frac{\partial}{\partial z_j}$, $1 \le j \le n$, are the distributional partial derivatives, is a maximal submanifold on which $\mu = \Re\omega_X$ vanishes. Such a submanifold is called a real Lagrangian or ν-Lagrangian. If $\Psi\colon T^*X \to T^*X$ is a local holomorphic symplectic map i.e. $\Psi^*\omega_X = \omega_X$, then it is also real symplectic, so $M^\flat = \Psi(M)$ is real Lagrangian. Assume that $M^\flat$ is transverse to the fibres of the projection map Π_X. $M^\flat$ is then a graph of a $(1,0)$-form whose real part is d-closed by the real Lagrangian condition, and so locally is equal to $du^\flat$ where $u^\flat$, the real-valued function determined up to a constant, is the symplectic transform of u.

In [Lem.2], Lempert used a global version of the above procedure to construct interesting solutions to the HCMAE. It turned out that the graphs of ∂u that arise from solutions of HCMAE can be characterized in terms of the natural holomorphic symplectic structure on the complex cotangent bundle. In particular, this characterization is invariant under symplectic biholomorphisms. Let us make this precise. Let ρ denote the section of T^*X defined by ∂u. Then $\partial u = \rho^*\alpha_X$ and $\overline{\partial}\partial u = d\partial u = \rho^*\omega_X = \rho^*\Psi^*(\omega_X\lceil M^\flat) =$

$(\Pi_X \circ \Psi \circ \rho)^*(\overline{\partial}\partial u^\flat)$ where $\omega_X \lceil M^\flat$ is the restriction of ω_X to $M^\flat$. Thus the null vectors of $\overline{\partial}\partial u$ correspond to those of $\overline{\partial}\partial u^\flat$ under $\Pi_X \circ \Psi \circ \rho$. If the Levi form is non-degenerate, there are no further invariants.

To formulate his results, consider a pair of complex manifolds (X, Y) of equal complex dimensions, $\dim_{\mathbb{C}} X = \dim_{\mathbb{C}} Y = n$. Let (T^*X, T^*Y) be their respective holomorphic cotangent bundles, (α_X, α_Y) and (ω_X, ω_Y) their canonical Liouville 1-forms, respectively symplectic forms. Transform $u \in C^1(X, \mathbb{R})$, (where C^k denotes the class of k-times continuously differentiable functions for, $1 \leq k \leq \infty$) with the help of the symplectic biholomorphic mapping $\Psi: T^*X \to T^*Y$. First observe that $d(\Psi^*\alpha_Y - \alpha_X) = \Psi^*\omega_Y - \omega_X = 0$, so that $\Psi^*\alpha_Y - \alpha_X$ is a d-closed $(1,0)$-form on T^*X. Locally there is a holomorphic function $h: T^*X \to \mathbb{C}$ with $dh = \Psi^*\alpha_Y - \alpha_X$. Assume $h: T^*X \to \mathbb{C}$ exists globally then for $u \in C^1(X, \mathbb{R})$ the gradient mapping $g_u: X \to T^*X$ is a continuous section given locally by $g_u(x) := (\partial u)_x \in T_x^*X$, for $x \in X$ i.e., $g_u(x) = (z_1, \ldots, z_n; \frac{\partial u}{\partial z_1}(x), \ldots, \frac{\partial u}{\partial z_n}(x)) \in T_x^*X$ in a coordinate patch. If now $\Pi_X: T^*X \to X$ and $\Pi_Y: T^*Y \to Y$ are the natural projections, define the continuous mapping $\Phi: X \to Y$ by

$$\Phi := \Pi_Y \circ \Psi \circ g_u. \tag{1.5}$$

which we assume throughout to be invertible. Fix $y \in Y$, set $x := \Phi^{-1}(y)$ and define $u' \in C(Y, \mathbb{R})$ by

$$u'(y) := u(x) + 2\Re h(g_u(x)). \tag{1.6}$$

Lempert proved the following

THEOREM 1.1. ([Lem.2]) *Let $u \in C^2(X, \mathbb{R})$. Then $\overline{\partial}\partial u = \Phi^*\overline{\partial}\partial u'$. In particular, if u satisfies HCMAE, i.e. $(\overline{\partial}\partial u)^n = 0$, then $(\overline{\partial}\partial u')^n = 0$. Furthermore, if the symplectic biholomorphic mapping Ψ satisfies $\Psi^*\alpha_Y = \alpha_X$, i.e. $h \equiv 0$, then $\partial u = \Phi^*\partial u'$.*

In this paper, we extend the above theorem to the larger class of absolutely continuous real-valued functions on lines (in short, ACL) defined on a complex manifold X of complex dimension n denoted by $ACL(X, \mathbb{R})$ (for the definition of ACL and its key properties, which we borrow from the theory of quasiconformal mappings, see [Ri.1]). For $u \in ACL(X, \mathbb{R})$, the gradient mapping $g_u: X \to T^*X$ which is a measurable section is given by $g_u(x) := (\partial u)_x \in T_x^*X$ for a.e $x \in X$ i.e. $g_u(x) := (z_1, \ldots, z_n, \frac{\partial u(x)}{\partial z_1}, \ldots, \frac{\partial u(x)}{\partial z_n}) \in T_x^*X$

for a.e x in a coordinate patch in X. We generalize Lempert's construction to obtain the Lempert transformation $u'(y) = u(x) + 2\Re h(g_u(x))$ where $x := \Phi^{-1}(y)$ and $\Phi: X \to Y$ the mapping defined in (1.7), is assumed throughout to be a bilipschitz homeomorphism almost everywhere (with respect to Lebesgue measure) on X.

Let $u \in ACL(X, \mathbb{R})$ and let its gradient mapping $g_u: X \to T^*X$ which is a measurable section be defined by $g_u := (\partial u)_x \in T^*_x X$ for a.e. $x \in X$. i.e. for a.e x in a coordinate patch in X, $g_u(x) := (z_1, \dots, z_n, \frac{\partial u(x)}{\partial z_1}, \dots, \frac{\partial u(x)}{\partial z_n}) \in T^*_x X$. If we let $\Pi_X: T^*X \to X$ and $\Pi_Y: T^*Y \to Y$ denote the canonical projections, then we can define a mapping $\Phi: X \to Y$ by

$$\Phi := \Pi_Y \circ \Psi \circ g_u, \qquad \text{a.e. on } X, \tag{1.7}$$

which we assume is a bilipschitz homeomorphism almost everywhere. Recall that a mapping $\Phi: X \to Y$ is lipschitz if there exits a constant $C > 0$ such that

$$ds_Y(\Phi(x), \Phi(y)) \leq C ds_X(x, y)$$

for all x and y in the domain of definition of Φ, where $ds_X(\cdot, \cdot)$ and $ds_Y(\cdot, \cdot)$ are the respective Hermitian metrics on X and Y and bilipschitz if Φ is a homeomorphism and both Φ and Φ^{-1} are Lipschitz. From the point of view of real analysis, the condition of being Lipschitz should be viewed as a weakened version of differentiability. In fact, from classical theorem of Rademacher (see [Whi.1], p. 272), we have that if $f: \mathcal{U} \to \mathbb{R}^m$ is a Lipschitz function defined on an open subset $\mathcal{U}$ of $\mathbb{R}^n$, then the distributional partial derivatives $\frac{\partial f_j}{\partial_{x_k}}$, $(1 \leq j \leq m, 1 \leq k \leq n)$ are all given by functions in $\mathcal{L}^\infty(\mathcal{U})$ (with respect to the Lebesgue measure dm). An immediate consequence of this is that, local bilipschitz homeomorphism preserve the class of Lebesgue measure.

In this paper, we are also interested in the pull backs of differential forms and currents [see Bo.1] under special mappings, chief among these are the gradient mappings $g_u: X \to T^*X$ which are measurable sections for every $u \in ACL(X, \mathbb{R})$ and Lipschitz mappings $\Phi: X \to Y$ between complex manifolds. If $(z_1, \dots, z_n)$ are given local coordinates on X and α is the differential form given by $\alpha = \sum_{j=1}^n \xi_j (dz_j)_x$, $\forall x \in X$ then the pull-back $g_u^*(\alpha) = \sum_{j=1}^n \xi_j \circ g_u d(z_j \circ g_u)$. Since g_u is a measurable section we easily see that $d(z_j \circ g_u) = dz_j \in C^\infty$. Thus $g_u^*(\alpha)$ is well-defined. A similar situation obtains for the symplectic 2-form $\omega_X = \sum_{j=1}^n d\xi_j \wedge dz_j$. We have

$g_u^*\omega_X = \sum_{j=1}^n g_u^*(d\xi_j)\wedge d(z_j\circ g_u) = \sum_{j=1}^n g_u^*(d\xi_j)\wedge dz_j = \sum_{j=1}^n dg_{u,n+j}\wedge dz_j$, $j = 1, 2, \ldots, n$.

We generalize these to Lipschitz mappings $\Phi: X \to Y$ between complex manifolds. Let $\Omega \subset \mathbb{C}^n$ be an open subset and $\Phi: \Omega \to \mathbb{C}^m$ a lipschitz map. Define the Jacobian matrix of Φ as the function $d\Phi \in L^\infty(\Omega, M_{n,m}(\mathbb{C}))$ given by $d\Phi(x) = (\frac{\partial \Phi_i(x)}{\partial z_j})$, where $M_{n,m}(\mathbb{C})$ is the space of complex $n \times m$ matrices. Let $U \subset \mathbb{C}^n$, $V \subset \mathbb{C}^m$, $W \subset \mathbb{C}^k$ be open subsets and $\Phi: U \to V$, $\Psi: V \to W$ be Lipschitz maps, then $\Psi\circ\Phi$ is also a Lipschitz map and we have almost everywhere $x \in U$; $d(\Psi \circ \Phi)(x) = d\Psi(\Phi(x)) \circ d\Phi(x)$ (cf. [Tel.1]). If Φ, Φ^{-1} are Lipschitz homeomorphisms between two open subsets $U \subset \mathbb{C}^n$, $V \subset \mathbb{C}^m$, then the class of Lebesgue measures is conserved by Φ (this follows from a consequence of the Rademacher result). Let $\Phi: U \to \mathbb{C}^n$ be a Lipschitz map and ω a measurable map $\omega: \mathbb{C}^n \to \Lambda_{\mathbb{C}}(\mathbb{C}^m)$ with $\Lambda_{\mathbb{C}}(\mathbb{C}^m)$ the complexified bundle of the exterior algebra of $\mathbb{C}^m$. By the result of Rademacher, we can pull back $\Phi^*(\omega)$ on U as follows. Suppose that $\omega(y) = a(y)\, dw_{i_1} \wedge dw_{i_2} \wedge \cdots \wedge dw_{i_m}$, then $\Phi^*(\omega)(x) = a(\Phi(x))\Phi^*(dw_{i_1}) \wedge \cdots \wedge \Phi^*(dw_{i_m})$, where $\Phi^*(dw_i) = \sum \frac{\partial \Phi_i}{\partial z_j} dz_j$ is in $L^\infty(U, \Lambda^1_{\mathbb{C}}(\mathbb{C}^m))$. In particular, we get a continuous linear map $\Phi^*: L^2(\mathbb{C}^n, \Lambda_{\mathbb{C}}(\mathbb{C}^m)) \to L^2(U, \Lambda_{\mathbb{C}}(\mathbb{C}^m))$. Let $\omega \in L^2(U, \Lambda_{\mathbb{C}}(\mathbb{C}^m))$ considered as a current on U be given by the formula

$$\langle \omega, \alpha \rangle = \int_U \omega \wedge \alpha$$

where $\alpha \in C^\infty{}_c(U, \Lambda_{\mathbb{C}}(\mathbb{C}^m))$. The exterior derivative of ω is the current defined by

$$\langle d\omega, \alpha \rangle = \epsilon \int_U \omega \wedge d\alpha$$

where $\epsilon = \pm 1$, $\alpha \in C^\infty{}_c(U, \Lambda_{\mathbb{C}}(\mathbb{C}^m))$ is homogeneous. We define $\Omega_d(U)$ to be the subspace of $L^2(U, \Lambda_{\mathbb{C}}(\mathbb{C}^m))$ consisting of those forms ω for which the current $d\omega$ is again a square integrable differential form. $\Omega_d(U)$ is the maximal domain of d.

THEOREM 1.2. ([Tel.1]) *Let $\Omega_d(V) \subset L^2(\boldsymbol{C}^n, \Lambda_{\mathbb{C}}(\mathbb{C}^m)$. Then we have $\Phi^*(\Omega_d(V)) \subset \Omega_d(U)$ and for any $\alpha \in \Omega_d(U)$*

$$\Phi^*(d\alpha) = d(\Phi^*(\alpha)) \qquad \textit{a.e. in } U$$

The above theorem is proved in H. Whitney [Whi.1] Theorem 9C, p. 305 for Lipschitz mappings F and flat forms ω, where a form ω is flat if ω and

$d\omega$ have bounded measurable components. A fortiori, the formula in the theorem holds for Lipschitz mappings and smooth forms.

Let $\Phi: X \to Y$ be the mapping defined in (1.7) which we assume is a bilipschitz homeomorphism almost everywhere on X. For $u' \in ACL(Y, \mathbb{R})$ we use Theorem 1.2 to define the current

$$\Phi^*(\overline{\partial}\partial u') := \Phi^*(d\partial u') = d(\Phi^*(\partial u')) \tag{1.8}$$

$$\text{and in distribution sense } \Phi^*(du') := d(\Phi^*(u')).$$

The contents of this paper are as follows. In section 2 we give the definition of the complex Monge-Ampère operator and the proofs of our main results, Theorems 2.3, 2.4 and 2.5. In section 3 we consider an example of the Lempert transform and apply this to prove a regularity result for an exterior nonlinear Dirichlet problem for HCMAE which arises naturally in pluricomplex potential theory.

§2. The Complex Monge-Ampére Equation

Let X be an n-dimensional complex manifold. Denote by $d = \partial + \overline{\partial}$ the usual decomposition of the exterior differentiation in terms of its $(1,0)$ and $(0,1)$ components and let $d^c = i(\overline{\partial} - \partial)$. Let u be a plurisubharmonic function on X and let T be a closed positive current of bidimension (p,q), i.e. of bidegree $(n-p, n-q)$. Our objective is to define the wedge product $dd^c u \wedge T$ even when neither u nor T are smooth. In particular, we are interested in the case of real-valued functions u which are absolutely continuous on lines in X. A priori, in either of these cases this product does not make sense because $dd^c u$ and T have measure coefficients and measures cannot be multiplied. There is no feasible way of defining $dd^c u \wedge T$ as a closed positive current without imposing additional hypotheses on u. If we now assume for simplicity, that u is a locally bounded plurisubharmonic function then the current uT is well-defined since u is a locally bounded Borel function and T has measure coefficients. According to E. Bedford and B. A. Taylor [B-T.2], also J.-P. Demailly [De.1], we define $dd^c u \wedge T := dd^c(uT)$ where $dd^c(\cdot)$ is taken in the sense of distribution or current theory. It is shown in [B-T.2, De.1] that the wedge product $dd^c u \wedge T$ is again a closed positive current. Given locally bounded plurisubharmonic functions $u_1, \ldots, u_q$, we define

$$dd^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T := dd^c(u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T),$$

inductively. This product by [B-T.2, De.1] is a closed positive current. In particular, when u is a locally bounded plurisubharmonic function, the bidegree (n, n) current $(dd^c u)^n$ is well-defined and is representable by a positive Borel measure. If u is of class C^2, an easy computation in local coordinates on the complex manifold X yields

$$(dd^c u)^n = \det \left[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right] \cdot 4^n n!\, i\, dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge i\, dz_n \wedge d\bar{z}_n.$$

Next, we need the following monotone continuity theorem.

THEOREM 2.1. ([B-T.2]) *If $u_1, \ldots, u_q$ are locally bounded plurisubharmonic functions and $u_1^k, \ldots, u_q^k$ are decreasing sequences of plurisubharmonic functions converging pointwise to $u_1, \ldots, u_q$ with $\{T_k := T\}$ a constant sequence of closed positive currents, then*

(i) $u_1^k dd^c u_2^k \wedge \ldots \wedge dd^c u_q^k \wedge T \to u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T$ *weakly.*

(ii) $dd^c u_1^k \wedge dd^c u_2^k \wedge \ldots \wedge dd^c u_q^k \wedge T \to dd^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T$ *weakly.*

THEOREM 2.2. *Let X be an n-dimensional complex manifold and $u \in PSH(X) \cap ACL(X, \mathbb{R})$. Let $(z_1, \ldots, z_n)$ be the holomorphic coordinates in a coordinate patch $\Omega \subset X$. Suppose $dd^c u = 2i \sum_{j,k=1}^n \alpha_{j,\bar{k}}\, dz_j \wedge d\bar{z}_k$, where $[\alpha_{j,\bar{k}}] \in L^q_{\mathrm{loc}}(\Omega)$, $1 \leq q \leq n$. Then $(dd^c u)^q$ is a current of bidegree (q, q) and order zero which has locally integrable coefficients given by $[2i \sum_{j,k=1}^n \alpha_{j,\bar{k}} dz_j \wedge d\bar{z}_k]^q$.*

Proof. Since the result is local, if u satisfies the hypothesis of the theorem, then we can use a convolution with a family of C^∞ regularizing sequences ρ_ϵ, $\epsilon > 0$ to find a decreasing sequence $u_\epsilon = u * \rho_\epsilon$ of smooth plurisubharmonic functions converging pointwise to $u \in ACL(\Omega, \mathbb{R})$ such that $\frac{\partial^2 u_\epsilon}{\partial z_j \partial \bar{z}_k}$ converges to $\alpha_{j,\bar{k}} = \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}$ locally in L^q. The result now follows from the weak continuity of $(dd^c u)^n$ from Theorem 2.1. □

THEOREM 2.3. *Let X and Y be two complex manifolds of equal complex dimensions n. Let $\Phi: X \to Y$ be the mapping defined in (1.7) which we assume is a bilipschitz homeomorphism almost everywhere on X. Let $u \in ACL(X, \mathbb{R})$. Choose a point $y \in Y$, set $x := \Phi^{-1}(y)$ and define the Lempert transformation of u by:*

$$u'(y) := u(x) + 2\Re h(g_u(x)). \tag{2.1}$$

Then $\bar{\partial}\partial u = \Phi^* \bar{\partial}\partial u'$. *Furthermore, if the symplectic biholomorphism* Ψ: $T^*X \to T^*Y$ *satisfies* $\Psi^*\alpha_Y = \alpha_X$, *then* $\partial u = \Phi^* \partial u'$ *in the sense of distributions.*

Proof. The proof proceeds in four steps.

First Step:

We show that

(i) $\partial u = g_u^* \alpha_X$

(ii) $du = g_u^*(\alpha_X + \bar{\alpha}_X)$ where $\bar{\alpha}_X = \sum_{j=1}^n \bar{\xi}_j d\bar{z}_j$, is a $(0,1)$-form on T^*X and also that

(iii) $\bar{\partial}\partial u = g_u^* \omega_X$.

Let $(z_1, \ldots, z_n)$ be given local coordinates on X such that $(z_1, \ldots, z_n, \xi_1, \ldots, \xi_n)$ is the corresponding local coordinates on T^*X. The gradient mapping $g_u : X \to T^*X$ which is a measurable section almost everywhere on X, has the form $g_u(z_1, \ldots, z_n) = \left(z_1, \ldots, z_n, \frac{\partial u}{\partial z_1}, \ldots, \frac{\partial u}{\partial z_n}\right)$ a.e. Now $g_u^*(\alpha_X)$ is well-defined as observed previously and has a sense, since g_u is a measurable section a.e, on X and $g_u^*(dz_j) = d(z_j \circ g_u) = dz_j \in C^\infty$. Hence

$$g_u^*(\alpha_X) = g_u^*\left(\sum_j^n \xi_j \, dz_j\right)$$
$$= \sum_j^n \xi_j \circ g_u \, d(z_j \circ g_u) = \sum_j^n \xi_j \circ g_u \, dz_j = \sum_j^n \frac{\partial u}{\partial z_j} \, dz_j = \partial u.$$

Similarly since the function u is real-valued function, we have that $\bar{\partial} u = g_u^* \bar{\alpha}_X$. But $du = \partial u + \bar{\partial} u$, so that

$$du = g_u^*(\alpha_X) + g_u^*(\bar{\alpha}_X) = \alpha_X \circ g_u + \bar{\alpha}_X \circ g_u = g_u^*(\alpha_X + \bar{\alpha}_X).$$

Also

$$\bar{\partial}\partial u = d\partial u = d g_u^* \alpha_X = d(\alpha_X \circ g_u) = g_u^* d\alpha_X = g_u^* \omega_X.$$

Second Step:

We show that if $g : X \to T^*X$ is a measurable section almost everywhere such that $du = g^*(\alpha_X + \bar{\alpha}_X)$, then g must be the gradient mapping $g_u : X \to$

T^*X almost everywhere of $u \in ACL(X, \mathbb{R})$. Indeed in local coordinates as in the first step g has the local representation

$$g(z_1, \ldots, z_n) = (g_1(z_1, \ldots, z_n), \ldots, g_n(z_1, \ldots, z_n), \ldots, g_{2n}(z_1, \ldots, z_n)).$$

Since $g_j(z_1, \ldots, z_n) = z_j$, $\forall j = 1, \ldots, n$ and as g is a measurable section, then $g^*(\alpha_X + \overline{\alpha}_X) = \sum_j^n (g_{n+j} dz_j + \overline{g}_{n+j} d\overline{z}_j)$. For this to be the same as du, we must have $g_{n+j} = \frac{\partial u}{\partial z_j}$ a.e. $\forall j = 1, \ldots, n$.

Third Step:

We show that if $\Psi: T^*X \to T^*Y$, $g_u: X \to T^*X$; $g_{u'}: Y \to T^*Y$ and $\Phi: X \to Y$ are given maps then $\Psi \circ g_u = g_{u'} \circ \Phi$. To see this define a mapping $g: Y \to T^*Y$ by the equation: $g := \Psi \circ g_u \circ \Phi^{-1}$. We need to show that $g = g_{u'}$. By equation (1.7), $\Pi_Y \circ g$ is an identity mapping on Y, so that g is a section of T^*Y. To show that $g = g_{u'}$, by the second step, it is enough to verify that $g^*(\alpha_Y + \overline{\alpha}_Y) = du'$. This is the same as showing that $\Phi^* g^*(\alpha_Y + \overline{\alpha}_Y) = \Phi^* du'$. But

$$\begin{aligned}\Phi^* g^*(\alpha_Y + \overline{\alpha}_Y) &= g_u^* \Psi^*(\alpha_Y + \overline{\alpha}_Y) = g_u^*(\alpha_X + \overline{\alpha}_X + dh + d\overline{h}) \\ &= du + 2d\Re h \circ g_u = d(u' \circ \Phi) = \Phi^* du'.\end{aligned}$$

Finally

Fourth Step:

Using the first and the third steps we can write $\Phi^* \overline{\partial}\partial u' := d(\Phi^* \partial u') = d\Phi^*(g_{u'}^* \alpha_Y) = \Phi^* g_{u'}^* \, d\alpha_Y = \Phi^* g_{u'}^* \omega_Y = \Phi^*(\Phi^{-1})^* g_u^* \Psi^* \omega_Y = g_u^* \omega_X = \overline{\partial}\partial u$, where the first equality is by the definition (1.8). Furthermore, if Ψ is such that $\Psi^* \alpha_Y = \alpha_X$ then

$$\begin{aligned}\Phi^* \partial u' &= \Phi^* g_{u'}^* \alpha_Y = \Phi^*(\Phi^{-1})^* g_u^* \Psi^* \alpha_Y \\ &= g_u^* \alpha_X = \partial u.\end{aligned}$$

This completes the proof of the theorem. □

The next theorem is a natural corollary of the theorem above.

THEOREM 2.4. *Let $u \in ACL(X, \mathbb{R})$ and suppose that $u' \in ACL(Y, \mathbb{R})$ with $u'(y) = u(x) + 2\Re h(g_u(x))$. Suppose $\Phi: X \to Y$, is as in Theorem 2.3. Let $g_{u'}: Y \to T^*Y$ be the gradient mapping of u' a.e, such that $g'_{u'} \in L^n_{\mathrm{loc}}(Y)$, $(g_{u'} \circ \Phi)' \in L^n_{\mathrm{loc}}(X)$, ($L^p(M)$, $1 \le p \le \infty$ denotes the usual L^p-spaces with* *respect to Lebesque measure and $L^{\infty}_{\mathrm{loc}}(M)$ is the space of all locally bounded measurable functions on the space M), with the coefficients of $d(g_{u'} \circ \Phi) \in L^n_{\mathrm{loc}}(X)$, where $g'_{u'}$ and $(g_{u'} \circ \Phi)'$ are the derivatives. Then if $u \in ACL(X, \mathbb{R})$ satisfies HCMAE, i.e. $(\overline{\partial}\partial u)^n = 0$, so also is $(\overline{\partial}\partial u')^n = 0$.*

Proof. From Step four in the proof of the theorem above we have

$$\begin{aligned}\Phi^*(\overline{\partial}\partial u') &= \Phi^* g^*_{u'}\omega_Y = \Phi^* g^*_{u'} \sum_{j=1}^{n} d\eta_j \wedge dw_j \\ &= \Phi^* \sum_{j=1}^{n} dg_{u',n+j} \wedge dw_j = \sum_{j=1}^{n} d(g_{u',n+j} \circ \Phi) \wedge d\Phi_j.\end{aligned}$$

Since $\overline{\partial}\partial u'$ is now a differential form and the coefficients of $d(g_{u'} \circ \Phi) \in L^n_{\mathrm{loc}}(X)$, we deduce that $\partial u' \wedge \overline{\partial}\partial u'$ is a differential form with coefficients which are functions in $L^n_{\mathrm{loc}}(X)$. Hence $\Phi^*(\partial u' \wedge \overline{\partial}\partial u')$ is well-defined. So we can form $d\Phi^*(\partial u' \wedge (\overline{\partial}\partial u') = \overline{\partial}\partial u \wedge \overline{\partial}\partial u$. Now by iterating this process n times we obtain $(\overline{\partial}\partial u)^n = \Phi^*(\overline{\partial}\partial u')^n$. Thus if $(\overline{\partial}\partial u)^n = 0$ then we deduce that $(\overline{\partial}\partial u')^n = 0$. □

THEOREM 2.5. *Let X and Y be two n-dimensional complex manifolds and let $\Phi: X \to Y$ defined for $u \in PSH(X) \cap ACL(X, \mathbb{R})$ be as in Theorem 2.3. Choose $y \in Y$, set $x := \Phi^{-1}(y)$ and denote the Lempert transformation of u by $u'(y) = u(x) + 2\Re h(g_u(x))$. Then*

(i) $\overline{\partial}\partial u = \Phi^*(\overline{\partial}\partial u')$,

(ii) *If $\Psi^*\alpha_Y = \alpha_X$ we have $\partial u = \Phi^*\partial u'$ in the sense of distributions and*

(iii) *If the coefficients of $\overline{\partial}\partial u'$ are in $L^n_{\mathrm{loc}}(X)$ and if $(\overline{\partial}\partial u)^n = 0$ then $(\overline{\partial}\partial u')^n = 0$.*

Proof. The proof follows the same line of reasoning as in that of the preceeding Theorems 2.3 and 2.4. □

§3. Applications of Lempert Transformations

In this final section we give an example of Lempert transformation [Lem.2] which is applied to an exterior nonlinear Dirichlet problem arising in pluricomplex potential theory [KL.1]. We also introduce a class of bounded strictly linearly convex domains in $\mathbb{C}^{n+1}$ and $\mathbb{P}_n(\mathbb{C})$ and recall that Lempert in [Lem.1] and [Lem.4] studied the exponential map for the Kobayashi metric which he showed is a Finsler metric. The main result of [Lem.1], used here is that the exponential map can be normalized to define a smooth diffeomorphism from the unit infinitesimal Kobayashi ball minus the origin onto the domain minus the base point and that its restriction to each line through the origin is holomorphic. We now give the definitions of these domains.

DEFINITION. A bounded domain $\Omega \subset \mathbb{P}_n(\mathbb{C})$ is called linearly convex if for every point z in its complement $\mathbb{P}_n(\mathbb{C}) \setminus \Omega$ there is a nonempty set of complex hyperplanes through z which is disjoint from Ω, i.e. Ω is linearly convex if its complement $\mathbb{P}_n(\mathbb{C}) \setminus \Omega$ can be written as a union of complex hyperplanes.

DEFINITION. $\Omega \subset \mathbb{P}_n(\mathbb{C})$ is said to be strictly linearly convex if it is bounded by a C^2 boundary and its C^2 perturbations are linearly convex. If ρ is a C^2 defining function for Ω a strictly linearly convex bounded domain $\Omega = \{z \in \mathbb{P}_n(\mathbb{C}); \rho(z) < 0\}$, $\partial\Omega = \{\rho = 0\}$ and $d\rho \neq 0$ on $\partial\Omega$ then for all $z \in \partial\Omega$

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho(z) \overline{w}_j w_k}{\partial \overline{z}_j \partial z_k} > \left| \sum_{j,k=1}^{n} \frac{\partial^2 \rho(z) w_j w_k}{\partial z_j \partial z_k} \right|$$

holds for every nonzero vector $w = (w_j)_{j=1}^n \in T_z^{\mathbb{C}}(\partial\Omega)$.

The complex tangent hyperplane $T_z^{\mathbb{C}}(\mathbb{C})$ has a unique point of contact $\{z\}$ with $\partial\Omega$ which is no higher than first order of contact.

EXAMPLE 3.1. Let $X = Y = \mathbb{C}^n$ and $T^*X = T^*Y = T^*\mathbb{C}^n$. Let $z = (z_1, \ldots, z_n)$ be the standard holomorphic coordinates on $\mathbb{C}^n$ and $\xi = (\xi_1, \ldots, \xi_n)$ the fibre coordinates on $T^*\mathbb{C}^n$. Define the symplectic biholomorphic mapping $\Psi: T^*\mathbb{C}^n \to T^*\mathbb{C}^n$: $\Psi(z_j, \xi_j) = (w_j, \eta_j)$ by setting $w_j = \xi_j$, and $\eta_j = -z_j$. Then

$$\sum_{j=1}^{n} \eta_j \, dw_j = -\sum_{j=1}^{n} z_j \, d\xi_j = \sum_{j=1}^{n} \xi_j \, dz_j - d \sum_{j=1}^{n} \xi_j z_j.$$

This gives the holomorphic function $h: T^*\mathbb{C}^n \to \mathbb{C}$ as $h(z_j, \xi_j) = -\sum_{j=1}^{n} \xi_j z_j$.

Now let $u \in ACL(\mathbb{C}^n, \mathbb{R})$ and set $w_j = \left(\frac{\partial u(z)}{\partial z_j}\right)$ almost everywhere. Then

$$u'(w) = u(z) - 2\Re \sum_{j=1}^{n} z_j \frac{\partial u(z)}{\partial z_j}$$

almost everywhere, is the Lempert transformation $\mathcal{L}\colon u \to u'$ given by $\mathcal{L}_u(z) = \left(\frac{\partial u(z)}{\partial z_j}\right)$ a.e. We obtain the following corollary to our Main Theorem.

COROLLARY 3.2. *Let $\mathcal{L}_u(z) = \left(\frac{\partial u(z)}{\partial z_j}\right)$ a.e. Then $\overline{\partial}\partial u = \mathcal{L}_u^* \overline{\partial}\partial u'$.*

Nonlinear Dirichlet problem.

Find a real-valued function U defined on the complement, $\mathbb{C}^n \setminus \Omega$ of Ω, where Ω is a strictly linearly convex bounded domain, with the following properties:

$$\text{(EDP)} \qquad \begin{cases} U \in PSH(\mathbb{C}^n \setminus \overline{\Omega}) & \\ \det\left[\frac{\partial^2 U}{\partial \overline{z}_j \partial z_k}\right] = 0 & \text{in } \mathbb{C}^n \setminus \overline{\Omega} \\ U(z) = \log \|z\| + O(1) & \text{as } \|z\| \to \infty \\ U(z) = 0 \text{ if } z \in \partial\Omega. & \end{cases}$$

THEOREM 3.3. *Suppose $\Omega \subset \mathbb{C}^n \subset \mathbb{P}_n(\mathbb{C})$ is a bounded strictly linearly convex domain with C^2 boundary $\partial\Omega$. Then the problem* (EDP) *admits a unique C^2 solution $U\colon \mathbb{C}^n \setminus \Omega \to \mathbb{R}$.*

Remarks on Extremal Mappings.

Let $\mathbb{D} \subset \mathbb{C}$ be the unit disk with the usual hyperbolic metric which we denote by *hyp*. This metric, as is well-known, is invariant under biholomorphic self-mappings of $\mathbb{D} := \{z \in \mathbb{C}; |z| < 1\}$. As in [Lem.4] consider $\Omega \subset \mathbb{C}^n \hookrightarrow \mathbb{P}_n(\mathbb{C})$ a C^∞ strictly linearly convex bounded domain. Given $z, w \in \Omega$, $z \neq w$, a holomorphic mapping $f\colon \mathbb{D} \to \Omega$ is called an extremal mapping with respect to the points z, w in Ω if $f(0) = z$, $f(\xi) = w$ with $0 < \xi < 1$ and if for any holomorphic mapping $g\colon \mathbb{D} \to \Omega$ such that $g(0) = z$, $g(\eta) = w$ with $0 < \eta < 1$ we have $\xi \leq \eta$, or infinitesimally, given $z \in \Omega$ and a direction vector $v \in \mathbb{C}^n$, $v \neq 0$ we call a holomorphic mapping $f\colon \mathbb{D} \to \Omega$, extremal with respect to z, v, if $f(0) = z$, $f'(0) = \lambda v$ for some $\lambda > 0$ and if for any $g\colon \mathbb{D} \to \Omega$ such that $g(0) = z$, $g'(0) = \mu v$, $\mu > 0$, we have $\lambda \geq \mu$.

We now consider pseudodistances on arbitrary domains in $\mathbb{C}^n$. Of particular interest are the Carathéodory and Kobayashi pseudodistances. Let

Ω be a bounded domain in $\mathbb{C}^n$ we define the Carathéodory pseudodistance, [Lem.5], on Ω as

$$C_\Omega(z,w) := \sup\{\mathrm{hyp}(f(z), f(w));\; f\colon \Omega \to \mathbb{D},\ \text{holomorphic mapping}\}, \tag{3.4}$$

where $C_\Omega(z,w) < \infty$, $\forall z, w \in \Omega$. As the Carathéodory pseudodistance on the domain Ω is defined by means of mappings from Ω into $\mathbb{D}$ it is natural to look at mappings from $\mathbb{D}$ into Ω. For points z and w in a domain Ω we let

$$\delta_\Omega(z,w) := \inf\{\mathrm{hyp}(\xi, \omega);\; \exists \text{ holomorphic map}, f\colon \mathbb{D} \to \Omega, f(\xi) = z, f(\omega) = w\}. \tag{3.5}$$

This definition presupposes the existence of holomorphic mapping $f\colon \mathbb{D} \to \Omega$ whose range contains both z and w. We define the Kobayashi pseudodistance as the largest pseudodistance which is smaller than δ_Ω.

DEFINITION. The Kobayashi pseudodistance for a domain Ω, K_Ω, is defined as

$$\inf\left\{\sum_{j=1}^{n} \delta_\Omega(w_j, w_{j+1}, \{z = w_1, \ldots, w_{n+1} = w\} \subset \Omega\right\}. \tag{3.6}$$

The extremal mappings f in (3.5) are characterized implicitly by the following result.

THEOREM 3.7. ([Lem.4]) *A holomorphic mapping $f\colon \mathbb{D} \to \Omega$ is extremal for the variational problem* (3.5) *if and only if it satisfies the following three conditions:*

(i) *f smoothly extends to the closed disk $\overline{\mathbb{D}}$,*

(ii) $f(\partial\mathbb{D}) \subset \partial\Omega$

(iii) *The family $T^{\mathbb{C}}_{f(\zeta)}(\partial\Omega)$, $f(\zeta) \in \partial\Omega$ of complex tangent hyperplanes to $\partial\Omega$ at $f(\zeta)$ can be included in a smooth, holomorphic family $\{\mathcal{E}_{f(\zeta)}; \zeta \in \overline{\mathbb{D}}\}$ of complex hyperplanes in $\mathbb{C}^n$.*

Condition (iii) means precisely that there is a smooth family $\{\mathcal{E}_{f(\zeta)}; \zeta \in \overline{\mathbb{D}}\}$ of complex hyperplanes in $\mathbb{C}^n$ such that $\mathcal{E}_{f(\zeta)}$ is tangent to $\partial\Omega$ at $f(\zeta)$ and therefore coincides with $T^{\mathbb{C}}_{f(\zeta)}(\partial\Omega)$. It is easy to see that in this case $f(\zeta) \in \mathcal{E}_{f(\zeta)}$ for all $\zeta \in \overline{\mathbb{D}}$, so that $\mathcal{E}_{f(\zeta)}$ are the fibres of a smooth subbundle of the restricted bundle $T(\mathbb{C}^n)\lceil f(\overline{\mathbb{D}})$, where $T(\mathbb{C}^n)$ is the holomorphic tangent bundle of $\mathbb{C}^n$.

COROLLARY 3.8. ([Lem.4]) *Let $\Omega \subset \mathbb{C}^n \hookrightarrow \mathbb{P}_n(\mathbb{C})$ be a strictly linearly convex domain. Then $K_\Omega = C_\Omega$.*

The Theorem 3.7 above holds for domains Ω with only C^2 boundaries $\partial\Omega$. To see this, it is enough to approximate Ω in the C^2 topology by a sequence of C^∞ strictly linearly convex bounded domains $(\Omega_k)_{k=1}^\infty$ and to recall that by Lempert's Hölder estimates, [Lem.4], the 1/2-Hölder norms of the corresponding extremal mappings $f_k: \mathbb{D} \to \Omega_k$ and the associated mappings $\tilde{f}_k$ defined as follows: First let $\nu(z) \in \mathbb{C}^n$ denote the exterior normal vector to $\partial\Omega$ at $z \in \partial\Omega$, then there exists a positive smooth function $p: \partial\mathbb{D} \to \mathbb{R}_+$ such that the mapping

$$\partial\mathbb{D} \ni \zeta \mapsto \zeta p(\zeta)\overline{\nu(\tilde{f}_k(\zeta)} \in \mathbb{C}^n$$

can be extended to a C^∞ mapping $\tilde{f}_k: \overline{\mathbb{D}} \to \mathbb{C}^n$ that is holomorphic in $\mathbb{D}$. The maps $\tilde{f}_k$ are uniformly bounded and furthermore, the norms of $|\tilde{f}_k(\zeta)|$, for $\zeta \in \partial\mathbb{D}$ are bounded away from zero. Thus a subsequence of the mappings $(f_k)_{k=1}^\infty$ converges uniformly to $f: \mathbb{D} \to \Omega$ and the corresponding subsequence of the mappings $(\tilde{f}_k)_{k=1}^\infty$ converge uniformly to $\tilde{f}: \mathbb{D} \to \mathbb{C}^n$ which satisfy the conditions (i)–(iii) of the theorem. Therefore, f is an extremal map with respect to the given data. Now, since extremal maps are unique, this means that an extremal mapping with respect to some given data satisfies (i)–(iii) even when the boundary of the domains Ω in question are only of the class C^2. Further important property of extremal maps is that if $f: \mathbb{D} \to \Omega$ is an extremal map, and $\zeta \neq \omega$ are arbitrary points in $\mathbb{D}$, then the extremal map with respect to $f(\zeta)$ and $f(\omega)$ is just $f \circ \alpha$, where $\alpha: \mathbb{D} \to \mathbb{D}$ is an automorphism that sends ζ to 0 and ω to a positive number. In particular, $f(\zeta) \neq f(\omega)$. More generally, f is a homeomorphic embedding of $\overline{\mathbb{D}}$ into $\overline{\Omega}$. In a similar fashion, the extremal with respect to the point $f(\zeta)$ and the direction $f'(\zeta)$ is again $f \circ \alpha$ with a suitable automorphism α of $\mathbb{D}$. Thus to prove our Theorem 3.3 we can consider bounded strictly linearly convex domains with C^∞ boundaries and C^∞ solutions which can then be approximated in the C^2 topology to the required situation in the theorem. In the following proof we think of everything as in the class C^∞ and then the final results are a consequence of the approximation procedure explained above.

Proof of Theorem 3.3. Reduce the problem to a nonlinear interior Dirichlet problem in the following way. First embed $\mathbb{C}^n$ in $\mathbb{P}_n(\mathbb{C})$ by identifying a point $z \in \mathbb{C}^n$ with the class $[(1,z)] \in \mathbb{P}_n(\mathbb{C})$. Next consider the dual complement $C^*\Omega \subset \mathbb{P}_n^*(\mathbb{C})$, the dual domain to Ω. $C^*\Omega$ is a domain with a preferred point $0^* \in C^*\Omega$ in its interior which corresponds to the complex hyperplane at infinity. The boundary $\partial C^*\Omega$ of $C^*\Omega$ consists of those complex hyperplanes in $\mathbb{P}_n(\mathbb{C})$ that are tangent to $\partial\Omega$.

LEMMA 3.9. *$C^*\Omega$ is a C^2 bounded strictly linearly convex domain.*

Proof. Consider a C^2 mapping $\gamma\colon \partial\Omega \to \partial C^*\Omega$ given by $\gamma(z) = z^*$ where z^* is the complex tangent hyperplane to $\partial\Omega$ at the point $z \in \partial\Omega$. Ω a strictly linearly convex bounded domain implies that the complex tangent hyperplanes to $\partial\Omega$ are disjoint from the closure $\overline{\Omega}$ of the domain Ω except at the unique point of contact z and that these complex tangent hyperplanes have no higher than first order of contact with $\partial\Omega$, hence the C^2 correspondence $z \mapsto z^*$ is a one-to-one and onto mapping. Now $\partial\Omega$ is of class C^2 and to see that the boundary $\partial C^*\Omega$ of $C^*\Omega$ is of the class C^2, we let $\rho \in C^2$ be the defining function of the domain Ω such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho(z)}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k > \left| \sum_{j,k=1}^{n} \frac{\partial^2 \rho(z)}{\partial z_j \partial z_k} w_j w_k \right|$$

holds for every nonzero vector $w = (w_j)_{j=1}^n \in T_z^{\mathbb{C}}(\partial\Omega)$, and for every $z \in \partial\Omega$. But, $\gamma(z) = z^*$ if and only if z is a critical point of ρ restricted to the complex hyperplane z^*. Since the critical point is nondegenerate for strictly linearly convex bounded domains Ω we can define a C^2 mapping γ^* in the neighbourhood of the boundary $\partial C^*\Omega$ of $C^*\Omega$ by setting $\gamma^*(z^*)$ to be equal to the critical point of the restriction of ρ to the complex hyperplane z^*. This then implies that $\gamma^* \circ \gamma$ is the identity mapping on $\partial\Omega$, which in turn implies that the boundary $\partial C^*\Omega$ is of class C^2. Finally we show that $\partial C^*\Omega$ is strictly linearly convex. For this, we first assume that Ω is the projective image of a ball and introduce a system of affine coordinates so that in that system Ω is the ball centred at the origin. Clearly in this case $C^*\Omega$ is a ball and hence strictly linearly convex. The general case is obtained by taking a point $z \in \partial\Omega$ and inscribing a ball $\mathbb{B}$ inside Ω tangent to $\partial\Omega$ at the point $z \in \partial\Omega$. The dual complements $C^*\mathbb{B}$ and $C^*\Omega$ are then tangent at the point z^*, and the complex tangent hyperplanes of $\partial C^*\mathbb{B}$ and $\partial C^*\Omega$ coincide at z^*. Since $C^*\mathbb{B}$ is strictly linearly convex, the complex tangent hyperplane is then disjoint from $C^*\Omega \setminus \{z^*\}$ and has precisely a first order of contact with $\partial C^*\Omega$. $\square$

Now consider $C^*\Omega$ contained in $(\mathbb{C}^n)^* \hookrightarrow \mathbb{P}_n^*(\mathbb{C})$ because the origin of $\mathbb{C}^n$ is contained in the interior of Ω so that the hyperplane at infinity is contained in the interior of $C^*\Omega$ and furthermore, $C^*\Omega$ does not contain any complex hyperplanes arising from points in the interior of Ω. We can think of $C^*\Omega$ as given by

$$C^*\Omega := \{w \in (\mathbb{C}^n)^*; \langle z, w\rangle \neq 1, z \in \overline{\Omega}\}$$

where $\langle\cdot,\cdot\rangle: \mathbb{C}^n \times (\mathbb{C}^n)^* \to \mathbb{C}$ is the usual pairing. We now return to the non-linear Dirichlet problem (EDP) and consider the interior Dirichlet problem for a function $u^*: C^*\overline{\Omega} \setminus \{0^*\} \to \mathbb{R}$:

$$\text{(IDP)} \qquad \begin{array}{ll} u^* \in PSH(C^*\Omega \setminus \{0^*\}) & \\ \det\left[\frac{\partial^2 u^*}{\partial \bar{z}_k \partial z_k}\right] = 0 & \text{in } C^*\Omega \setminus \{0^*\} \\ u^*(w) = \log\|w\| + O(1) & \text{as } w \to 0^* \\ u^*(w) = 0 & \text{if } w \in \partial C^*\Omega, \end{array}$$

with $0(1) = \eta(w/|w|) + 0(|w|)$, where $\eta \in C^2(\mathbb{P}_{n-1}(\mathbb{C})$. The problem (IDP) admits a unique C^2 solution $u^*: C^*\Omega \setminus \{0^*\} \to \mathbb{R}$. To show this we define the transformation $\mathcal{L}_{u^*}(w) = z$ if

$$z_j = \left[\frac{\dfrac{\partial u^*(w)}{\partial w_j}}{\displaystyle\sum_{\nu}^{n} w_\nu \frac{\partial u^*(w)}{\partial w_\nu}}\right], \tag{3.10}$$

where $\frac{\partial u^*(w)}{\partial w} = \left(\frac{\partial u^*(w)}{\partial w_1}, \ldots, \frac{\partial u^*(w)}{\partial w_n}\right)$. We claim that (3.10) is a C^2 diffeomorphism of $C^*\Omega \setminus \{0^*\}$ onto $\mathbb{C}^n \setminus \Omega$. This is a consequence of the following lemma.

LEMMA 3.11. *The sublevel sets*

$$C^*\Omega_c := \{w \in C^*\Omega; u^*(w) < c\}$$

are C^2 bounded strictly linearly convex sets for any $c \leq 0$.

We assume the lemma is true for the moment and proceed with the proof of the problem (IDP). It is clear that the denominator of the expression in (3.10) does not vanish in $C^*\Omega \setminus \{0^*\}$. This implies that $\mathcal{L}_{u^*} \in C^2(C^*\Omega \setminus \{0^*\})$. To see that $\mathcal{L}_{u^*}$ is a diffeomorphism of $C^*\Omega \setminus \{0^*\}$ onto $\mathbb{C}^n \setminus \Omega$, we note that $\mathcal{L}_{u^*}(w) = z$ if and only if w is a critical point of the restriction of u^* to the complex hyperplane $\{\zeta^*; \langle \zeta^*, z\rangle = 1\} = H_z$ contained in $(\mathbb{C}^n)^*$. Lemma 3.11 then implies that on any complex hyperplane that intersects $C^*\overline{\Omega}$, the restriction of u^* to H_z has exactly one critical point. Now $H_z \cap C^*\overline{\Omega} \neq 0^*$ if and only if $z \in \mathbb{C}^n \setminus \Omega$. This further, implies that $\mathcal{L}_{u^*}$ is one-to-one from $C^*\Omega \setminus \{0^*\}$ onto $\mathbb{C}^n \setminus \Omega$. The mapping $\mathcal{L}_{u^*}^{-1}: z^* \mapsto z$ is C^2, since Lemma 3.11 implies that the critical point z^* of the restriction of u^* to the complex hyperplane H_z is nondegenerate and depends smoothly on the complex hyperplane H_z and hence on the point z. This implies that $\mathcal{L}_{u^*}$ is a diffeomorphism as claimed.

Next define $u: \mathbb{C}^n \setminus \Omega \to \mathbb{R}$ by $u = u^* \circ \mathcal{L}_{u^*}^{-1}$ and set $U = -u$. In an equivalent way we can set $u(z)$ to be equal to the critical value of the restriction of u^* to the complex hyperplane H_z. Then clearly both u and U are of the class C^2 and by Corollary 3.2 satisfy the homogeneous complex Monge-Ampère equation. Now because $\mathcal{L}_*(\partial C^*\Omega) = \partial\Omega$ we easily see that $u = U = 0$ on $\partial\Omega$. The distance of the complex hyperplane H_z to 0 is $\|z\|^{-1}$. Thus $U(z) = -\log\|z\| + O(1)$.

Finally $\Omega_c = \{z \in \mathbb{C}^n \setminus \Omega; U(z) < c\} \cup \Omega$ are strictly linearly convex as these are the dual of the sets $C^*\Omega_c = \{w \in C^*\Omega; u^*(w) < -c\}$ and $\Omega_c = \{z \in \mathbb{C}^n; \sup|\langle w, z\rangle|, w \in C^*\Omega_c\}$. Lemma 3.9 now applies with Ω and $C^*\Omega$ replaced by $C^*\Omega_c$ and Ω_c. The linear convexity of the sublevel sets of U and the fact that U satisfies $\det\left[\frac{\partial^2 U}{\partial \bar{z}_j \partial z_k}\right] = 0$ implies that U is a plurisubharmonic function. The uniqueness assertion follows from the minimum principle of E. Bedford and B. A. Taylor [B-T.1].

Proof of Lemma 3.11. We call $C^*\Omega$, Ω and u^*, u. In [Lem.1] Lempert constructs a C^2-diffeomorphism $\Phi_0: \overline{\Omega} \setminus \{0\} \to \overline{\mathbb{B}_n} \setminus \{0\} := \{z \in \mathbb{C}^n; 0 < \|z\| \leq 1\}$ such that $u = \log\|\Phi_0\|$. The existence of such a C^2 map Φ_0 implies that the sublevel sets of u are C^2. Next we show that the sublevel sets $\{u <$

constant} are strictly linearly convex. Take $z \in \partial\Omega_c := \{u(z) = c; c \leq 0\}$ the boundary of the sublevel set $\Omega_c := \{u < c\}$. From [Lem.2] there exists a unique holomorphic mapping f from the closed unit disk $\overline{\mathbb{D}}$ into $\overline{\Omega}$ with $\xi = \exp c$ such that $f(0) = 0$, $f(\xi) = z$ and f depends continuously on $z \in \overline{\Omega} \setminus \{0\}$. Further f is an embedding. In particular, f is transverse to $\partial\Omega$. In [Lem.2] Lempert further constructs a holomorphic left inverse $F: \overline{\Omega} \to \overline{\mathbb{D}}$ to f with the following properties:

(i) Its fibres $F^{-1}(\xi)$, $\xi \in \overline{\mathbb{D}}$ are hyperplanes restricted to the neighbourhood of Ω

(ii) $|F(z)| < 1$ if $z \in \overline{\Omega} \setminus f(\partial\mathbb{D})$

(iii) $dF \neq 0$ on $\overline{\Omega}$.

We claim that $F^{-1}(\xi)$ is a complex tangent hyperplane to $\partial\Omega_c$ at z. Since $\xi = F(z)$ the fact that $f(\xi) = z$ implies that z lies on the complex hyperplane $F^{-1}(\xi)$. To verify the claim it is therefore, enough to show that no other point $w \in \overline{\Omega_c}$ can lie on $F^{-1}(\xi)$. Now for any such $w \in \overline{\Omega_c}$ there exists a holomorphic mapping $g: \overline{\mathbb{D}} \to \overline{\Omega}$ such that $g(0) = 0$ and $g(\omega) = w$ with $\omega = \exp u(w) \leq \xi$. We apply Schwartz's Lemma to $F \circ g: \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ to give $\xi \geq \omega \geq |F(g(\omega))| = |F(w)|$ and $\xi = F(w)$ i.e., $w \in F^{-1}(\xi)$ can hold only if $F \circ g = id_{\mathbb{D}}$. This by property (2) and the uniqueness of the mapping f can happen only if $f = g$, that is to say, $w = z$. Thus $F^{-1}(\xi)$ is indeed the complex tangent hyperplane to $\partial\Omega_c$. Now it remains to show the order of contact between $\partial\Omega_c$ and $F^{-1}(\xi)$. Let $z(t)$, $0 \leq t \leq 1$ be a smooth curve on $\partial\Omega_c$ tangent to $F^{-1}(\xi)$ at $z = z(0)$. Then it is not tangent to $f(\overline{\mathbb{D}})$. Therefore, if f_t, $0 \leq t \leq 1$ is the holomorphic mapping $f_t: \overline{\mathbb{D}} \to \overline{\Omega}$ such that $f_t(0) = 0$ and $f_t(\xi) = z(t)$, then we have $\operatorname{dist}(z(t), f(\overline{\mathbb{D}})) \geq C|t|$, with $C > 0$ and $\operatorname{dist}(f_t(\overline{\mathbb{D}}), f(\overline{\mathbb{D}})) \geq C|t|$ in view of the continuous dependence of f on z. In particular, for $\xi \in \partial\overline{\mathbb{D}}$, $\operatorname{dist}(f_t(\xi), f(\mathbb{D})) \geq C|t|$. However, the complex hyperplanes $F^{-1}(\xi)$ have precisely first order of contact with $\partial\Omega$, so that $|F(f_t(\xi))| \leq 1 - Ct^2$. Schwartz's Lemma applied again to $F \circ f_t$ now shows that $|F(f_t(\xi))| \leq \xi(1 - Ct^2)$, that is to say, $|F(z(t))| \leq \xi - Ct^2$. This implies that $z(t)$ has no more than first order of contact with $F^{-1}(\xi)$. $\square$

The interior problem (IDP) can be formulated in a slightly more general way. Let Ω be a strictly linearly convex bounded domain and $w \in \Omega$ a fixed point. Consider all the extremal mappings $f: \overline{\mathbb{D}} \to \overline{\Omega}$ such that $f(0) = w$. Along these extremal maps we push-forward the Green function of the unit disk $\mathbb{D} \subset \mathbb{C}$, i.e. $\log|\zeta|$, with pole at 0 to get a smooth function $u: \overline{\Omega} \to \mathbb{R}$. This function can be taken as the Green function of $\overline{\Omega}$ for the pluricomplex potential theory associated with the complex Monge-Ampère operator $(dd^c\cdot)^n$ studied in [B-T.2] as demonstrated in the following theorem.

THEOREM 3.12. *Let $\Omega \subset \mathbb{C}^n$ be a strictly linearly convex bounded domain with C^2 boundary and let $w \in \Omega$ be a given fixed point. Then the problem*

$$(\mathrm{DP}) \qquad \begin{cases} u \in PSH(\Omega \setminus \{w\}) & \\ \det\left[\dfrac{\partial^2 u(z)}{\partial z_j \partial \bar{z}_k}\right] = 0 & \text{if } z \in \Omega \setminus \{w\} \\ u(z) = \log\|z - w\| + O(1) & \text{as } z \to w \\ u(z) = 0 & \text{if } z \in \partial\Omega \end{cases}$$

admits a unique C^2 solution $u: \Omega \setminus \{w\} \to \mathbb{R}$.

Proof. Here briefly are the ideas Lempert used in [Lem.1] to prove the above theorem for strictly convex bounded domain, i.e. a domain Ω, where the normal curvatures of $\partial\Omega$ are everywhere positive. The same ideas apply in our case. Associated to the solution u of the theorem, there is a foliation of $\Omega \setminus \{w\}$ which we call the Lempert foliation of $\Omega \setminus \{w\}$. Let $\mu = e^{2u}$. Assume $\tau: \Omega^* \to \Omega$ is the blow-up of Ω at the point w. Then $u^* = \mu \circ \tau \in C^\infty(\Omega^*)$. Each leaf of the foliation is an extremal disk for the Kobayashi metric through w. The key idea of the proof is that extremal mappings exist for strictly linearly convex bounded domains, are unique for any given directions and extend smoothly up to the boundary [Lem.4] and [CHL.1]. The solution is then obtained by pushing forward $\log|\zeta|^2$ from the unit disk. Since extremal disks are transversal to the holomorphic tangent bundle to $\partial\Omega$, it is then easy to see that the function u defined in this way and suitably extended outside of $\overline{\Omega}$ is a defining function for Ω and is thus a plurisubharmonic function. The details of the proof are then exactly as in [Lem.1] and so are omitted here. $\square$

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