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A COMBINATORIAL IDENTITY FOR THE DERIVATIVE OF A THETA SERIES OF A FINITE TYPE ROOT LATTICE

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Abstract. Let $\mathfrak{g}$ be a (not necessarily simply laced) finite-dimensional complex simple Lie algebra with $\mathfrak{h}$ the Cartan subalgebra and $Q \subset \mathfrak{h}^*$ the root lattice. Denote by $\Theta_Q(q)$ the theta series of the root lattice Q of $\mathfrak{g}$. We prove a curious "combinatorial" identity for the derivative of $\Theta_Q(q)$, i.e. for $q\frac{d}{dq}\Theta_Q(q)$, by using the representation theory of an affine Lie algebra.

§1. Introduction

Let $\mathfrak{g} = \mathfrak{g}(X_N)$ be a finite-dimensional complex simple Lie algebra of type X_N, where $X = A, D, E, C, B, F, G$ and $N \in \mathbb{Z}_{\geq 1}$. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ (note that $\dim_{\mathbb{C}} \mathfrak{h} = N$). Denote by $\Delta \subset \mathfrak{h}^*$ the set of roots, by Δ_+ (resp. Δ_-) the set of positive (resp. negative) roots, and by $\Pi = \{\alpha_i\}_{i=1}^N$ (resp. $\Pi^\vee = \{h_i\}_{i=1}^N$) the set of simple roots (resp. coroots). Also we set $\rho := (1/2) \cdot \sum_{\alpha \in \Delta_+} \alpha$ (the Weyl vector) and $Q := \sum_{i=1}^N \mathbb{Z}\alpha_i$ (the root lattice). For a dominant integral weight $\lambda \in P_+ := \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0},\ 1 \leq i \leq N\}$, we denote by $L(\lambda)$ the irreducible highest weight $\mathfrak{g}$-module of highest weight λ, and set $d(\lambda) := \dim_{\mathbb{C}} L(\lambda)$.

Let us normalize the Killing form $(\,\cdot\,|\,\cdot\,)$ on $\mathfrak{g}$ in such a way that $(\alpha|\alpha) = 2$ for all long roots $\alpha \in \Delta_{long}$. Then the theta series $\Theta_Q(q)$ of the root lattice $Q \subset \mathfrak{h}^*$ is defined by

$$\Theta_Q(q) := \sum_{\alpha \in Q} q^{\frac{r}{2}(\alpha|\alpha)},$$

where the number r is given by:

$$r = \begin{cases} 1 & \text{if } X = A, D, E, \\ 2 & \text{if } X = C, B, F, \\ 3 & \text{if } X = G. \end{cases}$$

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Our main result in this paper is the following theorem.

THEOREM. *Let $Q = \sum_{i=1}^{N} \mathbb{Z}\alpha_i \subset \mathfrak{h}^*$ be the root lattice of type X_N, $(\cdot|\cdot)$ the normalized Killing form on $\mathfrak{h}^*$, and $\Theta_Q(q) = \sum_{\alpha\in Q} q^{\frac{r}{2}(\alpha|\alpha)}$ the theta series of Q. Then we have*

$$\begin{aligned} &2r^{-1}(1+h^\vee)\, q\frac{d}{dq}\Theta_Q(q) \\ &\quad = \sum_{\lambda\in Q\cap P_+} d(\lambda)(\lambda+2\rho|\lambda) q^{\frac{r}{2}(\lambda|\lambda)} \prod_{\alpha\in\Delta_+}\Big(1-q^{r(\lambda+\rho|\alpha)}\Big). \end{aligned}$$

Here r is as above and $h^\vee$ is the dual Coxeter number given below.

The dual Coxeter number $h^\vee$ (see [K4, Chap. 6]) is given by:

$$h^\vee = \begin{cases} N+1 & \text{if } X_N = A_N,\ r=1, \\ 2N-2 & \text{if } X_N = D_N,\ r=1, \\ 12 & \text{if } X_N = E_6,\ r=1, \\ 18 & \text{if } X_N = E_7,\ r=1, \\ 30 & \text{if } X_N = E_8,\ r=1, \\ 2N & \text{if } X_N = C_N, B_N,\ r=2, \\ 12 & \text{if } X_N = F_4,\ r=2, \\ 6 & \text{if } X_N = G_2,\ r=3. \end{cases}$$

We should note that in the cases where $X_N = A_N, D_N, E_N$, $h^\vee$ is the dual Coxeter number of the generalized Cartan matrix of type $X_N^{(1)}$, and in the cases where $X_N = C_L, B_L, F_4, G_2$, $h^\vee$ is the dual Coxeter number of the generalized Cartan matrix of type $A_{2L-1}^{(2)}$, $D_{L+1}^{(2)}$, $E_6^{(2)}$, $D_4^{(3)}$, respectively.

Remark. For $\lambda \in P_+$, the dimension $d(\lambda)$ of $L(\lambda)$ is given by the Weyl dimension formula:

$$d(\lambda) = \prod_{\alpha\in\Delta_+} \frac{(\lambda+\rho|\alpha)}{(\rho|\alpha)}.$$

Remark. We also have an expression for the theta series $\Theta_Q(q)$ itself of the root lattice Q (see Remark 3.4 and Proposition 4.4.3). However, this expression (at least) in the cases where $X = A, D, E$ is already known, and similar identities can be found in [K2, Remark (d) below Proposition 2] and

[KT, Remark 5.2], while the expression for the derivative of $\Theta_Q(q)$ given in Theorem is new. It seems to us that identities of this kind are, even in a special case, not reduced to well-known ones in the classical literature (cf. Example 3.3).

We prove our theorem by using the representation theory of affine Lie algebras. Let $\widehat{\mathfrak{g}} = \mathfrak{g}(X_N^{(r)})$ be the affine Lie algebra of type $X_N^{(r)}$, where $X_N^{(r)} = A_N^{(1)}, D_N^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, A_{2L-1}^{(2)}, D_{L+1}^{(2)}, E_6^{(2)}, D_4^{(3)}$, and let $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the Cartan subalgebra, where c is the canonical central element and d the scaling element. Denote by $V := \widehat{L}(\widehat{\Lambda}_0)$ the irreducible highest weight $\widehat{\mathfrak{g}}$-module (the basic representation) of highest weight $\widehat{\Lambda}_0 \in (\widehat{\mathfrak{h}})^*$, where $\widehat{\Lambda}_0$ is the basic fundamental weight given by: $\widehat{\Lambda}_0(\mathfrak{h}) := 0$, $\widehat{\Lambda}_0(c) := 1$, and $\widehat{\Lambda}_0(d) := 0$. We can give a $\mathbb{Z}$-gradation (called the basic gradation) of V by setting

$$V_m := \{v \in V \mid dv = -mv\} \quad \text{for } m \in \mathbb{Z}.$$

Then our proof is carried out by calculating the graded trace

$$g(q) := \sum_{m \in \mathbb{Z}} \mathrm{Tr}(\Omega|_{V_m})\, q^m$$

of the Casimir element $\Omega \in Z(U(\mathfrak{g}))$ on $V = \widehat{L}(\widehat{\Lambda}_0)$ in two different ways.

This paper is organized as follows. In Section 2, we calculate in one way the graded trace $g(q)$ above on the (general) irreducible highest weight $\widehat{\mathfrak{g}}$-module $\widehat{L}(\Lambda)$ of dominant integral highest weight Λ in the cases where $X = A, D, E$. In Section 3, we prove our main theorem in the cases where $X = A, D, E$ by calculating $g(q)$ in another way, using some well-known results of Kac. In Section 4, we prove our main theorem in the cases where $X = C, B, F, G$ by arguments similar to those in the A, D, E cases.

Throughout this paper, we assume that the reader is familiar with most of Kac [K4], especially with Chapters 6, 7, 8, and 12.

§2. Graded trace of the Casimir element

2.1. Nontwisted affine Lie algebras

Here we recall from [K4, Chaps. 6 and 7] some standard notation and facts about nontwisted affine Lie algebras.

Let $\mathfrak{g} = \mathfrak{g}(X_N)$ be a finite-dimensional complex simple Lie algebra of type X_N, where $X = A, D, E$ and $N \in \mathbb{Z}_{\geq 1}$. Fix a Cartan subalgebra $\mathfrak{h}$

of $\mathfrak{g}$ with $\dim_{\mathbb{C}} \mathfrak{h} = N$, and denote by $\Delta \subset \mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ the set of roots, by Δ_+ (resp. Δ_-) the set of positive (resp. negative) roots, and by $\Pi = \{\alpha_i\}_{i=1}^N$ (resp. $\Pi^\vee = \{h_i\}_{i=1}^N$) the set of simple roots (resp. coroots). We normalize the Killing form $(\,\cdot\,|\,\cdot\,)$ on $\mathfrak{g}$ in such a way that

$$(\alpha|\alpha) = 2 \quad \text{for all (long) roots } \alpha \in \Delta.$$

Let us denote by $\widehat{\mathfrak{g}} = \mathfrak{g}(X_N^{(1)})$ a (nontwisted) affine Lie algebra of type $X_n^{(1)}$ over $\mathbb{C}$, i.e.,

$$\widehat{\mathfrak{g}} = \widehat{\mathcal{L}}(\mathfrak{g}) = (\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}) \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where $\mathbb{C}[t, t^{-1}]$ is the algebra of Laurent polynomials in t, c the canonical central element, and d the scaling element. Notice that the Lie algebra $\mathfrak{g}$ can be identified with the subalgebra $\mathbb{C}t^0 \otimes_{\mathbb{C}} \mathfrak{g}$ of $\widehat{\mathfrak{g}}$.

We denote the Cartan subalgebra of $\widehat{\mathfrak{g}}$ by:

$$\widehat{\mathfrak{h}} = (\mathbb{C}t^0 \otimes_{\mathbb{C}} \mathfrak{h}) \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

and introduce an element $\delta \in (\widehat{\mathfrak{h}})^*$ (the null root) defined by: $\delta(\mathfrak{h} \oplus \mathbb{C}c) = 0$, $\delta(d) = 1$. Then the set $\widehat{\Delta}_+ \subset (\widehat{\mathfrak{h}})^*$ of positive roots is described as:

$$\widehat{\Delta}_+ = \{j\delta \mid j \in \mathbb{Z}_{\geq 1}\} \sqcup \{j\delta + \alpha \mid j \in \mathbb{Z}_{\geq 1},\, \alpha \in \Delta\} \sqcup \Delta_+,$$

where an element $\alpha \in \mathfrak{h}^*$ is regarded as an element of $(\widehat{\mathfrak{h}})^*$ by putting: $\alpha(c) = \alpha(d) = 0$. Moreover, the root spaces $\widehat{\mathfrak{g}}_\gamma$, $\gamma \in \widehat{\Delta}_+$, are written as:

$$\widehat{\mathfrak{g}}_{j\delta} = \mathbb{C}t^j \otimes_{\mathbb{C}} \mathfrak{h}, \;\; \widehat{\mathfrak{g}}_{j\delta+\alpha} = \mathbb{C}t^j \otimes_{\mathbb{C}} \mathfrak{g}_\alpha, \quad j \in \mathbb{Z},\, \alpha \in \Delta,$$

where $\mathfrak{g}_\alpha$ is the root space of $\mathfrak{g}$ corresponding to a root $\alpha \in \Delta$. Also we denote by $\widehat{\Pi} = \{\widehat{\alpha}_i\}_{i=0}^N \subset \widehat{\Delta}_+$ the set of simple roots of $\widehat{\mathfrak{g}}$, and by $\widehat{\Pi}^\vee = \{\widehat{h}_i\}_{i=0}^N \subset \widehat{\mathfrak{h}}$ the set of simple coroots of $\widehat{\mathfrak{g}}$. (See [K4, Chap. 7] for the explicit construction of $\widehat{\Pi}$ and $\widehat{\Pi}^\vee$.)

The normalized Killing form $(\,\cdot\,|\,\cdot\,)$ on $\mathfrak{g}$ can be extended to the normalized invariant form (see [K4, Chap. 6]) $(\,\cdot\,|\,\cdot\,)$ on $\widehat{\mathfrak{g}}$ by:

$$\begin{cases} (t^m \otimes x | t^n \otimes y) = \delta_{m+n,0}(x|y), \quad x, y \in \mathfrak{g},\, m, n \in \mathbb{Z}; \\ (\mathbb{C}c \oplus \mathbb{C}d | \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}) = 0; \\ (c|c) = (d|d) = 0; \\ (c|d) = 1. \end{cases}$$

The restriction of this bilinear form $(\,\cdot\,|\,\cdot\,)$ to the Cartan subalgebra $\widehat{\mathfrak{h}}$ induces a nondegenerate symmetric bilinear form $(\,\cdot\,|\,\cdot\,)$ on $\mathfrak{h}^*$. Note that in this case, for every root $\alpha \in \Delta \subset \mathfrak{h}^* \subset (\widehat{\mathfrak{h}})^*$, we have $(\alpha|\alpha) = 2$.

2.2. Casimir operators for $\mathfrak{g}$ and $\widehat{\mathfrak{g}}$

The Casimir element Ω for $\mathfrak{g}$ is an element of the center $Z(U(\mathfrak{g}))$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ defined by:

$$\Omega = \sum_{i=1}^{M} u_i u^i,$$

where $\{u_i\}_{i=1}^M$ and $\{u^i\}_{i=1}^M$ with $M := \dim_{\mathbb{C}} \mathfrak{g}$ are arbitrary dual bases of $\mathfrak{g}$ with respect to the normalized Killing form $(\,\cdot\,|\,\cdot\,)$ on $\mathfrak{g}$. Notice that the element $\Omega \in Z(U(\mathfrak{g}))$ is independent of the choice of dual bases, and that Ω acts on each irreducible highest weight $\mathfrak{g}$-module $L(\lambda)$ of highest weight $\lambda \in \mathfrak{h}^*$ by the scalar $(\lambda + 2\rho|\lambda)$, where $\rho = (1/2)\cdot\sum_{\alpha\in\Delta_+}\alpha \in \mathfrak{h}^*$ is the Weyl vector for $\mathfrak{g}$.

Recall from [K4, Chaps. 2 and 12] the definition and construction of the (generalized) Casimir operator $\widehat{\Omega}$ for $\widehat{\mathfrak{g}}$, which is a well-defined operator on a $\widehat{\mathfrak{g}}$-module V such that for each $v \in V$, $\widehat{\mathfrak{g}}_\gamma v = 0$ for all but a finite number of positive roots $\gamma \in \widehat{\Delta}_+$. Then we know that the operator $\widehat{\Omega}$ can be expressed in the following form:

$$\widehat{\Omega} = \Omega + 2(c + h^\vee)d + 2\sum_{i=1}^{M}\sum_{n\geq 1}(t^{-n}\otimes u_i)(t^n \otimes u^i),$$

where the scalar $h^\vee$, called the dual Coxeter number, is given by:

$$h^\vee = \begin{cases} N+1 & \text{if } X_N = A_N, \\ 2N-2 & \text{if } X_N = D_N, \\ 12 & \text{if } X_N = E_6, \\ 18 & \text{if } X_N = E_7, \\ 30 & \text{if } X_N = E_8. \end{cases}$$

Remark 2.2.1. It is easily checked that

$$M = \dim_{\mathbb{C}} \mathfrak{g} = N(1 + h^\vee)$$

in all the cases where $X = A, D, E$.

Moreover, we know that the operator $\widehat{\Omega}$ acts on the irreducible highest weight $\widehat{\mathfrak{g}}$-module $\widehat{L}(\Lambda)$ of highest weight $\Lambda \in (\widehat{\mathfrak{h}})^*$ by the scalar $(\Lambda + 2\widehat{\rho}|\Lambda)$, where the element $\widehat{\rho} \in (\widehat{\mathfrak{h}})^*$ (the Weyl vector for $\widehat{\mathfrak{g}}$) is defined by: $\widehat{\rho}(\widehat{h}_i) = 1$ for all $0 \leq i \leq N$, and $\widehat{\rho}(d) = 0$.

2.3. Calculation of the graded trace of Ω

Let

$$\widehat{P}_+ := \{\Lambda \in (\widehat{\mathfrak{h}})^* \mid \Lambda(\widehat{h}_i) \in \mathbb{Z}_{\geq 0},\ 0 \leq i \leq N\}$$

be the set of dominant integral weights. Fix $\Lambda \in \widehat{P}_+$ such that $\Lambda(d) = 0$, and put $k := \Lambda(c) \in \mathbb{Z}_{\geq 0}$ (the level of Λ). Let $V := \widehat{L}(\Lambda)$ be the irreducible highest weight $\widehat{\mathfrak{g}}$-module of highest weight Λ. We give a $\mathbb{Z}$-gradation, called the basic gradation, of V by setting:

$$V_m = \{v \in V \mid dv = -mv\} \quad \text{for } m \in \mathbb{Z}.$$

Then we have (see [K4, Chap. 12])

$$V = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} V_m$$

with $V_{-m} = \{0\}$ for $m > 0$ and $\dim_{\mathbb{C}} V_m < +\infty$ for all $m \geq 0$. Note that each homogeneous subspace V_m for $m \in \mathbb{Z}_{\geq 0}$ is stable under the action of $\mathfrak{g} \cong \mathbb{C}t^0 \otimes_{\mathbb{C}} \mathfrak{g} \hookrightarrow \widehat{\mathfrak{g}}$ since $[d, \mathbb{C}t^0 \otimes_{\mathbb{C}} \mathfrak{g}] = 0$. In particular, we have

$$\Omega V_m \subset V_m \quad \text{for each } m \in \mathbb{Z}_{\geq 0}.$$

Thus we can define a formal power series $g(q)$, called the graded trace of Ω on $V = \widehat{L}(\Lambda)$, by

$$g(q) := \sum_{m \in \mathbb{Z}_{\geq 0}} \mathrm{Tr}(\Omega|_{V_m})\, q^m,$$

which is the generating function of the traces $\mathrm{Tr}(\Omega|_{V_m})$, $m \in \mathbb{Z}_{\geq 0}$.

The following elementary fact in linear algebra will play an essential role in the calculation of the graded trace $g(q)$ in this subsection.

Lemma 2.3.1. *Let X, Y be finite-dimensional vector spaces over $\mathbb{C}$, and let $A : X \to Y$, $B : Y \to X$ be linear maps. Then we have*

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA).$$

Set

$$c(k) := \frac{k(\dim_{\mathbb{C}} \mathfrak{g})}{k + h^\vee} \in \mathbb{Q}_{>0}.$$

We now define the following formal power series in q:

$$\phi(q) := \prod_{n=1}^{\infty} (1 - q^n),$$

$$H(q) := -c(k) \cdot \sum_{n \geq 1} \log(1 - q^n), \tag{2.3.1}$$

$$h(q) := \exp(H(q)).$$

Remark 2.3.2. We often write $h(q) = \phi(q)^{-c(k)}$ and $H(q) = \log(h(q))$.

The following lemma immediately follows from the definition of $h(q)$ above.

Lemma 2.3.3. *We have*

$$\frac{d}{dq}h(q) = h(q) \cdot \frac{d}{dq}H(q).$$

Furthermore, we can show the following:

Lemma 2.3.4. *We have*

$$q\frac{d}{dq}H(q) = c(k) \cdot \sum_{n \geq 1} n \sum_{j \geq 1} q^{nj}.$$

Proof. By differentiating the right-hand side of (2.3.1) by terms, we obtain

$$\frac{d}{dq}H(q) = c(k) \cdot \sum_{n \geq 1} \frac{nq^{n-1}}{1 - q^n}.$$

Thus, multiplying both sides by q, we have

$$q\frac{d}{dq}H(q) = c(k) \cdot \sum_{n \geq 1} \frac{nq^n}{1 - q^n}.$$

Since, for each $n \in \mathbb{Z}_{\geq 1}$,

$$\frac{q^n}{1 - q^n} = \sum_{j \geq 1} q^{nj},$$

we deduce that

$$q\frac{d}{dq}H(q) = c(k) \cdot \sum_{n \geq 1} n \sum_{j \geq 1} q^{nj}.$$

This proves the lemma. ∎

Now we recall that the Casimir operator $\widehat{\Omega}$ for $\widehat{\mathfrak{g}}$ can be written in the form:

$$\widehat{\Omega} = \Omega + 2(c + h^{\vee})d + 2\sum_{i=1}^{M}\sum_{n\geq 1}(t^{-n} \otimes u_i)(t^n \otimes u^i),$$

as an operator on $V = \widehat{L}(\Lambda)$, and that $\widehat{\Omega}$ acts on $\widehat{L}(\Lambda)$ by the scalar $(\Lambda + 2\widehat{\rho}|\Lambda)$. Since $\widehat{L}(\Lambda)$ is a highest weight $\widehat{\mathfrak{g}}$-module, we see that the canonical central element $c \in \widehat{\mathfrak{g}}$ acts on $\widehat{L}(\Lambda)$ by the scalar $k = \Lambda(c)$. Also, by definition, the scaling element $d \in \widehat{\mathfrak{g}}$ acts on each homogeneous subspace V_m by the scalar $-m$ for $m \in \mathbb{Z}_{\geq 0}$. In addition, it follows from the commutation relation $[d, t^n \otimes x] = nt^n \otimes x$ for $x \in \mathfrak{g}$, $n \in \mathbb{Z}$ that

$$(t^{-n} \otimes u_i)(t^n \otimes u^i)V_m \subset (t^{-n} \otimes u_i)V_{m-n} \subset V_m$$

for $m \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{\geq 1}$. Hence we deduce that for each $m \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} \operatorname{Tr}(\Omega|_{V_m}) = (\Lambda + 2\widehat{\rho}|\Lambda)(\dim_{\mathbb{C}} V_m) + 2(k + h^{\vee})m(\dim_{\mathbb{C}} V_m) \\ - 2\sum_{i=1}^{M}\sum_{n\geq 1}\operatorname{Tr}((t^{-n} \otimes u_i)(t^n \otimes u^i)|_{V_m}). \end{aligned} \tag{2.3.2}$$

Proposition 2.3.5. *For each $1 \leq i \leq M$, $n \in \mathbb{Z}_{\geq 1}$, we have*

$$\operatorname{Tr}((t^{-n} \otimes u_i)(t^n \otimes u^i)|_{V_m}) = kn \cdot \sum_{j\geq 1}\dim_{\mathbb{C}} V_{m-nj}.$$

Here we understand $\dim_{\mathbb{C}} V_{-m} = 0$ for $m < 0$. In particular, the trace above does not depend on $1 \leq i \leq M$.

Proof. First we note that for $1 \leq i \leq M$, $n \geq 1$,

$$(t^n \otimes u^i)V_m \subset V_{m-n}, \quad (t^{-n} \otimes u_i)V_{m-n} \subset V_m$$

by the commutation relation $[d, t^n \otimes x] = nt^n \otimes x$ for $x \in \mathfrak{g}$, $n \in \mathbb{Z}$. Thus we have

$$(t^{-n} \otimes u_i)(t^n \otimes u^i)V_m \subset V_m, \quad (t^n \otimes u^i)(t^{-n} \otimes u_i)V_{m-n} \subset V_{m-n}.$$

Hence, by Lemma 2.3.1, we see that

$$\operatorname{Tr}((t^{-n} \otimes u_i)(t^n \otimes u^i)|_{V_m}) = \operatorname{Tr}((t^n \otimes u^i)(t^{-n} \otimes u_i)|_{V_{m-n}}). \tag{2.3.3}$$

Here we recall the commutation relation:

$$\begin{aligned}[t^n \otimes u^i, t^{-n} \otimes u_i] &= t^0 \otimes [u^i, u_i] + n(u^i|u_i)c \\ &= t^0 \otimes [u^i, u_i] + nc.\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}\mathrm{Tr}((t^n \otimes u^i)(t^{-n} \otimes u_i)|_{V_{m-n}}) &= \mathrm{Tr}((t^{-n} \otimes u_i)(t^n \otimes u^i)|_{V_{m-n}}) \\ &\quad + \mathrm{Tr}([u^i, u_i]|_{V_{m-n}}) + kn(\dim_{\mathbb{C}} V_{m-n}).\end{aligned}$$

Since

$$\begin{aligned}\mathrm{Tr}([u^i, u_i]|_{V_{m-n}}) &= \mathrm{Tr}(u^i|_{V_{m-n}} \circ u_i|_{V_{m-n}} - u_i|_{V_{m-n}} \circ u^i|_{V_{m-n}}) \\ &= \mathrm{Tr}(u^i|_{V_{m-n}} u_i|_{V_{m-n}}) - \mathrm{Tr}(u_i|_{V_{m-n}} u^i|_{V_{m-n}}) \\ &= 0\end{aligned}$$

again by Lemma 2.3.1, we get

$$\begin{aligned}&\mathrm{Tr}((t^n \otimes u^i)(t^{-n} \otimes u_i)|_{V_{m-n}}) \\ &\quad = \mathrm{Tr}((t^{-n} \otimes u_i)(t^n \otimes u^i)|_{V_{m-n}}) + kn(\dim_{\mathbb{C}} V_{m-n}).\end{aligned} \tag{2.3.4}$$

By combining (2.3.3) and (2.3.4), we obtain a recurrence relation:

$$\begin{aligned}&\mathrm{Tr}((t^{-n} \otimes u_i)(t^n \otimes u^i)|_{V_m}) \\ &\quad = \mathrm{Tr}((t^{-n} \otimes u_i)(t^n \otimes u^i)|_{V_{m-n}}) + kn(\dim_{\mathbb{C}} V_{m-n}).\end{aligned}$$

Note that $V_{m-nj} = \{0\}$ for sufficiently large $j \in \mathbb{Z}_{\geq 1}$. Hence it follows from the recurrence relation above that

$$\mathrm{Tr}((t^{-n} \otimes u_i)(t^n \otimes u^i)|_{V_m}) = kn \cdot \sum_{j \geq 1} \dim_{\mathbb{C}} V_{m-nj}.$$

This proves the proposition. □

By (2.3.2) and Proposition 2.3.5, we obtain

$$\begin{aligned}\mathrm{Tr}(\Omega|_{V_m}) &= (\Lambda + 2\widehat{\rho}|\Lambda)(\dim_{\mathbb{C}} V_m) + 2(k + h^\vee)m(\dim_{\mathbb{C}} V_m) \\ &\quad - 2kM \sum_{n \geq 1} n \sum_{j \geq 1} \dim_{\mathbb{C}} V_{m-nj}.\end{aligned} \tag{2.3.5}$$

Here we introduce the following formal power series, called the graded dimension of V,

$$f(q) := \sum_{m\geq 0} (\dim_{\mathbb{C}} V_m)\, q^m,$$

which is the generating function of the dimensions $\dim_{\mathbb{C}} V_m$, $m \in \mathbb{Z}_{\geq 0}$. If we set

$$F(q) := f(q) \cdot h(q)^{-1} = f(q) \cdot \phi(q)^{c(k)},$$

then we have

(2.3.6)
$$\begin{aligned}
\frac{d}{dq} f(q) &= \Big(\frac{d}{dq} F(q)\Big) \cdot h(q) + F(q) \cdot \Big(\frac{d}{dq} h(q)\Big) \\
&= \Big(\frac{d}{dq} F(q)\Big) \cdot h(q) + F(q) \cdot \Big(h(q) \cdot \frac{d}{dq} H(q)\Big) \quad \text{by Lemma 2.3.3} \\
&= \Big(\frac{d}{dq} F(q)\Big) \cdot h(q) + f(q) \cdot \frac{d}{dq} H(q).
\end{aligned}$$

Now we calculate the graded trace $g(q) = \sum_{m\geq 0} \mathrm{Tr}(\Omega|_{V_m})\, q^m$. By (2.3.5), we have

$$\begin{aligned}
g(q) &= \sum_{m\geq 0} \mathrm{Tr}(\Omega|_{V_m})\, q^m \\
&= (\Lambda + 2\widehat{\rho}|\Lambda) f(q) + 2(k + h^{\vee})\, q \frac{d}{dq} f(q) \\
&\qquad - 2kM \sum_{m\geq 0} \Big(\sum_{n\geq 1} n \sum_{j\geq 1} \dim_{\mathbb{C}} V_{m-nj} \Big) q^m.
\end{aligned}$$

We further deduce that

$$\begin{aligned}
\sum_{m\geq 0} \Big(\sum_{n\geq 1} n \sum_{j\geq 1} \dim_{\mathbb{C}} V_{m-nj} \Big) q^m &= \sum_{m\geq 0} \sum_{\substack{n\geq 1 \\ j\geq 1}} n (\dim_{\mathbb{C}} V_{m-nj})\, q^m \\
&= \sum_{\substack{n\geq 1 \\ j\geq 1}} \sum_{m\geq 0} n (\dim_{\mathbb{C}} V_{m-nj})\, q^m \\
&= \sum_{\substack{n\geq 1 \\ j\geq 1}} \sum_{m\geq 0} n (\dim_{\mathbb{C}} V_m)\, q^{m+nj}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m\geq 0} \sum_{\substack{n\geq 1\\ j\geq 1}} n(\dim_{\mathbb{C}} V_m)\, q^m \cdot q^{nj} \\
&= \Big(\sum_{m\geq 0} (\dim_{\mathbb{C}} V_m)\, q^m \Big) \cdot \Big(\sum_{n\geq 1} n \sum_{j\geq 1} q^{nj} \Big) \\
&= f(q)\cdot c(k)^{-1} q \frac{d}{dq} H(q) \quad \text{by Lemma 2.3.4.}
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
g(q) &= (\Lambda + 2\widehat{\rho}|\Lambda) f(q) + 2(k+h^{\vee})\, q \frac{d}{dq} f(q) - 2kMc(k)^{-1} f(q)\cdot q\frac{d}{dq}H(q) \\
&= (\Lambda + 2\widehat{\rho}|\Lambda) f(q) + 2(k+h^{\vee})\, q \Big\{ \frac{d}{dq} f(q) - f(q)\cdot \frac{d}{dq}H(q) \Big\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by the definition of } c(k) \\
&= (\Lambda + 2\widehat{\rho}|\Lambda) h(q) F(q) + 2(k+h^{\vee})\, q \Big\{ h(q)\cdot \frac{d}{dq} F(q) \Big\} \quad \text{by (2.3.6)} \\
&= h(q)\cdot \Big\{ (\Lambda + 2\widehat{\rho}|\Lambda) F(q) + 2(k+h^{\vee})\, q \frac{d}{dq} F(q) \Big\} \\
&= \frac{(\Lambda + 2\widehat{\rho}|\Lambda) F(q) + 2(k+h^{\vee})\, q \frac{d}{dq} F(q)}{\prod_{n\geq 1} (1-q^n)^{c(k)}}.
\end{aligned}
$$

Thus we have proved the following.

THEOREM 2.3.6. *Let $\widehat{\mathfrak{g}} = \mathfrak{g}(X_N^{(1)})$ be the affine Lie algebra of type $X_N^{(1)}$ with $X = A, D, E$, and let $V = \widehat{L}(\Lambda)$ be the irreducible highest weight $\widehat{\mathfrak{g}}$-module of dominant integral highest weight $\Lambda \in (\widehat{\mathfrak{h}})^*$ (such that $\Lambda(d) = 0$) given the basic gradation $V = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} V_m$. Then the graded trace $g(q) = \sum_{m\geq 0} \mathrm{Tr}(\Omega|_{V_m})\, q^m$ of the Casimir element Ω for the finite-dimensional simple Lie algebra $\mathfrak{g} = \mathfrak{g}(X_N)$ of type X_N is expressed in the following form:*

$$
g(q) = \frac{(\Lambda + 2\widehat{\rho}|\Lambda) F(q) + 2(k+h^{\vee})\, q \frac{d}{dq} F(q)}{\prod_{n\geq 1} (1-q^n)^{\frac{k(\dim_{\mathbb{C}} \mathfrak{g})}{k+h^{\vee}}}},
$$

where

$$
F(q) = \Big(\prod_{n\geq 1} (1-q^n)^{\frac{k(\dim_{\mathbb{C}} \mathfrak{g})}{k+h^{\vee}}} \Big) \cdot \sum_{m\geq 0} (\dim_{\mathbb{C}} V_m)\, q^m.
$$

Remark 2.3.7. If $\Lambda \in (\widehat{\mathfrak{h}})^*$ is a dominant integral weight such that $k = \Lambda(c) = 1$ and $\Lambda(d) = 0$, then we know from [K4, Chap. 12]

$$(\Lambda|\Lambda)h^\vee = 2(\widehat{\rho}|\Lambda).$$

So we have

$$(\Lambda + 2\widehat{\rho}|\Lambda) = (\Lambda|\Lambda) \cdot (1 + h^\vee).$$

Also, since $M = \dim_{\mathbb{C}} \mathfrak{g} = N(1 + h^\vee)$ by Remark 2.2.1, we have

$$c(1) = \frac{\dim_{\mathbb{C}} \mathfrak{g}}{1 + h^\vee} = N.$$

Hence we obtain

$$g(q) = \frac{(1 + h^\vee)\Big\{(\Lambda|\Lambda)F(q) + 2q\dfrac{d}{dq}F(q)\Big\}}{\displaystyle\prod_{n\ge 1}(1 - q^n)^N}.$$

In particular, if Λ is the basic fundamental weight $\widehat{\Lambda}_0 \in (\widehat{\mathfrak{h}})^*$ defined by $\widehat{\Lambda}_0(\mathfrak{h}) := 0$, $\widehat{\Lambda}_0(c) := 1$, $\widehat{\Lambda}_0(d) := 0$, then we have

$$g(q) = \frac{2(1 + h^\vee)\, q\dfrac{d}{dq}F(q)}{\displaystyle\prod_{n\ge 1}(1 - q^n)^N}$$

since $(\widehat{\Lambda}_0|\widehat{\Lambda}_0) = 0$.

Remark 2.3.8. Recall from [K4, Chap. 12] that a dominant integral weight $\Lambda \in (\widehat{\mathfrak{h}})^*$ such that $k = \Lambda(c) = 1$ and $\Lambda(d) = 0$ is of the form $\Lambda = \widehat{\Lambda}_0$ or $\Lambda = \widehat{\Lambda}_0 + \bar{\Lambda}_i$ with $1 \le i \le N$ such that $\widehat{a}_i^\vee = 1$, where $\{\bar{\Lambda}_i\}_{i=1}^N \subset \mathfrak{h}^* \subset (\widehat{\mathfrak{h}})^*$ are the fundamental weights of $\mathfrak{g} = \mathfrak{g}(X_N)$ and $c = \sum_{i=0}^N \widehat{a}_i^\vee \widehat{h}_i$ is the canonical central element.

§3. Identity for the derivative of a theta series of type A, D, E

In this section, we assume that $\Lambda \in (\widehat{\mathfrak{h}})^*$ is a dominant integral weight such that $k = \Lambda(c) = 1$ and $\Lambda(d) = 0$.

Recall from [K4, Chap. 6] that we have an orthogonal direct sum:

$$(\widehat{\mathfrak{h}})^* = \mathfrak{h}^* \oplus (\mathbb{C}\delta + \mathbb{C}\widehat{\Lambda}_0).$$

For an element $\Lambda \in (\widehat{\mathfrak{h}})^*$, we denote by $\bar{\Lambda} \in \mathfrak{h}^*$ the orthogonal projection of Λ on $\mathfrak{h}^*$. Note that we have

$$\Lambda = \bar{\Lambda} + \Lambda(c)\widehat{\Lambda}_0 + \Lambda(d)\delta,$$

and hence $\Lambda = \bar{\Lambda} + \widehat{\Lambda}_0$ (cf. Remark 2.3.8). In particular, $(\Lambda|\Lambda) = (\bar{\Lambda}|\bar{\Lambda})$ since $(\widehat{\Lambda}_0|\widehat{\Lambda}_0) = 0$.

We know the following fact due to Kac (see [K4, Chap. 12]).

FACT 1. *The graded dimension $f(q) = \sum_{m\geq 0}(\dim_{\mathbb{C}} V_m)\, q^m$ of the irreducible highest weight $\widehat{\mathfrak{g}}$-module $V = \widehat{L}(\Lambda)$ of highest weight Λ with the basic gradation is given by*:

$$\begin{aligned} f(q) &= \sum_{m\geq 0}(\dim_{\mathbb{C}} V_m)\, q^m \\ &= \frac{q^{-\frac{1}{2}(\bar{\Lambda}|\bar{\Lambda})} \cdot \displaystyle\sum_{\alpha\in\bar{\Lambda}+Q} q^{\frac{1}{2}(\alpha|\alpha)}}{\displaystyle\prod_{n\geq 1}(1-q^n)^N}, \end{aligned}$$

where $Q := \sum_{i=1}^N \mathbb{Z}\alpha_i \subset \mathfrak{h}^*$ is the root lattice of $\mathfrak{g} = \mathfrak{g}(X_N)$ and $(\,\cdot\,|\,\cdot\,)$ is the normalized Killing form on $\mathfrak{h}^*$.

By Fact 1, we have

$$\begin{aligned} F(q) &= f(q) \cdot \prod_{n\geq 1}(1-q^n)^N \\ &= q^{-\frac{1}{2}(\bar{\Lambda}|\bar{\Lambda})} \cdot \sum_{\alpha\in\bar{\Lambda}+Q} q^{\frac{1}{2}(\alpha|\alpha)} \end{aligned}$$

since $c(1) = \frac{M}{1+h^\vee} = N$. We set

$$\Theta_{Q,\bar{\Lambda}}(q) := \sum_{\alpha\in\bar{\Lambda}+Q} q^{\frac{1}{2}(\alpha|\alpha)}.$$

Then we deduce that

$$q\frac{d}{dq}F(q) = -\frac{1}{2}(\bar{\Lambda}|\bar{\Lambda}) \cdot F(q) + q^{-\frac{1}{2}(\bar{\Lambda}|\bar{\Lambda})} \cdot q\frac{d}{dq}\Theta_{Q,\bar{\Lambda}}(q).$$

Hence we obtain by Remark 2.3.7 that

$$g(q) = \frac{2(1+h^\vee)q^{-\frac{1}{2}(\bar{\Lambda}|\bar{\Lambda})} \cdot q\dfrac{d}{dq}\Theta_{Q,\bar{\Lambda}}(q)}{\displaystyle\prod_{n\geq 1}(1-q^n)^N} \tag{3.1}$$

since $(\Lambda|\Lambda) = (\bar{\Lambda}|\bar{\Lambda})$.

Here we recall that each homogeneous subspace V_m of V is a finite-dimensional $\mathfrak{g}\,(\hookrightarrow \widehat{\mathfrak{g}})$-module for $m \in \mathbb{Z}_{\geq 0}$. Hence it decomposes into a direct sum of irreducible highest weight $\mathfrak{g}$-modules $L(\lambda)$ with $\lambda \in P_+ := \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0},\, 1 \leq i \leq N\}$. For each $\lambda \in P_+$, we denote by $\Phi(\Lambda,\lambda)_m$ the multiplicity of $L(\lambda)$ in V_m:

$$V_m = \bigoplus_{\lambda \in P_+} \Phi(\Lambda,\lambda)_m L(\lambda), \tag{3.2}$$

and set

$$\Phi(\Lambda,\lambda)(q) := \sum_{m\geq 0} \Phi(\Lambda,\lambda)_m q^m.$$

Then we know the following fact due to Kac (see [K4, Chap. 12]).

Fact 2. *Let $\lambda \in P_+$. If $\lambda \notin \bar{\Lambda} + Q$, then we have $\Phi(\Lambda,\lambda)(q) = 0$. If $\lambda \in \bar{\Lambda} + Q$, then we have*

$$\Phi(\Lambda,\lambda)(q) = \frac{q^{\frac{1}{2}\{(\lambda|\lambda)-(\bar{\Lambda}|\bar{\Lambda})\}} \cdot \displaystyle\prod_{\alpha\in\Delta_+}\left(1-q^{(\lambda+\rho|\alpha)}\right)}{\displaystyle\prod_{n\geq 1}(1-q^n)^N}.$$

Since the Casimir element $\Omega \in Z(U(\mathfrak{g}))$ acts on $L(\lambda)$ by the scalar $(\lambda+2\rho|\lambda)$, we see from the decomposition (3.2) that for each $m \in \mathbb{Z}_{\geq 0}$,

$$\mathrm{Tr}(\Omega|_{V_m}) = \sum_{\lambda\in(\bar{\Lambda}+Q)\cap P_+} d(\lambda)(\lambda+2\rho|\lambda)\Phi(\Lambda,\lambda)_m,$$

where $d(\lambda) = \dim_{\mathbb{C}} L(\lambda)$. Therefore we deduce, by using Fact 2, that

$$\begin{aligned}
g(q) &= \sum_{m\geq 0} \mathrm{Tr}(\Omega|_{V_m})\, q^m \\
&= \sum_{m\geq 0} \sum_{\lambda\in(\bar{\Lambda}+Q)\cap P_+} d(\lambda)(\lambda+2\rho|\lambda)\Phi(\Lambda,\lambda)_m q^m \\
&= \sum_{\lambda\in(\bar{\Lambda}+Q)\cap P_+} d(\lambda)(\lambda+2\rho|\lambda)\Big(\sum_{m\geq 0}\Phi(\Lambda,\lambda)_m q^m\Big) \\
&= \sum_{\lambda\in(\bar{\Lambda}+Q)\cap P_+} d(\lambda)(\lambda+2\rho|\lambda)\Phi(\Lambda,\lambda)(q) \\
&= \sum_{\lambda\in(\bar{\Lambda}+Q)\cap P_+} d(\lambda)(\lambda+2\rho|\lambda)\Big(h(q)q^{\frac{1}{2}\{(\lambda|\lambda)-(\bar{\Lambda}|\bar{\Lambda})\}}\prod_{\alpha\in\Delta_+}(1-q^{(\lambda+\rho|\alpha)})\Big) \\
&= q^{-\frac{1}{2}(\bar{\Lambda}|\bar{\Lambda})}h(q)\cdot \sum_{\lambda\in(\bar{\Lambda}+Q)\cap P_+} d(\lambda)(\lambda+2\rho|\lambda)q^{\frac{1}{2}(\lambda|\lambda)}\prod_{\alpha\in\Delta_+}\Big(1-q^{(\lambda+\rho|\alpha)}\Big),
\end{aligned}$$

where $h(q) = \prod_{n\geq 1}(1-q^n)^{-N}$. By comparing this equality with (3.1), we obtain

$$\begin{aligned}
&2(1+h^\vee)\, q\frac{d}{dq}\Theta_{Q,\bar{\Lambda}}(q) \\
&\quad= \sum_{\lambda\in(\bar{\Lambda}+Q)\cap P_+} d(\lambda)(\lambda+2\rho|\lambda)q^{\frac{1}{2}(\lambda|\lambda)}\prod_{\alpha\in\Delta_+}\Big(1-q^{(\lambda+\rho|\alpha)}\Big).
\end{aligned}$$

Thus we have proved the following.

THEOREM 3.1. *Let $\mathfrak{g} = \mathfrak{g}(X_N)$ be a finite-dimensional simple Lie algebra of type X_N with $X = A, D, E$, and let $\Lambda = \bar{\Lambda} + \widehat{\Lambda}_0 \in (\widehat{\mathfrak{h}})^*$ with $\bar{\Lambda} \in \mathfrak{h}^*$ be a dominant integral weight. Then we have*

$$\begin{aligned}
&2(1+h^\vee)\, q\frac{d}{dq}\Theta_{Q,\bar{\Lambda}}(q) \\
&\quad= \sum_{\lambda\in(\bar{\Lambda}+Q)\cap P_+} d(\lambda)(\lambda+2\rho|\lambda)q^{\frac{1}{2}(\lambda|\lambda)}\prod_{\alpha\in\Delta_+}\Big(1-q^{(\lambda+\rho|\alpha)}\Big),
\end{aligned}$$

where $\Theta_{Q,\bar{\Lambda}}(q) = \sum_{\alpha\in\bar{\Lambda}+Q} q^{\frac{1}{2}(\alpha|\alpha)}$ and $d(\lambda) = \dim_{\mathbb{C}} L(\lambda)$ for $\lambda \in P_+$.

Remark 3.2. For $\lambda \in P_+$, the dimension $d(\lambda) = \dim_{\mathbb{C}} L(\lambda)$ is given by the Weyl dimension formula:

$$d(\lambda) = \prod_{\alpha \in \Delta_+} \frac{(\lambda + \rho|\alpha)}{(\rho|\alpha)}.$$

Example 3.3. Let $\mathfrak{g}$ be a simple Lie algebra of type A_2, i.e.,

$$\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C}) = \{X \in M(3, \mathbb{C}) \mid \mathrm{Tr}(X) = 0\},$$

and $\bar{\Lambda} = 0$. Then we have

$$\Pi = \{\alpha_1, \alpha_2\}, \;\; \Delta_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}, \;\; \rho = \alpha_1 + \alpha_2, \;\; h^\vee = 3,$$
$$Q = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 = \{k\alpha_1 + m\alpha_2 \mid k, m \in \mathbb{Z}\},$$
$$(\alpha_1|\alpha_1) = (\alpha_2|\alpha_2) = 2, \quad (\alpha_1|\alpha_2) = -1,$$
$$P_+ \cap Q = \{k\alpha_1 + m\alpha_2 \mid 2k \geq m \geq 0, \, 2m \geq k \geq 0, \, k, m \in \mathbb{Z}\}.$$

Also, for $\lambda = k\alpha_1 + m\alpha_2 \in P_+ \cap Q$, we have

$$d(\lambda) = \frac{1}{2}(2k - m + 1)(2m - k + 1)(k + m + 2)$$

by Remark 3.2. Thus we can write the identity in Theorem 3.1 as follows:

$$\begin{aligned} 8 \cdot & \sum_{k,m \in \mathbb{Z}} (k^2 - km + m^2)\, q^{k^2 - km + m^2} \\ &= \sum_{\substack{2k \geq m \geq 0 \\ 2m \geq k \geq 0 \\ k,m \in \mathbb{Z}}} (2k - m + 1)(2m - k + 1)(k + m + 2)(k^2 - km + m^2 + k + m) \\ &\qquad \times q^{k^2 - km + m^2}(1 - q^{2k - m + 1})(1 - q^{2m - k + 1})(1 - q^{k + m + 2}). \end{aligned}$$

Remark 3.4. It immediately follows from the decomposition (3.2) that for each $m \in \mathbb{Z}_{\geq 0}$,

$$\dim_{\mathbb{C}} V_m = \sum_{\lambda \in (\bar{\Lambda} + Q) \cap P_+} d(\lambda) \Phi(\Lambda, \lambda)_m.$$

Therefore, as above, we can easily deduce by using Fact 2 that

$$\begin{aligned} f(q) &= \sum_{m \geq 0} (\dim_{\mathbb{C}} V_m)\, q^m \\ &= q^{-\frac{1}{2}(\bar{\Lambda}|\bar{\Lambda})} \prod_{n \geq 1} (1 - q^n)^{-N} \cdot \sum_{\lambda \in (\bar{\Lambda} + Q) \cap P_+} d(\lambda) q^{\frac{1}{2}(\lambda|\lambda)} \prod_{\alpha \in \Delta_+} \left(1 - q^{(\lambda + \rho|\alpha)}\right). \end{aligned}$$

By comparing this with Fact 1, we obtain

$$\begin{aligned}\Theta_{Q,\bar{\Lambda}}(q) &= \sum_{\alpha\in\bar{\Lambda}+Q} q^{\frac{1}{2}(\alpha|\alpha)} \\ &= \sum_{\lambda\in(\bar{\Lambda}+Q)\cap P_+} d(\lambda)\, q^{\frac{1}{2}(\lambda|\lambda)} \prod_{\alpha\in\Delta_+} \Big(1-q^{(\lambda+\rho|\alpha)}\Big).\end{aligned}$$

§4. Results in the C, B, F, G cases

4.1. Twisted affine Lie algebras

Here we recall from [K4, Chaps. 6 and 8] (and also [W]) some standard notation and facts about twisted affine Lie algebras.

Let $\mathfrak{g} = \mathfrak{g}(X_N)$ be a finite-dimensional complex simple Lie algebra of type X_N, where $X_N = A_{2L-1}$ ($L \geq 3$), D_{L+1} ($L \geq 2$), E_6, or D_4 (recall the notation of Section 2.1). Also we denote by $\mu : \mathfrak{g} \to \mathfrak{g}$ the Lie algebra automorphism induced by a Dynkin diagram automorphism $\mu : \{1, \ldots, N\} \to \{1, \ldots, N\}$ of order r.

Remark 4.1.1. In the case where $X_N = D_4$ above, we take one of two Dynkin diagram automorphisms of order 3. In the case where $X_N = D_{L+1}$ with $L = 3$ above, we take one of three Dynkin diagram automorphisms of order 2. In each of other cases above, there is only one nontrivial Dynkin diagram automorphism, which is of order 2. Thus $r = 2$ if $X_N = A_{2L-1}$, D_{L+1}, E_6, and $r = 3$ if $X_N = D_4$.

Let $\zeta := \exp\big(\frac{2\pi\sqrt{-1}}{r}\big) \in \mathbb{C}^*$ be a primitive r-th root of unity. Since $\mu^r = \mathrm{id}$, we have μ-eigenspace decompositions of $\mathfrak{g}$ and $\mathfrak{h}$:

$$\mathfrak{g} = \bigoplus_{\bar{k}\in\mathbb{Z}/r\mathbb{Z}} \mathfrak{g}_{\bar{k}}, \quad \mathfrak{g}_{\bar{l}} := \{x \in \mathfrak{g} \mid \mu(x) = \zeta^l x\}.$$

$$\mathfrak{h} = \bigoplus_{\bar{l}\in\mathbb{Z}/r\mathbb{Z}} \mathfrak{h}_{\bar{l}}, \quad \mathfrak{h}_{\bar{l}} := \mathfrak{g}_{\bar{l}} \cap \mathfrak{h},$$

where $\bar{l} := l + r\mathbb{Z} \in \mathbb{Z}/r\mathbb{Z}$ denotes the residue class of $l \in \mathbb{Z}$. It is known (see [K4, Chap. 8]) that the fixed point subalgebra $\mathfrak{g}_{\bar{0}}$ of $\mathfrak{g}$ is, in fact, a finite-dimensional simple Lie algebra of type Y_L with Cartan subalgebra $\mathfrak{h}_{\bar{0}}$, where Y_L is given by:

$$Y_L = \begin{cases} C_L & \text{if } X_N = A_{2L-1},\ r = 2, \\ B_L & \text{if } X_N = D_{L+1},\ r = 2, \\ F_4 & \text{if } X_N = E_6,\ r = 2, \\ G_2 & \text{if } X_N = D_4,\ r = 3. \end{cases}$$

Furthermore, for each $\bar{l} \in \mathbb{Z}/r\mathbb{Z}$, $\mathfrak{g}_{\bar{l}}$ admits a weight space decomposition with respect to the Cartan subalgebra $\mathfrak{h}_{\bar{0}}$ of $\mathfrak{g}_{\bar{0}}$:

$$\mathfrak{g}_{\bar{l}} = \mathfrak{h}_{\bar{l}} \oplus \bigoplus_{\alpha \in \Delta_{\bar{l}}} \mathfrak{g}_{\bar{l}}.$$

In particular, $\Delta_{\bar{0}} \subset (\mathfrak{h}_{\bar{0}})^*$ is the set of roots of $\mathfrak{g}_{\bar{0}} = \mathfrak{g}(Y_L)$.

Let $\widetilde{\mathfrak{g}} = \mathfrak{g}(X_N^{(r)})$ be a twisted affine Lie algebra of type $X_N^{(r)}$ over $\mathbb{C}$, where $X_N^{(r)} = A_{2L-1}^{(2)}$, $D_{L+1}^{(2)}$, $E_6^{(2)}$, $D_4^{(3)}$. Namely, $\widetilde{\mathfrak{g}}$ is the following subalgebra of $\widehat{\mathfrak{g}}$:

$$\widetilde{\mathfrak{g}} = \widehat{\mathcal{L}}(\mathfrak{g}, \mu, r) = \left(\bigoplus_{j \in \mathbb{Z}} \mathbb{C}t^j \otimes_{\mathbb{C}} \mathfrak{g}_{\bar{j}} \right) \oplus \mathbb{C}K \oplus \mathbb{C}D,$$

where $K := rc$ is the canonical central element and $D := d$ is the scaling element.

Remark 4.1.2. There are some misprints on the canonical central element and the scaling element of the twisted affine Lie algebra $\widetilde{\mathfrak{g}} = \mathfrak{g}(X_N^{(r)})$ in [K4, Section 8.3] and [W, Section 7.2]. Note also that $a_0 = 1$ in the notation therein unless $X_N^{(r)} = A_{2L}^{(2)}$.

The Cartan subalgebra of $\widetilde{\mathfrak{g}}$ is the following subalgebra of $\widehat{\mathfrak{h}}$:

$$\widetilde{\mathfrak{h}} = (\mathbb{C}t^0 \otimes_{\mathbb{C}} \mathfrak{h}_{\bar{0}}) \oplus \mathbb{C}K \oplus \mathbb{C}D.$$

The set $\widetilde{\Delta}_+ \subset (\widetilde{\mathfrak{h}})^*$ of positive roots of $\widetilde{\mathfrak{g}}$ is described as:

$$\widetilde{\Delta}_+ = \{j\delta \mid j \in \mathbb{Z}_{\geq 1}\} \sqcup \{j\delta + \alpha \mid j \in \mathbb{Z}_{\geq 1},\, \alpha \in \Delta_{\bar{j}}\} \sqcup (\Delta_{\bar{0}})_+,$$

where $\delta \in (\widetilde{\mathfrak{h}})^*$ is the restriction of the null root $\delta \in (\widehat{\mathfrak{h}})^*$ of $\widehat{\mathfrak{g}}$ to the subalgebra $\widetilde{\mathfrak{h}} \subset \widehat{\mathfrak{h}}$ and $(\Delta_{\bar{0}})_+ \subset (\mathfrak{h}_{\bar{0}})^*$ is the set of positive roots of $\mathfrak{g}_{\bar{0}} = \mathfrak{g}(Y_L)$ regarded (as usual) as a subset of $(\widetilde{\mathfrak{h}})^*$. Moreover, the root spaces $\widetilde{\mathfrak{g}}_\gamma$, $\gamma \in \widetilde{\Delta}_+$, are written as:

$$\widetilde{\mathfrak{g}}_{j\delta} = \mathbb{C}t^j \otimes_{\mathbb{C}} \mathfrak{h}_{\bar{j}}, \;\; \widetilde{\mathfrak{g}}_{j\delta+\alpha} = \mathbb{C}t^j \otimes_{\mathbb{C}} \mathfrak{g}_{\bar{j},\alpha}, \quad j \in \mathbb{Z},\, \alpha \in \Delta_{\bar{j}}.$$

Also we denote by $\widetilde{\Pi} = \{\widetilde{\alpha}_i\}_{i=0}^L \subset \widetilde{\Delta}_+$ the set of simple roots of $\widetilde{\mathfrak{g}}$, and by $\widetilde{\Pi}^\vee = \{\widetilde{h}_i\}_{i=0}^L \subset \widetilde{\mathfrak{h}}$ the set of simple coroots of $\widetilde{\mathfrak{g}}$. (See [K4, Chap. 8] for the explicit construction of $\widetilde{\Pi}$ and $\widetilde{\Pi}^\vee$.)

Remark 4.1.3. The finite-dimensional simple Lie algebra $\mathfrak{g}_{\bar{0}} = \mathfrak{g}(Y_L)$ of type Y_L can be identified with the subalgebra $\mathbb{C}t^0 \otimes_{\mathbb{C}} \mathfrak{g}_{\bar{0}}$ of $\widetilde{\mathfrak{g}} = \mathfrak{g}(X_N^{(r)})$. In fact, the Dynkin diagram of $\mathfrak{g}(X_N^{(r)})$ with the 0-th vertex (enumerated as in [K4, Chap. 4]) removed is nothing but the Dynkin diagram of $\mathfrak{g}(Y_L)$. Thus the simple roots of $\mathfrak{g}(Y_L) = \mathfrak{g}_{\bar{0}}$ are the restrictions of the $\widetilde{\alpha}_i$'s, $1 \leq i \leq L$, to $\mathfrak{h}_{\bar{0}} \subset \widetilde{\mathfrak{h}}$. So

$$\dot{Q} := \sum_{i=1}^{L} \mathbb{Z}\widetilde{\alpha}_i \subset (\mathfrak{h}_{\bar{0}})^*$$

is the root lattice of $\mathfrak{g}_{\bar{0}} = \mathfrak{g}(Y_L)$.

The normalized Killing form $(\,\cdot\,|\,\cdot\,)$ on $\mathfrak{g} = \mathfrak{g}(X_N)$ can be extended to the normalized invariant form (see [K4, Chap. 6]) $\langle\,\cdot\,|\,\cdot\,\rangle$ on $\widetilde{\mathfrak{g}} = \mathfrak{g}(X_N^{(r)})$ by:

$$\begin{cases} \langle t^i \otimes x | t^j \otimes y \rangle = r^{-1}\delta_{i+j,0}(x|y), \;\; i,j \in \mathbb{Z},\, x \in \mathfrak{g}_{\bar{i}},\, y \in \mathfrak{g}_{\bar{j}}; \\ \langle \mathbb{C}K \oplus \mathbb{C}D | \mathbb{C}t^j \otimes_{\mathbb{C}} \mathfrak{g}_{\bar{j}} \rangle = 0, \;\; j \in \mathbb{Z},\, x \in \mathfrak{g}_{\bar{j}}; \\ \langle K|K \rangle = \langle D|D \rangle = 0; \\ \langle K|D \rangle = r\langle c|d \rangle = 1. \end{cases}$$

(We note that there are misprints in [K4, Eq. (8.3.8) on p. 131] and in [W, Corollary 7.2E].) Namely, the normalized invariant form $\langle\,\cdot\,|\,\cdot\,\rangle$ on $\widetilde{\mathfrak{g}} = \mathfrak{g}(X_N^{(r)})$ is the restriction of the normalized invariant form $(\,\cdot\,|\,\cdot\,)$ on $\widehat{\mathfrak{g}} = \mathfrak{g}(X_n^{(1)})$ multiplied by r^{-1}. Let $x, y \in \mathfrak{g}_{\bar{0}}$. Then $(x|y)$ is defined since $\mathfrak{g}_{\bar{0}} \subset \mathfrak{g}$, and $\langle x|y \rangle$ is also defined since $\mathfrak{g}_{\bar{0}} \cong \mathbb{C}t^0 \otimes_{\mathbb{C}} \mathfrak{g}_{\bar{0}} \subset \widetilde{\mathfrak{g}}$. By the definition of $\langle\,\cdot\,|\,\cdot\,\rangle$ above, we have

$$\langle x|y \rangle = r^{-1}(x|y).$$

Hence, for $\lambda, \mu \in (\mathfrak{h}_{\bar{0}})^* \subset (\widetilde{\mathfrak{h}})^* \cap \mathfrak{h}^*$, we have

$$\langle \lambda|\mu \rangle = r(\lambda|\mu). \tag{4.1.1}$$

Remark 4.1.4. It is easily checked (see [K4, Chaps. 6 and 8]) that the restriction of the normalized Killing form $(\,\cdot\,|\,\cdot\,)$ on $\mathfrak{g} = \mathfrak{g}(X_N)$ satisfies the condition:

$$(\alpha|\alpha) = 2 \quad \text{for all long roots } \alpha \in (\Delta_{\bar{0}})_{long} \subset (\mathfrak{h}_{\bar{0}})^* \subset \mathfrak{h}^*.$$

Hence the restriction of the normalized Killing form $(\,\cdot\,|\,\cdot\,)$ on $\mathfrak{g} = \mathfrak{g}(X_N)$ to the fixed point subalgebra $\mathfrak{g}_{\bar{0}}$ coincides with the Killing form on $\mathfrak{g}_{\bar{0}} = \mathfrak{g}(Y_L)$ normalized in such a way that the square length of every long root is 2. So we denote this normalized Killing form on $\mathfrak{g}(Y_L) = \mathfrak{g}_{\bar{0}}$ also by $(\,\cdot\,|\,\cdot\,)$.

4.2. Casimir operators for $\mathfrak{g}_{\bar{0}}$ and $\widetilde{\mathfrak{g}}$

The Casimir element $\dot{\Omega} \in Z(U(\mathfrak{g}_{\bar{0}}))$ for $\mathfrak{g}_{\bar{0}}$ and the Casimir operator $\widetilde{\Omega}$ for $\widetilde{\mathfrak{g}}$ are defined in the same way as Ω for $\mathfrak{g}$ and $\widehat{\Omega}$ for $\widehat{\mathfrak{g}}$ in Section 2.2, respectively. Furthermore, using the explicit descriptions of the set of positive roots $\widetilde{\Delta}_+$ of $\widetilde{\mathfrak{g}}$ and the corresponding root spaces $\widetilde{\mathfrak{g}}_\gamma$, $\gamma \in \widetilde{\Delta}_+$, we can show that the Casimir operator $\widetilde{\Omega}$ can be expressed in the following form (we need to be careful about the normalizations of the bilinear forms $\langle \cdot | \cdot \rangle$ and $(\cdot | \cdot)$):

$$\widetilde{\Omega} = r\dot{\Omega} + 2(K + h^\vee)D + 2r \sum_{\bar{l} \in \mathbb{Z}/r\mathbb{Z}} \sum_{\substack{n \geq 1 \\ \bar{n} = \bar{l}}} \sum_{i=1}^{\dim_{\mathbb{C}} \mathfrak{g}_{\bar{l}}} (t^{-n} \otimes u(\bar{n})^i)(t^n \otimes u(\bar{n})_i). \tag{4.2.1}$$

Here, for each $n \in \mathbb{Z}_{\geq 1}$, $\{u(\bar{n})_i \mid 1 \leq i \leq \dim_{\mathbb{C}} \mathfrak{g}_{\bar{n}}\}$ and $\{u(\bar{n})^i \mid 1 \leq i \leq \dim_{\mathbb{C}} \mathfrak{g}_{\bar{n}}\}$ are bases of $\mathfrak{g}_{\bar{n}}$ and $\mathfrak{g}_{-\bar{n}}$ consisting of weight vectors with respect to the adjoint action of $\mathfrak{h}_{\bar{0}}$ such that

$$\begin{cases} (u(\bar{n})_i | u(\bar{n})^j) = \delta_{ij}, \quad 1 \leq i, j \leq \dim_{\mathbb{C}} \mathfrak{g}_{\bar{n}}, \\ \displaystyle\sum_{i=1}^{\dim_{\mathbb{C}} \mathfrak{g}_{\bar{n}}} [u(\bar{n})_i, u(\bar{n})^i] = 0 \in \mathfrak{h}_{\bar{0}}, \end{cases} \tag{4.2.2}$$

and the dual Coxeter number $h^\vee$ is given by:

$$h^\vee = \begin{cases} 2L & \text{if } X_N = A_{2L-1},\ r = 2, \\ 2L & \text{if } X_N = D_{L+1},\ r = 2, \\ 12 & \text{if } X_N = E_6,\ r = 2, \\ 6 & \text{if } X_N = D_4,\ r = 3. \end{cases}$$

Remark 4.2.1. In all the cases where $X_N^{(r)} = A_{2L-1}^{(2)}, D_{L+1}^{(2)}, E_6^{(2)}, D_4^{(3)}$, we can check by direct computation that

$$\dim_{\mathbb{C}} \mathfrak{g}_{\bar{n}} = (1 + h^\vee) \dim_{\mathbb{C}} \mathfrak{h}_{\bar{n}}$$

for all $n \in \mathbb{Z}_{\geq 0}$.

4.3. Graded trace of the Casimir element $\dot{\Omega}$ for $\mathfrak{g}_{\bar{0}}$

Let $\Lambda \in (\widetilde{\mathfrak{h}})^*$ be a dominant integral weight, i.e., $\Lambda(\widetilde{h}_i) \in \mathbb{Z}_{\geq 0}$ for all $0 \leq i \leq L$. We assume that $\Lambda(D) = 0$. Put $k := \Lambda(K) \in \mathbb{Z}_{\geq 0}$, and

$$c_l(k) := \frac{k(\dim_{\mathbb{C}} \mathfrak{g}_{\bar{l}})}{k + h^\vee} \in \mathbb{Q}_{>0} \quad \text{for } 0 \leq l \leq r - 1.$$

Let $V := \widetilde{L}(\Lambda)$ be the ireducible highest weight $\widetilde{\mathfrak{g}}$-module of highest weight $\Lambda \in (\widetilde{\mathfrak{h}})^*$ given the basic gradation:

$$V = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} V_m, \quad V_m := \{v \in V \mid Dv = -mv\}.$$

Recall from [K4, Chap. 12] that $\dim_{\mathbb{C}} V_m < +\infty$ for all $m \in \mathbb{Z}_{\geq 0}$. Also note that each homogeneous subspace V_m for $m \in \mathbb{Z}_{\geq 0}$ is stable under the action of $\mathfrak{g}_{\bar{0}} \cong \mathbb{C}t^0 \otimes_{\mathbb{C}} \mathfrak{g}_{\bar{0}} \hookrightarrow \widetilde{\mathfrak{g}}$ since $[D, \mathbb{C}t^0 \otimes_{\mathbb{C}} \mathfrak{g}_{\bar{0}}] = 0$, and hence that

$$\dot{\Omega} V_m \subset V_m \quad \text{for each } m \in \mathbb{Z}_{\geq 0}.$$

We set

$$f(q) := \sum_{m \geq 0} (\dim_{\mathbb{C}} V_m)\, q^m,$$
$$g(q) := \sum_{m \geq 0} \mathrm{Tr}(\dot{\Omega}|_{V_m})\, q^m.$$

Now we define the following formal power series in q for $0 \leq l \leq r-1$:

$$\phi_l(q) := \prod_{\substack{n \geq 1 \\ n \equiv l \pmod r}} (1 - q^n),$$

$$H_l(q) := -c_l(k) \cdot \sum_{\substack{n \geq 1 \\ n \equiv l \pmod r}} \log(1 - q^n),$$

$$h_l(q) := \exp(H_l(q)).$$

Remark 4.3.1. We often write

$$h_l(q) = \phi_l(q)^{-c_l(k)} = \prod_{\substack{n \geq 1 \\ n \equiv l \pmod r}} (1 - q^n)^{-c_l(k)}$$

and $H_l(q) = \log(h_l(q))$.

We get the following lemmas in the same way as Lemmas 2.3.3 and 2.3.4.

Lemma 4.3.2. *For $0 \le l \le r-1$, we have*

$$\frac{d}{dq} h_l(q) = h_l(q) \cdot \frac{d}{dq} H_l(q).$$

Lemma 4.3.3. *For $0 \le l \le r-1$, we have*

$$q \frac{d}{dq} H_l(q) = c_l(k) \cdot \sum_{\substack{n \ge 1 \\ n \equiv l \pmod r}} n \sum_{j \ge 1} q^{nj}.$$

Furthermore, we can show the following proposition.

Proposition 4.3.4. *For $0 \le l \le r-1$ and $n \in \mathbb{Z}_{\ge 1}$ such that $n \equiv l \pmod r$, we have*

$$\begin{aligned} &\mathrm{Tr}\Bigg(\sum_{i=1}^{\dim_{\mathbb{C}} \mathfrak{g}_{\bar{l}}} (t^{-n} \otimes u(\bar{n})^i)(t^n \otimes u(\bar{n})_i)|_{V_m} \Bigg) \\ &\qquad = r^{-1} (\dim_{\mathbb{C}} \mathfrak{g}_{\bar{l}}) kn \cdot \sum_{j \ge 1} \dim_{\mathbb{C}} V_{m-nj}. \end{aligned}$$

Proof. First we note that for $0 \le l \le r-1$, $n \in \mathbb{Z}_{\ge 1}$ such that $n \equiv l \pmod r$, and $1 \le i \le \dim_{\mathbb{C}} \mathfrak{g}_{\bar{l}}$, we have the following commutation relation (since $c = r^{-1} K$):

$$\begin{aligned} [t^n \otimes u(\bar{n})_i, t^{-n} \otimes u(\bar{n})^i] &= t^0 \otimes [u(\bar{n})_i, u(\bar{n})^i] + n (u(\bar{n})_i | u(\bar{n})^i) c \\ &= 1 \otimes [u(\bar{n})_i, u(\bar{n})^i] + r^{-1} n K. \end{aligned}$$

Hence, by (4.2.2), we have

$$\sum_{i=1}^{\dim_{\mathbb{C}} \mathfrak{g}_{\bar{l}}} [t^n \otimes u(\bar{n})_i, t^{-n} \otimes u(\bar{n})^i] = r^{-1} (\dim_{\mathbb{C}} \mathfrak{g}_{\bar{l}}) n K.$$

Using this equality, we obtain a recurrence relation by an argument similar to the one in the proof of Proposition 2.3.5:

$$\begin{aligned} &\mathrm{Tr}\Bigg(\sum_{i=1}^{\dim \mathfrak{g}_{\bar{l}}} (t^{-n} \otimes u(\bar{n})^i)(t^n \otimes u(\bar{n})_i)|_{V_m} \Bigg) \\ &\quad = \mathrm{Tr}\Bigg(\sum_{i=1}^{\dim \mathfrak{g}_{\bar{l}}} (t^{-n} \otimes u(\bar{n})^i)(t^n \otimes u(\bar{n})_i)|_{V_{m-n}} \Bigg) \\ &\qquad + r^{-1} (\dim_{\mathbb{C}} \mathfrak{g}_{\bar{l}}) kn (\dim_{\mathbb{C}} V_{m-n}). \end{aligned}$$

Therefore, we deduce that

$$\mathrm{Tr}\Bigg(\sum_{i=1}^{\dim_{\mathbb{C}}\mathfrak{g}_{\bar{l}}}(t^{-n}\otimes u(\bar{n})^i)(t^n\otimes u(\bar{n})_i)|_{V_m}\Bigg)=r^{-1}(\dim_{\mathbb{C}}\mathfrak{g}_{\bar{l}})kn\cdot\sum_{j\geq 1}\dim_{\mathbb{C}}V_{m-nj}.$$

This proves the proposition. ∎

Here we recall that the Casimir operator $\widetilde{\Omega}$ acts on $\widetilde{L}(\Lambda)$ by the scalar $\langle\Lambda+2\widetilde{\rho}|\Lambda\rangle$, where $\widetilde{\rho}$ is an element (called the Weyl vector) of $(\widetilde{\mathfrak{h}})^*$ defined by: $\widetilde{\rho}(\widetilde{h_i})=1$ for all $0\leq i\leq L$, and $\widetilde{\rho}(D)=0$. Hence, from the expression (4.2.1) of the Casimir operator $\widetilde{\Omega}$, we deduce in the same way as in Section 2.3 that for each $m\in\mathbb{Z}_{\geq 0}$,

$$\begin{aligned}
&\langle\Lambda+2\widetilde{\rho}|\Lambda\rangle(\dim_{\mathbb{C}}V_m)\\
&\quad=r\,\mathrm{Tr}(\dot{\Omega}|_{V_m})-2(k+h^\vee)m(\dim_{\mathbb{C}}V_m)\\
&\qquad+2r\sum_{l=0}^{r-1}\sum_{\substack{n\geq 1\\ n\equiv l\ (\mathrm{mod}\ r)}}\mathrm{Tr}\Bigg(\sum_{i=1}^{\dim_{\mathbb{C}}\mathfrak{g}_{\bar{l}}}(t^{-n}\otimes u(\bar{n})^i)(t^n\otimes u(\bar{n})_i)|_{V_m}\Bigg).
\end{aligned}$$

Furthermore, by Proposition 4.3.4, we obtain

$$\begin{aligned}
r\,\mathrm{Tr}(\dot{\Omega}|_{V_m})&=\langle\Lambda+2\widetilde{\rho}|\Lambda\rangle(\dim_{\mathbb{C}}V_m)+2(k+h^\vee)m(\dim_{\mathbb{C}}V_m)\\
&\qquad-2k\sum_{l=0}^{r-1}(\dim_{\mathbb{C}}\mathfrak{g}_{\bar{l}})\sum_{\substack{n\geq 1\\ n\equiv l\ (\mathrm{mod}\ r)}}n\sum_{j\geq 1}\dim_{\mathbb{C}}V_{m-nj}.
\end{aligned}$$

Consequently, the graded trace $g(q)$ of the Casimir element $\dot{\Omega}$ on $V=\bigoplus_{m\in\mathbb{Z}_{\geq 0}}V_m$ can be calculated as in Section 2.3:

(4.3.1)

$$\begin{aligned}
g(q)&=\sum_{m\geq 0}\mathrm{Tr}(\dot{\Omega}|_{V_m})\,q^m\\
&=r^{-1}\langle\Lambda+2\widetilde{\rho}|\Lambda\rangle f(q)+2r^{-1}(k+h^\vee)\,q\frac{d}{dq}f(q)\\
&\qquad-2r^{-1}k\sum_{l=0}^{r-1}(\dim_{\mathbb{C}}\mathfrak{g}_{\bar{l}})c_l(k)^{-1}f(q)\cdot q\frac{d}{dq}H_l(q)\quad\text{by Lemma 4.3.3}\\
&=r^{-1}\langle\Lambda+2\widetilde{\rho}|\Lambda\rangle+2r^{-1}(k+h^\vee)\,q\Bigg\{\frac{d}{dq}f(q)-f(q)\cdot\sum_{l=0}^{r-1}\frac{d}{dq}H_l(q)\Bigg\}.
\end{aligned}$$

If we set

$$F(q) := f(q) \cdot \prod_{l=0}^{r-1} h_l(q)^{-1} = f(q) \cdot \prod_{l=0}^{r-1} \prod_{\substack{n \geq 1 \\ n \equiv l \ (\mathrm{mod}\ r)}} (1-q^n)^{c_l(k)},$$

then we have

$$\begin{aligned} \frac{d}{dq} f(q) &= \Bigl(\frac{d}{dq}F(q)\Bigr) \cdot \prod_{l=0}^{r-1} h_l(q) + F(q) \cdot \Biggl\{ \sum_{i=0}^{r-1} (\frac{d}{dq} h_i(q)) \cdot \prod_{\substack{0 \leq l \leq r-1 \\ l \neq i}} h_l(q) \Biggr\} \\ &= \Bigl(\frac{d}{dq}F(q)\Bigr) \cdot \prod_{l=0}^{r-1} h_l(q) + F(q) \cdot \Biggl\{ \sum_{i=0}^{r-1} (\frac{d}{dq} H_i(q)) \cdot \prod_{l=0}^{r-1} h_l(q) \Biggr\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{by Lemma 4.3.2} \\ &= \Bigl(\frac{d}{dq}F(q)\Bigr) \cdot \prod_{l=0}^{r-1} h_l(q) + f(q) \cdot \sum_{i=0}^{r-1} \frac{d}{dq} H_i(q) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\text{by the definition of } F(q). \end{aligned}$$

Combining this equality with (4.3.1), we obtain

$$\begin{aligned} g(q) &= r^{-1}\langle \Lambda + 2\widetilde{\rho} | \Lambda \rangle f(q) + 2r^{-1}(k+h^{\vee})\, q \Biggl\{ \Bigl(\frac{d}{dq}F(q)\Bigr) \cdot \prod_{l=0}^{r-1} h_l(q) \Biggr\} \\ &= \frac{r^{-1}\langle \Lambda + 2\widetilde{\rho} | \Lambda \rangle F(q) + 2r^{-1}(k+h^{\vee})\, q \dfrac{d}{dq} F(q)}{\displaystyle\prod_{l=0}^{r-1} \prod_{\substack{n \geq 1 \\ n \equiv l \ (\mathrm{mod}\ r)}} (1-q^n)^{c_l(k)}}. \end{aligned}$$

Recall from [K4, Chap. 6] that we have an orthogonal direct sum:

$$(\widetilde{\mathfrak{h}})^* = (\mathfrak{h}_{\bar{0}})^* \oplus (\mathbb{C}\delta + \mathbb{C}\widetilde{\Lambda}_0),$$

where $\widetilde{\Lambda}_0 \in (\widetilde{\mathfrak{h}})^*$ is the basic fundamental weight defined by: $\widetilde{\Lambda}_0(\mathfrak{h}_{\bar{0}}) := 0$, $\widetilde{\Lambda}_0(K) := 1$, $\widetilde{\Lambda}_0(D) := 0$. For an element $\Lambda \in (\widetilde{\mathfrak{h}})^*$, we denote by $\bar{\Lambda} \in (\mathfrak{h}_{\bar{0}})^*$ the orthogonal projection of Λ on $(\mathfrak{h}_{\bar{0}})^*$. Since $\Lambda(D) = 0$, we have $\Lambda = \bar{\Lambda} + \Lambda(K)\widetilde{\Lambda}_0 = \bar{\Lambda} + k\widetilde{\Lambda}_0$. Also we know that $\widetilde{\rho} = \dot{\rho} + h^{\vee}\widetilde{\Lambda}_0$, where

$\dot{\rho} = (1/2) \cdot \sum_{\alpha \in (\Delta_{\bar{0}})_+} \alpha \in (\mathfrak{h}_{\bar{0}})^*$ is the Weyl vector for $\mathfrak{g}_{\bar{0}}$. Hence, by (4.1.1), we have

$$\langle \Lambda + 2\widetilde{\rho} | \Lambda \rangle = \langle \bar{\Lambda} + 2\dot{\rho} | \bar{\Lambda} \rangle = r(\bar{\Lambda} + 2\dot{\rho} | \bar{\Lambda})$$

since $\langle \widetilde{\Lambda}_0 | \widetilde{\Lambda}_0 \rangle = 0$. Thus we have proved the following.

THEOREM 4.3.5. *Let $\widetilde{\mathfrak{g}} = \mathfrak{g}(X_N^{(r)})$ be the twisted affine Lie algebra of type $X_N^{(r)}$ with $X_N^{(r)} = A_{2L-1}^{(2)}, D_{L+1}^{(2)}, E_6^{(2)}, D_4^{(3)}$, and let $V = \widetilde{L}(\Lambda)$ be the irreducible highest weight $\widetilde{\mathfrak{g}}$-module of dominant integral highest weight $\Lambda \in (\widetilde{\mathfrak{h}})^*$ (such that $\Lambda(D) = 0$) given the basic gradation $V = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} V_m$. Then the graded trace $g(q) = \sum_{m \geq 0} \mathrm{Tr}(\dot{\Omega}|_{V_m})\, q^m$ of the Casimir element $\dot{\Omega}$ for the finite-dimensional simple Lie algebra $\mathfrak{g}_{\bar{0}} = \mathfrak{g}(Y_L)$ of type Y_L (with $Y_L = C_L, B_L, F_4, G_2$, respectively) is expressed in the following form*:

$$g(q) = \frac{r^{-1} \langle \Lambda + 2\widetilde{\rho} | \Lambda \rangle F(q) + 2r^{-1}(k + h^\vee)\, q \dfrac{d}{dq} F(q)}{\displaystyle\prod_{l=0}^{r-1} \prod_{\substack{n \geq 1 \\ n \equiv l \pmod{r}}} (1 - q^n)^{\frac{k(\dim_{\mathbb{C}} \mathfrak{g}_{\bar{l}})}{k + h^\vee}}},$$

where

$$F(q) = \left(\prod_{l=0}^{r-1} \prod_{\substack{n \geq 1 \\ n \equiv l \pmod{r}}} (1 - q^n)^{\frac{k(\dim_{\mathbb{C}} \mathfrak{g}_{\bar{l}})}{k + h^\vee}} \right) \cdot \sum_{m \geq 0} (\dim_{\mathbb{C}} V_m)\, q^m.$$

Remark 4.3.6. If $\Lambda \in (\widetilde{\mathfrak{h}})^*$ is a dominant integral weight such that $k = \Lambda(K) = 1$ and $\Lambda(D) = 0$, then we know from [K4, Chap. 12] that $\langle \Lambda | \Lambda \rangle h^\vee = 2 \langle \widetilde{\rho} | \Lambda \rangle$. So we have

$$\langle \Lambda + 2\widetilde{\rho} | \Lambda \rangle = \langle \Lambda | \Lambda \rangle \cdot (1 + h^\vee).$$

Also, since $\dim_{\mathbb{C}} \mathfrak{g}_{\bar{l}} = (1 + h^\vee) \dim_{\mathbb{C}} \mathfrak{h}_{\bar{l}}$ for $0 \leq l \leq r-1$ by Remark 4.2.1, we have

$$c_l(k) = \frac{\dim_{\mathbb{C}} \mathfrak{g}_{\bar{l}}}{1 + h^\vee} = \dim_{\mathbb{C}} \mathfrak{h}_{\bar{l}} \quad \text{for } 0 \leq l \leq r-1.$$

Hence we obtain

$$g(q) = \frac{r^{-1}(1 + h^\vee) \Big\{ \langle \Lambda | \Lambda \rangle F(q) + 2q \dfrac{d}{dq} F(q) \Big\}}{\displaystyle\prod_{l=0}^{r-1} \prod_{\substack{n \geq 1 \\ n \equiv l \pmod{r}}} (1 - q^n)^{\dim_{\mathbb{C}} \mathfrak{h}_{\bar{l}}}}.$$

In particular, if Λ is the basic fundamental weight $\widetilde{\Lambda}_0 \in (\widetilde{\mathfrak{h}})^*$, then we get

$$g(q) = \frac{2r^{-1}(1+h^\vee)\, q\dfrac{d}{dq}F(q)}{\displaystyle\prod_{l=0}^{r-1} \prod_{\substack{n\geq 1 \\ n\equiv l \pmod r}} (1-q^n)^{\dim_{\mathbb{C}} \mathfrak{h}_{\bar{l}}}}$$

since $\langle \widetilde{\Lambda}_0 | \widetilde{\Lambda}_0 \rangle = 0$.

Remark 4.3.7. Recall from [K4, Chap. 12] that a dominant integral weight $\Lambda \in (\widetilde{\mathfrak{h}})^*$ such that $k = \Lambda(K) = 1$ and $\Lambda(D) = 0$ is of the form $\Lambda = \widetilde{\Lambda}_0$ or $\Lambda = \widetilde{\Lambda}_0 + \dot{\Lambda}_i$ with $1 \leq i \leq L$ such that $\widetilde{a}_i^\vee = 1$, where $\{\dot{\Lambda}_i\}_{i=1}^L \subset (\mathfrak{h}_{\bar{0}})^* \subset (\widetilde{\mathfrak{h}})^*$ are the fundamental weights of $\mathfrak{g}_{\bar{0}} = \mathfrak{g}(Y_L)$ and $K = \sum_{i=0}^L \widetilde{a}_i^\vee \widetilde{h}_i$ is the canonical central element.

4.4. Identity for the derivative of a theta series of type C, B, F, G

In this section, we assume that $\Lambda \in (\widetilde{\mathfrak{h}})^*$ is a dominant integral weight such that $k = \Lambda(K) = 1$ and $\Lambda(D) = 0$. Hence we have $\Lambda = \bar{\Lambda} + \widetilde{\Lambda}_0$ with $\bar{\Lambda} \in (\mathfrak{h}_{\bar{0}})^*$ (cf. Remark 4.3.7). In particular, $\langle \Lambda | \Lambda \rangle = \langle \bar{\Lambda} | \bar{\Lambda} \rangle = r(\bar{\Lambda}|\bar{\Lambda})$ by (4.1.1).

We know the following fact due to Kac (see [K4, Chap. 12]).

Fact 3. *The graded dimension $f(q) = \sum_{m\geq 0} (\dim_{\mathbb{C}} V_m)\, q^m$ of the irreducible highest weight $\widetilde{\mathfrak{g}}$-module $V = \widetilde{L}(\Lambda)$ of highest weight Λ with the basic gradation is given by*:

$$\begin{aligned} f(q) &= \sum_{m\geq 0} (\dim_{\mathbb{C}} V_m)\, q^m \\ &= \frac{q^{-\frac{r}{2}(\bar{\Lambda}|\bar{\Lambda})} \cdot \displaystyle\sum_{\alpha \in \bar{\Lambda} + \dot{Q}} q^{\frac{r}{2}(\alpha|\alpha)}}{\displaystyle\prod_{l=1}^{r-1} \prod_{\substack{n\geq 1 \\ n\equiv l \pmod r}} (1-q^n)^{\dim_{\mathbb{C}} \mathfrak{h}_{\bar{l}}}}, \end{aligned}$$

where $\dot{Q} = \sum_{i=1}^L \mathbb{Z}\widetilde{\alpha}_i \subset (\mathfrak{h}_{\bar{0}})^*$ is the root lattice of $\mathfrak{g}_{\bar{0}} = \mathfrak{g}(Y_L)$ and $(\cdot|\cdot)$ is the normalized Killing form on $(\mathfrak{h}_{\bar{0}})^*$.

By Fact 3, we have

$$\begin{aligned} F(q) &= f(q) \cdot \prod_{l=0}^{r-1} \prod_{\substack{n \geq 1 \\ n \equiv l \pmod r}} (1-q^n)^{c_l(k)} \\ &= q^{-\frac{r}{2}(\bar{\Lambda}|\bar{\Lambda})} \cdot \sum_{\alpha \in \bar{\Lambda} + \dot{Q}} q^{\frac{r}{2}(\alpha|\alpha)} \end{aligned}$$

since $c_l(1) = \dim_{\mathbb{C}} \mathfrak{h}_{\bar{l}}$ for $0 \leq l \leq r-1$. We set

$$\Theta_{\dot{Q},\bar{\Lambda}}(q) := \sum_{\alpha \in \bar{\Lambda} + \dot{Q}} q^{\frac{r}{2}(\alpha|\alpha)}.$$

Then we get

$$q \frac{d}{dq} F(q) = -\frac{r}{2}(\bar{\Lambda}|\bar{\Lambda}) \cdot F(q) + q^{-\frac{r}{2}(\bar{\Lambda}|\bar{\Lambda})} \cdot q \frac{d}{dq} \Theta_{\dot{Q},\bar{\Lambda}}(q),$$

and hence from Remark 4.3.6

$$g(q) = \frac{2r^{-1}(1+h^{\vee}) q^{-\frac{r}{2}(\bar{\Lambda}|\bar{\Lambda})} \cdot q \frac{d}{dq} \Theta_{\dot{Q},\bar{\Lambda}}(q)}{\displaystyle\prod_{l=0}^{r-1} \prod_{\substack{n \geq 1 \\ n \equiv l \pmod r}} (1-q^n)^{\dim_{\mathbb{C}} \mathfrak{h}_{\bar{l}}}} \tag{4.4.1}$$

since $\langle \Lambda|\Lambda \rangle = r(\bar{\Lambda}|\bar{\Lambda})$ by (4.1.1).

Now, for $\lambda \in \dot{P}_+ := \{\lambda \in (\mathfrak{h}_{\bar{0}})^* \mid \lambda(\widetilde{h}_i) \in \mathbb{Z}_{\geq 0},\ 1 \leq i \leq L\}$, we denote by $\dot{L}(\lambda)$ the irreducible highest weight $\mathfrak{g}_{\bar{0}}$-module of highest weight λ, and by $\Phi(\Lambda, \lambda)_m$ the multiplicity of $\dot{L}(\lambda)$ in the homogeneous subspace V_m of V viewed as a $\mathfrak{g}_{\bar{0}}$-module:

$$V_m = \bigoplus_{\lambda \in \dot{P}_+} \Phi(\Lambda, \lambda)_m \dot{L}(\lambda).$$

Further we set

$$\Phi(\Lambda, \lambda)(q) := \sum_{m \geq 0} \Phi(\Lambda, \lambda)_m q^m.$$

Then we know the following fact due to Kac (see [K4, Chap. 12]).

FACT 4. *Let* $\lambda \in \dot{P}_+$. *If* $\lambda \notin \bar{\Lambda} + \dot{Q}$, *then we have* $\Phi(\Lambda, \lambda)(q) = 0$. *If* $\lambda \in \bar{\Lambda} + \dot{Q}$, *then we have*

$$\Phi(\Lambda, \lambda)(q) = \frac{q^{\frac{r}{2}\{(\lambda|\lambda) - (\bar{\Lambda}|\bar{\Lambda})\}} \cdot \displaystyle\prod_{\alpha \in (\Delta_{\bar{0}})_+} \left(1 - q^{r(\lambda + \dot{\rho}|\alpha)}\right)}{\displaystyle\prod_{l=0}^{r-1} \prod_{\substack{n \geq 1 \\ n \equiv l \ (\mathrm{mod}\ r)}} (1 - q^n)^{\dim_{\mathbb{C}} \mathfrak{h}_{\bar{l}}}}.$$

Using Fact 4 instead of Fact 2, we deduce as in Section 3:

$$\begin{aligned}
g(q) &= \sum_{m \geq 0} \mathrm{Tr}(\dot{\Omega}|_{V_m})\, q^m \\
&= \sum_{m \geq 0} \sum_{\lambda \in (\bar{\Lambda} + \dot{Q}) \cap \dot{P}_+} \dot{d}(\lambda)(\lambda + 2\dot{\rho}|\lambda)\Phi(\Lambda, \lambda)_m q^m \\
&= q^{-\frac{r}{2}(\bar{\Lambda}|\bar{\Lambda})} \left(\prod_{l=0}^{r-1} h_l(q) \right) \\
&\qquad \times \sum_{\lambda \in (\bar{\Lambda} + \dot{Q}) \cap \dot{P}_+} \dot{d}(\lambda)(\lambda + 2\dot{\rho}|\lambda) q^{\frac{r}{2}(\lambda|\lambda)} \prod_{\alpha \in (\Delta_{\bar{0}})_+} \left(1 - q^{r(\lambda + \dot{\rho}|\alpha)}\right),
\end{aligned}$$

where $\dot{d}(\lambda) := \dim_{\mathbb{C}} \dot{L}(\lambda)$ for $\lambda \in (\bar{\Lambda} + \dot{Q}) \cap \dot{P}_+$ and

$$h_l(q) = \prod_{\substack{n \geq 1 \\ n \equiv l \ (\mathrm{mod}\ r)}} (1 - q^n)^{-\dim_{\mathbb{C}} \mathfrak{h}_{\bar{l}}}$$

for $0 \leq l \leq r - 1$. Comparing this equality with (4.4.1), we obtain the following.

THEOREM 4.4.1. *Let* $\mathfrak{g}_{\bar{0}} = \mathfrak{g}(Y_L)$ *be a finite-dimensional simple Lie algebra of type* Y_L *with* $Y_L = C_L, B_L, F_4, G_2$, *and let* $\Lambda = \bar{\Lambda} + \widetilde{\Lambda}_0 \in (\widetilde{\mathfrak{h}})^*$ *with* $\bar{\Lambda} \in (\mathfrak{h}_{\bar{0}})^*$ *be a dominant integral weight. Then we have*

$$\begin{aligned}
&2r^{-1}(1 + h^\vee)\, q \frac{d}{dq} \Theta_{\dot{Q}, \bar{\Lambda}}(q) \\
&\quad = \sum_{\lambda \in (\bar{\Lambda} + \dot{Q}) \cap \dot{P}_+} \dot{d}(\lambda)(\lambda + 2\dot{\rho}|\lambda)\, q^{\frac{r}{2}(\lambda|\lambda)} \prod_{\alpha \in (\Delta_{\bar{0}})_+} \left(1 - q^{r(\lambda + \dot{\rho}|\alpha)}\right),
\end{aligned}$$

where $\Theta_{\dot{Q}, \bar{\Lambda}}(q) = \sum_{\alpha \in \bar{\Lambda} + \dot{Q}} q^{\frac{r}{2}(\alpha|\alpha)}$. *Here* $r = 2$ *if* $Y = C, B, F$ *and* $r = 3$ *if* $Y = G$.

Remark 4.4.2. For $\lambda \in \dot{P}_+$, the dimension $\dot{d}(\lambda) = \dim_{\mathbb{C}} \dot{L}(\lambda)$ is given by the Weyl dimension formula:

$$\dot{d}(\lambda) = \prod_{\alpha \in (\Delta_{\bar{0}})_+} \frac{(\lambda + \dot{\rho}|\alpha)}{(\dot{\rho}|\alpha)}.$$

By using Facts 3 and 4 instead of Facts 1 and 2, respectively, we can show the following proposition as in Remark 3.4 (this identity is new, while the identity in Remark 3.4 is already known).

PROPOSITION 4.4.3. *We have the following identity.*

$$\begin{aligned}\Theta_{\dot{Q},\bar{\Lambda}}(q) &= \sum_{\alpha \in \bar{\Lambda} + \dot{Q}} q^{\frac{r}{2}(\alpha|\alpha)} \\ &= \sum_{\lambda \in (\bar{\Lambda} + \dot{Q}) \cap \dot{P}_+} \dot{d}(\lambda)\, q^{\frac{r}{2}(\lambda|\lambda)} \prod_{\alpha \in (\Delta_{\bar{0}})_+} \left(1 - q^{r(\lambda + \dot{\rho}|\alpha)}\right),\end{aligned}$$

where $r = 2$ if $Y = C, B, F$ and $r = 3$ if $Y = G$.

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