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NILPOTENCY AND TRIVIALITY OF MOD p MORITA-MUMFORD CLASSES OF MAPPING CLASS GROUPS OF SURFACES

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Abstract. This paper is concerned with mod p Morita-Mumford classes $e_n^{(p)} \in H^{2n}(\Gamma_g, \mathbb{F}_p)$ of the mapping class group Γ_g of a closed oriented surface of genus $g \geq 2$, especially triviality and nontriviality of them. It is proved that $e_n^{(p)}$ is nilpotent if $n \equiv -1 \pmod{p-1}$, while the stable mod p Morita-Mumford class $e_n^{(p)} \in H^{2n}(\Gamma_\bullet, \mathbb{F}_p)$ is proved to be nontrivial and not nilpotent if $n \not\equiv -1 \pmod{p-1}$. With these results in mind, we conjecture that $e_n^{(p)}$ vanishes whenever $n \equiv -1 \pmod{p-1}$, and obtain a few pieces of supporting evidence.

§1. Introduction

Let Σ_g be a closed oriented surface of genus $g \geq 2$ and let Γ_g be the mapping class group of Σ_g. The cohomology of Γ_g is one of the most important objects in topology as well as in algebraic geometry. Indeed, any cohomology classes of Γ_g can be considered as characteristic classes of surface bundles, while the rational cohomology of Γ_g is naturally isomorphic to that of the moduli space of compact Riemann surfaces of genus g (see [11], [34] for instance).

Morita [28] and Mumford [35] independently introduced a series of certain cohomology classes $e_n \in H^{2n}(\Gamma_g, \mathbb{Z})$ $(n \geq 0)$ which are called Morita-Mumford classes of Γ_g (see §2 for the definition). Concerning rational Morita-Mumford classes, Miller [27] and Morita [28] independently proved that the natural homomorphism

$$\mathbb{Q}[e_1, e_2, e_3, \dots] \longrightarrow H^*(\Gamma_g, \mathbb{Q})$$

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is injective in dimensions less than $2g/3$ (see [1] for an elementary proof of this fact). It is conjectured that the above map is actually an isomorphism in dimensions less than $2g/3$, and there are many pieces of evidence which support the conjecture (see [34]). On the contrary, little is known about integral or mod p Morita-Mumford classes apart from a recent work of Kawazumi and Uemura [21] which shows that integral Morita-Mumford classes can play an interesting rôle in a study of cohomological properties of finite subgroups of Γ_g. For further results concerning Morita-Mumford classes, see [9], [11], [30], [31], [32], [33], [34] and references therein.

The purpose of this paper is to investigate mod p Morita-Mumford classes $e_n^{(p)} \in H^{2n}(\Gamma_g, \mathbb{F}_p)$, especially triviality and nontriviality of them, and thereby fill the lack of knowledge of mod p Morita-Mumford classes. Let us introduce the content of this paper briefly. In §2, we will recall relevant definitions and facts concerning mapping class groups, Morita-Mumford classes, and Harer's stability theorem. In §3, we will review results in [1], [21] concerning Morita-Mumford classes on finite subgroups of Γ_g. §4 is devoted to prove the following theorem which is the main result of this paper.

THEOREM. *If $n \equiv -1 \pmod{p-1}$, then $e_n^{(p)} \in H^{2n}(\Gamma_g, \mathbb{F}_p)$ is nilpotent. In particular, $e_n^{(2)} \in H^{2n}(\Gamma_g, \mathbb{F}_2)$ is nilpotent for all $n \geq 0$. Conversely, if $n \not\equiv -1 \pmod{p-1}$, then $e_n^{(p)} \in H^{2n}(\Gamma_\bullet, \mathbb{F}_p)$ is nontrivial and is not nilpotent.*

Here $H^*(\Gamma_\bullet, \mathbb{F}_p) = \lim_{g\to\infty} H^*(\Gamma_g, \mathbb{F}_p)$ is the stable mod p cohomology of mapping class groups which exists by virtue of Harer's stability theorem and $e_n^{(p)} \in H^{2n}(\Gamma_\bullet, \mathbb{F}_p)$ is the stable mod p Morita-Mumford class (see §2). In view of the theorem, it is reasonable to make the following conjecture.

CONJECTURE 1. *If $n \equiv -1 \pmod{p-1}$, then $e_n^{(p)} \in H^{2n}(\Gamma_g, \mathbb{F}_p)$ vanishes.*

Passing to the stable mod p cohomology, Conjecture 1 together with the prescribed theorem implies the following conjecture.

CONJECTURE 2. *The stable mod p Morita-Mumford class $e_n^{(p)} \in H^{2n}(\Gamma_\bullet, \mathbb{F}_p)$ vanishes if and only if $n \equiv -1 \pmod{p-1}$.*

In §5 and §6, we will give a few pieces of evidence which support the conjectures. In particular, Conjecture 1 is settled in the affirmative for the following cases: (i) $n = 1$, (ii) $g = 2$, (iii) $g = 3$ and $p = 2$, (iv) $g = (p-1)/2$ and $n \geq p-3$.

In §7, we will deal with oriented Σ_g-bundles. Any oriented Σ_g-bundle $E \to X$ is determined by the holonomy homomorphism $h : \pi_1(X) \to \Gamma_g$, and its mod p Morita-Mumford classes $e_n^{(p)} \in H^{2n}(X, \mathbb{F}_p)$ are nothing but the pull-back of $e_n^{(p)} \in H^{2n}(\Gamma_g, \mathbb{F}_p)$ by h. In the context of oriented Σ_g-bundles, there is yet another affirmative evidence for the conjectures. In particular, we will prove the following result which is a consequence of the Grothendieck Riemann-Roch theorem.

THEOREM. *Let p be a prime and $E \to X$ an oriented Σ_g-bundle over a closed oriented $2n$-manifold X. If n is odd and satisfies $n \equiv -1 \pmod{p-1}$, then $e_n^{(p)} = 0$ in $H^{2n}(X, \mathbb{F}_p)$.*

The last section is devoted to applications. For any oriented Σ_g-bundle $E \to \Sigma_h$ over a closed oriented surface Σ_h, the total space E is oriented null-cobordant if and only if the signature $\sigma(E)$ of E vanishes (see Remark 6). Meyer [26] showed that $\sigma(E) = 0$ whenever $g = 2$, and showed that for every $g \geq 3$ there exists an integer $h \geq 0$ and an oriented Σ_g-bundle $E \to \Sigma_h$ with $\sigma(E) \neq 0$. Together with results of Igusa [18] and Faber [8], the vanishing of mod 2 Morita-Mumford classes of Γ_2 and Γ_3 which will be proved in §6 implies the following result which extends results of Meyer for $g = 2$ and 3.

THEOREM. *Let $E \to X$ be an oriented Σ_g-bundle over a closed oriented manifold X.* (i) *If $g = 2$, then E is oriented null-cobordant.* (ii) *If $g = 3$, $\dim X \geq 3$, and all the rational Pontryagin classes of X vanish, then E is oriented null-cobordant.*

Notation. Given a prime p, the field with p elements is denoted by $\mathbb{F}_p$. Given a group G and its subgroup H, the *restriction* is denoted by $\mathrm{res}_H^G : H^*(G, \mathbb{Z}) \to H^*(H, \mathbb{Z})$. When H is of finite index in G, the *transfer* (or the *corestriction* in the literature) is denoted by $\mathrm{Tr}_H^G : H^*(H, \mathbb{Z}) \to H^*(G, \mathbb{Z})$. Given an element $\gamma \in G$, the cyclic subgroup generated by γ is denoted by $\langle \gamma \rangle$. Given a closed manifold X, its mod 2 fundamental class is denoted by $[X]_2$. When X is oriented, its fundamental class is denoted by $[X]$.

§2. Review of mapping class groups and Morita-Mumford classes

Let Σ_g be a closed oriented surface of genus $g \geq 2$. Let $\mathrm{Diff}_+\Sigma_g$ be the group of orientation-preserving diffeomorphisms of Σ_g equipped with C^∞-topology. The *mapping class group* Γ_g of Σ_g is defined to be the group of connected components of $\mathrm{Diff}_+\Sigma_g$:

$$\Gamma_g := \pi_0(\mathrm{Diff}_+\Sigma_g).$$

Let $\mathrm{Diff}_+(\Sigma_g, *)$ be the subgroup of $\mathrm{Diff}_+\Sigma_g$ consisting of all the orientation-preserving diffeomorphisms which fix the distinguished base point $* \in \Sigma_g$. The group

$$\Gamma_g^1 := \pi_0(\mathrm{Diff}_+(\Sigma_g, *))$$

is called the mapping class group of Σ_g relative to the base point $* \in \Sigma_g$. The natural homomorphism $\pi : \Gamma_g^1 \to \Gamma_g$ gives rise to an extension

$$1 \longrightarrow \pi_1(\Sigma_g, *) \longrightarrow \Gamma_g^1 \xrightarrow{\pi} \Gamma_g \longrightarrow 1.$$

Let $D \subset \Sigma_g$ be the fixed embedded disk and write $\Sigma_{g,1} = \Sigma_g \setminus \mathrm{Int} D$. Let $\mathrm{Diff}_+(\Sigma_{g,1}, \partial\Sigma_{g,1})$ be the group of all the orientation-preserving diffeomorphisms of $\Sigma_{g,1}$ which fix the boundary $\partial\Sigma_{g,1}$ pointwise. The group

$$\Gamma_{g,1} := \pi_0(\mathrm{Diff}_+(\Sigma_{g,1}, \partial\Sigma_{g,1}))$$

is called the mapping class group of $\Sigma_{g,1}$ relative to the one boundary component. The natural homomorphism $\Gamma_{g,1} \to \Gamma_g^1$ gives rise to a central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Gamma_{g,1} \longrightarrow \Gamma_g^1 \longrightarrow 1, \tag{2.1}$$

where the central subgroup $\mathbb{Z}$ is generated by the Dehn twist along a separating simple closed curve parallel to the boundary $\partial\Sigma_{g,1}$.

The inclusion $\Sigma_{g,1} \hookrightarrow \Sigma_{g+1,1}$ induces a homomorphism $\Gamma_{g,1} \to \Gamma_{g+1,1}$. *Harer's stability theorem* [13] as improved by Ivanov [19] asserts that homomorphisms $\Gamma_{g,1} \to \Gamma_{g+1,1}$ and $\Gamma_{g,1} \to \Gamma_g$ induce isomorphisms

$$H_n(\Gamma_{g,1}, \mathbb{Z}) \xrightarrow{\cong} H_n(\Gamma_{g+1,1}, \mathbb{Z}) \text{ and } H_n(\Gamma_{g,1}, \mathbb{Z}) \xrightarrow{\cong} H_n(\Gamma_g, \mathbb{Z})$$

when $n < g/2$. Hence we may define the stable cohomology of mapping class groups by

$$H^*(\Gamma_\bullet, \mathbb{Z}) := \lim_{g\to\infty} H^*(\Gamma_{g,1}, \mathbb{Z}).$$

Let $e \in H^2(\Gamma_g^1, \mathbb{Z})$ be the Euler class of the central extension (2.1). The n-th *Morita-Mumford class* $e_n \in H^{2n}(\Gamma_g, \mathbb{Z})$ $(n \geq 0)$ is then defined by

$$e_n = \pi_!(e^{n+1}) \in H^{2n}(\Gamma_g, \mathbb{Z}),$$

where $\pi_! : H^*(\Gamma_g^1, \mathbb{Z}) \to H^{*-2}(\Gamma_g, \mathbb{Z})$ is the Gysin homomorphism. Note that $e_0 = \chi(\Sigma_g) = 2 - 2g$. Given a prime number p, the *mod p Morita-Mumford class* $e_n^{(p)} \in H^*(\Gamma_g, \mathbb{F}_p)$ is defined to be the mod p reduction of $e_n \in H^*(\Gamma_g, \mathbb{Z})$. By the naturality of the Gysin homomorphism with respect to the mod p reduction, one has

$$e_n^{(p)} = \pi_!((e^{(p)})^{n+1}) \in H^*(\Gamma_g, \mathbb{F}_p),$$

where $e^{(p)} \in H^2(\Gamma_g^1, \mathbb{F}_p)$ is the mod p reduction of $e \in H^2(\Gamma_g^1, \mathbb{Z})$ which is referred as the mod p Euler class of the central extension (2.1).

Let $e_n \in H^{2n}(\Gamma_{g,1}, \mathbb{Z})$ be the image of $e_n \in H^{2n}(\Gamma_g, \mathbb{Z})$ under the homomorphism induced by $\Gamma_{g,1} \to \Gamma_g$ (we use the same symbol). As was shown by Miller [27] and Morita [28], e_n is preserved by the homomorphisms induced by $\Gamma_{g,1} \to \Gamma_{g+1,1}$ and $\Gamma_{g,1} \to \Gamma_g$. Hence the *stable Morita-Mumford classes* $e_n \in H^{2n}(\Gamma_\bullet, \mathbb{Z})$ make sense. We close this section by proving nontriviality of mod p Euler class $e^{(p)} \in H^2(\Gamma_g^1, \mathbb{F}_p)$ along the lines of Harer [16, Theorem 7.1] and Morita [28, p. 559].

PROPOSITION 1. *For any prime p, the mod p Euler class $e^{(p)} \in H^2(\Gamma_g^1, \mathbb{F}_p)$ of the central extension* (2.1) *is nontrivial if g is sufficiently large.*

Proof. Since $\Gamma_{g,1} \to \Gamma_g$ is the composition of the homomorphisms $\Gamma_{g,1} \to \Gamma_g^1$ and $\Gamma_g^1 \to \Gamma_g$, Harer's stability theorem implies that the induced homomorphism $H^n(\Gamma_g^1, \mathbb{F}_p) \to H^n(\Gamma_{g,1}, \mathbb{F}_p)$ is surjective for $n < g/2$. This shows that the Gysin sequence

$$\cdots \to H^{n-2}(\Gamma_g^1, \mathbb{F}_p) \overset{\cup e^{(p)}}{\to} H^n(\Gamma_g^1, \mathbb{F}_p) \to H^n(\Gamma_{g,1}, \mathbb{F}_p) \to H^n(\Gamma_g^1, \mathbb{F}_p) \to \cdots$$

splits into the short exact sequence

$$0 \longrightarrow H^{n-2}(\Gamma_g^1, \mathbb{F}_p) \xrightarrow{\cup e^{(p)}} H^n(\Gamma_g^1, \mathbb{F}_p) \longrightarrow H^n(\Gamma_{g,1}, \mathbb{F}_p) \longrightarrow 0$$

for $n < g/2$ and the proposition follows. □

The last proposition suggests the notriviality of the conjectures stated in the introduction.

Remark 1. When $p \geq 5$, the mod p Euler class $e^{(p)} \in H^2(\Gamma_g^1, \mathbb{F}_p)$ is actually nontrivial for all $g \geq 3$. See Corollary 2.

§3. Morita-Mumford classes on finite subgroups

Let G be a finite subgroup of Γ_g. By the affirmative solution of the Nielsen realization problem due to Kerckhoff [22], G is realized as a group of automorphisms of a suitable compact Riemann surface R of genus g. For each $x \in R$, let G_x be the isotropy subgroup at x. Since G preserves the orientation, G_x is necessarily cyclic. Set $\mathcal{S} = \{x \in R : G_x \neq 1\}$, and let $\mathcal{S}/G$ be a set of representatives of G-orbits of elements of $\mathcal{S}$. For each $x \in \mathcal{S}/G$, let ξ_x be the flat complex line bundle over the classifying space BG_x of G_x associated to the action of G_x on the holomorphic tangent space T_xR, and let $c_1(\xi_x) \in H^2(G_x, \mathbb{Z})$ be its first Chern class. Among other things, Kawazumi and Uemura proved the following result.

Theorem 1. ([21]) *In the situation stated above, one has*

$$\operatorname{res}_G^{\Gamma_g} e_n = \sum_{x \in \mathcal{S}/G} \operatorname{Tr}_{G_x}^G (c_1(\xi_x)^n) \in H^{2n}(G, \mathbb{Z}) \tag{3.1}$$

for all $n \geq 0$.

Now we deal with finite cyclic subgroups of Γ_g. Let $\gamma \in \Gamma_g$ be an element of order m. As before, choose a compact Riemann surface R of genus g for which $\langle \gamma \rangle$ is a group of automorphisms. Let $\{x_i\}_{1 \leq i \leq q}$ be a set of representatives of the singular orbits of $\langle \gamma \rangle$, and α_i the order of the isotropy subgroup at x_i. Let β_i be an integer such that $\gamma^{\beta_i m/\alpha_i}$ acts on $T_{x_i}R$ by $z \mapsto \exp(2\pi\sqrt{-1}/\alpha_i)z$ with respect to a suitable local coordinate z at x_i. The number β_i is well-defined modulo α_i and is prime to α_i. The *fixed point data* of γ is then the collection

$$\langle g, m \mid \beta_1/\alpha_1, \ldots, \beta_q/\alpha_q \rangle.$$

The rational numbers $\beta_1/\alpha_1, \ldots, \beta_q/\alpha_q$ are unique up to order, if we consider them as elements in $\mathbb{Q}/\mathbb{Z}$. The fixed point data satisfies

$$\sum_{i=1}^{q} \frac{\beta_i}{\alpha_i} \in \mathbb{Z} \tag{3.2}$$

and the Riemann-Hurwitz equation

$$2g - 2 = m(2h - 2) + m \sum_{i=1}^{q} \Big(1 - \frac{1}{\alpha_i}\Big) \tag{3.3}$$

for some integer $h \geq 0$. According to Symonds [37], the fixed point data is independent of various choices made and hence is well defined for $\gamma \in \Gamma_g$. For later use, we recall the following fact concerning a realization of fixed point data.

PROPOSITION 2. *Let p be a prime and $\beta_1, \ldots, \beta_q$ integers prime to p. Then $\langle g, p \mid \beta_1/p, \ldots, \beta_q/p \rangle$ can be realized as the fixed point data of an element of Γ_g of order p if and only if it satisfies* (3.2) *and* (3.3).

Proof. See [10]. ∎

Now let $\gamma \in \Gamma_g$ be an element of order m having the fixed point data $\langle g, m \mid \beta_1/\alpha_1, \ldots, \beta_q/\alpha_q \rangle$ as before. Let ξ be the complex line bundle over the classifying space $B\langle\gamma\rangle$ of $\langle\gamma\rangle$ associated to the representation $\langle\gamma\rangle \to U(1)$ defined by $\gamma \mapsto \exp(2\pi i/m)$, and let $u_\gamma \in H^2(\langle\gamma\rangle, \mathbb{Z})$ be the first Chern class of ξ. As is known, one has

$$H^*(\langle\gamma\rangle, \mathbb{Z}) \cong \mathbb{Z}[u_\gamma]/(m u_\gamma).$$

Under these conventions, Theorem 1 reduces to the following result.

PROPOSITION 3. *Let $\gamma \in \Gamma_g$ be an element of order m having the fixed point data $\langle g, m \mid \beta_1/\alpha_1, \ldots, \beta_q/\alpha_q \rangle$. Then*

$$\mathrm{res}^{\Gamma_g}_{\langle\gamma\rangle} e_n = \sum_{i=1}^{q} \frac{m}{\alpha_i} (\beta_i^*)^n u_\gamma^n \in H^{2n}(\langle\gamma\rangle, \mathbb{Z}),$$

where β_i^* is an integer satisfying $\beta_i^* \beta_i \equiv 1 \pmod{\alpha_i}$.

Proof. See [1]. ∎

PROPOSITION 4. *Let G be a finite subgroup of Γ_g whose Sylow p-subgroup is cyclic. Then*

$$\mathrm{res}^{\Gamma_g}_{G} e_n = 0 \quad \text{in} \quad H^{2n}(G, \mathbb{F}_p)$$

holds for all $n \equiv -1 \pmod{p-1}$.

Proof. Let G_p be a Sylow p-subgroup of G generated by an element $\gamma \in G_p$ having the fixed point data

$$\langle g, p^t \mid \beta_1/p^{t_1}, \dots, \beta_q/p^{t_q} \rangle.$$

Choose an integer β_i^* satisfying $\beta_i \beta_i^* \equiv 1 \pmod{p^{t_i}}$. Since $(\beta_i^*)^{p-1} \equiv 1 \pmod{p}$, we see that, if $n \equiv -1 \pmod{p-1}$ then $(\beta_i^*)^n \equiv \beta_i \pmod{p}$. On the other hand, it follows from (3.2) that

$$\sum_{i=1}^{q} p^{t-t_i} \beta_i \equiv 0 \pmod{p}.$$

Applying Proposition 3, we see that, if $n \equiv -1 \pmod{p-1}$ then

$$\mathrm{res}_{G_p}^{\Gamma_g} e_n = \sum_{i=1}^{q} p^{t-t_i} (\beta_i^*)^n u_\gamma^n = \sum_{i=1}^{q} p^{t-t_i} \beta_i u_\gamma^n = 0$$

in $H^{2n}(G_p, \mathbb{F}_p)$. As $\mathrm{res}_{G_p}^{G} : H^*(G, \mathbb{F}_p) \to H^*(G_p, \mathbb{F}_p)$ is injective, this completes the proof. $\square$

Remark 2. It was proved in [1] that, for any cyclic subgroup $C \subset \Gamma_g$ of order m, $\mathrm{res}_C^{\Gamma_g} e_n \in H^{2n}(C, \mathbb{Z})$ vanishes whenever $n \equiv -1 \mod \phi(m)$, where ϕ is the Euler function.

§4. Proof of the main result

The purpose of this section is to prove the main result of this paper which was mentioned in the introduction. More precisely, we will prove the following two theorems.

THEOREM 2. *If $n \equiv -1 \pmod{p-1}$, then $e_n^{(p)} \in H^{2n}(\Gamma_g, \mathbb{F}_p)$ is nilpotent. In particular, $e_n^{(2)} \in H^{2n}(\Gamma_g, \mathbb{F}_2)$ is nilpotent for all $n \geq 0$.*

THEOREM 3. *If $n \not\equiv -1 \pmod{p-1}$, then $(e_n^{(p)})^k \in H^{2nk}(\Gamma_g, \mathbb{F}_p)$ is nontrivial for any $k \geq 1$ satisfying $k < g/4n$.*

Passing to the stable mod p cohomology $H^*(\Gamma_\bullet, \mathbb{F}_p)$, Theorem 3 together with Harer's stability theorem implies the following result which was mentioned in the introduction.

Corollary 1. *The stable mod p Morita-Mumford class $e_n^{(p)} \in H^{2n}(\Gamma_\bullet, \mathbb{F}_p)$ is nontrivial and is not nilpotent whenever $n \not\equiv -1 \pmod{p-1}$.*

Let p be a prime number. Recall that a finite group E is called an *elementary abelian p-group of rank m* if $E \cong (\mathbb{Z}/p\mathbb{Z})^m$. Let us denote $\mathrm{rank}_p E = m$. We begin with the following lemma which may be well-known.

Lemma. *Let p be a prime number and E an elementary abelian p-group with $\mathrm{rank}_p E \geq 2$. For any cyclic subgroup $C \subset E$ of order p, the transfer*

$$\mathrm{Tr}_C^E : H^*(C, \mathbb{Z}) \longrightarrow H^*(E, \mathbb{Z})$$

is the zero homomorphism for $* > 0$.

Proof. It is easy to see that the restriction $\mathrm{res}_C^E : H^2(E, \mathbb{Z}) \to H^2(C, \mathbb{Z})$ is surjective. Since $H^*(C, \mathbb{Z}) \cong \mathbb{Z}[x]/(px)$ with $\deg x = 2$, it follows that $\mathrm{res}_C^E : H^*(E, \mathbb{Z}) \to H^*(C, \mathbb{Z})$ is also surjective. Given an element $u \in H^*(C, \mathbb{Z})$, choose $v \in H^*(E, \mathbb{Z})$ with $u = \mathrm{res}_C^E v$, and one has

$$\mathrm{Tr}_C^E u = \mathrm{Tr}_C^E \circ \mathrm{res}_C^E v = (E : C) \cdot v,$$

where $(E : C)$ is the index of C in E. It follows that $pv = 0$ since $H^2(E, \mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^m$ with $m = \mathrm{rank}_p E$ and v can be chosen as a product of elements of $H^2(E, \mathbb{Z})$. As p divides $(E : C)$, this proves the lemma. ∎

Proposition 5. *Let p be a prime number. For any elementary abelian p-subgroups E of Γ_g with $\mathrm{rank}_p E \geq 2$, one has*

$$\mathrm{res}_E^{\Gamma_g} e_n = 0 \quad \text{in} \quad H^{2n}(E, \mathbb{Z})$$

for all $n \geq 1$.

Proof. According to Theorem 1, $\mathrm{res}_E^{\Gamma_g} e_n$ is the sum of elements each of which belongs to the image of the transfer from a cyclic subgroup of order p. In view of the lemma, this completes the proof. ∎

Remark 3. Proposition 5 should be compared with a result of Morita [29] which asserts that, for every amenable subgroup $A \subset \Gamma_g$, one has $\mathrm{res}_A^{\Gamma_g} e_n = 0$ in $H^{2n}(A, \mathbb{Q})$ for all $n \geq 1$. Consequently, if $A \subset \Gamma_g$ is a free abelian subgroup of finite rank, then

$$\mathrm{res}_A^{\Gamma_g} e_n = 0 \quad \text{in} \quad H^{2n}(A, \mathbb{Z})$$

for all $n \geq 1$.

Now we are ready to prove Theorem 2. The key ingredient for the proof of Theorem 2 is the celebrated F-isomorphism theorem of Quillen:

THEOREM 4. ([36]) *Let* Γ *be a group having finite virtual cohomological dimension. If* $u \in H^*(\Gamma, \mathbb{F}_p)$ *restricts to zero for every elementary abelian* p*-subgroups, then* u *is nilpotent.*

Proof of Theorem 2. Recall that the virtual cohomological dimension of Γ_g (written $\operatorname{vcd}\Gamma_g$) is finite. Actually, $\operatorname{vcd}\Gamma_g = 4g-5$ according to Harer [14]. In view of Proposition 4 and 5, Theorem 2 follows from Theorem 4. ∎

Proof of Theorem 3. Given an odd prime number p, choose a primitive root r_p $(1 < r_p \leq p-1)$ which is a generator of the multiplicative group $\mathbb{F}_p^\times = \mathbb{F}_p \setminus \{0\}$. Choose an integer $k \geq 1$ satisfying

$$h := pk + \frac{(p-1)(p-r_p-1)}{2} \geq g.$$

Let $\gamma \in \Gamma_h$ be an element having the fixed point data

$$\langle h, p \mid \underbrace{1/p, \ldots, 1/p}_{p-r_p}, r_p/p \rangle.$$

Such $\gamma \in \Gamma_h$ exists by virtue of Proposition 2. Applying Proposition 3 to $\gamma \in \Gamma_h$, we see that

$$\operatorname{res}^{\Gamma_h}_{\langle\gamma\rangle} e_n = (p - r_p + (r_p^*)^n) u_\gamma \quad \text{in} \quad H^{2n}(\langle\gamma\rangle, \mathbb{Z}),$$

where r_p^* is an integer satisfying $r_p^* r_p \equiv 1 \pmod p$. Since r_p is a primitive root, $p - r_p + (r_p^*)^n \equiv 0 \pmod p$ if and only if $n \equiv -1 \pmod{p-1}$. As

$$H^*(\langle\gamma\rangle, \mathbb{F}_p) \cong \mathbb{F}_p[u_\gamma, v]/(v^2)$$

with $\deg v = 1$, we conclude that the mod p Morita-Mumford class $e_n^{(p)} \in H^{2n}(\Gamma_h, \mathbb{F}_p)$ is nontrivial and is not nilpotent whenever $n \not\equiv -1 \pmod{p-1}$. Now the theorem follows from Harer's stability theorem. ∎

§5. First Morita-Mumford classes

The natural action of Γ_g on the first cohomology $H^1(\Sigma_g, \mathbb{Z})$ of Σ_g preserves the symplectic form on it given by the cup product. Hence if we choose a symplectic basis for $H^1(\Sigma_g, \mathbb{Z})$, we obtain a homomorphism

$$\Gamma_g \longrightarrow Sp(2g, \mathbb{Z})$$

where $Sp(2g, \mathbb{Z})$ is the Siegel modular group of degree g. This induces a homomorphism

$$\Gamma_g \longrightarrow Sp(2g, \mathbb{R}).$$

Now the maximal compact subgroup of $Sp(2g, \mathbb{R})$ is isomorphic to $U(g)$. Hence passing to the classifying spaces we obtain a continuous map

$$B\Gamma_g \longrightarrow BU(g).$$

Let $c_n \in H^{2n}(\Gamma_g, \mathbb{Z})$ be the pull-back of the universal Chern class $c_n \in H^{2n}(BU(g), \mathbb{Z})$. According to Morita [28] and Mumford [35], one has

$$e_{k-1} - e_{k-2}c_1 + \cdots + (-1)^g e_{k-g-1}c_g = 0 \ \text{ in } \ H^{2(k-1)}(\Gamma_g, \mathbb{Z}) \tag{5.1}$$

for all $k \geq g$, where we understand $e_{-1} = 0$.

Proposition 6. *For all $g \geq 2$, one has $e_1 = -12c_1$ in $H^2(\Gamma_g, \mathbb{Z})$.*

Proof. Over the rationals, one has

$$e_1 = -12c_1 \ \text{ in } \ H^2(\Gamma_g, \mathbb{Q}). \tag{5.2}$$

This can be proved by applying the Atiyah-Singer index theorem for families of elliptic operators or the Grothendieck Riemann-Roch theorem (see [28], [35] and §7 below). On the other hand, Harer [12], [15] showed that $H^2(\Gamma_g, \mathbb{Z})$ is generated by the first Chern class c_1 and that

$$H^2(\Gamma_g, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/10\mathbb{Z}, & g = 2 \\ \mathbb{Z}, & g \geq 3. \end{cases} \tag{5.3}$$

The proposition for $g \geq 3$ follows from (5.2) and (5.3). When $g = 2$, (5.1) for $k = 2$ implies $e_1 = -2c_1$ in $H^2(\Gamma_2, \mathbb{Z})$ since $e_0 = -2$. In view of (5.3), this proves the proposition. ∎

Proposition 6 and the previously mentioned results of Harer imply the following corollary which is consistent with the conjectures.

Corollary 2. *For all* $g \geq 2$, $e_1^{(2)} \in H^2(\Gamma_g, \mathbb{F}_2)$ *and* $e_1^{(3)} \in H^2(\Gamma_g, \mathbb{F}_3)$ *vanish. Furthermore, for all* $p \geq 5$ *and* $g \geq 3$, $e_1^{(p)} \in H^2(\Gamma_g, \mathbb{F}_p)$ *is nontrivial.*

Now we prove nilpotency of first mod p Morita-Mumford classes and thereby show that the converse of Theorem 2 does not hold.

Proposition 7. *For all* $g \geq 3$, *the first mod* p *Morita-Mumford class* $e_1^{(p)} \in H^2(\Gamma_g, \mathbb{F}_p)$ *is nontrivial and is nilpotent whenever* $p > 4g + 2$.

Proof. If Γ_g has no elements of order p, then $H^n(\Gamma_g, \mathbb{F}_p)$ vanishes for all $n > \operatorname{vcd}\Gamma_g = 4g - 5$. This can be proved by inspection of the Farrell-Tate cohomology of Γ_g (see [4] for Farrell-Tate cohomology). On the other hand, as was proved by Wiman a hundred years ago (see [17]), if $p > 4g+2$ then Γ_g has no elements of order p. In summary, we have

$$H^n(\Gamma_g, \mathbb{F}_p) = 0 \quad \text{if } p > 4g + 2 \text{ and } n > 4g - 5. \tag{5.4}$$

Together with Corollary 2, this completes the proof. ∎

As for higher mod p Morita-Mumford classes, Theorem 3 and (5.4) imply the following result.

Proposition 8. *If* $p > 4g + 2$, *then* $e_n^{(p)} \in H^{2n}(\Gamma_g, \mathbb{F}_p)$ *is nontrivial and is nilpotent for all* $n < g/4$ *with* $n \not\equiv -1 \pmod{p-1}$.

§6. Various calculations

The purpose of this section is to give a few pieces of evidence which support the conjectures (Propositions 9, 10 and 11).

Proposition 9. *For* $g = 2$ *or* 3, *the mod* 2 *Morita-Mumford class* $e_n^{(2)} \in H^{2n}(\Gamma_g, \mathbb{F}_2)$ *vanishes for all* $n \geq 0$.

Proof. Applying the mod 2 reduction to (5.1), we obtain

$$e_{k-1}^{(2)} + e_{k-2}^{(2)} w_2 + \cdots + e_{k-g-1}^{(2)} w_{2g} = 0 \quad \text{in} \quad H^{2(k-1)}(\Gamma_g, \mathbb{F}_2) \tag{6.1}$$

for all $k \geq g$, where $w_{2i} \in H^{2i}(\Gamma_g, \mathbb{F}_2)$ is the pull-back of the symplectic Stiefel-Whitney class $w_{2i} \in H^{2i}(BSp(2g, \mathbb{R}), \mathbb{F}_2)$ by $\Gamma_g \to Sp(2g, \mathbb{R})$. When $g = 2$, we have

$$e^{(2)}_{k-1} + e^{(2)}_{k-2} w_2 + e^{(2)}_{k-3} w_4 = 0 \quad \text{in} \quad H^{2(k-1)}(\Gamma_2, \mathbb{F}_2) \tag{6.2}$$

for all $k \geq 2$. Since $e^{(2)}_1 = e^{(2)}_0 = 0$ by Corollary 2, the iterated use of (6.2) implies the proposition for $g = 2$. When $g = 3$, we have

$$e^{(2)}_{k-1} + e^{(2)}_{k-2} w_2 + e^{(2)}_{k-3} w_4 + e^{(2)}_{k-4} w_6 = 0 \quad \text{in} \quad H^{2(k-1)}(\Gamma_3, \mathbb{F}_2) \tag{6.3}$$

for all $k \geq 3$. Applying $k = 3$ to (6.3), we see that $e^{(2)}_2$ vanishes by Corollary 2. Hence we have $e^{(2)}_2 = e^{(2)}_1 = e^{(2)}_0 = 0$. By the iterated use of (6.3), the proposition for $g = 3$ is proved. □

Proposition 10. *For $p = 3$ or 5, the mod p Morita-Mumford class $e^{(p)}_n \in H^{2n}(\Gamma_2, \mathbb{F}_p)$ of Γ_2 vanishes if and only if $n \equiv -1 \pmod{p-1}$. Furthermore, if $n \not\equiv -1 \pmod{p-1}$ then $e^{(p)}_n \in H^{2n}(\Gamma_2, \mathbb{F}_p)$ is not nilpotent.*

Proof. We have

$$e^{(3)}_{k-1} + e^{(3)}_{k-2} c_1 + e^{(3)}_{k-3} c_2 = 0 \quad \text{in} \quad H^{2(k-1)}(\Gamma_2, \mathbb{F}_3)$$

for all $k \geq 2$. Here the mod 3 reduction of $c_i \in H^{2i}(\Gamma_3, \mathbb{Z})$ is denoted by the same symbol. Since $10e^{(3)}_1 = 10c_1 = 0$ by (5.3), we have $e^{(3)}_1 = c_1 = 0$ in $H^*(\Gamma_2, \mathbb{F}_3)$ and hence

$$e^{(3)}_{k-1} = -e^{(3)}_{k-3} c_2 \quad \text{in} \quad H^{2(k-1)}(\Gamma_2, \mathbb{F}_3) \tag{6.4}$$

for all $k \geq 2$. By the iterated use of (6.4), we conclude that $e^{(3)}_n \in H^{2n}(\Gamma_2, \mathbb{F}_3)$ vanishes for every odd integer $n \geq 1$.

Conversely, let $\gamma \in \Gamma_2$ be an element of order 3 having the fixed point data $\langle 2, 3 \mid 1/3, 1/3, 2/3, 2/3 \rangle$. We have

$$\operatorname{res}^{\Gamma_2}_{\langle\gamma\rangle} e_n = (1 + 1 + 2^n + 2^n) u_\gamma = 2(1 + 2^n) u_\gamma \quad \text{in} \quad H^*(\langle\gamma\rangle, \mathbb{Z}).$$

As in the proof of Theorem 3, it follows that $e^{(3)}_n \in H^{2n}(\Gamma_2, \mathbb{F}_3)$ is nontrivial and is not nilpotent for every even integer $n > 0$, which verifies the proposition for $p = 3$.

Finally, we will prove the argument for $p = 5$. Let $\gamma \in \Gamma_2$ be an element of order 5 having the fixed point data $\langle 2, 5 \mid 1/5, 1/5, 3/5 \rangle$. We have

$$\mathrm{res}^{\Gamma_2}_{\langle\gamma\rangle} e_n = (1 + 1 + 2^n) u_\gamma = 2(1 + 2^{n-1}) u_\gamma \quad \text{in} \quad H^*(\langle\gamma\rangle, \mathbb{Z}).$$

Hence $e_n^{(5)} \in H^*(\Gamma_2, \mathbb{F}_5)$ is nontrivial and is not nilpotent if $n \not\equiv -1 \pmod 4$. According to Cohen [5] (see also [20]), there is a cyclic subgroup $C \subset \Gamma_2$ of order 5 such that the inclusion $C \hookrightarrow \Gamma_2$ induces the isomorphism $\mathrm{res}^{\Gamma_2}_C : H^*(\Gamma_2, \mathbb{F}_5) \cong H^*(C, \mathbb{F}_5)$ of the mod 5 cohomology. In view of Proposition 4, we see that $e_n^{(5)} \in H^*(\Gamma_2, \mathbb{F}_5)$ vanishes when $n \equiv -1 \pmod 4$, which proves the proposition for $p = 5$. ∎

Remark 4. Actually, the subgroup $C \subset \Gamma_2$ which appeared in the last paragraph is generated by an element of Γ_2 having the fixed point data $\langle 2, 5 \mid 1/5, 1/5, 3/5 \rangle$. See [20].

Remark 5. According to a result of Lee and Weintraub [24], the mod p cohomology of Γ_2 is nontrivial if and only if $p = 2, 3$, or 5. Hence all the nontrivial mod p Morita-Mumford classes of Γ_2 are determined by the last two propositions.

PROPOSITION 11. *Let p be a prime. Then $e_n^{(p)} \in H^{2n}(\Gamma_{(p-1)/2}, \mathbb{F}_p)$ vanishes whenever $n \geq p - 3$ and $n \equiv -1 \pmod{p-1}$.*

Proof. Let $\mathcal{C}_p$ be the set of representatives of conjugacy classes of cyclic subgroups of order p of $\Gamma_{(p-1)/2}$. According to results of Xia [39], the natural homomorphism

$$\prod_{C \in \mathcal{C}_p} \mathrm{res}^{\Gamma_{(p-1)/2}}_C : \widehat{H}^n(\Gamma_{(p-1)/2}, \mathbb{Z})_{(p)} \longrightarrow \prod_{C \in \mathcal{C}_p} \widehat{H}^n(C, \mathbb{Z})$$

is injective, where $\widehat{H}^n(-, \mathbb{Z})$ is the Farrell-Tate cohomology and $\widehat{H}^n(-, \mathbb{Z})_{(p)}$ is the p-primary component of $\widehat{H}^n(-, \mathbb{Z})$. Since $\widehat{H}^n(\Gamma_{(p-1)/2}, \mathbb{Z})$ is a torsion group, the mod p reduction $\widehat{H}^n(\Gamma_{(p-1)/2}, \mathbb{Z}) \to \widehat{H}^n(\Gamma_{(p-1)/2}, \mathbb{F}_p)$ factors through $\widehat{H}^n(\Gamma_{(p-1)/2}, \mathbb{Z})_{(p)}$ (cf. [4, p. 290]). Consequently,

$$\prod_{C \in \mathcal{C}_p} \mathrm{res}^{\Gamma_{(p-1)/2}}_C : \widehat{H}^n(\Gamma_{(p-1)/2}, \mathbb{F}_p) \longrightarrow \prod_{C \in \mathcal{C}_p} \widehat{H}^n(C, \mathbb{F}_p)$$

is injective on the image of $\widehat{H}^n(\Gamma_{(p-1)/2}, \mathbb{Z}) \to \widehat{H}^n(\Gamma_{(p-1)/2}, \mathbb{F}_p)$, because the mod p reduction $\widehat{H}^*(C, \mathbb{Z}) \to \widehat{H}^*(C, \mathbb{F}_p)$ is injective as it is easily seen. According to fundamental properties of Farrell-Tate cohomology, there is a commutative diagram

$$\begin{tikzcd}
H^n(\Gamma_{(p-1)/2}, \mathbb{Z}) \arrow[r] \arrow[d] & H^n(\Gamma_{(p-1)/2}, \mathbb{F}_p) \arrow[r] \arrow[d] & \prod_{C \in \mathcal{C}_p} H^n(C, \mathbb{F}_p) \arrow[d] \\
\widehat{H}^n(\Gamma_{(p-1)/2}, \mathbb{Z}) \arrow[r] & \widehat{H}^n(\Gamma_{(p-1)/2}, \mathbb{F}_p) \arrow[r] & \prod_{C \in \mathcal{C}_p} \widehat{H}^n(C, \mathbb{F}_p)
\end{tikzcd}$$

in which all the vertical arrows are isomorphisms for $n > \operatorname{vcd} \Gamma_{(p-1)/2} = 2p-7$ (cf. [4, p. 278]). In view of Proposition 4, this completes the proof. ∎

§7. Morita-Mumford classes for Σ_g-bundles

In this section, we consider mod p Morita-Mumford classes in the context of oriented Σ_g-bundles and thereby obtain affirmative evidence for Conjecture 1. A smooth fiber bundle $\pi : E \to X$ with fiber Σ_g is called a *Σ_g-bundle*. Let $T_{E/X}$ be the *relative tangent bundle* (or the tangent bundle along the fiber) of π. Namely it is the subbundle of the tangent bundle of E consisting of those vectors which are tangent to the fibers of the bundle. If $T_{E/X}$ is orientable and an orientation is given on it, we say that the Σ_g-bundle $\pi : E \to X$ is *oriented*. Given an oriented Σ_g-bundle $\pi : E \to X$, let

$$e = e(T_{E/X}) \in H^2(E, \mathbb{Z})$$

be the Euler class of $T_{E/X}$. The n-th Morita-Mumford class $e_n \in H^{2n}(X, \mathbb{Z})$ of π is then defined by

$$e_n = \pi_!(e^{n+1}) \in H^{2n}(X, \mathbb{Z}),$$

where $\pi_! : H^*(E, \mathbb{Z}) \to H^{*-2}(X, \mathbb{Z})$ is the Gysin homomorphism. As is known,

$$\pi_!(x \cup \pi^*(y)) = \pi_!(x) \cup y \tag{7.1}$$

holds for any $x \in H^*(E, \mathbb{Z})$ and $y \in H^*(X, \mathbb{Z})$. According to a result of Earle and Eells [6], the classifying space $B\mathrm{Diff}_+\Sigma_g$ of oriented Σ_g-bundles is the Eilenberg-MacLane space $K(\Gamma_g, 1)$ of Γ_g. It follows that the isomorphism class of an oriented Σ_g-bundle $\pi : E \to X$ is completely determined by the *holonomy homomorphism*

$$h : \pi_1(X) \longrightarrow \Gamma_g$$

induced by the classifying map $X \to K(\Gamma_g, 1)$. In addition, the Morita-Mumford classes $e_n \in H^{2n}(X, \mathbb{Z})$ of π are identified with the pull-back of $e_n \in H^{2n}(\Gamma_g, \mathbb{Z})$ by the holonomy homomorphism. See [28] for further detail. We will prove the following theorem.

THEOREM 5. *Let p be a prime, $n_1, n_2, \ldots, n_k$ positive integers, and $n = \sum_i n_i$. Let $E \to X$ be an oriented Σ_g-bundle over a closed oriented manifold X whose $2n$-th cohomology group $H^{2n}(X, \mathbb{Z})$ is torsion-free. If n_i is odd and satisfies $n_i \equiv -1 \pmod{p-1}$ for some i, then*

$$e_{n_1}^{(p)} e_{n_2}^{(p)} \cdots e_{n_k}^{(p)} = 0 \quad \textit{in} \quad H^{2n}(X, \mathbb{F}_p).$$

As an immediate consequence, we obtain the following result.

COROLLARY 3. *Let p be a prime and $E \to X$ an oriented Σ_g-bundle over a closed oriented $2n$-manifold X. If n is odd and satisfies $n \equiv -1 \pmod{p-1}$, then*

$$e_n^{(p)} = 0 \quad \textit{in} \quad H^{2n}(X, \mathbb{F}_p).$$

To prove Theorem 5, we recall a certain relation among Morita-Mumford classes and Chern classes which was proved by Morita [28] along the line of Atiyah [2]. Let $\pi : E \to X$ be a Σ_g-bundle over a closed oriented manifold X. For each fiber $E_x = \pi^{-1}(x)$ $(x \in X)$, consider the real cohomology $H^1(E_x, \mathbb{R})$. The natural projection

$$\eta : \bigcup_{x \in X} H^1(E_x, \mathbb{R}) \longrightarrow X$$

gives rise to a $2g$-dimensional real vector bundle over X. A choice of a fiber metric on π yields a structure of g-dimensional complex vector bundle on η. Let $c_n(\eta) \in H^{2n}(X, \mathbb{Z})$ be the n-th Chern class of η. Note that $c_n(\eta)$ coincides with the pull-back of the n-th Chern class $c_n \in H^{2n}(\Gamma_g, \mathbb{Z})$ of Γ_g by the holonomy homomorphism $h : \pi_1(X) \to \Gamma_g$. Now the Atiyah-Singer index theorem for families of elliptic operators or the Grothendieck Riemann-Roch theorem implies

$$e_{2n-1} = (-1)^n \frac{2n}{B_{2n}} s_{2n-1}(\eta) \quad \text{in} \quad H^{4n-2}(X, \mathbb{Q}) \tag{7.2}$$

for all n, where B_{2n} is the $2n$-th Bernoulli number and $s_n(\eta) \in H^{2n}(X,\mathbb{Z})$ is the characteristic class of η corresponding to the formal sum $\sum_k x_k^n$ (it is an integral polynomial of Chern classes of η and is called the n-th Newton class of η). See [28] for further detail. Now we will prove Theorem 5.

Proof of Theorem 5. Without loss of generality, we may assume n_1 is odd and satisfies $n_1 \equiv -1 \pmod{p-1}$. For simplicity, set $2m-1 = n_1$. It follows from (7.2) that

$$e_{2m-1}\prod_{i=2}^{k} e_{n_i} = (-1)^m \frac{2m}{B_{2m}} s_{2m-1}(\eta) \prod_{i=2}^{k} e_{n_i} \quad \text{in} \quad H^{2n}(X,\mathbb{Q}).$$

Let $\mathrm{num}(B_{2m}/2m)$ (*resp.* $\mathrm{den}(B_{2m}/2m)$) be the numerator (*resp.* denominator) of $B_{2m}/2m$. Then the previous equality leads to

$$\mathrm{num}\Big(\frac{B_{2m}}{2m}\Big) e_{2m-1}\prod_{i=2}^{k} e_{n_i} = (-1)^m \mathrm{den}\Big(\frac{B_{2m}}{2m}\Big) s_{2m-1}(\eta) \prod_{i=2}^{k} e_{n_i} \tag{7.3}$$

in $H^{2n}(X,\mathbb{Q})$. Observe that both sides of (7.3) are integral. In other words, they belong to the image of the homomorphism $H^*(X,\mathbb{Z}) \to H^*(X,\mathbb{Q})$ induced by $\mathbb{Z} \hookrightarrow \mathbb{Q}$. Since $H^{2n}(X,\mathbb{Z})$ is torsion-free, we conclude that (7.3) holds in $H^{2n}(X,\mathbb{Z})$. As $2m \equiv 0 \pmod{p-1}$, von Staudt's theorem implies that $\mathrm{den}(B_{2m}/2m)$ is divisible by p (see [3] for instance). Applying the mod p reduction to (7.3), we have

$$\mathrm{num}\Big(\frac{B_{2m}}{2m}\Big) e_{n_1}^{(p)} e_{n_2}^{(p)} \cdots e_{n_k}^{(p)} = 0 \quad \text{in} \quad H^{2n}(X,\mathbb{F}_p).$$

Since $\mathrm{num}(B_{2m}/2m)$ is prime to p, the theorem is proved. □

Now let $s_n \in H^{2n}(\Gamma_g,\mathbb{Z})$ be the n-th Newton class of Γ_g which is defined to be the characteristic class corresponding to the formal sum $\sum_k x_k^n$ of the g-dimensional complex vector bundle associated with the continuous map $B\Gamma_g \to BU(g)$ introduced in §5. Then the Grothendieck Riemann-Roch theorem implies

$$e_{2n-1} = (-1)^n \frac{2n}{B_{2n}} s_{2n-1} \quad \text{in} \quad H^{4n-2}(\Gamma_g,\mathbb{Q})$$

for all $n \geq 1$. In view of the proof of Theorem 5, it is valuable to propose the following conjecture.

Conjecture 3.

$$\mathrm{num}\Big(\frac{B_{2n}}{2n}\Big)e_{2n-1} = (-1)^n\mathrm{den}\Big(\frac{B_{2n}}{2n}\Big)s_{2n-1} \quad in \quad H^{4n-2}(\Gamma_g, \mathbb{Z})$$

holds for all $n \geq 1$.

According to Proposition 6, Conjecture 3 is affirmative for $n = 1$. By virtue of the proof of Theorem 5, the affirmative solution of Conjecture 3 implies those of Conjectures 1 and 2 for mod p Morita-Mumford classes of *odd* indices. We will deal with Conjecture 3 in the forthcoming paper with N. Kawazumi.

§8. Applications to structure of oriented Σ_g-bundles

The purpose of this section is to investigate structure of oriented Σ_g-bundles for $g = 2$ or 3 by applying Theorem 9. Let $\pi : E \to X$ be an oriented Σ_g-bundle over a closed manifold X which is not necessarily orientable. Then the tangent bundle $T_*(E)$ of E decomposes itself into

$$T_*(E) = T_{E/X} \oplus \pi^*(T_*(X)).$$

Hence the total Stiefel-Whitney class $w.(E)$ of E is expressed as

$$w.(E) = (1 + e(T_{E/X})) \cdot \pi^*(w.(X)) \quad \text{in} \quad H^*(E, \mathbb{F}_2).$$

As (7.1) remains valid over $\mathbb{F}_2$ and $\langle u, [E]_2\rangle = \langle \pi_!(u), [X]_2\rangle$ holds for all $u \in H^{\dim E}(E, \mathbb{F}_2)$, we see that all the Stiefel-Whitney numbers of E are expressed by the Stiefel-Whitney classes of X and the mod 2 Morita-Mumford classes of π. In particular, if $g = 2$ or 3 then it follows from Theorem 9 that all the Stiefel-Whitney numbers of E vanish. Hence we obtain the following result.

Proposition 12. *Let $E \to X$ be an oriented Σ_g-bundle over a closed manifold X which is not necessarily orientable. If $g = 2$ or 3, then E is unoriented null-cobordant.*

Now let $\pi : E \to X$ be an oriented Σ_g-bundle over a closed *oriented* manifold X so that the total space E admits a natural orientation. Then the total rational Pontryagin class $p.(E)$ of E is expressed as

$$p.(E) = (1 + e(T_{E/X})^2) \cdot \pi^*(p.(X)) \quad \text{in} \quad H^*(E, \mathbb{Q}).$$

Since (7.1) remains valid over the rationals and $\langle u, [E]\rangle = \langle \pi_!(u), [X]\rangle$ holds for all $u \in H^{\dim E}(E, \mathbb{Q})$, we see that all the Pontryagin numbers of E are expressed by the rational Pontryagin classes of X and the rational Morita-Mumford classes of *odd* indices of π. In particular, we obtain the following result.

Corollary 4. *Let $E \to X$ be an oriented Σ_2-bundle over a closed oriented manifold X. Then E is oriented null-cobordant.*

Proof. Since Γ_2 is $\mathbb{Q}$-acyclic by a result of Igusa [18], we see that all the rational Morita-Mumford classes of π vanish and hence all the Pontryagin numbers of E vanish. ∎

Let $E \to X$ be an oriented Σ_3-bundle over a closed oriented manifold X. If $\dim X \not\equiv 2 \pmod 4$, then E is oriented null-cobordant by Proposition 12. In case $\dim X \equiv 2 \pmod 4$, we obtain the following result.

Corollary 5. *Let $E \to X$ be an oriented Σ_3-bundle over a closed oriented manifold X with $\dim X = 4n + 2$ for some $n \geq 1$. Suppose that, for any positive integers $n_1, n_2, \ldots, n_k$ satisfying $n = \sum_i n_i$,*

$$p_{n_1}(X)\, p_{n_2}(X) \cdots p_{n_k}(X) = 0 \quad \textit{in} \quad H^{4n}(X, \mathbb{Q}),$$

where $p_{n_i}(X)$ is the n_i-th rational Pontryagin class of X. Then E is oriented null-cobordant.

Proof. It was proved by Faber [8] that $e_1^2 = 0$ and $e_n = 0$ for $n \geq 2$ in $H^*(\Gamma_3, \mathbb{Q})$. Indeed, according to Looijenga [25], one has

$$H^*(\Gamma_3, \mathbb{Q}) \cong \mathbb{Q}[e_1]/(e_1^2) + \mathbb{Q}u$$

where u is an element of degree 6. Hence all the Pontryagin numbers of E vanish by the assumption. ∎

As an immediate consequence, we obtain the following result.

Corollary 6. *Let $E \to X$ be an oriented Σ_3-bundle over a closed oriented manifold X with $\dim X \geq 3$. If all the rational Pontryagin classes of X vanish, then E is oriented null-cobordant.*

In case $E \to X$ is an oriented Σ_g-bundle over a closed oriented surface X, the signature $\sigma(E)$ of the total space E was studied by Meyer [26]. He showed that $\sigma(E) = 0$ whenever $g = 2$. Corollary 4 is a generalization of this fact. He also showed that for every $g \geq 3$ and every $n \in 4\mathbb{Z}$ there exists an oriented Σ_g-bundle $E \to X$ over a closed oriented surface X such that $\sigma(E) = n$ (see [2], [7] and [23] for explicit constructions of such Σ_g-bundles). Hence Corollary 6 is not valid when $\dim X = 2$.

Remark 6. Since $e_1^{(2)} = 0$ for all $g \geq 2$ by Corollary 2, the total space of any oriented Σ_g-bundle over a closed surface which is not necessarily orientable is unoriented null-cobordant.

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References

[1] T. Akita, N. Kawazumi and T. Uemura, *Periodic surface automorphisms and algebraic independence of Morita-Mumford classes*, J. Pure Appl. Algebra, **160** (2001), 1–11.

[2] M. F. Atiyah, The signature of fibre-bundles, Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 73–84.

[3] A. I. Borevich and I. R. Shafarevich, Number theory, Academic Press, New York, 1966.

[4] K. S. Brown, Cohomology of Groups, Springer-Verlag, New York, 1982.

[5] F. R. Cohen, *Homology of mapping class groups for surfaces of low genus*, The Lefschetz centennial conference, Part II (Mexico City, 1984), Contemp. Math. **58**, II, Amer. Math. Soc., Providence (1987), pp. 21–30.

[6] C. J. Earle and J. Eells, *A fibre bundle description of Teichmüller theory*, J. Differential Geometry, **3** (1969), 19–43.

[7] H. Endo, *A construction of surface bundles over surfaces with non-zero signature*, Osaka J. Math., **35** (1998), 915–930.

[8] C. Faber, *Chow rings of moduli spaces of curves. I. The Chow ring of* $\overline{\mathcal{M}}_3$, Ann. of Math. (2), **132** (1990), 331–419.

[9] C. Faber and E. Looijenga (eds.), Moduli of Curves and Abelian Varieties, The Dutch Intercity Seminar on Moduli, Aspects of Mathematics, **E33**, Friedr. Vieweg & Sohn, Braunschweig, 1999.

[10] H. H. Glover, G. Mislin and Y. Xia, *On the Yagita invariant of mapping class groups*, Topology, **33** (1994), 557–574.

[11] R. Hain and E. Looijenga, *Mapping class groups and moduli spaces of curves*, Algebraic geometry–Santa Cruz 1995, Proc. Sympos. Pure Math., **62**, Part 2, Amer. Math. Soc., Providence (1997), pp. 97–142.

[12] J. L. Harer, *The second homology group of the mapping class group of an orientable surface*, Invent. Math., **72** (1983), 221–239.

[13] ———, *Stability of the homology of the mapping class groups of orientable surfaces*, Ann. of Math. (2), **121** (1985), 215–249.

[14] ———, *The virtual cohomological dimension of the mapping class group of an orientable surface*, Invent. Math., **84** (1986), 157–176.

[15] ———, *The cohomology of the moduli space of curves*, Theory of moduli (Montecatini Terme, 1985), Lecture Notes in Math., **1337**, Springer, Berlin (1988), pp. 138–221.

[16] ———, *The third homology group of the moduli space of curves*, Duke Math. J., **63** (1991), 25–55.

[17] W. J. Harvey, *Cyclic groups of automorphisms of a compact Riemann surface*, Quart. J. Math. Oxford Ser. (2), **17** (1966), 86–97.

[18] J.-I. Igusa, *Arithmetic variety of moduli for genus two*, Ann. of Math. (2), **72** (1960), 612–649.

[19] N. V. Ivanov, *On the homology stability for Teichmüller modular groups: closed surfaces and twisted coefficients*, Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991), Contemp. Math., **150**, Amer. Math. Soc., Providence (1993), pp. 149–194.

[20] N. Kawazumi, *Homology of hyperelliptic mapping class groups for surfaces*, Topology Appl., **76** (1997), 203–216.

[21] N. Kawazumi and T. Uemura, *Riemann-Hurwitz formula for Morita-Mumford classes and surface symmetries*, Kodai Math. J., **21** (1998), 372–380.

[22] S. P. Kerckhoff, *The Nielsen realization problem*, Ann. of Math. (2), **117** (1983), 235–265.

[23] K. Kodaira, *A certain type of irregular algebraic surfaces*, J. Analyse Math., **19** (1967), 207–215.

[24] R. Lee and S. H. Weintraub, *Cohomology of* $\mathrm{Sp}_4(\mathbf{Z})$ *and related groups and spaces*, Topology, **24** (1985), 391–410.

[25] E. Looijenga, *Cohomology of* $\mathcal{M}_3$ *and* $\mathcal{M}_3^1$, Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991), Contemp. Math., **150**, Amer. Math. Soc., Providence (1993), pp. 205–228.

[26] W. Meyer, *Die Signatur von Flächenbündeln*, Math. Ann., **201** (1973), 239–264.

[27] E. Y. Miller, *The homology of the mapping class group*, J. Differential Geom., **24** (1986), 1–14.

[28] S. Morita, *Characteristic classes of surface bundles*, Invent. Math., **90** (1987), 551–577.

[29] ———, *Characteristic classes of surface bundles and bounded cohomology*, A fête of topology, Academic Press, Boston (1988), pp. 233–257.

[30] ———, *Mapping class groups of surfaces and three-dimensional manifolds*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), Math. Soc. Japan, Tokyo (1991), pp. 665–674.

[31] ———, *The structure of the mapping class group and characteristic classes of surface bundles*, Mapping class groups and moduli spaces of Riemann surfaces (Göttingen,

1991/Seattle, WA, 1991), Contemp. Math., **150**, Amer. Math. Soc., Providence (1993), pp. 303–315.

[32] ———, *Characteristic classes of surface bundles and the Casson invariant*, Sugaku Expositions, **7** (1994), 59–79.

[33] ———, *Problems on the structure of the mapping class group of surfaces and the topology of the moduli space of curves*, Topology, geometry and field theory, World Sci. Publishing, River Edge (1994), pp. 101–110.

[34] ———, *Structure of the mapping class groups of surfaces: a survey and a prospect*, Proceedings of the Kirbyfest (Berkeley, CA, 1998), Geometry & Topology Monographs **2**, Geometry & Topology, Coventry (1999), 349–406.

[35] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, Arithmetic and geometry, Vol. II, Progr. Math., **36**, Birkhäuser Boston, Boston (1983), pp. 271–328.

[36] D. Quillen, *The spectrum of an equivariant cohomology ring. II*, Ann. of Math. (2), **94** (1971), 573–602.

[37] P. Symonds, *The cohomology representation of an action of C_p on a surface*, Trans. Amer. Math. Soc., **306** (1988), 389–400.

[38] T. Uemura, *Morita-Mumford classes on finite cyclic subgroups of the mapping class group of closed surfaces*, Hokkaido Math. J., **28** (1999), 597–611.

[39] Y. Xia, *The p-torsion of the Farrell-Tate cohomology of the mapping class group $\Gamma_{(p-1)/2}$*, Topology '90 (Columbus, OH, 1990), Ohio State Univ. Math. Res. Inst. Publ., **1**, de Gruyter, Berlin (1992), pp. 391–398.

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