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AN ANALOGUE OF PITMAN'S $2M - X$ THEOREM FOR EXPONENTIAL WIENER FUNCTIONALS

PART II: THE ROLE OF THE GENERALIZED INVERSE GAUSSIAN LAWS

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Abstract. In Part I of this work, we have shown that the stochastic process $Z^{(\mu)}$ defined by (8.1) below is a diffusion process, which may be considered as an extension of Pitman's $2M - X$ theorem. In this Part II, we deduce from an identity in law partly due to Dufresne that $Z^{(\mu)}$ is intertwined with Brownian motion with drift μ and that the intertwining kernel may be expressed in terms of Generalized Inverse Gaussian laws.

Foreword

Although the results found in this paper (Part II) are closely related to those in Part I [19], the methods used in each part are quite different; in particular, Part II may be read independently from Part I. However, it may be convenient for a reader of Part II to have Part I nearby.

§8. Introduction to Part II

The main result presented in Part I (Theorem 1.6) of this study consists in the following: if $B^{(\mu)} = \{B_t^{(\mu)}, t \geqq 0\}$ denotes a Brownian motion with constant drift μ, then the stochastic process

$$Z_t^{(\mu)} \equiv \exp\bigl(-B_t^{(\mu)}\bigr) A_t^{(\mu)}, \quad t \geqq 0 \tag{8.1}$$

is a diffusion, where $\{A_t^{(\mu)}, t \geqq 0\}$ is the quadratic variation process of the associated geometric Brownian motion $\{e_t^{(\mu)} \equiv \exp\bigl(B_t^{(\mu)}\bigr), t \geqq 0\}$ given by

$$A_t^{(\mu)} = \int_0^t \exp\bigl(2B_s^{(\mu)}\bigr)\, ds.$$

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The infinitesimal generator and some properties of $Z^{(\mu)} = \{Z_t^{(\mu)},\, t \geqq 0\}$, in particular, its relation to the generalized (upward and downward) Bessel processes, were discussed in detail in Part I.

For the reader's convenience, we repeat the precise statements of the main result and of the key proposition.

THEOREM 1.6. *Let $\mu \in \mathbf{R}$.* (i) *$Z^{(\mu)}$ is a diffusion process whose natural filtration $\{\mathscr{Z}_t^{(\mu)},\, t \geqq 0\}$ is strictly contained in that of $B^{(\mu)}$, that is, for any $t > 0$,*

$$\mathscr{Z}_t^{(\mu)} \subsetneqq \sigma\{B_s^{(\mu)}\,;\, s \leqq t\}.$$

Furthermore, the infinitesimal generator of $Z^{(\mu)}$ is given by

$$\frac{1}{2} z^2 \frac{d^2}{dz^2} + \left\{ \left(\frac{1}{2} - \mu\right) z + \left(\frac{K_{1+\mu}}{K_\mu}\right)\left(\frac{1}{z}\right) \right\} \frac{d}{dz}.$$

(ii) *The diffusion processes $Z^{(\mu)}$ and $Z^{(-\mu)}$ have the same distribution.*

PROPOSITION 1.7. *Let $\mu \in \mathbf{R}$. For any $t > 0$, the conditional distribution of $B_t^{(\mu)}$ given $\mathscr{Z}_t^{(\mu)}$ is*

$$P\big(B_t^{(\mu)} \in dx | \mathscr{Z}_t^{(\mu)}, Z_t^{(\mu)} = z\big) = \frac{\exp(\mu x)}{2K_\mu(1/z)} \exp\left(-\frac{\cosh(x)}{z}\right) dx. \tag{8.2}$$

The same formula holds when t is replaced by any $\big(\mathscr{Z}_t^{(\mu)}\big)$-stopping time T.

In the present Part II, we show that these results may be derived as a consequence of the following extension given in [17] of Dufresne's identity [9].

THEOREM 8.1. *Let $\mu > 0$. Then the following identity in law holds*:

$$\left\{ \frac{1}{A_t^{(-\mu)}},\, t > 0 \right\} \overset{\text{(law)}}{=} \left\{ \frac{1}{A_t^{(\mu)}} + \frac{1}{\widetilde{A}_\infty^{(-\mu)}},\, t > 0 \right\}, \tag{8.3}$$

where $\widetilde{A}_\infty^{(-\mu)}$ on the right hand side denotes a copy of $A_\infty^{(-\mu)} = \lim_{t\to\infty} A_t^{(-\mu)}$, independent of $\{A_t^{(\mu)},\, t \geqq 0\}$.

Remark 8.1. It is known ([8], [28]) that $A_\infty^{(-\mu)}$, $\mu > 0$, is distributed as $1/2\gamma_\mu$, where γ_μ is a Gamma(μ) random variable whose law is given by

$$P\big(\gamma_\mu \in dt\big) = \frac{1}{\Gamma(\mu)} t^{\mu-1} e^{-t}\, dt.$$

The derivation of Theorem 1.6 from Theorem 8.1 is made in Section 10, after giving, in the next Section 9, some necessary preliminaries on the generalized inverse Gaussian laws (GIG laws in short), which play important roles in Section 10 and, indeed, throughout this whole Part II paper. As a result of their interpretation in this Brownian motion context (which is partially new; see Vallois [24]), we obtain further properties of the GIG laws in Section 13.

In Section 11 we derive from the conditional laws obtained in Section 10 some further semigroup intertwining results, in particular, between the semigroups of $B^{(\mu)}$ and $Z^{(\mu)}$. In Section 12 we study the diffusion process $Z^{(\mu)}$ under a Brownian bridge. Furthermore, in Section 13 we develop the parallel between the study of Rogers–Pitman [21] and ours. In Section 14 we study the processes ${}^{+}Z^{(\mu)}$ and ${}^{+}\xi^{(\mu)}$ for $\mu < 0$, which are defined similarly to $Z^{(\mu)}$ and $\xi^{(\mu)}$ (see Part I for details on $\xi^{(\mu)}$) but are looking into the future.

§9. Generalized Inverse Gaussian Distributions: Basic Results

The GIG laws constitute a three parameter family of probability distributions on $\mathbf{R}_+$ which are given by

$$(9.1)\qquad GIG(\mu;a,b)(dt) = \Big(\frac{b}{a}\Big)^{\mu} \frac{t^{\mu-1}}{2K_\mu(ab)} \exp\bigg(-\frac{1}{2}(a^2/t + b^2 t)\bigg)\, dt$$

for $\mu \in \mathbf{R}$, $a, b > 0$. We follow the notation in recent papers by Barndorff–Nielsen [2] and Barndorff–Nielsen–Shephard [4]. The reader should beware of the different notations for GIG laws used in Seshadri [22] and Vallois [24].

Owing to the historical survey in Seshadri [22], the GIG laws would deserve to be named the Halphen laws. They have been introduced by Good [10] and have been widely discussed by Barndorff–Nielsen et al [2], [3] and Vallois [24] among others.

In this paper it will be convenient to denote by $I_{a,b}^{(\mu)}$ a random variable with law $GIG(\mu; a, b)$. We note the following elementary properties.

$$(9.2)\qquad 1/I_{a,b}^{(\mu)} \stackrel{\text{(law)}}{=} I_{b,a}^{(-\mu)}\ ;$$

$$(9.3)\qquad c^2 I_{a,b}^{(\mu)} \stackrel{\text{(law)}}{=} I_{ca,b/c}^{(\mu)} \text{ for } c > 0 \text{ and, consequently}$$

$$\Big(\frac{b}{a}\Big) I_{a,b}^{(\mu)} \stackrel{\text{(law)}}{=} I_{\sqrt{ab},\sqrt{ab}}^{(\mu)}\ ;$$

$$P(I^{(\nu)}_{a,b} \in dt) = t^{\nu-\mu} P(I^{(\mu)}_{a,b} \in dt)/C^{\nu,\mu}_{a,b}, \text{ where} \tag{9.4}$$
$$C^{\nu,\mu}_{a,b} = \Big(\frac{b}{a}\Big)^{\mu-\nu} \frac{K_\nu(ab)}{K_\mu(ab)}.$$

Beside these elementary properties, the following one lies deeper and has been the subject of a number of studies (see, in particular, Vallois [24]):

$$I^{(\mu)}_{a,b} \overset{(\mathrm{law})}{=} I^{(-\mu)}_{a,b} + \frac{2}{b^2}\gamma_\mu \quad \text{for } \mu > 0, \tag{9.5}$$

where γ_μ denotes a Gamma(μ) random variable independent of $I^{(-\mu)}_{a,b}$.

A companion family to the GIG laws are the hyperbolic distributions which are the laws of the random variables:

$$H^{(\mu)}_{a,b} = N\sqrt{I^{(\mu)}_{a,b}} \tag{9.6}$$

for a standard normal random variable N independent of $I^{(\mu)}_{a,b}$. It is easily shown that the distribution of $H^{(\mu)}_{a,b}$ is given by

$$\begin{aligned} &P\big(H^{(\mu)}_{a,b} \in dx\big) \\ &\quad = \frac{(b/a)^\mu}{\sqrt{2\pi} b^{\mu-1/2} K_\mu(ab)} \big(x^2+a^2\big)^{(\mu-1/2)/2} K_{\mu-1/2}\big(b\sqrt{a^2+x^2}\,\big)\,dx \end{aligned} \tag{9.7}$$

(see, e.g., [4], formula (63)).

Vallois [24] has extensively discussed the realizations of (9.5) in terms of diffusion processes. The case where $\mu = 1/2$ is an easy, although important particular one. In this case (9.5) may be understood as a consequence of the following, which is an instance of application of the strong Markov property:

$$\Lambda^{(-b)}_{a\to 0} \overset{(\mathrm{law})}{=} T^{(-b)}_{a\to 0} + \Lambda^{(-b)}_{0\to 0}, \tag{9.8}$$

where

$$\Lambda^{(-b)}_{\alpha\to\beta} = \sup\big\{t\,;\, \alpha + B^{(-b)}_t = \beta\big\}, \quad T^{(-b)}_{\alpha\to\beta} = \inf\big\{t\,;\, \alpha + B^{(-b)}_t = \beta\big\},$$

and the two random variables on the right hand side of (9.8) are independent.

More generally, (9.5) may be understood in a similar manner starting from a downward $BES(-\mu, \delta\!\downarrow)$ process and considering the additive decomposition of Λ, its last hitting time of 0, as $T + (\Lambda - T)$ with an obvious notation. We will provide a more detailed discussion in Section 13.

§10. Second proofs of the Markov property of $Z^{(\mu)}$ and of Proposition 1.7

We shall proceed with the help of the following two lemmas. We set

$$\mathscr{B}_t^{(\nu)} = \sigma\{B_s^{(\nu)}\ ;\ s \leqq t\} \quad \text{and} \quad \mathscr{Z}_t^{(\nu)} = \sigma\{Z_s^{(\nu)}\ ;\ s \leqq t\}.$$

LEMMA 10.1. *Let $\mu > 0$. Then $A_\infty^{(-\mu)}$ is independent of $Z^{(-\mu)}$ and it holds that*

(10.1) $$\mathscr{B}_\infty^{(-\mu)} = \mathscr{Z}_\infty^{(-\mu)} \vee \sigma\big(A_\infty^{(-\mu)}\big).$$

More generally, $\mathscr{B}_t^{(-\mu)} = \mathscr{Z}_t^{(-\mu)} \vee \sigma\big(A_t^{(-\mu)}\big)$.

Remark 10.1. Combining this lemma with Proposition 1.7, we see that $\mathscr{Z}_t^{(-\mu)}$ and $A_t^{(-\mu)}$ are conditionally independent given $Z_t^{(-\mu)}$.

Proof. Taking derivatives (with respect to t) on both hand sides of (8.3), we obtain the following reinforcement of (8.3):

$$\big\{\big(Z_t^{(-\mu)}, A_t^{(-\mu)}\big), t \geqq 0\big\} \overset{\text{(law)}}{=} \left\{\left(Z_t^{(\mu)}, \frac{A_t^{(\mu)}\widetilde{A}_\infty^{(-\mu)}}{A_t^{(\mu)} + \widetilde{A}_\infty^{(-\mu)}}\right), t \geqq 0\right\}.$$

In particular, letting $t \to \infty$ for the quantities which involve $A^{(\pm\mu)}$ found on both hand sides, we obtain

$$\big(\big\{Z_t^{(-\mu)}, t \geqq 0\big\}, A_\infty^{(-\mu)}\big) \overset{\text{(law)}}{=} \big(\big\{Z_t^{(\mu)}, t \geqq 0\big\}, \widetilde{A}_\infty^{(-\mu)}\big),$$

which proves the first assertion of the lemma.

The second and third statements follow from

$$\frac{1}{A_t^{(-\mu)}} = \frac{1}{A_s^{(-\mu)}} - \int_s^t \frac{\exp\big(2B_u^{(-\mu)}\big)}{\big(A_u^{(-\mu)}\big)^2}\,du = \frac{1}{A_s^{(-\mu)}} - \int_s^t \frac{du}{\big(Z_u^{(-\mu)}\big)^2},$$

which holds for $0 < s \leqq t \leqq \infty$. □

Remark 10.2. The last argument shows equally well that, for $0 < s < t < \infty$,

$$\mathscr{B}_t^{(\nu)} = \mathscr{Z}_t^{(\nu)} \vee \sigma\big(A_s^{(\nu)}\big) = \mathscr{Z}_t^{(\nu)} \vee \sigma\big(B_s^{(\nu)}\big)$$

holds for any $\nu \in \mathbf{R}$. Consequently, one has

(10.2) $$\mathscr{B}_t^{(\nu)} = \mathscr{B}_s^{(\nu)} \vee \mathscr{Z}_t^{(\nu)},$$

but the reader should beware that, however tempting this may be (i.e., letting $s \downarrow 0$), the equality (10.2) does not yield the equality between $\mathscr{B}_t^{(\nu)}$ and $\mathscr{Z}_t^{(\nu)}$, as discussed in Theorem 1.6 and explained in Proposition 1.7. For the discussion of a similar situation in the framework of Tsirel'son's celebrated stochastic differential equation, see, e.g., Yor [29].

LEMMA 10.2. *Let μ be positive, $Q_t^{\omega,z}$ be the regular conditional distribution of $B^{(-\mu)}$ given $\mathscr{Z}_t^{(-\mu)}$, where $z = Z_t^{(-\mu)}$, and γ_μ be a Gamma(μ) random variable which is independent of $B^{(-\mu)}$. Then, for any bounded Borel function f on $[0,\infty)$, it holds that*

$$E^{Q_t^{\omega,z}}\big[f\big(ze_t^{(-\mu)} + \big(e_t^{(-\mu)}\big)^2/2\gamma_\mu\big)\big] = E[f(1/2\gamma_\mu)]. \tag{10.3}$$

More generally, this identity extends with t replaced by any $\big(\mathscr{Z}_t^{(-\mu)}\big)$-stopping time.

Proof. As was remarked above, the identity in law

$$A_\infty^{(-\mu)} = \int_0^\infty \exp\big(2B_s^{(-\mu)}\big)\, ds \overset{\text{(law)}}{=} \frac{1}{2\gamma_\mu}$$

holds for any $\mu > 0$. Moreover we have (still under $Q_t^{\omega,z}$)

$$A_\infty^{(-\mu)} = A_t^{(-\mu)} + \int_0^\infty \exp\big(B_{t+s}^{(-\mu)}\big)\, ds = A_t^{(-\mu)} + \exp\big(2B_t^{(-\mu)}\big)\widetilde{A}_\infty^{(-\mu)},$$

so that finally, under $Q_t^{\omega,z}$, we get

$$A_\infty^{(-\mu)} = e_t^{(-\mu)} Z_t^{(-\mu)} + \big(e_t^{(-\mu)}\big)^2 \widetilde{A}_\infty^{(-\mu)} \overset{\text{(law)}}{=} \frac{1}{2\gamma_\mu} \tag{10.4}$$

by virtue of Lemma 10.1, where $\widetilde{A}_\infty^{(-\mu)}$ is independent of $\mathscr{B}_t^{(-\mu)}$ and is distributed as $1/2\gamma_\mu$. This yields the assertion of the lemma. ∎

Denote by $E^{(\mu)}$ the expectation with respect to the law of $\big\{B_t^{(\mu)},\, t \geqq 0\big\}$ on the canonical path space and let $\mathscr{Z}_t$ be the σ-field generated by the Z-process up to time t. Then, by the Cameron–Martin theorem, we get

$$E^{(-\mu)}[F_t\, f(e_t)] = E^{(-1/2)}\bigg[F_t\, f(e_t)(e_t)^{-\mu+1/2} \exp\bigg(\Big(\frac{1}{8} - \frac{1}{2}\mu^2\Big)t\bigg)\bigg]$$

for any $\mathscr{Z}_t$-measurable non-negative functional F_t and for any bounded non-negative Borel function f. Moreover, from Bayes rule (cf., e.g., [16], Chapter 7), it holds that

$$E^{(-\mu)}[f(e_t) \mid \mathscr{Z}_t] = \frac{E^{(-1/2)}[f(e_t)(e_t)^{-\mu+1/2} \mid \mathscr{Z}_t]}{E^{(-1/2)}[(e_t)^{-\mu+1/2} \mid \mathscr{Z}_t]}. \tag{10.5}$$

Now we are in a position to prove both Proposition 1.7 and Theorem 1.6.

Second proof of Proposition 1.7. We shall show, using the notation

$$\rho_z^{(\mu)}(u) = \frac{u^{\mu-1}}{2K_\mu(1/z)} \exp\Big(-\frac{1}{2z}\Big(u + \frac{1}{u}\Big)\Big),$$

that the probability law of $e_t^{(\mu)}$ under $Q_t^{\omega,z}$ is $\rho_z^{(\mu)}(u)\,du$. Note that

$$\rho_z^{(\mu)}(u)\,du = GIG\big(\mu; 1/\sqrt{z}, 1/\sqrt{z}\,\big)(du).$$

At first we consider the case $\mu = -1/2$. Since

$$K_{-1/2}(z) = K_{1/2}(z) = \Big(\frac{\pi}{2z}\Big)^{1/2} e^{-z},$$

the Laplace transform of $\rho_z^{(-1/2)}(u)$ is computed as follows:

$$\begin{aligned}
&\int_0^\infty e^{-\alpha u} \rho_z^{(-1/2)}(u)\,du \\
&\quad = e^{1/z} \int_0^\infty \frac{1}{\sqrt{2\pi u^3 z}} \exp\Big(-\Big(\frac{1}{2zu} + \Big(\alpha + \frac{1}{2z}\Big)u\Big)\Big)\,du \\
&\quad = \exp\Big(-\sqrt{\frac{2\alpha}{z} + \frac{1}{z^2}} + \frac{1}{z}\Big).
\end{aligned}$$

Now we use (10.3) by setting $f(u) = \exp(-\lambda^2 u/2)$, $\lambda > 0$. Then, noting that

$$E\big[\exp(-\lambda^2/4\gamma_{1/2})\big] = e^{-\lambda},$$

we obtain

$$\begin{aligned}
&E^{Q_t^{\omega,z}}\Big[\exp\Big(-\frac{\lambda^2}{2} e_t^{(-1/2)} z - \frac{\lambda^2}{2} \frac{(e_t^{(-1/2)})^2}{2\gamma_\mu}\Big)\Big] \\
&\quad = E^{Q_t^{\omega,z}}\Big[\exp\Big(-\Big(\frac{\lambda^2}{2} z + \lambda\Big) e_t^{(-1/2)}\Big)\Big] \\
&\quad = e^{-\lambda}.
\end{aligned}$$

Moreover, setting $\lambda + \lambda^2 z/2 = \alpha$, we get

$$E^{Q_t^{\omega,z}}\big[\exp(-\alpha e_t^{(-1/2)})\big] = \exp\bigg(-\sqrt{\frac{2\alpha}{z}+\frac{1}{z^2}}+\frac{1}{z}\bigg).$$

Therefore we have proved that the distribution of $e_t^{(-1/2)}$ under $Q_t^{\omega,z}$ is given by $\rho_z^{(-1/2)}(u)\,du$, that is,

$$P\big(e_t^{(-1/2)} \in du | \mathscr{Z}_t^{(-1/2)}, Z_t^{(-1/2)} = z\big) = \rho_z^{(-1/2)}(u)\,du.$$

Furthermore, by the integral representation of the Macdonald function (cf. Lebedev [13], p. 119)

$$K_\nu(y) = \frac{1}{2}\Big(\frac{y}{2}\Big)^\nu \int_0^\infty e^{-t-(y^2/4t)} t^{-\nu-1}\,dt, \tag{10.6}$$

a straightforward calculation shows

$$E\big[\big(e_t^{(-1/2)}\big)^{-\mu+1/2} | \mathscr{Z}_t^{(-1/2)}, Z_t^{(-1/2)} = z\big] = \frac{1}{\sqrt{2\pi z}} K_\mu(1/z) e^{1/z}.$$

Finally, by using (10.5), we obtain

$$\begin{aligned} &P\big(e_t^{(-\mu)} \in du | \mathscr{Z}_t^{(-\mu)}, Z_t^{(-\mu)} = z\big) \\ &\quad = \frac{\sqrt{2\pi z}}{2K_\mu(1/z)e^{1/z}} u^{-\mu+1/2} \rho_z^{(-1/2)}(u)\,du = \rho_z^{(-\mu)}(u)\,du. \end{aligned} \tag{10.7}$$

(8.2) is now easily verified from (10.7) by a change of variable and the proof of Proposition 1.7 is completed. □

Second proof of Theorem 1.6. By Itô's formula, we deduce from the definition of $Z^{(-\mu)}$

$$Z_t^{(-\mu)} = -\int_0^t Z_s^{(-\mu)}\,dB_s + \Big(\frac{1}{2}+\mu\Big)\int_0^t Z_s^{(-\mu)}\,ds + \int_0^t \exp\big(B_s^{(-\mu)}\big)\,ds. \tag{10.8}$$

We consider

$$E\big[\exp(B_t^{(-\mu)}) \mid \mathscr{Z}_t^{(-\mu)}\big] \equiv E\big[e_t^{(-\mu)} \mid \mathscr{Z}_t^{(-\mu)}\big]$$

and use (10.7). Then, by the integral representation (10.6) of the Macdonald function and the identity $K_\nu = K_{-\nu}$, we can easily show

$$E\big[e_t^{(-\mu)} \mid \mathscr{Z}_t^{(-\mu)}, Z_t^{(-\mu)} = z\big] = \Big(\frac{K_{1-\mu}}{K_{-\mu}}\Big)\Big(\frac{1}{z}\Big) \equiv \Big(\frac{K_{1-\mu}}{K_\mu}\Big)\Big(\frac{1}{z}\Big).$$

Therefore, "projecting" the right hand side of (10.8) on the filtration $\mathscr{Z}_t^{(-\mu)}$ (cf. [16], Chapter 7), we obtain the existence of a $(\mathscr{Z}_t^{(-\mu)})$-Brownian motion $\{\beta_t,\, t \geqq 0\}$ such that

$$Z_t^{(-\mu)} = \int_0^t Z_s^{(-\mu)}\, d\beta_s + \Big(\frac{1}{2} + \mu\Big) \int_0^t Z_s^{(-\mu)}\, ds + \int_0^t \Big(\frac{K_{1-\mu}}{K_\mu}\Big)\Big(\frac{1}{Z_s^{(-\mu)}}\Big)\, ds$$

and the proof of Theorem 1.6 is completed. $\square$

§11. More Intertwinings

As discussed in Alili–Dufresne–Yor [1], the stochastic process $\hat{\xi}^{(\mu)} = \big\{\hat{\xi}_t^{(\mu)},\, t \geqq 0\big\}$ defined by

$$\hat{\xi}_t^{(\mu)} = \exp\big(-B_t^{(\mu)}\big)\Big\{\hat{\xi}_0 + \int_0^t \exp\big(B_s^{(\mu)}\big)\, d\gamma_s\Big\}, \quad t \geqq 0,$$

is an $\mathbf{R}$-valued diffusion, where $\{\gamma_s,\, s \geqq 0\}$ is a standard Brownian motion independent of $B^{(\mu)}$. In particular, if $\hat{\xi}_0 = 0$, $\big\{\hat{\xi}_t^{(0)},\, t \geqq 0\big\}$ is distributed as $\{\sinh(B_t),\, t \geqq 0\}$ and, from this, simple time reversal considerations lead to Bougerol's identity

$$\sinh(B_t) \overset{\text{(law)}}{=} \gamma_{A_t} \quad \text{for any fixed } t.$$

We now prove the following.

THEOREM 11.1. *The diffusions $\hat{\xi}^{(\mu)}$ and $Z^{(\mu)}$ are intertwined as follows: letting $\big\{\widehat{P}_t^{(\mu)}\big\}$ be the semigroup of $\hat{\xi}^{(\mu)}$ and $\widehat{\mathbb{K}}^{(\mu)}$ be the Markov kernel given by*

$$\widehat{\mathbb{K}}^{(\mu)}\varphi(z) = E\big[\varphi\big(H_{1,1/z}^{(-\mu)}\big)\big]$$

for a hyperbolic random variable $H_{1,1/z}^{(-\mu)}$ defined by (9.6), *it holds that*

$$Q_t^{(\mu)}\widehat{\mathbb{K}}^{(\mu)} = \widehat{\mathbb{K}}^{(\mu)}\widehat{P}_t^{(\mu)}. \tag{11.1}$$

Proof. It is quite similar to the proof of the intertwining relationship between $\xi^{(\mu)}$ and $Z^{(\mu)}$ which was presented in Theorem 1.7. Indeed we need only to consider

$$E\big[\varphi\big(\hat{\xi}_t^{(\mu)}\big)|\mathscr{Z}_t^{(\mu)}, Z_t^{(\mu)} = z\big] = E\Big[\varphi\Big(N\sqrt{\xi_t^{(\mu)}}\Big)|\mathscr{Z}_t^{(\mu)}, Z_t^{(\mu)} = z\Big] \tag{11.2}$$

$$= E\Big[\varphi\Big(N\sqrt{zI_{1/\sqrt{z},1/\sqrt{z}}^{(-\mu)}}\Big)|Z_t^{(\mu)} = z\Big] \tag{11.3}$$

$$= E\big[\varphi\big(H_{1,1/z}^{(-\mu)}\big)\big]. \tag{11.4}$$

The first equality (11.2) comes from the independence of γ and $B^{(\mu)}$, (11.3) follows from Theorem 1.7 and (9.3) and, finally, (11.4) is a consequence of the definition of $H^{(-\mu)}$. ∎

We also present another intertwining result, which is even simpler to notice. However it would be a pity to forget this relationship, since it relates $B^{(\mu)}$ and $Z^{(\mu)}$ via intertwinings.

THEOREM 11.2. *The geometric Brownian motion* $G^{(\mu)} = \{G_t^{(\mu)} = \exp(B_t^{(\mu)}),\, t \geqq 0\}$, *and the diffusion process* $Z^{(\mu)} = \{Z_t^{(\mu)},\, t \geqq 0\}$ *are intertwined as follows*:

$$Q_t^{(\mu)} \widetilde{\mathbb{K}}^{(\mu)} = \widetilde{\mathbb{K}}^{(\mu)} \widetilde{P}_t^{(\mu)}, \tag{11.5}$$

where $\{\widetilde{P}_t^{(\mu)}\}$ *is the semigroup of* $G^{(\mu)}$ *and the Markov kernel* $\widetilde{\mathbb{K}}^{(\mu)}$ *is given by*

$$\widetilde{\mathbb{K}}^{(\mu)}\varphi(z) = E\big[\varphi\big(I^{(\mu)}_{1/\sqrt{z},1/\sqrt{z}}\big)\big].$$

Proof. This presents no difficulty, since we have shown in Proposition 1.6 that

$$E\big[\varphi\big(\exp\big(B_t^{(\mu)}\big)\big)|\mathscr{Z}_t^{(\mu)}, Z_t^{(\mu)} = z\big] = E\big[\varphi\big(I^{(\mu)}_{1/\sqrt{z},1/\sqrt{z}}\big)\big].$$

∎

Remark 11.1. Of course, instead of the intertwining relation (11.5) between $G^{(\mu)}$ and $Z^{(\mu)}$, we can equivalently state an intertwining relation between $B^{(\mu)}$ and $Z^{(\mu)}$ which can be deduced from (11.5).

Remark 11.2. In the case where $\mu = 0$, it seems most interesting to compare the results from Theorems 11.1 and 11.2 because, as we already remarked, $\hat{\xi}^{(0)}$ is nothing else but $\{\sinh(B_t),\, t \geqq 0\}$. Thus, is it true that Theorem 11.2 is equivalent to Theorem 11.1 when $\mu = 0$?

§12. The Process $Z^{(\mu)}$ under a Brownian Bridge

In this section we present a formula which expresses the conditional law of $\{Z_s^{(\mu)},\, 0 \leqq s \leqq t\}$ given $B_t^{(\mu)} = x$ in terms of the law of $\{Z_s^{(\mu)},\, 0 \leqq s \leqq t\}$. Precisely we show the following.

PROPOSITION 12.1. *For any* $\mathbf{R}$*-valued bounded measurable functional* F_t *on* $C([0,t];\mathbf{R}_+)$*, one has*

$$\begin{aligned}(12.1)\qquad &\frac{1}{\sqrt{2\pi t}}\exp\Big(-\frac{1}{2}\Big(\frac{x^2}{t}+\mu^2 t\Big)\Big)E[F_t(Z)|B_t=x]\\ &\quad = E\bigg[F_t\big(Z^{(\mu)}\big)\frac{1}{2K_\mu\big(1/Z_t^{(\mu)}\big)}\exp\bigg(-\frac{\cosh(x)}{Z_t^{(\mu)}}\bigg)\bigg].\end{aligned}$$

Remark 12.1. Concerning the left hand side of (12.1), we note that

$$E\big[F_t\big(Z^{(\mu)}\big)|B_t^{(\mu)}=x\big]=E\big[F_t(Z)|B_t=x\big],$$

since the bridge of $\{B_u^{(\mu)},\, 0\leqq u\leqq t\}$ given $B_t^{(\mu)}=x$ does not depend on μ.

By virtue of Proposition 1.7, Proposition 12.1 is a particular case of the following more general:

PROPOSITION 12.2. *Let* $\widetilde{Z}=\{\widetilde{Z}_t,\, t\geqq 0\}$ *and* $\widetilde{B}=\{\widetilde{B}_t,\, t\geqq 0\}$ *be two stochastic processes taking values in* $\mathbf{R}$ *which are strongly intertwined with the kernel*

$$\Lambda : f\longmapsto \Lambda f(z)=\int_{\mathbf{R}}\lambda(z,x)f(x)\,dx\,;$$

more precisely, the process $\widetilde{Z}$ *is adapted with respect to* $\widetilde{B}$*, i.e.,* $\widetilde{\mathscr{Z}}_t\subseteq\widetilde{\mathscr{B}}_t$ *and, moreover,*

$$E[f(\widetilde{B}_t)\mid \widetilde{\mathscr{Z}}_t]=\Lambda f(\widetilde{Z}_t)$$

for every $t>0$ *and every non-negative Borel function* f*. Then it holds that*

$$\tilde{p}_t(x)E\big[F_t(\widetilde{Z})|\widetilde{B}_t=x\big]=E\big[F_t(\widetilde{Z})\lambda(\widetilde{Z}_t,x)\big]$$

for dx *almost all* x *and for every bounded functional* $F_t(\widetilde{Z})=F\big(\{\widetilde{Z}_s,\, 0\leqq s\leqq t\}\big)$*, where* $\tilde{p}_t(x)$ *is the probability density for* $\widetilde{B}_t$ *with respect to the Lebesgue measure* dx*.*

Proof. This is a simple consequence of the fact that $E\big[F_t(\widetilde{Z})f(\widetilde{B}_t)\big]$ can be written in two forms:

$$E\big[F_t(\widetilde{Z})f(\widetilde{B}_t)\big]=\int_{\mathbf{R}}\tilde{p}_t(x)f(x)E\big[F_t(\widetilde{Z})|\widetilde{B}_t=x\big]\,dx$$

and

$$E\big[F_t(\widetilde{Z})f(\widetilde{B}_t)\big] = E\big[F_t(\widetilde{Z})\Lambda f(\widetilde{Z}_t)\big] = \int_{\mathbf{R}} E\big[F_t(\widetilde{Z})\lambda(\widetilde{Z}_t,x)\big]f(x)\,dx.$$

∎

An application of Proposition 12.2 to the pair $\big(\widetilde{B}_t, \widetilde{Z}_t = 2\widetilde{M_t} - \widetilde{B}_t\big)$, where $\widetilde{B}_t$ is a Brownian motion and $\widetilde{M_t} = \sup_{s\leqq t} \widetilde{B}_s$, yields the following well known result.

Corollary 12.3. ([6], [7]) *Let $\{b_u,\, 0 \leqq u \leqq t\}$ be the Brownian bridge of length t such that $b_0 = b_t = 0$. Denote by $\{\ell_u,\, 0 \leqq u \leqq t\}$ its local time at* 0 *and set $\sigma_u = \sup_{s\leqq u} b_s$. Then both processes $\{\sigma_u - b_u,\, 0 \leqq u \leqq t\}$ and $\{\ell_u + |b_u|,\, 0 \leqq u \leqq t\}$ are distributed as the Brownian meander of length t.*

For a more general statement involving Brownian bridge ending at $x \neq 0$, see, e.g., Revuz–Yor [20], Chapter XII, Exercise (4.25), p. 510.

Although Proposition 12.1 may be considered as some adequate substitute concerning the process $Z^{(\mu)}$ under Brownian bridges, we miss an interpretation of the "right hand side" of the formula (12.1) as the law of an "adequate" meander, that is, we miss Imhof's result in the $BES(3)$–Brownian motion case. Again see, e.g., Biane–Yor [7], Revuz–Yor [20] and the original papers by Imhof [11], [12].

Nonetheless, we derive the following formulae from Proposition 12.1.

Corollary 12.4. (i) *The law of $Z_t^{(\mu)}$ may be characterized as follows*:

$$E\bigg[\frac{1}{2K_\mu(1/Z_t^{(\mu)})}\exp\Big(-\frac{\cosh(x)}{Z_t^{(\mu)}}\Big)\bigg] = \frac{1}{\sqrt{2\pi t}}\exp\Big(-\frac{1}{2}\Big(\frac{x^2}{t}+\mu^2 t\Big)\Big).$$

(ii) *It holds that*

$$\int_0^\infty \theta_u(t)\exp(-u\cosh(x))\frac{du}{u} = \frac{1}{\sqrt{2\pi t}}\exp(-x^2/2t), \tag{12.2}$$

where $\theta_r(t)$ has been introduced in formula (4.8) *in Part I* [19].

Proof. (i) follows immediately from formula (12.1) in which we take $F_t \equiv 1$. (ii) follows from (i) in the case $\mu = 0$ and the formula which gives the probability density of $Z_t^{(\mu)}$ in terms of $\theta_{1/z}(t)$. ∎

Remark 12.2. Again, formula (12.2) provides a nice check on our results; indeed, taking the Laplace transform (in $\lambda^2/2$) with respect to t on both hand sides of (12.2), we obtain

$$\int_0^\infty I_\lambda(u)\exp(-u\cosh(x))\frac{du}{u} = \frac{1}{\lambda}e^{-\lambda x} \quad (\lambda > 0). \tag{12.3}$$

Now this formula is a well-known particular case of the more general Lipschitz-Hankel formula (see, e.g., Watson [25], p. 386, formula (7), and also the formulae (2.f) and (2.7') in Yor [27].)

§13. Some Complements to the GIG relations and a Parallel with the Rogers-Pitman Study

Our main result in this Part II, the Markov property of $Z^{(\mu)}$, has been obtained as a consequence of two essential ingredients.

(*i*) Generalized Dufresne's identity in law

We know

$$\left\{\frac{1}{A_t^{(-\mu)}},\ t \geqq 0\right\} \overset{\text{(law)}}{=} \left\{\frac{1}{A_t^{(\mu)}} + \frac{1}{\widetilde{A}_\infty^{(-\mu)}},\ t \geqq 0\right\} \tag{13.1}$$

for $\mu > 0$, which was recalled in Theorem 8.1. Dufresne [9] has shown this identity for fixed time, hence we used the above terminology. In fact, for our purpose in this section, it is more convenient to rewrite (13.1) in the following equivalent form:

$$\frac{1}{A_\infty^{(-\mu)}} \text{ is independent of } \left\{\frac{1}{A_t^{(-\mu)}} - \frac{1}{A_\infty^{(-\mu)}},\ t \geqq 0\right\} \text{ and} \tag{13.2}$$
$$\text{the latter process is distributed as } \left\{\frac{1}{A_t^{(\mu)}},\ t \geqq 0\right\}.$$

Moreover $1/A_\infty^{(-\mu)}$ is distributed as $2\gamma_\mu$, where γ_μ is a Gamma(μ) random variable.

(*ii*) Quadratic Random Equation

Our second essential ingredient has been to establish the quadratic equation:

$$AX^2 + zX \overset{\text{(law)}}{=} A, \tag{13.3}$$

where $A \stackrel{\text{(law)}}{=} 1/2\gamma_\mu$, $X = e_t^{(-\mu)}$ is independent of A and the left hand side of (13.3) is considered under $Q_t^{\omega,z}$. We showed that, for $\mu = 1/2$, there is only one "distribution solution" X to (13.3). Although we can conclude from there in our framework (cf. Proposition 1.7), it would be interesting to know whether (13.3) admits only one "distribution solution" for any μ.

As we were trying to solve this problem, our attention was drawn to Theorem 3.10, page 105 of Seshadri [22], where the following is shown (see also Letac–Seshadri [14]). Let A be as above and consider the equation

$$\frac{1}{X} \stackrel{\text{(law)}}{=} X + \frac{z}{A}, \tag{13.4}$$

where the two random variables on the right hand side are assumed to be independent. Then (13.4) admits only one solution given by

$$X \stackrel{\text{(law)}}{=} I^{(-\mu)}_{1/\sqrt{z},1/\sqrt{z}}.$$

Firstly we note that (13.4) is nothing else but the equation version of (9.5). Secondly we prove that $X = I^{(-\mu)}_{1/\sqrt{z},1/\sqrt{z}}$ satisfies in fact a two-dimensional identity in law which unifies (13.3) and (13.4).

Proposition 13.1. *The random variable* $X = I^{(-\mu)}_{1/\sqrt{z},1/\sqrt{z}}$ *satisfies*

$$\Big(AX, X + \frac{z}{A}\Big) \stackrel{\text{(law)}}{=} \Big(AX, \frac{1}{X}\Big), \tag{13.5}$$

where $A \stackrel{\text{(law)}}{=} 1/2\gamma_\mu$ *and is independent of* X. *As a consequence, both* (13.3) *and* (13.4) *hold.*

Proof. Using the explicit form (9.1) of the density of X, it is easy to show that

$$E\big[X^\mu e^{-uX} g((1+2zu)X)\big] = E\big[X^\mu e^{-uX} g(X^{-1})\big] \tag{13.6}$$

holds for every bounded Borel function g on $\mathbf{R}_+$ and every $u > 0$. Then, it is also easy to show that (13.6) implies

$$\Big(\frac{\gamma_\mu}{X}, X + 2z\gamma_\mu\Big) \stackrel{\text{(law)}}{=} \Big(\frac{\gamma_\mu}{X}, \frac{1}{X}\Big),$$

which is an equivalent assertion to (13.5). ∎

Remark 13.1. (1) Letac–Seshadri [14] reinforce (13.4) in the form of a continued fraction representation

$$X \overset{\text{(law)}}{=} \frac{1}{\Gamma_1+}\frac{1}{\Gamma_2+}\frac{1}{\Gamma_3+}$$

where $\{\Gamma_i\}_{i=1}^{\infty}$ is an i.i.d. sequence distributed as $2z\gamma_\mu$ and the notation $\frac{1}{a+}b$ means $\frac{1}{a+b}$.

(2) In the opposite direction, we can reinforce the quadratic equation (13.3) as follows. Set $\Gamma = 2z\gamma_\mu$. Then (13.3) is rewritten as

$$\frac{1}{\Gamma} \overset{\text{(law)}}{=} X + \frac{X^2}{\Gamma}.$$

From this we can deduce

$$\frac{1}{\Gamma} \overset{\text{(law)}}{=} X_1 + X_1^2X_2 + X_1^2X_2^2X_3 + \cdots + X_1^2X_2^2\cdots X_n^2X_{n+1} + \cdots, \tag{13.7}$$

where $\{X_i\}_{i=1}^{\infty}$ is an i.i.d. sequence distributed as $e_t^{(-\mu)}$ under $Q_t^{\omega,z}$, i.e., as $I^{(-\mu)}_{1/\sqrt{z},1/\sqrt{z}}$, and the right hand side of (13.7) converges almost surely. In order to see this convergence, it suffices to note

$$E[(X_i)^\alpha] = \frac{K_{\mu-\alpha}(1/z)}{K_\mu(1/z)} < 1 \quad \text{if } 0 < \alpha < 2\mu.$$

(3) Letac–Wesolowski [15] have recently shown that, if $X = I_{a,b}^{(\mu)}$ and Y is a Gamma random variable whose distribution is given by

$$P(Y \in dy) = \gamma_{p,2a^{-1}}(dy) = \frac{1}{\Gamma(p)}\Big(\frac{a}{2}\Big)^p y^{p-1}\exp\Big(-\frac{1}{2}ay\Big)\,dy, \quad y > 0,$$

and if X and Y are independent, then $(X+Y)^{-1}$ and $X^{-1} - (X+Y)^{-1}$ are independent and distributed as $I_{a,b}^{(-\mu)}$ and $\gamma_{p,2b^{-1}}$, respectively. This result is an extension of (13.5), which also presents a converse to this independence property.

The following complements Proposition 13.1.

PROPOSITION 13.2. *The joint law of* (AX, X) *is given by*

$$\begin{aligned} &P(AX \in du,\, X \in dx) \\ &\quad = \frac{1}{\Gamma(\mu)K_\mu(1/z)(2u)^{\mu+1}x}\exp\Big[-\frac{1}{2}\Big(\Big(\frac{1}{u}+\frac{1}{z}\Big)x + \frac{1}{zx}\Big)\Big]\,dudx. \end{aligned} \tag{13.8}$$

Consequently, conditioning on $AX = u$, one has

$$\left(1+\frac{z}{u}\right)X \overset{\text{(law)}}{=} X, \tag{13.9}$$

$$\begin{aligned} &P(X \in dx | AX = u) \\ &\qquad = \frac{1}{C_z(u)x} \exp\left[-\frac{1}{2}\left(\left(\frac{1}{u}+\frac{1}{z}\right)x + \frac{1}{zx}\right)\right], \\ &C_z(u) = 2K_0\left(\sqrt{z^{-1}(u^{-1}+z^{-1})}\right), \end{aligned} \tag{13.10}$$

$$P(AX \in du) = \frac{C_z(u)}{\Gamma(\mu)(2u)^{\mu+1}K_\mu(1/z)}\, du. \tag{13.11}$$

Proof. (13.8) is elementary. (13.9) follows immediately from (13.5). (13.10) and (13.11) are also shown easily if we use the integral representation

$$K_\lambda(\sqrt{ab}) = \frac{1}{2}\left(\frac{a}{b}\right)^{\lambda/2} \int_0^\infty x^{\lambda-1} \exp\left(-\frac{1}{2}(ax+b/x)\right) dx, \quad a,b>0.$$

∎

Remark 13.2. It should be noted that the conditional law in (13.10) does not depend on μ; in fact, this law is $GIG(0;a,b)$, where $a^2 = z^{-1}$, $b^2 = u^{-1}+z^{-1}$. Moreover the right hand side of (13.11) is quite reminiscent of the probability law (9.7) of $H_{a,b}^{(\mu)}$.

We now discuss how (i) and (ii) are seen in the limit regime already discussed after Theorem 1.1_μ in order to pass from $Z^{(\mu)}$ to $2M^{(\mu)} - B^{(\mu)}$, and in fact in most part of Section 1 in our part I paper.

$(i)_\infty$ Williams' Theorem

In (13.2) above, we replace μ and the time t by μc and t/c^2 $(c>0)$, respectively, take logarithms and let c tend to 0. Then, setting

$$A^{(\mu)}(s,u;c) = \int_s^u \exp\left(\frac{2}{c}B_\tau^{(\mu)}\right) d\tau,$$

we find

$$\left\{c \log \frac{A^{(-\mu)}(t,\infty;c)}{A^{(-\mu)}(0,t;c)A^{(-\mu)}(0,\infty;c)}\right\}_{t\geqq 0} \overset{\text{(law)}}{=} \left\{-c\log A^{(\mu)}(0,t;c)\right\}_{t\geqq 0}.$$

Now, using an elementary Laplace method argument, we obtain the following analogue of (13.2).

THEOREM 13.3. *Assume $\mu > 0$. Then $\sup_{u \geqq 0} B_u^{(-\mu)}$ obeys the exponential distribution with parameter 2μ and is independent of*

$$\Sigma_t^{(-\mu)} = \begin{cases} \sup_{0 \leqq u \leqq t} B_u^{(-\mu)}, & t \leqq \rho, \\ 2 \sup_{u \geqq 0} B_u^{(-\mu)} - \sup_{u \geqq t} B_u^{(-\mu)}, & t > \rho, \end{cases} \tag{13.12}$$

where $\rho = \inf\{t \,;\, B_t^{(-\mu)} = \sup_{u \geqq 0} B_u^{(-\mu)}\}$. Moreover it holds that

$$\{\Sigma_t^{(-\mu)},\, t \geqq 0\} \overset{\text{(law)}}{=} \{\sup_{0 \leqq u \leqq t} B_u^{(\mu)},\, t \geqq 0\}.$$

In fact, in the limiting procedure described above, $\{\Sigma_t^{(-\mu)},\, t \geqq 0\}$ arises from

$$\sup_{0 \leqq u \leqq t} B_u^{(-\mu)} + \sup_{u \geqq 0} B_u^{(-\mu)} - \sup_{u \geqq t} B_u^{(-\mu)}, \quad t > 0,$$

which is easily shown to coincide with the right hand side of (13.12).

The above theorem is a part of a theorem due to Williams [26] and has been discussed by Rogers–Pitman [21] in their Corollary 2.

$(ii)_\infty$ "Quadratic" Random Equations in the Rogers–Pitman Framework

We now describe an analogue of the equation (13.3) in the Rogers–Pitman framework [21] for the process $Y_t^{(-\mu)} \equiv 2M_t^{(-\mu)} - B_t^{(-\mu)}$, where $M_t^{(-\mu)} = \sup_{s \leqq t} B_s^{(-\mu)}$.

We begin with the decomposition

$$M_\infty^{(-\mu)} = M_t^{(-\mu)} \vee \left(\sup_{u \geqq t} B_u^{(-\mu)}\right) \tag{13.13}$$

under $Q_t^{\omega,y}$, the conditional law given $\mathscr{Y}_t^{(-\mu)} = \sigma\{Y_s^{(-\mu)} \,;\, s \leqq t\}$. Recall (cf. [21] and Theorem 13.3 above) that $M_\infty^{(-\mu)} \overset{\text{(law)}}{=} \mathbf{e}/2\mu$, where $\mathbf{e}$ is a standard exponential random variable, and that $M_\infty^{(-\mu)}$ is independent of $\mathscr{Y}_\infty^{(-\mu)}$. Then, denoting $X = B_t^{(-\mu)}$, we can deduce easily from (13.13) that the analogue of (13.3) is

$$\frac{\mathbf{e}}{\mu} \overset{\text{(law)}}{=} (X + y) \vee \left(\frac{\mathbf{e}}{\mu} + 2X\right).$$

Taking exponentials on both hand sides, we obtain

$$A \overset{\text{(law)}}{=} \left(z\widehat{X}\right) \vee \left(A\widehat{X}^2\right), \tag{13.14}$$

where $A \stackrel{\text{(law)}}{=} \exp(\mathbf{e}/\mu)$. (13.14) explains the term "quadratic" in the above subtitle. It follows from the Rogers-Pitman results that the following $X_{\mu,y}$ is a distribution solution to (13.13):

$$P(X_{\mu,y} \in dx) = \frac{\mu e^{\mu x}}{2\sinh \mu y} I_{(-y,y)}(x)\, dx.$$

As in (*ii*) above, it would be interesting to prove that (13.13) admits only this distribution solution.

§14. The Forward Processes ${}^{+}\boldsymbol{Z}^{(-\mu)}$ and ${}^{+}\boldsymbol{\xi}^{(-\mu)}$

Following Proposition 13.1, we got interested in the realization, in terms of $B^{(-\mu)}$, of the random variable AX, which clearly corresponds to

$${}^{+}Z_t^{(-\mu)} \equiv \exp\bigl(-B_t^{(-\mu)}\bigr) A_{(t,\infty)}^{(-\mu)}, \quad A_{(t,\infty)}^{(-\mu)} = \int_t^\infty \exp\bigl(2B_s^{(-\mu)}\bigr)\, ds$$

under $Q_t^{\omega,z}$. We call these stochastic processes "forward" and, obviously, it is also natural to define

$${}^{+}\xi_t^{(-\mu)} \equiv \exp\bigl(-2B_t^{(-\mu)}\bigr) A_{(t,\infty)}^{(-\mu)}.$$

To insist upon the "forward" and "backward" characters, we denote $Z^{(-\mu)}$ and $\xi^{(-\mu)}$ by ${}^{-}Z^{(-\mu)}$ and ${}^{-}\xi^{(-\mu)}$, respectively, in this section.

Then we show the following.

THEOREM 14.1. *For $\mu > 0$, the processes ${}^{-}Z^{(-\mu)}$ and ${}^{+}Z^{(-\mu)}$ satisfy the following identity*:

$$\begin{aligned}(14.1) \qquad & \bigl\{\bigl({}^{-}Z_t^{(-\mu)}, {}^{+}Z_t^{(-\mu)}, A_\infty^{(-\mu)}\bigr),\ t \geqq 0\bigr\} \\ & \stackrel{\text{(law)}}{=} \bigl\{\bigl({}^{-}Z_t^{(\mu)}, \exp\bigl(-B_t^{(\mu)}\bigr)\widetilde{A}_\infty^{(-\mu)}, \widetilde{A}_\infty^{(-\mu)}\bigr),\ t \geqq 0\bigr\},\end{aligned}$$

where $\widetilde{A}_\infty^{(-\mu)}$ is a copy of $A_\infty^{(-\mu)}$, independent of $\{B_t^{(\mu)}\}$.

Proof. This is easily obtained from algebraic manipulations of the generalized Dufresne identity (13.1) and (13.2). ∎

We should note that (14.1) gives a nice probabilistic explanation of the identity (13.5), since we find the pair of *independent* random variables $\bigl({}^{-}Z_t^{(\mu)}, \exp\bigl(-B_t^{(\mu)}\bigr)\bigr)$ and $\widetilde{A}_\infty^{(-\mu)}$ on the right hand side of (14.1).

As a companion to Theorem 14.1, we have the following.

Theorem 14.2. *Let $\mu > 0$. Then the processes* $\{{}^+\xi_t^{(-\mu)}, t \geqq 0\}$ *and* $\{{}^-\xi_t^{(\mu)}, t \geqq 0\}$ *satisfy the following relation*:

$$\{{}^+\xi_t^{(-\mu)}, t \geqq 0\} \stackrel{\text{(law)}}{=} \{{}^-\xi_t^{(\mu)} + \exp(-2B_t^{(\mu)})\widetilde{A}_\infty^{(-\mu)}, t \geqq 0\}. \tag{14.2}$$

Proof. Although (14.2) also follows by elementary algebraic manipulations starting from (14.1), we give a few details. Adding the first two components on both hand sides of (14.1), we obtain

$$\begin{aligned}&\{\exp(-B_t^{(-\mu)})A_\infty^{(-\mu)}, t \geqq 0\} \\ &\quad\stackrel{\text{(law)}}{=} \{\exp(-B_t^{(\mu)})(A_t^{(\mu)} + \widetilde{A}_\infty^{(-\mu)}), t \geqq 0\}.\end{aligned} \tag{14.3}$$

Of course, this holds jointly with (14.1).

Now using (14.1) and (14.3) in conjunction, we obtain that $\{{}^+\xi_t^{(-\mu)} = {}^+Z_t^{(-\mu)} \exp(-B_t^{(-\mu)}), t \geqq 0\}$ is distributed as the right hand side of (14.2). □

§15. Extension to Matrices

After the interpretation of Rogers-Pitman result and ours via quadratic random equations given in Section 13, it does not seem unreasonable to expect some multi-dimensional extensions (in a different direction from that suggested by Question 1 given in Part I [19], Section 7), which might be developed from the identity:

$$\mu_{\lambda,M,N} = \mu_{-\lambda,M,N} * W_d(2\lambda, M^{-1}). \tag{15.1}$$

(15.1) is a convolution equation on $\mathscr{P}_d$, the set of real symmetric $d \times d$ positive definite matrices ($M, N \in \mathscr{P}_d$), where $W_d(n, \Sigma)$ denotes the (two parameter) family of Wishart laws and $\mu_{-\lambda,M,N}$ denotes the generalized inverse Gaussian distribution on $\mathscr{P}_d$. For details, see Seshadri [22], Exercise 3.11, page 116, Bernadac [5], Terras [23] formula (2.56), page 83, but the reader should beware of the many typos in these references!

(15.1) is an extension of (13.4). Moreover Letac–Wesolowski ([15], Theorem 3.1) have shown that, if $M = N$, then

$$(X, Y) \stackrel{\text{(law)}}{=} ((X+Y)^{-1}, X^{-1} - (X+Y)^{-1})$$

holds for a generalized inverse Gaussian X and a Wishart random variable Y under the assumption that they are independent. From this result, it is easy to deduce

$$XY^{-1}X + X \stackrel{\text{(law)}}{=} Y^{-1},$$

which is a quadratic equation which extends (13.3).

As a temporary conclusion, we cannot resist quoting the following sentence taken from Terras ([23], page 50), which is a tantalizing invitation to relate the present Markov process intertwinings with some classical group representations: "It is also useful to view Bessel and Whittaker functions in the light of the theory of the operators intertwining pairs of representations."

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