T. Mabuchi
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VECTOR FIELD ENERGIES AND CRITICAL METRICS ON KÄHLER MANIFOLDS

TOSHIKI MABUCHI

Abstract. Associated with a Hamiltonian holomorphic vector field on a compact Kähler manifold, a nice functional on a space of Kähler metrics will be constructed as an integration of the bilinear pairing in [FM] contracted with the Hamiltonian holomorphic vector field. As applications, we have functionals $\hat{\mu}$, $\hat{\nu}$ whose critical points are extremal Kähler metrics or "Kähler-Einstein metrics" in the sense of [M4], respectively. Finally, the same method as used by [G1] allows us to obtain, from the convexity of $\hat{\nu}$, the uniqueness of "Kähler-Einstein metrics" on nonsingular toric Fano varieties possibly with nonvanishing Futaki character.

§1. Introduction

The purpose of this paper is to define, with applications to the study of critical metrics, some functional associated with a Hamiltonian holomorphic vector field (see the key observation stated below). Throughout this paper, we fix once and for all an n-dimensional compact complex connected manifold M with a Kähler class $\kappa \in H^{1,1}(M, \mathbb{R})$. The Albanese map of M to the Albanese variety $\mathrm{Alb}(M)$ induces a complex Lie group homomorphism

$$a_M : \mathrm{Aut}^0(M) \longrightarrow \mathrm{Aut}^0(\mathrm{Alb}(M))(\cong \mathrm{Alb}(M))$$

between the identity components of the groups of holomorphic automorphisms of M and $\mathrm{Alb}(M)$. Then the identity component $G := \mathrm{Ker}^0\, a_M$ of the kernel of a_M is a linear algebraic group (see [Fj]). Let $\mathcal{K}$ be the set of all Kähler metrics on M in the Kähler class κ, where a Kähler metric and the associated Kähler form are used interchangeably. For each $\omega \in \mathcal{K}$, we write ω as

$$\omega = \sqrt{-1}\, \sum_{\alpha,\beta} g_{\alpha\bar{\beta}} dz^{\alpha} \wedge dz^{\bar{\beta}}$$

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in terms of a system $(z^1, z^2, \dots, z^n)$ of holomorphic local coordinates on M. Put $A_\kappa := \int_M \omega^n = \kappa^n[M]$. To each complex-valued smooth function φ on X, we associate a complex vector field $\operatorname{grad}^{\mathbb{C}}_\omega \varphi$ on M of type $(1,0)$ by

$$\operatorname{grad}^{\mathbb{C}}_\omega \varphi := \frac{1}{\sqrt{-1}} \sum_{\alpha,\beta} g^{\bar\beta\alpha} \frac{\partial\varphi}{\partial z^{\bar\beta}} \frac{\partial}{\partial z^\alpha}.$$

Consider the complex Lie subalgebra $\mathfrak{g}$ of $H^0(M, \mathcal{O}(TM))$ corresponding to the complex Lie subgroup G of $\operatorname{Aut}^0(M)$. Let $\tilde{\mathfrak{g}}_\omega$ be the space of all complex smooth functions $\varphi \in C^\infty(M)_{\mathbb{C}}$ on M such that $\operatorname{grad}^{\mathbb{C}}_\omega \varphi$ is a holomorphic vector field on M and that $\int_M \varphi\omega^n/A_\kappa = 0$. Then we have an isomorphism of complex Lie algebras

$$\iota_\omega : \tilde{\mathfrak{g}}_\omega \cong \mathfrak{g}, \qquad \varphi \longleftrightarrow \iota_\omega(\varphi) := \operatorname{grad}^{\mathbb{C}}_\omega \varphi,$$

where $\tilde{\mathfrak{g}}_\omega$ has a natural structure of a complex Lie algebra in terms of the Poisson bracket by ω. Put $\tilde{\mathfrak{k}}_\omega := \{\varphi \in \tilde{\mathfrak{g}}_\omega; \varphi \text{ is real-valued on } M\}$ and $\mathfrak{k}_\omega := \iota_\omega(\tilde{\mathfrak{k}}_\omega)$. Then the real Lie subgroup K_ω of G generated by the Lie subalgebra $\mathfrak{k}_\omega$ of $\mathfrak{g}$ is nothing but the identity component of the group of the isometries in G of the compact Kähler manifold (M, ω). Put $\mathcal{K}_V := \{\omega \in \mathcal{K}; V \in \mathfrak{k}_\omega\}$, $V \in \mathfrak{g}$. Fix an element ω in $\mathcal{K}_V$ by assuming $\mathcal{K}_V \neq \emptyset$. Put

$$\omega_\psi \ := \ \omega \ + \ \sqrt{-1}\, \partial\bar\partial\psi, \qquad \psi \in C^\infty(M)_{\mathbb{R}}.$$

By sending ψ to ω_ψ, we have a surjection of $\tilde{\mathcal{K}}_V := \{\psi \in \tilde{\mathcal{K}}; \omega_\psi \in \mathcal{K}_V\}$ onto $\mathcal{K}_V$, where $\tilde{\mathcal{K}}$ denotes the set of all $\psi \in C^\infty(M)_{\mathbb{R}}$ such that $\omega_\psi \in \mathcal{K}$. Given a one-parameter family $\psi_t \in \tilde{\mathcal{K}}_V$, $a \le t \le b$, we say that $\{\psi_t; a \le t \le b\}$ is a smooth path in $\tilde{\mathcal{K}}_V$, if the map of $M \times [a,b]$ to $\mathbb{R}$ sending (x,t) to $\psi_t(x)$ is C^∞. For such a smooth path $\{\psi_t; a \le t \le b\}$, we put $\dot\psi_t := (\partial/\partial t)(\psi_t)$ for simplicity. A key observation is[1]

Proposition A. *Let V be a holomorphic vector field belonging to $\mathfrak{g}$ such that $\mathcal{K}_V \neq \emptyset$. Then there exists a functional $\eta_V : \mathcal{K}_V \to \mathbb{R}$ satisfying the equality*

$$\frac{d}{dt}\eta_V(\omega_t) = \int_M \varphi_t \dot\psi_t \omega_t^n / A_\kappa, \qquad a \le t \le b, \tag{1.1}$$

for every smooth path $\{\psi_t; a \le t \le b\}$ in $\tilde{\mathcal{K}}_V$, where we set $\omega_t := \omega_{\psi_t}$, and the functions $\varphi_t \in \tilde{\mathfrak{k}}_{\omega_t}$, $a \le t \le b$, on M are such that $V = \operatorname{grad}^{\mathbb{C}}_{\omega_t} \varphi_t$.

[1]My sincere gratitude is due to Prof. Ryoichi Kobayashi who invited me to present this key observation in a lecture at Nagoya University in 1997. Arguments as in the proof of this were also used independently by [GC].

For $W_1,\ W_2\ \in\ \mathfrak{g}$, we put $(W_1, W_2)_\omega := \int_M \iota_\omega^{-1}(W_1)\iota_\omega^{-1}(W_2)\omega^n/A_\kappa$, which is independent of the choice of ω in $\mathcal{K}$, and will be denoted also by $(W_1, W_2)_\kappa$ (cf. [FM]). Such independence plays a crucial role in [FM], and Proposition A above gives some explanation for this independence (see (3.3)). Moreover, for V as above, η_V satisfies (cf. §3)

$$\frac{d}{dt}\{\eta_V(g_t^*\omega')\} = 2\,\mathrm{Im}\,(V, W)_{\mathcal{K}}, \quad \text{for all } W \in \mathfrak{z}(V) \text{ and } \omega' \in \mathcal{K}_V. \tag{1.2}$$

where $\mathfrak{z}(V)$ is the centralizer $\{W \in \mathfrak{g}; [W, V] = 0\}$ of V in $\mathfrak{g}$, and for any $z \in \mathbb{C}$, $\mathrm{Re}\, z$ and $\sqrt{-1}\ \mathrm{Im}\, z$ denote the real part and the imaginary part of $z = \mathrm{Re}\, z + \sqrt{-1}\ \mathrm{Im}\, z$, respectively. Let $\underline{\mathcal{K}}$ denote the nonempty subset of $\mathcal{K}$ consisting of all $\omega \in \mathcal{K}$ such that K_ω is maximal compact in G. Then Proposition A allows us to construct functionals, $\hat{\mu}_V : \mathcal{K}_V \to \mathbb{R}$, $\hat{\mu} : \underline{\mathcal{K}} \to \mathbb{R}$ and $\hat{\nu} : \underline{\mathcal{K}} \to \mathbb{R}$, such that[2]

(1) all critical points for $\hat{\mu}_V$ and $\hat{\mu}$ are both extremal Kähler metrics;

(2) the set of the critical points for $\hat{\nu}$ consists all "Kähler-Einstein metrics" on M,

where for the functional $\hat{\mu}$, the pair (M, κ) is assumed to be quantized (cf. §5), and for the functional $\hat{\nu}$, the cohomology class κ is assumed to be $2\pi c_1(M)_{\mathbb{R}}$. Note also that, in (2) above, M possibly has nonvanishing Futaki character, where the terminology "Kähler-Einstein metric" is used in the sense of [M4]. We also have (see Propositions 5.7 and 6.5 and nearby arguments):

THEOREM B. *The functionals $\hat{\mu}$ and $\hat{\nu}$ are G-invariant.*

From moduli-theoretic points of view, this G-invariance would be one of the most important properties featuring the functionals $\hat{\mu}$ and $\hat{\nu}$ above. By the convexity of $\hat{\nu}$, the method used by Guan in [G1] for extremal Kähler metrics now implies

THEOREM C. (see [M5] for a more general case) *Let M be a nonsingular toric Fano variety, defined over $\mathbb{C}$, possibly with nonvanishing Futaki character. Then "Kähler-Einstein metrics"* (cf. [M4]) *on M in the class $2\pi c_1(M)_{\mathbb{R}}$ is unique, if any, up to the action of $G = \mathrm{Aut}^0(M)$.*

[2] An important point is that both $\hat{\mu}$ and $\hat{\nu}$ are defined "globally" on $\underline{\mathcal{K}}$ without specifying any maximal compact subgroup of G. Such a condition of globality has never been studied seriously by any other authors.

§2. Proof of Proposition A

For each $V \in \mathfrak{g}$, let $V_{\mathbb{R}}$ denote the real vector field $V + \bar{V}$ on M corresponding to the holomorphic vector field V on M. Then the one-parameter group $\exp(tV_{\mathbb{R}})$, $t \in \mathbb{R}$, on M generated by the vector field $V_{\mathbb{R}}$ comes from the action on M of the one-parameter group $\exp tV$, $t \in \mathbb{R}$, in G. Hence, if there is no fear of confusion, we use $\exp tV$ and $\exp(tV_{\mathbb{R}})$ interchangeably. Assuming $\mathcal{K}_V \neq \emptyset$, let $\omega \in \mathcal{K}_V$. Then the one-parameter group $P_V := \{\exp(tV_{\mathbb{R}}); t \in \mathbb{R}\}$ has a compact closure $\bar{P}_V$ in G, since $\bar{P}_V$ is closed in the compact group K_ω. Therefore

$$\tilde{\mathcal{K}}_V = \{\psi \in \tilde{\mathcal{K}}\,; V_{\mathbb{R}}\psi = 0\} = \{\psi \in \tilde{\mathcal{K}}\,; \psi \text{ is } \bar{P}_V\text{-invariant}\}. \tag{2.1}$$

For ω as above, let $\sigma(\omega)$ and $\Box_\omega$ be respectively the corresponding scalar curvature and the Laplacian on functions defined by

$$\sigma(\omega) := \sum_{\alpha,\beta} g^{\bar{\beta}\alpha} R_{\alpha\bar{\beta}}, \qquad \Box_\omega = \sum_{\alpha,\beta} g^{\bar{\beta}\alpha} \frac{\partial^2}{\partial z^\alpha \partial z^{\bar{\beta}}},$$

where $\sum_{\alpha,\beta} R_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}}$ denotes the Ricci form $R(\omega) := \sqrt{-1}\bar{\partial}\partial \log \omega^n$ for ω. For each $\omega_\psi \in \mathcal{K}_V$, its scalar curvature $\sigma(\omega_\psi)$ and Laplacian $\Box_{\omega_\psi}$ are denoted sometimes by $\sigma(\psi)$ and $\Box_\psi$ respectively. To each pair $(\psi_1, \psi_2) \in \tilde{\mathcal{K}}_V \times \tilde{\mathcal{K}}_V$, we associate $E_V(\psi', \psi'') \in \mathbb{R}$ by

$$E_V(\psi', \psi'') := \int_a^b \left(\int_M \varphi_t \dot{\psi}_t \omega_{\psi_t}^n / A_\kappa \right) dt, \tag{2.2}$$

where $\{\psi_t; a \leq t \leq b\}$ is an arbitrary piecewise smooth path in $\tilde{\mathcal{K}}_V$ satisfying $\psi_a = \psi'$ and $\psi_b = \psi''$, and the functions $\varphi_t \in \tilde{\mathfrak{k}}_{\omega_t}$, $a \leq t \leq b$, on M are such that

$$V \;=\; \operatorname{grad}^{\mathbb{C}}_{\omega_t} \varphi_t$$

with $\omega_t := \omega_{\psi_t}$. Now by setting $\eta_V(\omega_\psi) := E_V(0, \psi)$, we can easily reduce the proof of Proposition A to showing the following theorem:

Theorem 2.3. *$E_V(\psi', \psi'')$ above is independent of the choice of the path $\{\psi_t; a \leq t \leq b\}$, in $\tilde{\mathcal{K}}_V$, and therefore well-defined. In particular,*

(2.4) $E_V(\psi, \psi') + E_V(\psi', \psi'') + E_V(\psi'', \psi) = 0$ *for all* $\psi, \psi', \psi'' \in \tilde{\mathcal{K}}_V$;

(2.5) $E_V(\psi, \psi + C) = 0$ *for all* $\psi \in \tilde{\mathcal{K}}_V$ *and all* $C \in \mathbb{R}$.

In view of the assumption $\omega \in \mathcal{K}_V$, we have $V = \mathrm{grad}^{\mathbb{C}}_{\omega}\,\phi$ for $\phi := \iota_{\omega}^{-1}(V) \in \tilde{\mathfrak{k}}_{\omega}$. Then the following lemma is essential in the proof of Theorem 2.3:

LEMMA 2.6. (cf. [FM;p.208]) *The equality $\varphi_t = \phi + \sqrt{-1}V\psi_t$ holds for all $a \leq t \leq b$.*

By using this lemma, we shall now prove Theorem 2.3.

Proof of Theorem 2.3. Define a map $\Psi = \Psi(s,t)$ of the rectangle $R := [0,1] \times [a,b]$ to $\tilde{\mathcal{K}}_V$ by $\Psi(s,t) := s\psi_t$ for $(s,t) \in [0,1] \times [a,b]$. Since $\{\psi_t; a \leq t \leq b\}$ is piecewise smooth, there exists a partition $a = a_0 < a_1 < a_2 < \ldots < a_r = b$ of the interval $[a,b]$ such that $\{\psi_t; a_{i-1} \leq t \leq a_i\}$ is smooth for each $i \in \{1,2,\ldots,r\}$. We then divide the proof of Theorem 2.3 into the following two steps:

Step 1: For simplicity, put $\omega_{s,t} := \omega_{\Psi(s,t)}$ for each $(s,t) \in R$. Then by Lemma 2.6, we have $V = \mathrm{grad}^{\mathbb{C}}_{\omega_{s,t}}\,\Phi(s,t)$, where $\Phi = \Phi(s,t)$ is defined by $\Phi(s,t) := \phi + \sqrt{-1}V\Psi(s,t) \in \tilde{\mathfrak{k}}_{\omega_{s,t}}$. Here, $\Phi(1,t) = \phi + \sqrt{-1}\,V\psi_t = \varphi_t$. The purpose of this step is to show that

$$(2.7) \qquad \int_{a_{i-1}}^{a_i} \left(\int_M \varphi_t \dot{\psi}_t \omega_\Psi^n / A_\kappa \right) dt = \int_0^1 \left(\int_M \Phi \frac{\partial \Psi}{\partial s} \omega_\Psi^n / A_\kappa \right) ds \Bigg|_{t=a_{i-1}}^{t=a_i}.$$

Let $\Theta := \left(\int_M \Phi\Psi_s \omega_\Psi^n/A_\kappa\right) ds + \left(\int_M \Phi\Psi_t\omega_\Psi^n/A_\kappa\right) dt$, where $\Psi_s := \partial\Psi/\partial s$ and $\Psi_t := \partial\Psi/\partial t$. Moreover, we put $\Phi_s := \partial\Phi/\partial s$ and $\Phi_t := \partial\Phi/\partial t$. For a suitable orientation of the rectangle R, its boundary ∂R is written as a sum $\gamma_1 + \gamma_2 - \gamma_3 - \gamma_4$, where

$$\gamma_1 := \{(s, a_{i-1}); 0 \leq s \leq 1\}, \qquad \gamma_2 := \{(1,t); a_{i-1} \leq t \leq a_i\},$$
$$\gamma_3 := \{(s, a_i); 0 \leq s \leq 1\}, \qquad \gamma_4 := \{(0,t); a_{i-1} \leq t \leq a_i\}.$$

Then by the Stokes theorem, $\int_R d\Theta = \int_{\partial R}\Theta = \int_{\gamma_1+\gamma_2-\gamma_3-\gamma_4}\Theta$. Moreover, the pullback of Θ to γ_4 vanishes. Hence, $\int_R d\Theta$ is just

$$-\int_{\gamma_3-\gamma_1}\Theta + \int_{\gamma_2}\Theta = -\int_0^1 \left(\int_M \Phi\Psi_s\omega_\Psi^n/A_\kappa\right) ds\Bigg|_{t=a_{i-1}}^{t=a_i} + \int_{a_{i-1}}^{a_i}\left(\int_M \varphi_t\dot{\psi}_t\omega_\Psi^n/A_\kappa\right) dt.$$

Thus the proof of (2.7) is reduced to showing the vanishing $d\Theta = 0$ on the rectangle R. In terms of a system of holomorphic local coordinates $(z^1, z^2, \dots, z^n)$, we write the Kähler metric $\omega_\Psi = \omega_{\Psi(s,t)} = \omega_{s,t}$ in the form

$$\omega_\Psi = \sqrt{-1}\sum_{\alpha,\beta} g_{\Psi\,\alpha\bar\beta} dz^\alpha \wedge dz^{\bar\beta}.$$

Then for $\zeta_1, \zeta_2 \in C^\infty(M)_{\mathbb{C}}$, we can define the Poisson bracket $[\zeta_1, \zeta_2]_\Psi$ of ζ_1 and ζ_2 relative to the Kähler metric ω_Ψ by

$$[\zeta_1, \zeta_2]_\Psi := \sqrt{-1}\sum_{\alpha,\beta} g_\Psi^{\bar\beta\alpha}\left(\frac{\partial\zeta_1}{\partial z^\alpha}\frac{\partial\zeta_2}{\partial z^{\bar\beta}} - \frac{\partial\zeta_1}{\partial z^{\bar\beta}}\frac{\partial\zeta_2}{\partial z^\alpha}\right).$$

Let $(\ ,\ )_\Psi : A^q(M)_{\mathbb{C}} \times A^q(M)_{\mathbb{C}} \to C^\infty(M)_{\mathbb{C}}$ be the pointwise Hermitian pairing associated with the Kähler metric ω_Ψ, where $A^q(M)_{\mathbb{C}}$ denotes the space of all complex-valued smooth q-forms on M. By a straightforward computation,

$$\begin{aligned} d\Theta &= ds \wedge dt \int_M \left\{\frac{\partial}{\partial s}\left(\Phi\Psi_t\omega_\Psi^n/A_\kappa\right) - \frac{\partial}{\partial t}\left(\Phi\Psi_s\omega_\Psi^n/A_\kappa\right)\right\} \\ &= ds \wedge dt \int_M \{(\Phi_s\Psi_t - \Phi_t\Psi_s) + \Phi\Psi_t(\Box_\Psi\Psi_s) - \Phi\Psi_s(\Box_\Psi\Psi_t)\}\,\omega_\Psi^n/A_\kappa \\ &= \sqrt{-1}ds \wedge dt \int_M \{\Psi_t(V\Psi_s) - \Psi_s(V\Psi_t)\}\,\omega_\Psi^n/A_\kappa \\ &\qquad + ds \wedge dt \int_M \left\{-\left(\bar\partial(\Phi\Psi_t), \bar\partial\Psi_s\right)_\Psi + \left(\bar\partial(\Phi\Psi_s), \bar\partial\Psi_t\right)_\Psi\right\}\omega_\Psi^n/A_\kappa. \end{aligned}$$

On the other hand, by $V = \operatorname{grad}^{\mathbb{C}}_\omega \Phi$, we obtain

$$\begin{aligned} &-\left(\bar\partial(\Phi\Psi_t), \bar\partial\Psi_s\right)_\Psi + \left(\bar\partial(\Phi\Psi_s), \bar\partial\Psi_t\right)_\Psi \\ &= \sqrt{-1}\Phi[\Psi_s, \Psi_t]_\Psi - \left(\Psi_t\bar\partial\Phi,\ \bar\partial\Psi_s\right)_\Psi + \left(\Psi_s\bar\partial\Phi,\ \bar\partial\Psi_t\right)_\Psi \\ &= \sqrt{-1}\{\Phi[\Psi_s, \Psi_t]_\Psi - \Psi_t(V\Psi_s) + \Psi_s(V\Psi_t)\}. \end{aligned}$$

These together with $V_{\mathbb{R}}\psi_t = 0$ (see (2.1)) show the vanishing of $d\Theta$ as follows:

$$\begin{aligned} d\Theta &= \sqrt{-1}ds \wedge dt \int_M \Phi[\Psi_s, \Psi_t]_\Psi\omega_\Psi^n/A_\kappa \\ &= \sqrt{-1}ds \wedge dt \int_M [\Phi, \Psi_s]_\Psi\Psi_t\omega_\Psi^n/A_\kappa \end{aligned}$$

$$= \sqrt{-1} ds \wedge dt \int_M (V_{\mathbb{R}} \Psi_s) \Psi_t \omega_\Psi^n / A_\kappa$$
$$= \sqrt{-1} ds \wedge dt \int_M (V_{\mathbb{R}} \psi_t) \Psi_t \omega_\Psi^n / A_\kappa = 0.$$

Step 2: Consider the equality (2.7) for $i = 1, 2, \ldots, r$. By adding them up, we obtain

$$\int_a^b \left(\int_M \varphi_t \dot{\psi}_t \omega_{\psi_t}^n / A_\kappa \right) dt = \int_0^1 \left(\int_M \Phi \psi_t \omega_\Psi^n / A_\kappa \right) ds \Big|_{t=a}^{t=b}.$$

Therefore, the left-hand side is independent of the choice of the piecewise smooth path $\{\psi_t ; a \leq t \leq b\}$ in $\tilde{\mathcal{K}}_V$, as long as $\psi_a = \psi'$ and $\psi_b = \psi''$. Then (2.4) is now immediate. For (2.5), let $\psi_t := \psi + tC$, where $t \in [0, 1]$. Put $\omega_t := \omega_{\psi_t}$ for simplicity. For each t, consider the associated $\varphi_t \in \tilde{\mathfrak{k}}_{\omega_t}$ satisfying $V = \mathrm{grad}^{\mathbb{C}}_{\omega_t} \varphi_t$. Then,

$$E(\psi, \psi + C) = \int_0^1 \int_M \varphi_t \dot{\psi}_t \omega_{\psi_t}^n / A_\kappa = C \int_0^1 \left(\int_M \varphi_t \omega_t^n / A_\kappa \right) = 0.$$

§3. An application to the study of the bilinear pairing $(\ ,\ )_\kappa$ on $\mathfrak{k}^{\mathbb{C}}$

Let $V \in \mathfrak{g}$ be such that $\omega \in \mathcal{K}_V \neq \emptyset$. We put $V^g := (g^{-1})_* V = \mathrm{Ad}(g^{-1}) V$ for all $g \in G$. Let ω_0 and ω_1 be arbitrary elements in $\mathcal{K}_V$. We choose a smooth path $\{\psi_t \in \tilde{\mathcal{K}}_V ; a \leq t \leq b\}$ in $\tilde{\mathcal{K}}_V$ such that the corresponding path $\omega_t := \omega_{\psi_t}$, $a \leq t \leq b$, connecting ω_0 and ω_1 in $\mathcal{K}_V$ satisfies

$$\int_M \dot{\psi}_t \omega_t^n / A_\kappa = 0 \qquad \text{for all } t.$$

For each t, we can write $V = \mathrm{grad}^{\mathbb{C}}_{\omega_t} \varphi_t$ for some unique $\varphi_t \in \tilde{\mathfrak{k}}_{\omega_t}$. On the other hand, for every $g \in G$, we see that $g^* \omega_0, g^* \omega_1 \in \mathcal{K}_{V^g}$, because the condition $V \in \mathfrak{k}_\omega$ always implies $V^g \in \mathfrak{k}_{g^* \omega}$. Now, $g^* \omega_t = g^* \omega + \sqrt{-1} \partial \bar{\partial} (g^* \psi_t)$, $a \leq t \leq b$, is a path in $\mathcal{K}_{V^g}$ connecting the metrics $g^* \omega_0$, $g^* \omega_1$ and satisfying $\int_M (g^* \dot{\psi}_t) g^* \omega_t^n / A_\kappa = 0$ for all t. In view of $V^g = \mathrm{grad}^{\mathbb{C}}_{g^* \omega_t} g^* \varphi_t \in \tilde{\mathfrak{k}}_{g^* \omega_t}$, we see that

$$(3.1) \qquad E_{V^g}(g^* \omega_0, g^* \omega_1) = \int_a^b \left(\int_M g^* \varphi_t \, g^* \dot{\psi}_t g^* \omega_t^n / A_\kappa \right) dt$$
$$= \int_a^b \left(\int_M \varphi_t \, \dot{\psi}_t \, \omega_t^n / A_\kappa \right) dt \ = \ E_V(\omega_0, \omega_1).$$

Consider the algebraic subgroup $Z(V) := \{g \in G; V^g = V\}$ of G. Obviously, $Z(V)$ has the Lie algebra $\mathfrak{z}(V)$. We now claim that

LEMMA 3.2. *$E_V(\omega_0, g^*\omega_0) = E_V(\omega_1, g^*\omega_1)$ for all $g \in Z(V)$ and ω_1, $\omega_2 \in \mathcal{K}_V$.*

Proof. By $g \in Z(V)$, we have $V^g = V$. Hence by (3.1), $E_V(g^*\omega_0, g^*\omega_1) = E_V(\omega_0, \omega_1) = E_V(\omega_0, g^*\omega_0) + E_V(g^*\omega_0, g^*\omega_1) - E_V(\omega_1, g^*\omega_1)$. Then the required equality $E_V(\omega_0, g^*\omega_0) = E_V(\omega_1, g^*\omega_1)$ follows immediately.

For a maximal compact subgroup K of G, let ω_0, $\omega_1 \in \mathcal{K}^K$, where $\mathcal{K}^K$ denotes the set of all K-invariant elements in $\mathcal{K}$. Let $\mathfrak{k}$ denote the Lie subalgebra of $\mathfrak{g}$ corresponding to the Lie subgroup K of G. Then $\mathfrak{k}_{\omega_0} = \mathfrak{k}_{\omega_1} = \mathfrak{k}$. Let V, $W \in \mathfrak{t}$, where $\mathfrak{t}$ is a maximal toral subalgebra of $\mathfrak{k}$. We first observe that ω_0, $\omega_1 \in \mathcal{K}_V$. Moreover, we can write

$$V \;=\; \operatorname{grad}^{\mathbb{C}}_{\omega_i} v_i \quad \text{and} \quad W \;=\; \operatorname{grad}^{\mathbb{C}}_{\omega_i} w_i, \qquad i = 0, 1,$$

for some v_i, $w_i \in \tilde{\mathfrak{k}}_{\omega_i}$. Put $g_t := \exp(t\sqrt{-1}\,W) = \exp\{t(\sqrt{-1}\,W)_{\mathbb{R}}\}$. This g_t belongs to $Z(V)$ for all $t \in \mathbb{R}$. Write $g_t^*\omega_i = \omega_i + \sqrt{-1}\,\partial\bar{\partial}\psi_{i,t}$ for some smooth one-parameter families $\{\psi_{i,t}; t \in \mathbb{R}\}$ of real-valued C^∞ functions on M. Note that

$$(\dot{\psi}_{i,t})_{|t=0} \;=\; 2w_i \;+\; C_i, \qquad i = 0, 1,$$

for some constants $C_i \in \mathbb{R}$. Now by Lemma 3.2, $E_V(\omega_0, g_t^*\omega_0) = E_V(\omega_1, g_t^*\omega_1)$ for all t. Differentiating this with repect to t at $t = 0$, we obtain

$$\int_M v_0 w_0 \omega_0^n / A_\kappa \;=\; \int_M v_1 w_1 \omega_1^n / A_\kappa. \tag{3.3}$$

Recall that the identity (3.3) is the key point in proving the well-definition of the bilinear pairing $\mathfrak{k}^{\mathbb{C}} \times \mathfrak{k}^{\mathbb{C}} \ni (V, W) \mapsto (V, W)_\kappa \in \mathbb{C}$ (cf. [FM];§1), where $\mathfrak{k}^{\mathbb{C}}$ denotes the complexification of $\mathfrak{k}$ in $\mathfrak{g}$.

Remark. Let $W \in \mathfrak{z}(V)$ and $V \in \mathfrak{g}$ with $\mathcal{K}_V \neq \emptyset$. Let ω be an arbitrary element of $\mathcal{K}_V$. Put $g_t := \exp tW_{\mathbb{R}}$ and $\omega_t := g_t^*\omega$, $t \in \mathbb{R}$. Then we can write

$$V \;=\; \operatorname{grad}^{\mathbb{C}}_{\omega_t} v_t, \quad \text{and} \quad W \;=\; \operatorname{grad}^{\mathbb{C}}_{\omega_t} w_t$$

for some $v_t \in \tilde{\mathfrak{k}}_{\omega_t}$ and $w_t \in \tilde{\mathfrak{g}}_{\omega_t}$. Write $\omega_t \;=\; \omega + \sqrt{-1}\,\partial\bar{\partial}\psi_t$ for some smooth one-parameter family $\{\psi_t; t \in \mathbb{R}\}$ of real-valued C^∞ functions on M. Then

$$\dot{\psi}_t \;=\; 2\,\mathrm{Im}\, w_t \;+\; C_t$$

for some real constant C_t. Then by the definition of the functional η_V, we have the following equality (see also (1.2)), which is an important ingredient of the proof of (3.3):

$$\frac{d}{dt}\eta_V(g_t^*\omega) \;=\; 2\,\mathrm{Im}\int_M v_t w_t \omega_t^n/A_\kappa \;=\; 2\,\mathrm{Im}(V,W)_\kappa, \qquad t\in\mathbb{R}.$$

Remark. Let $V \in \mathfrak{g}$ be such that $\omega \in \mathcal{K}_V \neq \emptyset$, and let $\mathbb{R}_+$ denote the multiplicative group of all positive real numbers. Put $e_V(g) := \exp(E_V(\omega, g^*\omega))$. Then $e_V : Z(V) \to \mathbb{R}_+$ defines a character of real Lie groups as follows:

$$\begin{aligned}
&\log(e_V(g_1g_2)) \;=\; E_V(\omega,(g_1g_2)^*\omega) \;=\; E_V(\omega, g_2^*g_1^*\omega)\\
&= E_V(\omega, g_1^*\omega) \;+\; E_V(g_1^*\omega, g_2^*g_1^*\omega) \;=\; E_V(\omega, g_1^*\omega) \;+\; E_V(\omega, g_2^*\omega)\\
&= \log e_V(g_1) \;+\; \log e_V(g_2),
\end{aligned}$$

i.e., $e_V(g_1g_2) = e_V(g_1)e_V(g_2)$ for all g_1, $g_2 \in Z(V)$. Thus, $e_V : Z(V) \to \mathbb{R}_+$ is a group character of real Lie groups.

§4. Functional $\hat{\mu}_V$ whose critical points are extremal Kähler metrics

In this section, we fix an element ω in $\underline{\mathcal{K}}$. Then the group K_ω (see §1) is maximal compact in G. The extremal Kähler vector field $\mathcal{V}_\omega \in \mathfrak{k}_\omega$ (cf. [FM]) is defined by

$$\mathcal{V}_\omega = \mathrm{grad}^{\mathbb{C}}_\omega(\mathrm{pr}_\omega\,\sigma(\omega)),$$

where $\mathrm{pr}_\omega : L^2(M,\omega)_{\mathbb{R}} \to \mathbb{R}\oplus\tilde{\mathfrak{k}}_\omega$ is the orthogonal projection from the space $L^2(M,\omega)_{\mathbb{R}}$ of all real-valued L^2-functions on the Kähler manifold (M,ω) onto its finite-dimensional subspace $\mathbb{R}\oplus\tilde{\mathfrak{k}}_\omega := \{\varphi \in C^\infty(M)_{\mathbb{R}}; \mathrm{grad}^{\mathbb{C}}_\omega \varphi \in \mathfrak{g}\}$. Then the orthogonal complement $(\mathbb{R}\oplus\tilde{\mathfrak{k}}_\omega)^\perp$ of $\mathbb{R}\oplus\tilde{\mathfrak{k}}_\omega$ in $L^2(M,\omega)_{\mathbb{R}}$ is exactly the kernel of pr_ω. In this section, we fix an element ω in $\underline{\mathcal{K}}$, and put

$$V := \mathcal{V}_\omega.$$

Then ω belongs to $\mathcal{K}_V$ obviously. Let $K^{\mathbb{C}}_\omega$ be the reductive algebraic subgroup of G obtained as the complexification of K_ω in G. The corresponding Lie subalgebra of $\mathfrak{g}$ will be denoted by $\mathfrak{k}^{\mathbb{C}}_\omega$. Obviously, $V \in \mathfrak{k}_\omega \subset \mathfrak{k}^{\mathbb{C}}_\omega \subset \mathfrak{z}(V)$. We first observe that

Lemma 4.1. *$Z(V)$ is connected.*

Proof. By the Chevalley decomposition of G, we write G as a semidirect product $K_\omega^{\mathbb{C}} \ltimes U$, where U is the unipotent radical of G. Let $\mathfrak{u}$ be the Lie subalgebra of $\mathfrak{g}$ corresponding to U. Then every element of $Z(V)$ is written as $k(\exp W)$ for some $k \in K_\omega^{\mathbb{C}}$ and $W \in \mathfrak{u}$. By $K_\omega^{\mathbb{C}} \subset Z(V)$, we see that $\exp W \in Z(V)$, i.e., $V = \{\exp \mathrm{ad}(W)\}V$. Then Jordan's normal form of the linear map $\mathrm{ad}(W)$ of $\mathfrak{g}$ onto itself allows us to obtain $W \in \mathfrak{z}(V)$. Now, $k \exp(tW) \in Z(V)$ for all $0 \leq t \leq 1$. Thus, $Z(V)$ is connected.

We now put $\mathcal{H}_V := \{\omega' \in \underline{\mathcal{K}}; \mathcal{V}_{\omega'} = V\}$, where $\mathcal{V}_{\omega'} \in \mathfrak{k}_{\omega'}$ denotes the extremal Kähler vector field of ω'. Then $\mathcal{H}_V$ is a nonempty subset of $\underline{\mathcal{K}}$ satisfying

$$\omega \;\in\; \{\omega' \in \mathcal{K}; \mathfrak{k}_{\omega'} = \mathfrak{k}_\omega\} \;\subset\; \mathcal{H}_V \;\subset\; \mathcal{K}_V.$$

Let $\tilde{\mathcal{H}}_V$ denote the set of all $\psi \in \tilde{\mathcal{K}}_V$ such that $\omega_\psi \in \mathcal{H}_V$. By a piecewise smooth path in $\tilde{\mathcal{H}}_V$, we mean a piecewise smooth path in $\tilde{\mathcal{K}}_V$ sitting in $\tilde{\mathcal{H}}_V$. For each $\psi \in \tilde{\mathcal{H}}_V$, we take an arbitrary piecewise smooth path $\{\psi_t; a \leq t \leq b\}$ in $\tilde{\mathcal{H}}_V$ such that $\psi_a = 0$ and $\psi_b = \psi$. Then the restriction to $\mathcal{H}_V$ of the K-energy map $\mu : \mathcal{K} \to \mathbb{R}$ (cf. [M1]) is given by

$$(4.2) \qquad \mu(\omega_\psi) := -\int_a^b \left\{ \int_M (\sigma(\omega_t) - C_\kappa)\dot{\psi}_t \omega_t^n / A_\kappa \right\} dt, \qquad \psi \in \tilde{\mathcal{H}}_V,$$

where we put $\omega_t := \omega_{\psi_t}$ for simplicity, and C_κ is the real constant $\int_M \sigma(\omega)\omega^n / A_\kappa$. The set of the critical points for μ just consists of all Kähler metrics in $\mathcal{K}_V$ of constant scalar curvature. Define $\hat{\mu}_V : \mathcal{H}_V \to \mathbb{R}$ by

$$(4.3) \qquad \hat{\mu}_V := \mu + \eta_V,$$

where η_V is as in the introduction. For each t, we write $V = \mathrm{grad}^{\mathbb{C}}_{\omega_t} \varphi_t$ for some unique $\varphi_t \in \tilde{\mathfrak{k}}_{\omega_t}$. By $\mathrm{pr}_{\omega_t} \sigma(\omega_t) \;=\; C_\kappa \;+\; \varphi_t$, we see from the equalities (1.1), (4.2), (4.3) that

$$(4.4) \qquad \hat{\mu}_V(\omega_\psi) = -\int_a^b \left(\int_M \{\sigma(\omega_t) - \mathrm{pr}_{\omega_t} \sigma(\omega_t)\} \, \dot{\psi}_t \omega_t^n / A_\kappa \right) dt, \quad \psi \in \tilde{\mathcal{H}}_V,$$

for $\{\psi_t; a \leq t \leq b\}$ as above. Let $\omega' \in \mathcal{H}_V$. Since $\sigma(\omega') - \mathrm{pr}_{\omega'} \sigma(\omega')$ is a $K_{\omega'}$-invariant function, ω' can be perturbed in $\mathcal{H}_V$ to the form $\omega' + \sqrt{-1}\varepsilon \, \partial\bar{\partial}\{\sigma(\omega') - \mathrm{pr}_{\omega'} \sigma(\omega')\}$, where $\varepsilon > 0$ is sufficiently small. Since the equality $\sigma(\omega') = \mathrm{pr}_{\omega'} \sigma(\omega')$ holds if and only if ω' is an extremal Kähler metric, (4.4) above implies that

PROPOSITION 4.5. *An element ω' of $\mathcal{H}_V$ is a critical point for the functional $\hat{\mu}_V : \mathcal{H}_V \to \mathbb{R}$ if and only if ω' is an extremal Kähler metric.*

Remark. The functional $\hat{\mu}_V$ above was obtained by the author in 1994, though the result was unpublished. A little afterwards, Simanca (see [S1]) obtained a similar result. Guan [G1] studied such a functional independently and successfully, applying it to the uniqueness (modulo connected group actions) of extremal Kähler metrics in a Kähler class of a nonsingular toric variety.

§5. Functional $\hat{\mu} : \underline{\mathcal{K}} \to \mathbb{R}$ for a quantized pair (M, κ)

Throughout this section, we assume that the pair (M, κ) is *quantized*, i.e., there exists a holomorphic line bundle L over M such that

(1) the Kähler class κ in the introduction is $2\pi c_1(L)_{\mathbb{R}}$;

(2) the G-action on M lifts to a holomorphic G-action on L preserving set-theoretically the image of the zero section of L.

For instance, if M is a Fano manifold, then the pair $(M, c_1(M)_{\mathbb{R}})$ is quantized by choosing the anticanonical bundle K_M^{-1} as L. The main purpose of this section is to define a functional $\hat{\mu} : \underline{\mathcal{K}} \to \mathbb{R}$ for each quantized pair (M, κ) from the functionals $\hat{\mu}_{V^g} : \mathcal{H}_{V^g} \to \mathbb{R}$, $g \in G$, (cf. §4) glued together.

Let $\mathfrak{u}$ be the Lie subalgebra of $\mathfrak{g}$ corresponding to the unipotent radical U of G, where we write G as a semi-direct product $K_\omega^{\mathbb{C}} \ltimes U$. Take a $\mathbb{C}$-basis $\{\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_m\}$ for $\mathfrak{u}$. Furthermore, let $\{\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_\ell\}$ be an $\mathbb{R}$-basis for $\mathfrak{k}_\omega$, which is naturally regarded as a $\mathbb{C}$-basis for $\mathfrak{k}_\omega^{\mathbb{C}}$. We choose $1 \ll k \in \mathbb{Z}$ such that $L^{\otimes k}$ is very ample. Let $\{\sigma_0, \sigma_1, \dots, \sigma_r\}$ be a $\mathbb{C}$-basis for $S := H^0(M, L^{\otimes k})$. Note that, via the U-action on L, the unipotent group U acts naturally on S, which induces an infinitesimal action of $\mathfrak{u}$ on S. Since U is unipotent, we may assume that

$$\mathcal{Y}_j \sigma_\lambda = \begin{cases} 0 & \text{if } 1 \le j \le m \text{ and } \lambda = 0; \\ \sum_{\mu=0}^{\lambda-1} b_{j,\lambda,\mu}\, \sigma_\mu & \text{if } 1 \le j \le m \text{ and } 1 \le \lambda \le r, \end{cases}$$

for some complex numbers $b_{j,\lambda,\mu} \in \mathbb{C}$. To each real number $0 < \varepsilon \ll 1$, we associate a Hermitian metric h_ε on L by

$$h_\varepsilon := \Big\{ \sum_{\lambda=0}^{r} \varepsilon^{2\lambda} \sigma_\lambda \bar{\sigma}_\lambda \Big\}^{-1} = \Big\{ \sum_{\lambda=0}^{r} (\varepsilon^\lambda \sigma_\lambda)(\varepsilon^\lambda \bar{\sigma}_\lambda) \Big\}^{-1}.$$

Let $\omega(\varepsilon)$ denote the Ricci form $R(h_\varepsilon) := \sqrt{-1}\bar\partial\partial \log h_\varepsilon$ for (L, h_ε). We then have $\omega(\varepsilon) \in \mathcal{K}$. The infinitesimal action of $\mathcal{Y}_j$ on h_ε is given by

$$\begin{aligned}(5.1)\qquad \mathcal{Y}_j h_\varepsilon &= -h_\varepsilon^2 \Big\{ \sum_{\lambda=0}^{r} \varepsilon^{2\lambda} (\mathcal{Y}_j \sigma_\lambda)\bar\sigma_\lambda \Big\} \\ &= -\varepsilon h_\varepsilon^2 \Big\{ \sum_{\lambda=1}^{r}\sum_{\mu=0}^{\lambda-1} \varepsilon^{\lambda-\mu-1} b_{j,\lambda,\mu}(\varepsilon^\mu \sigma_\mu)(\varepsilon^\lambda \bar\sigma_\lambda)\Big\}.\end{aligned}$$

For each $i \in \{1, 2, \dots, \ell\}$ and $j \in \{1, 2, \dots, m\}$, consider the functions $\xi_i \in \tilde{\mathfrak{k}}_{\omega(\varepsilon)}$ and $\eta_j \in \tilde{\mathfrak{g}}_{\omega(\varepsilon)}$ such that $\operatorname{grad}^{\mathbb{C}}_{\omega(\varepsilon)} \xi_i = \mathcal{X}_i$ and $\operatorname{grad}^{\mathbb{C}}_{\omega(\varepsilon)} \eta_j = \mathcal{Y}_j$. Then ξ_i is real-valued, where η_j is possibly complex-valued. By $\operatorname{grad}^{\mathbb{C}}_{\omega(\varepsilon)} h_\varepsilon^{-1}(\mathcal{Y}_j h_\varepsilon) = \sqrt{-1}\mathcal{Y}_j$ (cf. [M3]),

$$(5.2)\qquad \sqrt{-1}\,\eta_j \;=\; h_\varepsilon^{-1}(\mathcal{Y}_j h_\varepsilon) \;-\; \int_M h_\varepsilon^{-1}(\mathcal{Y}_j h_\varepsilon)\{\omega(\varepsilon)\}^n/A_\kappa.$$

Put $v_\lambda := \varepsilon^\lambda \sigma_\lambda$ and $C_0 := \sum_{j=1}^m \{\sum_{\lambda=1}^r \sum_{\mu=0}^{\lambda-1} |b_{j,\lambda,\mu}|^2\}^{1/2}$. Moreover, let $a_{j,\lambda,\mu}$ denote the complex number $\varepsilon^{\lambda-\mu-1} b_{j,\lambda,\mu}$ or 0, according as $\lambda > \mu$ or $\lambda \le \mu$. We then put $w_{j,\lambda} := \sum_{\mu=0}^r a_{j,\lambda,\mu} v_\mu$. In view of (5.1), the Cauchy-Schwarz inequality allows us to estimate the absolute value $|h_\varepsilon^{-1}(\mathcal{Y}_j h_\varepsilon)|$ of $h_\varepsilon^{-1}(\mathcal{Y}_j h_\varepsilon)$ as follows:

$$\begin{aligned}|h_\varepsilon^{-1}(\mathcal{Y}_j h_\varepsilon)|^2 \;&=\; \varepsilon^2 \frac{|\sum_{\lambda=1}^r w_{j,\lambda}\bar v_\lambda|^2}{(\sum_{\lambda=0}^r v_\lambda \bar v_\lambda)^2} \;\le\; \varepsilon^2 \frac{(\sum_{\lambda=1}^r w_{j,\lambda}\bar w_{j,\lambda})(\sum_{\lambda=1}^r v_\lambda \bar v_\lambda)}{(\sum_{\lambda=0}^r v_\lambda \bar v_\lambda)^2} \\ &\le\; \varepsilon^2 \frac{\sum_{\lambda=1}^r w_{j,\lambda}\bar w_{j,\lambda}}{\sum_{\lambda=0}^r v_\lambda \bar v_\lambda} \;\le\; \varepsilon^2 \sum_{\lambda=1}^r \sum_{\mu=0}^r |a_{j,\lambda,\mu}|^2 \;\le\; (C_0\varepsilon)^2.\end{aligned}$$

This together with (5.2) implies $|\eta_j| \;\le\; 2C_0\varepsilon$ for all j. Now for $i \in \{1, 2, \dots, \ell\}$ and $j,\ j' \in \{1, 2, \dots, m\}$, the bilinear pairings $(\mathcal{X}_i, \mathcal{Y}_j)_\kappa$, $(\mathcal{Y}_j, \mathcal{Y}_{j'})_\kappa$ on $\mathfrak{g}$ (cf. [FM]; p.208) are estimated by

$$\begin{aligned}|(\mathcal{X}_i, \mathcal{Y}_j)_\kappa|^2 &= \left| \int_M \xi_i \eta_j \{\omega(\varepsilon)\}^n/A_\kappa \right|^2 \\ &\le \int_M \xi_i^2 \{\omega(\varepsilon)\}^n/A_\kappa \int_M |\eta_j|^2 \{\omega(\varepsilon)\}^n/A_\kappa \\ &\le\; (\mathcal{X}_i, \mathcal{X}_i)_\kappa \int_M |\eta_j|^2\{\omega(\varepsilon)\}^n/A_\kappa \;\le\; 4C_0^2\varepsilon^2 (\mathcal{X}_i, \mathcal{X}_i)_\kappa, \\ |(\mathcal{Y}_i, \mathcal{Y}_{j'})_\kappa| &= \left| \int_M \eta_j \eta_{j'} \{\omega(\varepsilon)\}^n/A_\kappa \right| \le \int_M |\eta_j \eta_{j'}| \{\omega(\varepsilon)\}^n/A_\kappa \;\le\; 4C_0^2 \varepsilon^2,\end{aligned}$$

where $(\mathcal{X}_i, \mathcal{X}_i)_\kappa = \int_M \xi_i^2 \{\omega(\varepsilon)\}^n / A_\kappa > 0$ is independent of the choice of ε (cf. [FM;p.208]). By letting $\varepsilon \to 0$, we have $(\mathcal{X}_i, \mathcal{Y}_j)_\kappa = (\mathcal{Y}_j, \mathcal{Y}_{j'})_\kappa = 0$. Hence,

THEOREM 5.3. *$\mathfrak{u} = \{ Z \in \mathfrak{g}; (Z, W)_\kappa = 0$ for all $W \in \mathfrak{g} \}$.*

Remark. Let $\mathfrak{g}^{q+1} \ni (W_0, W_1, \dots, W_q) \mapsto (W_0, W_1, W_2, \dots, W_q)_\kappa \in \mathbb{C}$ be the symmetric $\mathbb{C}$-multilinear form as defined in [FM; p.209], where q is an arbitrary positive integer. Then by the same argument as above, we can easily show that $\mathfrak{u} = \{Z \in \mathfrak{g}; (Z, W_1, \dots, W_q)_\kappa = 0$ for all $(W_1, \dots, W_q) \in \mathfrak{g}^q\}$.

For each $\omega' \in \mathcal{K}$, let $f_{\omega'}$ denote the real-valued C^∞ function on M such that $\sigma(\omega') - C_\kappa = \Box_{\omega'} f_{\omega'}$. The associated Futaki character $F_\kappa : \mathfrak{g} \to \mathbb{C}$ is defined by

$$F_\kappa(W) := (\sqrt{-1})^{-1} \int_M (W f_{\omega'}) \omega'^n / A_\kappa, \qquad W \in \mathfrak{g}.$$

This F_κ depends only on κ and is independent of the choice of ω' in $\mathcal{K}$. Each element W in $\mathfrak{g}$ is written as $\operatorname{grad}^{\mathbb{C}}_{\omega'} \phi$ for some unique $\phi \in \tilde{\mathfrak{g}}_{\omega'}$. Then

$$F_\kappa(W) = \int_M (\sigma(\omega') - C_\kappa) \phi \omega'^n / A_\kappa, \tag{5.4}$$

in view of the computation in [FM; (2.1)] (see also [LS]). We now consider a one-parameter subgroup $g_t := \exp(t Z_{\mathbb{R}})$, $t \in \mathbb{R}$, of G, under the assumption that

$$\omega' \in \underline{\mathcal{K}} \quad \text{and} \quad Z \in \mathfrak{z}(V). \tag{5.5}$$

Since $\mathfrak{g}$ is a direct sum $\mathfrak{k}^{\mathbb{C}}_{\omega'} \oplus \mathfrak{u}$ as a vector space, Z is written as a sum $X + Y$ for some $X \in \mathfrak{k}^{\mathbb{C}}_{\omega'}$ and $Y \in \mathfrak{u}$, where there uniquely exist $\xi \in \tilde{\mathfrak{k}}^{\mathbb{C}}_{\omega'}$ and $\eta \in \tilde{\mathfrak{g}}_{\omega'}$ such that $X = \operatorname{grad}^{\mathbb{C}}_{\omega'} \xi$ and $Y = \operatorname{grad}^{\mathbb{C}}_{\omega'} \eta$. Note also that $\omega_t := g_t^* \omega'$ is written uniquely as ω_{ψ_t} for some smooth path $\{\psi_t; t \in \mathbb{R}\}$ in $\tilde{\mathcal{K}}$ satisfying $\int_M \dot{\psi}_t \omega_t^n = 0$ for all $t \in \mathbb{R}$. Then $\dot{\psi}_t = 2 \operatorname{Im}(\xi + \eta)$ at $t = 0$. Since $\sigma(\omega') - \operatorname{pr}_{\omega'} \sigma(\omega') \in (\mathbb{R} \oplus \tilde{\mathfrak{k}}_{\omega'})^\perp$, we obtain

$$\begin{aligned}
&\left(\int_M \{\sigma(\omega_t) - \operatorname{pr}_{\omega_t} \sigma(\omega_t)\} \dot{\psi}_t \omega_t^n / A_\kappa \right)_{|t=0} \\
&= 2 \operatorname{Im} \left(\int_M \{\sigma(\omega') - \operatorname{pr}_{\omega'} \sigma(\omega')\} (\xi + \eta) \omega'^n / A_\kappa \right) \\
&= 2 \operatorname{Im} \left(\int_M \{\sigma(\omega') - \operatorname{pr}_{\omega'} \sigma(\omega')\} \eta \omega'^n / A_\kappa \right).
\end{aligned}$$

On the other hand, by Theorem 5.3, $\int_M \mathrm{pr}_{\omega'}\sigma(\omega')\eta\omega'^n/A_\kappa = 0$. Further by (5.4) and [N1], $\int_M \sigma(\omega')\eta\omega'^n/A_\kappa = \int_M(\sigma(\omega') - C_\kappa)\eta\omega'^n/A_\kappa = F_\kappa(\mathcal{Y}) = 0$. Hence,

$$(5.6) \qquad \left(\int_M \{\sigma(\omega_t) - \mathrm{pr}_{\omega_t}\sigma(\omega_t)\}\,\dot{\psi}_t\omega_t^n/A_\kappa\right)_{|t=0} \;=\; 0.$$

Let V and $\hat{\mu}_V : \mathcal{H}_V \to \mathbb{R}$ be as in the previous section. For each $g \in G$, the extremal Kähler vector field for $g^*\omega$ is $V^g := (g^{-1})_* V = \mathrm{Ad}(g^{-1})V$. Replacing V by V^g in the definition of $\mathcal{H}_V$, we obtain

$$\mathcal{H}_{V^g} \;:=\; \{\omega' \in \underline{\mathcal{K}}; \mathcal{V}_{\omega'} = V^g\},$$

which is just the pullback $g^*\mathcal{H}_V$ of $\mathcal{H}_V$ via g. Then the corresponding functional which replaces $\hat{\mu}_V$ will be denoted by $\hat{\mu}^g : \mathcal{H}_{V^g} \to \mathbb{R}$. We can actually define $\hat{\mu}^g : \mathcal{H}_{V^g} \to \mathbb{R}$ by

$$\hat{\mu}^g(g^*\omega') \;:=\; \hat{\mu}_V(\omega') \qquad \text{for all } \ \omega' \in \mathcal{H}_V,$$

where by (3.1), the functionals $\hat{\mu}^g$ and $\hat{\mu}_{V^g} := \mu + \eta_{V^g}$ on $\mathcal{H}_{V^g}$ differ just by a constant. Hence, if $V^{g_1} = V^{g_2}$ for some g_1, $g_2 \in G$, the corresponding functionals $\hat{\mu}^{g_1}$, $\hat{\mu}^{g_2}$ differ by a constant. Obviously, $\hat{\mu}_e$ is just $\hat{\mu}_V$ if e is the unit of G. Note that $\mathcal{H}_{V^{g_1}} \cap \mathcal{H}_{V^{g_2}} = \emptyset$ if $V^{g_1} \neq V^{g_2}$. In view of

$$\underline{\mathcal{K}} \;=\; \bigcup_{g\in G} \mathcal{H}_{V^g},$$

the functionals $\hat{\mu}^g : \mathcal{H}_{V^g} \to \mathbb{R}$, $g \in G$, glue together to define a G-invariant functional $\hat{\mu} : \underline{\mathcal{K}} \to \mathbb{R}$ on $\underline{\mathcal{K}}$ satisfying the equality

$$\hat{\mu}_{|\mathcal{H}_{V^g}} \;=\; \hat{\mu}^g, \qquad \text{for all } g \in G,$$

if we can show Proposition 5.7 below. Here, the G-invariance of $\hat{\mu}$ means that the equality $\hat{\mu}(g^*\omega') = \hat{\mu}(\omega')$ holds for all pairs (g, ω') in $G \times \underline{\mathcal{K}}$.

Proposition 5.7. *If $g \in Z(V)$, then $\hat{\mu}^g = \hat{\mu}_V$.*

Proof. If $g \in Z(V)$, then $V^g = V$, and hence $\mathcal{H}_{V^g} = \mathcal{H}_V$. Let θ be an arbitrary element of $\mathcal{H}_V$. It then suffices to show $\hat{\mu}_V(g^*\theta) = \hat{\mu}_V(\theta)$ for all $g \in Z(G)$. Take an arbitrary element X in $\mathfrak{z}(V)$, and we put $h_t := \exp(tX_{\mathbb{R}})$

and $\omega_t := (h_t)^*\theta$ for each $t \in \mathbb{R}$. Since $Z(V)$ is connected, the proof is reduced to showing the following infinitesimal equality:

$$\frac{d}{dt}\hat{\mu}_V(\omega_t)_{|t=0} = 0.$$

For some smooth path $\{\psi_t; t \in \mathbb{R}\}$ in $\tilde{\mathcal{K}}$ satisfying $\int_M \dot{\psi}_t \omega_t^n / A_\kappa = 0$, $t \in \mathbb{R}$, the Kähler form ω_t above is written as ω_{ψ_t} for each t. Moreover, we write X as $\operatorname{grad}^{\mathbb{C}}_{\omega_t} \phi_t$ for some $\phi_t \in \tilde{\mathfrak{g}}_{\omega_t}$. Then by (4.4) and (5.6), we have the following identity as required:

$$\frac{d}{dt}\hat{\mu}_V(\omega_t)_{|t=0} \;=\; -\left(\int_M \{\sigma(\omega_t) - \operatorname{pr}_{\omega_t}\sigma(\omega_t)\}\, \dot{\psi}_t \omega_t^n / A_\kappa\right)_{|t=0} \;=\; 0.$$

For every quantized pair (M, κ), we can thus define a G-invariant functional $\hat{\mu} : \underline{\mathcal{K}} \to \mathbb{R}$ as above. By [C1], all extremal Kähler metrics in the cohomology class κ belong to $\underline{\mathcal{K}}$. On the other hand, the definition of $\hat{\mu}$ shows that

Theorem 5.8. *An element ω' of $\underline{\mathcal{K}}$ is a critical point for the functional $\hat{\mu} : \underline{\mathcal{K}} \to \mathbb{R}$ if and only if ω' is an extremal Kähler metric.*

Remark. In this remark, we delete the assumption that the pair (M, κ) is quantized. Suppose that the Kähler class κ admits an extremal Kähler metric ω. Let $V := \mathcal{V}_\omega$ be the associated extremal Kähler vector field. Then by [C1; (3.9)][3], the subgroups $Z(V)$ and $K^{\mathbb{C}}_\omega$ of G coincide. Hence, in this case, the functionals $\hat{\mu}^g : \mathcal{H}_{V^g} \to \mathbb{R}$, $g \in G$, glue together to define a G-invariant functional $\hat{\mu} : \underline{\mathcal{K}} \to \mathbb{R}$ such that Theorem 5.8 above is valid even when the pair (M, κ) is not necessarily quantized.

§6. Functional $\hat{\nu}$ whose critical points are "Kähler-Einstein metrics"

Throughout this section this section, we assume that the Kähler class κ in the introduction is $2\pi c_1(M)_{\mathbb{R}}$. Moreover, the anticanonical line bundle K_M^{-1} of M is chosen as the line bundle L in §5. Since the G-action on M naturally lifts to a G-action on K_M, the pair (M, κ) is quantized in the

[3] In the decomposition $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{k}' \oplus \mathfrak{m} \oplus \sum_{\lambda > 0} \mathfrak{h}_\lambda = \mathfrak{a} \oplus \mathfrak{h}'_0 \oplus \sum_{\lambda > 0} \mathfrak{h}_\lambda$ in [C1; (3.9)], note that the vector spaces $\mathfrak{k}' \oplus \mathfrak{m} \oplus \sum_{\lambda > 0} \mathfrak{h}_\lambda$, $\mathfrak{h}'_0$, $\mathfrak{k}' \oplus \mathfrak{m}$ are respectively $\mathfrak{g}$, $\mathfrak{z}(V)$, $\mathfrak{k}_\omega \oplus \sqrt{-1}\mathfrak{k}_\omega = \mathfrak{k}^{\mathbb{C}}_\omega$ in our notation.

sense of §5. By $\emptyset \neq \underline{\mathcal{K}} \subset \mathcal{K}$, we fix first of all an element ω in $\underline{\mathcal{K}}$. Then we have a unique element θ of $\underline{\mathcal{K}}$ such that $R(\theta) = \omega$. As in [BM1] (see also [BM2]), we assign to each pair $(\theta', \theta'') \in \mathcal{K} \times \mathcal{K}$ a real number $N(\theta', \theta'') \in \mathbb{R}$ by

$$N(\theta', \theta'') \;:=\; \int_a^b \left\{ \int_M (\Box_{\theta_t} \dot{u}_t)\; R(\theta_t)^n / A_\kappa \right\} dt,$$

where $\{u_t; a \leq t \leq b\}$ is an arbitrary piecewise smooth path in $\tilde{\mathcal{K}}$ such that the associated path $\theta_t := \omega_{u_t}$, $a \leq t \leq b$, in $\mathcal{K}$ satisfies $\theta' = \theta_a$ and $\theta'' = \theta_b$. Let $D_\omega : \tilde{\mathcal{K}} \to \mathbb{R}$ be the functional in [D1] (see also [DT]) defined by

$$\begin{aligned} D_\omega(\psi) \;:=\; & \sqrt{-1} \sum_{i=0}^{n-1} \frac{n-i}{n+1} \int_M \partial\psi \wedge \bar{\partial}\psi \wedge \omega^{n-1-i} \wedge \omega_\psi^i / A_\kappa \\ & - \int_M \psi \omega^n / A_\kappa \;-\; \log\left(\int_M e^{f_\omega - \psi} \omega^n / A_\kappa \right), \end{aligned}$$

where for each $\omega' \in \mathcal{K}$, the function $f_{\omega'} \in C^\infty(M)_{\mathbb{R}}$ is defined by the equalities $R(\omega') = \omega' + \sqrt{-1}\partial\bar{\partial} f_{\omega'}$ and $\int_M (1 - e^{f_{\omega'}}) \omega'^n / A_\kappa = 0$. For each $\psi \in \tilde{\mathcal{K}}$, let θ^ψ denote the unique element in $\mathcal{K}$ defined by $R(\theta^\psi) = \omega_\psi$. It is easy to check that

$$D_\omega(\psi) \;=\; N(\theta, \theta^\psi).$$

Define a functional $\nu : \mathcal{K} \to \mathbb{R}$ by setting $\nu(\omega_\psi) := D_\omega(\psi) = N(\theta, \theta^\psi)$ for each $\psi \in \tilde{\mathcal{K}}$. Given a pair $(\omega', \omega'') \in \mathcal{K} \times \mathcal{K}$, let us consider an arbitrary smooth path $\{\psi_t; a \leq t \leq b\}$ in $\tilde{\mathcal{K}}$ such that $\omega' = \omega_a$ and $\omega'' = \omega_a$, where we set $\omega_t := \omega_{\psi_t}$, $a \leq t \leq b$, for simplicity. Then

$$\text{(6.1)} \qquad \frac{d}{dt} \nu(\omega_t) \;=\; - \int_M (1 - e^{f_{\omega_t}}) \dot{\psi}_t \omega_t^n / A_\kappa, \qquad a \leq t \leq b,$$

and the set of the critical points for ν consists of all Kähler-Einstein metrics on M. Let $W \in \mathfrak{g}$ and $\omega' \in \mathcal{K}$. Then $W = \operatorname{grad}^{\mathbb{C}}_{\omega'} \phi$ for some $\phi \in \tilde{\mathfrak{g}}_{\omega'}$. By the same computation as in [M4; §2], we obtain

$$\text{(6.2)} \qquad \int_M (1 - e^{f_{\omega'}}) \phi \omega'^n / A_\kappa \;=\; \int_M (\sigma(\omega') - n) \phi \omega'^n / A_\kappa \;=\; F_\kappa(W),$$

where for κ as above, we have $C_\kappa = n$. As in the last section, let V denote the extremal Kähler vector field $\mathcal{V}_\omega$ of (M, ω). Define a functional $\hat{\nu}_V : \mathcal{H}_V \to \mathbb{R}$ by

$$\hat{\nu}_V \;:=\; \nu \;+\; \eta_V.$$

Then by [M4; 2.1], we see that $V = \mathrm{grad}^{\mathbb{C}}_{\omega_t} \mathrm{pr}_{\omega_t} \sigma(\omega_t) = \mathrm{grad}^{\mathbb{C}}_{\omega_t} \mathrm{pr}_{\omega_t}(1-e^{f_{\omega_t}})$. It now follows from (1.1) and (6.1) that

$$(6.3) \qquad \frac{d}{dt}\hat{\nu}_V(\omega_t) \; = \; -\int_M \{(1-e^{f_{\omega_t}}) - \mathrm{pr}_{\omega_t}(1-e^{f_{\omega_t}})\}\dot{\psi}_t\omega_t^n/A_\kappa,$$

for all $a \leq t \leq b$. Recall that an element ω' of $\mathcal{K}$ is called a "Kähler-Einstein metric" if $1 - e^{f_{\omega'}} \in \tilde{\mathfrak{k}}_{\omega'}$ (cf. [M4]). We now obtain

Proposition 6.4. *An element ω' of $\mathcal{H}_V$ is a critical point for the functional $\hat{\nu}_V : \mathcal{H}_V \to \mathbb{R}$ if and only if ω' is a "Kähler-Einstein metric" in the sense of* [M4].

For each $g \in G$, the extremal Kähler vector field for $g^*\omega$ is $V^g := (g^{-1})_*V = \mathrm{Ad}(g^{-1})V$. Furthermore, $\mathcal{H}_{V^g} = \{\omega' \in \underline{\mathcal{K}}; \mathcal{V}_{\omega'} = V^g\} = g^*\mathcal{H}_V$. In view of (3.1), we can define the corresponding functional $\hat{\nu}^g : \mathcal{H}_{V^g} \to \mathbb{R}$ by

$$\hat{\nu}^g(g^*\omega') \; := \; \hat{\nu}_V(\omega') \qquad \text{for all } \; \omega' \in \mathcal{H}_V.$$

Then $\hat{\nu}^g$ depends smoothly on $g \in G$, where $\hat{\nu}^g$ coincides with $\hat{\nu}_V$ if g is the unit e of G. Moreover, if $V^{g_1} \neq V^{g_2}$, then $\mathcal{H}_{V^{g_1}} \cap \mathcal{H}_{V^{g_2}} = \emptyset$. In view of

$$\underline{\mathcal{K}} \; = \; \bigcup_{g\in G} \mathcal{H}_{V^g},$$

the functionals $\hat{\nu}^g : \mathcal{H}_{V^g} \to \mathbb{R}$, $g \in G$, glue together to define a G-invariant functional $\hat{\nu} : \underline{\mathcal{K}} \to \mathbb{R}$ on $\underline{\mathcal{K}}$ in such a way that

$$\hat{\nu}_{|\mathcal{H}_{V^g}} \; = \; \hat{\nu}^g, \qquad \text{for all } g \in G,$$

if we can show Proposition 6.5 below, where the G-invariance of $\hat{\nu}$ means that the equality $\hat{\nu}(g^*\omega') = \hat{\nu}(\omega')$ holds for all pairs (g, ω') in $G \times \underline{\mathcal{K}}$.

Proposition 6.5. *If $g \in Z(V)$, then $\hat{\nu}^g = \hat{\nu}_V$.*

Proof. Let $X \in \mathfrak{z}(V)$ and $\theta \in \mathcal{H}_V$. Put $\omega_t := (\exp tX_{\mathbb{R}})^*\theta$, $t \in \mathbb{R}$. As in the proof of Proposition 5.7, it suffices to show

$$\frac{d}{dt}\hat{\nu}_V(\omega_t)_{|t=0} \; = \; 0.$$

Here, ω_t is written as ω_{ψ_t} for some smooth path $\{\psi_t; t \in \mathbb{R}\}$ in $\tilde{\mathcal{K}}$, where $\int_M \dot{\psi}_t\omega_t^n/A_\kappa = 0$ for all t. Moreover, write X as $\mathrm{grad}^{\mathbb{C}}_{\omega_t} \phi_t$ for some $\phi_t \in \tilde{\mathfrak{g}}_{\omega_t}$.

By (6.2), the arguments deducing (5.6) is valid even when we replace $\sigma(\omega')$ and $\sigma(\omega_t)$ respectively by $1 - e^{f_{\omega'}}$ and $1 - e^{f_{\omega_t}}$. Therefore,

$$\left(\int_M \left\{ (1 - e^{f_{\omega_t}}) - \mathrm{pr}_{\omega_t}(1 - e^{f_{\omega_t}}) \right\} \dot{\psi}_t \omega_t^n / A_\kappa \right)_{|t=0} = 0. \tag{6.6}$$

Then by (6.3) and (6.6), we obtain the following required identity:

$$\frac{d}{dt}\hat{\nu}(\omega_t)_{|t=0} = -\left(\int_M \left\{ (1 - e^{f_{\omega_t}}) - \mathrm{pr}_{\omega_t}(1 - e^{f_{\omega_t}}) \right\} \dot{\psi}_t \omega_t^n / A_\kappa \right)_{|t=0} = 0.$$

Recall that all "Kähler-Einstein metrics" in the cohomology class κ belong to $\underline{\mathcal{K}}$ (cf. [M4;§4]). From the definition of the functional $\hat{\nu}$ above, we further obtain:

THEOREM 6.7. *An element ω' of $\underline{\mathcal{K}}$ is a critical point for the functional $\hat{\nu} : \underline{\mathcal{K}} \to \mathbb{R}$ if and only if ω' is a "Kähler-Einstein metric" in the sense of* [M4].

§7. Convexity of $\hat{\nu}$ applied to the proof of Theorem C

For each maximal compact subgroup K of G, let $\mathcal{K}^K$ and $\tilde{\mathcal{K}}^K$ denote the set of all K-invariant elements in $\mathcal{K}$ and $\tilde{\mathcal{K}}$, respectively (cf. §3). Then $\underline{\mathcal{K}}$ is written in the form

$$\underline{\mathcal{K}} = \cup_K \mathcal{K}^K,$$

where the union is taken over all maximal compact subgroups K of G. For such a K, we always have $\mathcal{K}^K \neq \emptyset$, and there exists an element ω of $\underline{\mathcal{K}}$ such that $K_\omega = K$. Let $\mathfrak{k}$ be the Lie subalgebra of $\mathfrak{g}$ corresponding to the Lie subgroup K of G. Then

$$\mathcal{K}^K = \{\omega_\psi ; \psi \in \tilde{\mathcal{K}}^K\} = \{\omega' \in \mathcal{K} ; \mathfrak{k}_{\omega'} = \mathfrak{k}\} \subset \mathcal{H}_V \subset \underline{\mathcal{K}},$$

where $V := \mathcal{V}_\omega$ is the extremal Kähler vector field of the Kähler manifold (M, ω). Note that, on $\mathcal{K}^K$, the functionals $\hat{\nu}$ and $\hat{\nu}_V$ coincide. We induce connections on $\mathcal{K}^K$ and $\tilde{\mathcal{K}}$ respectively from the connections (cf. [M2]) on $\mathcal{K}$ and $\tilde{\mathcal{K}}$. The purpose of this section is to show that the functional $\hat{\nu}$ is convex when restricted to $\mathcal{K}^K$. As an application of the convexity, we also show the uniqueness of "Kähler-Einstein metrics" (see [M4]) for toric Fano manifolds, modulo connected group actions, by the method as used by [G1] for extremal Kähler metrics.

Fix an arbitrary element ω_0 of $\mathcal{K}^K$. Let ζ be a K-invariant element in $C^\infty(M)_\mathbb{R}$ such that $\int_M \zeta\ \omega_0^n/A_\kappa = 0$. For an $0 < \varepsilon \ll 1$, choose a smooth path $\psi = \{\psi_t; -\varepsilon \le t \le \varepsilon\}$ in $\tilde{\mathcal{K}}^K$ such that $\dot\psi_{t|t=0} = \zeta$ and that $\int_M \dot\psi_t \omega_t^n/A_\kappa = 0$ for all t, where the associated path

$$\omega_t := \omega_{\psi_t}, \qquad -\varepsilon \le t \le \varepsilon, \tag{7.1}$$

in $\mathcal{K}^K$ passes through ω_0 at $t = 0$. We now consider the smooth one-parameter family $\dot\psi$ of C^∞ functions on M defined by

$$\dot\psi \ := \ \{\dot\psi_t; -\varepsilon \le t \le \varepsilon\}.$$

Let us write $\omega_t = \sum_{\alpha,\beta} (g_t)_{\alpha\bar\beta} dz^\alpha \wedge dz^{\bar\beta}$ by using a system $(z^1, \dots, z^n)$ of holomorphic local coordinates on M. To each smooth one-parameter family $\eta = \{\eta_t; -\varepsilon \le t \le \varepsilon\}$ of C^∞ functions on M, we put

$$(\frac{D}{\partial t}\eta)_t \ = \ \dot\eta_t \ - \ \frac{1}{2}\sum_{\alpha,\beta} (g_t)^{\bar\beta\alpha} \left(\frac{\partial\dot\psi_t}{\partial z^\alpha}\frac{\partial\eta_t}{\partial z^{\bar\beta}} \ + \ \frac{\partial\dot\psi_t}{\partial z^{\bar\beta}}\frac{\partial\eta_t}{\partial z^\alpha} \right), \qquad -\varepsilon \le t \le \varepsilon.$$

Then $\frac{D}{\partial t}\eta \ = \ \{(\frac{D}{\partial t}\eta)_t; -\varepsilon \le t \le \varepsilon\}$ is the smooth one-parameter family of C^∞ functions on M obtained as the covariant derivative of η along the path ψ (cf. [M2]). In tangential directions of $\mathcal{K}^K$, the Hessian $\mathrm{Hess}\,\hat\nu$ of $\hat\nu$ at ω_0 is given by

$$\begin{aligned}(\mathrm{Hess}\,\hat\nu)_{\omega_0}(\zeta,\zeta) &= \frac{d^2}{dt^2}\hat\nu_V(\omega_t)_{|t=0} \\ &+ \int_M \{(1 - e^{f_{\omega_0}}) - \mathrm{pr}_{\omega_0}(1 - e^{f_{\omega_0}})\}(\frac{D}{\partial t}\dot\psi)_{t=0}\omega_0^n/A_\kappa.\end{aligned} \tag{7.2}$$

For required convexity, it now suffices to show that $(\mathrm{Hess}\,\hat\nu)_{\omega_0}(\zeta,\zeta)$ above is always nonnegative. For smooth one-parameter families $\xi = \{\xi_t; -\varepsilon \le t \le \varepsilon\}$, $\eta = \{\eta_t; -\varepsilon \le t \le \varepsilon\}$ of C^∞ functions on M, we define

$$\langle\!\langle \xi, \eta \rangle\!\rangle_t \ = \ \int_M \xi_t\eta_t\, \omega_t^n,$$

where ω_t, $-\varepsilon \le t \le \varepsilon$, are as in (7.1). For the extremal Kähler vector field V, there exists a one-parameter family $\phi = \{\phi_t; -\varepsilon \le t \le \varepsilon\}$ of real-valued C^∞ functions on M such that $\int_M \phi_t\omega_t^n/A_\kappa = 0$ for all t, and that

$$V = \mathrm{grad}^{\mathbb{C}}_{\omega_t}\phi_t, \qquad -\varepsilon \le t \le \varepsilon.$$

Then by [FM],we have $\phi_t = \phi_0 + \sqrt{-1}\ V\psi_t$. Since ψ_t is a K-invariant function, and since $V \in \mathfrak{k}$, it follows that $(V + \bar{V})\dot{\psi}_t = V_{\mathbb{R}}\dot{\psi}_t = 0$, and therefore

$$\begin{aligned}(7.3)\qquad \dot{\phi}_t &= \sqrt{-1}\, V\dot{\psi}_t \\ &= \frac{1}{2}\sum_{\alpha,\beta}(g_t)^{\bar{\beta}\alpha}\left(\frac{\partial\dot{\psi}_t}{\partial z^\alpha}\frac{\partial\phi_t}{\partial z^{\bar{\beta}}} + \frac{\partial\dot{\psi}_t}{\partial z^{\bar{\beta}}}\frac{\partial\phi_t}{\partial z^\alpha}\right), \text{ i.e., } \frac{D}{\partial t}\phi = 0,\end{aligned}$$

(see [G1]). On the other hand, by $\omega_t \in \mathcal{K}^K$, we have $V = \mathrm{grad}^{\mathbb{C}}_{\omega_t}\mathrm{pr}_{\omega_t}(1 - e^{f_{\omega_t}})$. It is now easy to check that $\phi_t = \mathrm{pr}_{\omega_t}(1 - e^{f_{\omega_t}})$ for all t. For simplicity, let $1 - e^f$ denote the one-parameter family $\{1 - e^{f_{\omega_t}}; -\varepsilon \leq t \leq \varepsilon\}$ of C^∞ functions on M. Then by (6.3),

$$(7.4)\qquad \frac{d}{dt}\hat{\nu}_V(\omega_t) = -\langle\!\langle 1 - e^f - \phi, \dot{\psi}\rangle\!\rangle_t.$$

We now put $\varphi_t := \psi_t + C_t$, where each $C_t \in \mathbb{R}$ is a constant depending smoothly on t such that $\int_M \dot{\varphi}_t\widetilde{\omega_t^n} = 0$ for all t. Here, $\widetilde{\omega_t^n} := e^{f_{\omega_t}}\omega_t^n/A_\kappa$. We also let

$$\tilde{\square}_t := \square_{\omega_t} + \sum_{\alpha,\beta}(g_t)^{\bar{\beta}\alpha}\frac{\partial f_{\omega_t}}{\partial z^\alpha}\frac{\partial}{\partial z^{\bar{\beta}}}.$$

Consider the smooth one-parameter family $\dot{\varphi} := \{\dot{\varphi}_t; -\varepsilon \leq t \leq \varepsilon\}$ of C^∞ functions on M. Then by $\langle\!\langle 1 - e^f - \phi, \dot{\psi}\rangle\!\rangle_t = \langle\!\langle 1 - e^f - \phi, \dot{\varphi}\rangle\!\rangle_t$, replacing $\dot{\psi}$ by $\dot{\varphi}$ in (7.4) and differentiating this with respect to t, we obtain

$$(7.5)\qquad \frac{d^2}{dt^2}\hat{\nu}_V(\omega_t) = -\langle\!\langle 1 - e^f - \phi, \frac{D}{\partial t}\dot{\varphi}\rangle\!\rangle_t - \langle\!\langle \frac{D}{\partial t}(1 - e^f - \phi), \dot{\varphi}\rangle\!\rangle_t.$$

Therefore, it follows from (7.2), (7.3) and (7.5) that

$$\begin{aligned}(\mathrm{Hess}\,\hat{\nu})_{\omega_0}(\zeta,\zeta) &= \frac{d^2}{dt^2}\hat{\nu}_V(\omega_t)_{|t=0} + \langle\!\langle 1 - e^f - \phi, \frac{D}{\partial t}\dot{\psi}\rangle\!\rangle_{t=0} \\ &= \frac{d^2}{dt^2}\hat{\nu}_V(\omega_t)_{|t=0} + \langle\!\langle 1 - e^f - \phi, \frac{D}{\partial t}\dot{\varphi}\rangle\!\rangle_{t=0} \\ &= -\langle\!\langle \frac{D}{\partial t}(1 - e^f - \phi), \dot{\varphi}\rangle\!\rangle_{t=0} = -\langle\!\langle \frac{D}{\partial t}(1 - e^f), \dot{\varphi}\rangle\!\rangle_{t=0}.\end{aligned}$$

For simplicity, we put $f_t := f_{\omega_t}$. Recall that $\dot{f}_t = -\square_{\omega_t}\dot{\varphi}_t - \dot{\varphi}_t + B_t$ for some constant $B_t \in \mathbb{R}$ (cf. [F1]). Let $\mathrm{Re}(\dots)$ denote the real part. Then by

$$-\langle\!\langle \frac{D}{\partial t}(1 - e^f), \dot{\varphi}\rangle\!\rangle_t$$

$$= \int_M \left\{ \dot{f}_t - \frac{1}{2} \sum_{\alpha,\beta} (g_t)^{\bar{\beta}\alpha} \left(\frac{\partial \dot{\psi}_t}{\partial z^\alpha} \frac{\partial f_t}{\partial z^{\bar{\beta}}} + \frac{\partial \dot{\psi}_t}{\partial z^{\bar{\beta}}} \frac{\partial f_t}{\partial z^\alpha} \right) \right\} \dot{\varphi}_t \widetilde{\omega_t^n}$$

and by $\int_M \dot{\varphi}_t \widetilde{\omega_t^n} = 0$, we now obtain

$$\begin{aligned} -\langle\!\langle \frac{D}{\partial t}(1 - e^f), \dot{\varphi} \rangle\!\rangle_t &= \int_M \left\{ - \operatorname{Re}(\tilde{\Box}_t \dot{\varphi}_t) - \dot{\varphi}_t \right\} \dot{\varphi}_t \widetilde{\omega_t^n} \\ &= \operatorname{Re} \left\{ \int_M - \left(\tilde{\Box}_t \dot{\varphi}_t + \dot{\varphi}_t \right) \dot{\varphi}_t \widetilde{\omega_t^n} \right\} \ \geq \ 0, \end{aligned}$$

since the eigenvalues of $-\tilde{\Box}_t$ are all real, and its first positive eigenvalue is bouded from below by 1 (cf. [F2]). Thus $(\operatorname{Hess} \hat{\nu})_{\omega_0}(\zeta, \zeta) \geq 0$, as required.

Remark. Let M be a nonsingular toric variety defined over $\mathbb{C}$. By the convexity of $\hat{\mu}_V$ along $\mathcal{K}^K$, [G1] shows that the extremal Kähler metrics in each Kähler class are unique up to the action of $G = \operatorname{Aut}^0(M)$. By the convexity of $\hat{\nu}$ along $\mathcal{K}^K$ shown just above, we can similarly prove in (7.6) the uniqueness of "Kähler-Einstein metrics" up to the action of $G = \operatorname{Aut}^0(M)$ when M is a nonsingular toric Fano variety.

(7.6) *Proof of Theorem* C. Let $\mathcal{E}$ be the set of all "Kähler-Einstein metrics" (cf. [M4]) in the class $2\pi c_1(M)_{\mathbb{R}}$. It then suffices to show that $\mathcal{E}$ is connected. Let ω_0, $\omega_1 \in \mathcal{E}$. Replacing ω_1 by $g^*\omega_1$ for some $g \in G$ if necessary, we may assume that both ω_0 and ω_1 belong to $\mathcal{K}^K$ for some maximal compact subgroup K of G. Since M is toric, the arguments as in [G1] allows us to connect ω_0 and ω_1 by a geodesic ω_t, $0 \leq t \leq 1$, in $\mathcal{K}^K$. In view of the convexity of $\hat{\nu}$ along $\mathcal{K}^K$, we have

$$\begin{aligned} &\frac{d}{dt}\hat{\nu}(\omega_t)_{|t=0} \ = \ \frac{d}{dt}\hat{\nu}(\omega_t)_{|t=1} \ = \ 0; \\ &\frac{d^2}{dt^2}\hat{\nu}(\omega_t) \ \geq \ 0, \qquad 0 \leq t \leq 1. \end{aligned}$$

Therefore, $\hat{\nu}(\omega_t)$ is constant on the closed interval $\{0 \leq t \leq 1\}$. Then it is easily seen that $\hat{\nu}(\omega_t)$ is a critical point of $\hat{\nu}$ for all t, and hence $\mathcal{E}$ is connected. (In fact, the geodesic ω_t, $0 \leq t \leq 1$, can be written as[4]

$$\omega_t \ = \ \{\exp(tZ_{\mathbb{R}})\}^* \omega_0$$

[4] In relation to this expression, we here note that Theorem 3.5 in [M2] is true under the additional assumption that Y is in the center of $\mathfrak{k}_\omega^{\mathbb{C}}$, though it is incorrect without any such assumption.

for some $Z \in \sqrt{-1}\,\mathfrak{z}(\mathfrak{k})$, where $\mathfrak{z}(\mathfrak{k})$ denotes the center of $\mathfrak{k}$.)

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Department of Mathematics, Osaka University
Toyonaka, Osaka, 560-0043
Japan
mabuchi@math.wani.osaka-u.ac.jp