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VERY AMPLENESS OF ADJOINT LINEAR SYSTEMS ON SMOOTH SURFACES WITH BOUNDARY

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Abstract. Let M be a $\mathbb{Q}$-divisor on a smooth surface over $\mathbb{C}$. In this paper we give criteria for very ampleness of the adjoint of $\lceil M \rceil$, the round-up of M. (Similar results for global generation were given by Ein and Lazarsfeld and used in their proof of Fujita's Conjecture in dimension 3.) In §4 we discuss an example which suggests that this kind of criteria might also be useful in the study of linear systems on surfaces.

Notations

$\lceil \cdot \rceil$	round-up
$\lfloor \cdot \rfloor$	round-down
$\{\cdot\}$	fractional part
$f^{-1}D$	strict transform (proper transform)
f^*D	pull-back (total inverse image)
PLC	partially log-canonical (Definition 1.7)
$\equiv$	numerical equivalence
$\sim$	linear equivalence
$\underset{\mathbb{Q}}{\sim}$	$\mathbb{Q}$-linear equivalence

0. Introduction

Let S be a nonsingular projective surface over $\mathbb{C}$, and let H be a given line bundle on S. Consider the following natural questions regarding the complete linear system $|H|$:

(1) *Compute* $\dim |H|$.

(2) *Is* $|H|$ *base-point-free?*

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(3) *Is $|H|$ very ample?*

The answer to (1) is usually given in two parts: the Riemann-Roch theorem computes $\chi(S, H)$, and then we need estimates for $h^i(S, H)$, $i > 0$. In particular, we may ask the following question related to (1):

($1'$) *When are $h^1(S, H)$ and $h^2(S, H)$ equal to zero?*

One classical answer to ($1'$) is provided by Kodaira's vanishing theorem: if L is any ample line bundle on S, then $h^i(S, -L) = 0$ for all $i < 2$; therefore, by Serre duality, we have $h^i(S, K_S + L) = 0$ for all $i > 0$. To answer ($1'$), write $H = K_S + L$ (thus defining L as $H - K_S$); if L is ample, then $h^i(S, H) = 0$ for all $i > 0$.

For questions (2) and (3), Reider [Rei] gave an answer which again considers H in the form of an adjoint line bundle, $H = K_S + L$:

Proposition (cf. [Rei, Theorem 1]). *If L is a line bundle on S, $L^2 \geq 5$ and $L \cdot C \geq 2$ for every curve $C \subset S$, then $|K_S + L|$ is base-point-free. If $L^2 \geq 10$ and $L \cdot C \geq 3$ for every curve C, then $|K_S + L|$ is very ample.*

We note here that Kodaira's theorem holds in all dimensions. Reider's criterion was tentatively extended in higher dimensions in the form of Fujita's conjecture ([Fuj]): if X is a smooth projective variety of dimension n, and L is an ample line bundle on X, then $|K_X + mL|$ is base-point-free for $m \geq n + 1$ and very ample for $m \geq n + 2$. Fujita's conjecture for base-point-freeness was proved in dimension 3 by Ein and Lazarsfeld ([EL]) and in dimension 4 by Kawamata ([Kaw]); more precise statements, which resemble Reider's criterion more closely, were also obtained. Very ampleness, however, is still open, even in dimension 3.

Kodaira's vanishing theorem and Reider's criterion are already very useful as stated. However, the applicability of Kodaira's theorem was greatly extended, first on surfaces, by Mumford, Ramanujam, Miyaoka, and then in all dimensions by Kawamata and Viehweg, as follows. First, the ampleness condition for L can be relaxed to $L \cdot C \geq 0$ for every curve C and $L^2 > 0$ (L *nef* and *big*). Second, and most important, assume that L itself is not nef and big, but there is a nef and big $\mathbb{Q}$-divisor M on S ($M \in \mathrm{Div}(S) \otimes \mathbb{Q}$) such that $L = \lceil M \rceil$ (i.e. $L - M$ is an effective $\mathbb{Q}$-divisor B whose coefficients are all < 1). Then we have $h^i(S, K_S + L) = 0$ for all

$i > 0$, just as in Kodaira's theorem. ($\mathbb{Q}$-divisors were first considered in this context in connection with the Zariski decomposition of effective divisors.)

In dimension ≥ 3, the Kawamata–Viehweg vanishing theorem requires an extra hypothesis (the irreducible components of $\mathrm{Supp}(B)$ must cross normally); however, Sakai remarked that for surfaces this extra hypothesis is not necessary (see Proposition 1.2.1 in §1).

For base-point-freeness (question (2) above), Ein and Lazarsfeld ([EL]) proved a similar extension of Reider's criterion, expressing H as $K_S + \lceil M \rceil$ for a $\mathbb{Q}$-divisor M on S; if $M^2 > 4$ and $M \cdot C \geq 2$ for every curve C, then $|H|$ is base-point-free. They used this result in their proof of Fujita's conjecture for base-point-freeness in dimension 3. (In fact they used a more precise local version, involving the local multiplicities of $B = L - M$; see §1 below).

In this paper we give criteria for very ampleness of linear systems of the form $|K_S + B + M|$, $B = \lceil M \rceil - M$, as above. In particular, we prove the following result:

Theorem 1. *Let S, B and M be as above, and assume that*

(0.1) $M^2 > 2(\beta_2)^2$,

(0.2) $M \cdot C \geq 2\beta_1$ *for every irreducible curve* $C \subset S$,
where β_2, β_1 are positive numbers satisfying the following inequalities:

(0.3) $\beta_2 \geq 2$,

(0.4) $\beta_1 \geq \frac{\beta_2}{\beta_2 - 1}$.

Then $|K_S + B + M|$ is very ample.

An immediate consequence of Theorem 1 is the following:

Corollary 2. *Assume that (S, B) is as before, and M is an ample $\mathbb{Q}$-divisor on S such that $B = \lceil M \rceil - M$, $M^2 > (2 + \sqrt{2})^2$, and $M \cdot C > 2 + \sqrt{2}$ for every curve $C \subset S$. Then $|K_S + B + M|$ is very ample. In particular, if A is an ample divisor (with integer coefficients) on S, then $|K_S + \lceil aA \rceil|$ is very ample for every $a \in \mathbb{Q}$, $a > 2 + \sqrt{2}$.*

Note that Reider's criterion implies only that $|K_S + aA|$ is very ample for every *integer* $a \geq 4$.

As in [EL, §2] (where the analogue for base-point-freeness was proved), we prove a local version of Theorem 1, with the numerical conditions on M relaxed in terms of local multiplicities of B.

As we mentioned earlier, the result for base-point-freeness on surfaces with boundary (i.e. for $\mathbb{Q}$-divisors M) was used in [EL] in the proof of Fujita's Conjecture in dimension 3. Similarly, we expect that the proof of the analogous result for very ampleness in dimension 3 will use very ampleness for $\mathbb{Q}$-divisors on surfaces. However, a natural and interesting question is whether or not the results for $\mathbb{Q}$-divisors on surfaces have any useful applications to the study of linear systems on surfaces. An example we discuss in §4 seems to indicate an affirmative answer. While the results proved in §4 can be obtained with other methods, our example shows how our $\mathbb{Q}$-Reider theorem extends the applicability of Reider's original result in the same way the Kawamata–Viehweg vanishing theorem extends the range of applicability of Kodaira's vanishing theorem. The usefulness of considering local multiplicities of B is also evident in this example.

The paper is divided as follows: §1 is devoted to base-point-freeness. The results discussed in this section, with one exception, were proved in [EL]; I include a (slightly modified) proof to fix the ideas and notations for the later sections. As one might expect, separation of points is relatively easy (at least in principle); it is discussed in §2. Then we move on to separation of tangent directions in §3. This part is surprisingly delicate; in particular the "multiplier ideal" method of Ein–Lazarsfeld, or Kawamata's equivalent "log-canonical threshold" formalism, do not work in this context. We explain the geometric contents of our method in the beginning of §3. Theorem 1 follows from Proposition 4 in §2 and Proposition 5 in §3. Finally, §4 contains the example mentioned earlier.

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1. Base-point-freeness

(1.1) Let S be a smooth projective surface over $\mathbb{C}$, and $B = \sum b_i C_i$ a fixed effective $\mathbb{Q}$-divisor on S with $0 \le b_i < 1$ for all i. (The pair (S, B) is sometimes called a "surface with boundary", whence the title of this paper.) Let M be a $\mathbb{Q}$-divisor on S such that $B + M$ has integer coefficients.

We assume throughout this paper that M is nef and big, i.e. that $M \cdot C \geq 0$ for every curve $C \subset S$ and $M^2 > 0$.

(1.2) For convenience, we gather here two technical results which we use time and again in our proofs.

(1.2.1) We use the following variants of the Kawamata–Viehweg vanishing theorem, which hold on smooth surfaces:

THEOREM. (a) (cf. [EL, Lemma 1.1]) *Let S be a smooth projective surface over $\mathbb{C}$, and let M be a nef and big $\mathbb{Q}$-divisor on S. Then*

$$H^i(S, K_S + \lceil M \rceil) = 0, \qquad \forall i > 0.$$

(b) (cf. [EL, Lemma 2.4]) *Assume moreover that $C_1, \ldots, C_k$ are distinct irreducible curves on S which have integer coefficients in M. Assume that $M \cdot C_j > 0$ for all $j = 1, \ldots, k$. Then*

$$H^i(S, K_S + \lceil M \rceil + C_1 + \cdots + C_k) = 0, \qquad \forall i > 0.$$

□

(1.2.2) We use the following criterion for base-point-freeness, respectively very ampleness, on a complete Gorenstein curve (cf. [Har2]):

PROPOSITION. *Let D be a Cartier divisor on the integral projective Gorenstein curve C. Then*:

(a) $\deg(D) \geq 2 \Longrightarrow$ *the complete linear system $|K_C + D|$ is base-point-free*;

(b) $\deg(D) \geq 3 \Longrightarrow |K_C + D|$ *is very ample.*

Proof. See [Har2, §1] for the relevant definitions (generalized divisors on C, including 0-dimensional subschemes; degree; etc.)

We prove (b); the proof of (a) is similar. By [Har2, Proposition 1.5], it suffices to show that $h^0(C, K_C + D - Z) = h^0(C, K_C + D) - 2$ for every 0-dimensional subscheme $Z \subset C$ of length 2. Consider the exact sequence:

$$0 \longrightarrow \mathcal{O}_C(K_C + D - Z) \longrightarrow \mathcal{O}_C(K_C + D) \longrightarrow \mathcal{O}_C(K_C + D) \otimes \mathcal{O}_Z \longrightarrow 0.$$

As $\mathcal{O}_C(K_C + D) \otimes \mathcal{O}_Z \cong \mathcal{O}_Z$ has length 2, the conclusion will follow from the vanishing of $H^1(C, K_C + D - Z)$. By Serre duality (cf. [Har2, Theorem 1.4]), $H^1(C, K_C + D - Z) \cong H^0(C, Z - D)$, and $H^0(C, Z - D) = 0$ due to $\deg(Z - D) = 2 - \deg(D) < 0$. □

(1.3) Fix a point $p \in S$. In this section we give sufficient conditions for $|K_S + B + M|$ to be free at p.

(1.3.1) NOTATION. $\mu = \mathrm{ord}_p(B) \overset{\mathrm{def}}{=} \sum b_i \cdot \mathrm{mult}_p(C_i) \qquad (B = \sum b_i C_i)$.

PROPOSITION 3. *$|K_S + B + M|$ is free at p in each of the following cases:*

1. $\mu \geq 2$;
2. $0 \leq \mu < 2$; *$M^2 > (\beta_2)^2$, $M \cdot C \geq \beta_1$ for every irreducible curve $C \subset S$ such that $p \in C$, where β_2, β_1 are positive numbers which satisfy the inequalities:*

$$\beta_2 \geq 2 - \mu, \tag{1.3.2}$$

$$\beta_1 \geq \min\left\{(2-\mu); \frac{\beta_2}{\beta_2 - (1-\mu)}\right\}. \tag{1.3.3}$$

Remark. Explicitly, the minimum in (1.3.3) is given by:

$$\min\left\{(2-\mu); \frac{\beta_2}{\beta_2 - (1-\mu)}\right\} = \begin{cases} 2-\mu & \text{if } 1 \leq \mu < 2 \\ \frac{\beta_2}{\beta_2-(1-\mu)} & \text{if } 0 \leq \mu < 1. \end{cases}$$

In other words, when $0 \leq \mu < 2$, the inequalities $\beta_2 \geq 2 - \mu$ and $\beta_1 \geq 2 - \mu$ suffice. When $\mu < 1$ the inequality for β_1 can be relaxed to

$$\beta_1 \geq \frac{\beta_2}{\beta_2 - (1-\mu)}; \tag{1.3.4}$$

this last part (which is useful in applications, cf. §4) is not contained in [EL].

Proof of Proposition 3

(1.4) Let $f : S_1 \to S$ be the blowing-up of S at p, and let $E \subset S_1$ be the exceptional divisor of f. We have $f^*B = f^{-1}B + \mu E$; $\lfloor f^{-1}B \rfloor = 0$, and therefore

$$\begin{aligned} K_{S_1} + \lceil f^*M \rceil &= f^*K_S + E + \lceil f^*(B+M) - f^*B \rceil \\ &= f^*(K_S + B + M) + E - \lfloor f^*B \rfloor \\ &= f^*(K_S + B + M) - (\lfloor \mu \rfloor - 1)E. \end{aligned} \tag{1.4.1}$$

(1.5) case $t = \lfloor \mu \rfloor - 1$ is a positive integer; since f^*M is nef and big on S_1, the vanishing theorem (1.2.1)(a) yields

$$H^1(S_1, K_{S_1} + \lceil f^*M \rceil) = 0, \tag{1.5.1}$$

and therefore (using (1.4.1) and the projection formula)

$$H^1(S, \mathcal{O}_S(K_S + B + M) \otimes \mathfrak{m}_p{}^t) = 0, \tag{1.5.2}$$

where $\mathfrak{m}_p$ is the maximal ideal of $\mathcal{O}_S$ at p. The conclusion follows from the surjectivity of the restriction map

$$H^0(S, K_S + B + M) \longrightarrow H^0(S, \mathcal{O}_S(K_S + B + M) \otimes \mathcal{O}_S/\mathfrak{m}_p{}^t) \cong \mathcal{O}_S/\mathfrak{m}_p{}^t.$$

(1.5.3) *Remark.* In fact we proved that $|K_S + B + M|$ separates s-jets at p, if $\mu \stackrel{\text{def}}{=} \operatorname{ord}_p(B) \geq s + 2$.

(1.6) Now assume that $\mu < 2$, and $M^2 > (\beta_2)^2$ with $\beta_2 \geq 2 - \mu$, etc.

(1.6.1) Claim. We can find an effective $\mathbb{Q}$-divisor D on S such that $\operatorname{ord}_p(D) = 2 - \mu$ and $D \underset{\mathbb{Q}}{\sim} tM$ for some $t \in \mathbb{Q}$, $0 < t < \frac{2-\mu}{\beta_2}$. ($\underset{\mathbb{Q}}{\sim}$ denotes $\mathbb{Q}$-linear equivalence, i.e. mD and mtM have integer coefficients and are linearly equivalent for some suitably large and divisible integer m.)

Proof of (1.6.1). By Riemann–Roch, $\dim |nM|$ grows like $\frac{M^2}{2}n^2 > \frac{(\beta_2)^2}{2}n^2$ for n sufficiently large and divisible (such that nM has integer coefficients). Since $\dim(\mathcal{O}_{S,p}/\mathfrak{m}_p{}^n)$ grows like $\frac{n^2}{2}$, for suitable n we can find $G \in |nM|$ with $\operatorname{ord}_p(G) > \beta_2 n$.

Take $D = rG$, $r = \frac{2-\mu}{\operatorname{ord}_p(G)}$; then $\operatorname{ord}_p(D) = 2 - \mu$, and $D \underset{\mathbb{Q}}{\sim} tM$ for $t = rn < \frac{2-\mu}{\beta_2 n} n = \frac{2-\mu}{\beta_2}$. ∎

Note that $\frac{2-\mu}{\beta_2} \leq 1$, by (1.3.2), so that $t < 1$; therefore $M - D \underset{\mathbb{Q}}{\sim} (1-t)M$ is still nef and big.

(1.7) Recall that $B = \sum b_i C_i$, for distinct irreducible curves $C_i \subset S$. Write $D = \sum d_i C_i$ (we allow some coefficients b_i and d_i to be zero); $d_i \in \mathbb{Q}$, $d_i \geq 0$, and $\operatorname{ord}_p(D) = \sum d_i \cdot \operatorname{mult}_p(C_i) = 2 - \mu$.

Let $D_i = f^{-1}C_i \subset S_1$ be the strict transform of C_i; then $f^*B = \sum b_i D_i + \mu E$, $f^*D = \sum d_i D_i + (2 - \mu)E$, $K_{S_1} = f^*K_S + E$, and

$$K_{S_1} - f^*(K_S + B + D) = -E - \sum (b_i + d_i) D_i.$$

Definition. (S, B, D) is **partially log-canonical at** p (PLC at p) if $-(b_i + d_i) \geq -1$ (i.e. $b_i + d_i \leq 1$) for every i such that $p \in C_i$. (The general definition requires the coefficient of E to be ≥ -1, too; in our case that coefficient is equal to -1.)

Note that PLC is not the same as *log-canonical* (cf. [KMM, Definition 0-2-10]), because f is not an embedded resolution of $(S, B + D)$.

(1.8) If (S, B, D) is PLC at p, then the proof is almost as simple as in the case $\mu \geq 2$:

$$\begin{aligned}(1.8.1)\qquad K_{S_1} &+ \lceil f^*(M - D) \rceil \\ &= f^*K_S + E + f^*(B + M) - \lfloor f^*(B + D) \rfloor \\ &= f^*(K_S + B + M) + E - 2E - \sum \lfloor b_i + d_i \rfloor D_i \\ &= f^*(K_S + B + M) - E - \sum{}' D_i - N_1,\end{aligned}$$

where $\sum' D_i$ extends over those i for which $p \in C_i$ and $b_i + d_i = 1$ (if any), and N_1 is an effective divisor supported away from E.

$f^*M \cdot D_i = M \cdot C_i > 0$ if $p \in C_i$; therefore (1.2.1)(b) yields:

$$(1.8.2)\qquad H^1(S_1, f^*(K_S + B + M) - E - N_1) = 0.$$

Arguing as in (1.5), we can show that $p \notin \mathrm{Bs}\,|K_S + B + M - N|$, where $N = f_*N_1$; i.e., $\exists \Lambda \in |K_S + B + M - N|$ with $p \notin \mathrm{Supp}(\Lambda)$. Then $\Lambda + N \in |K_S + B + M|$ and $p \notin \mathrm{Supp}(\Lambda + N)$, as required.

Note that we haven't used (1.3.3) yet; all we needed so far was $\beta_1 > 0$.

(1.9) Finally, assume that (S, B, D) is not PLC at p. Then $b_j + d_j > 1$ for some j with $p \in C_j$. In fact, since $2 = \mathrm{ord}_p(B + D) = \sum (b_i + d_i) \cdot \mathrm{mult}_p(C_i)$, there can be at most one C_j through p with $b_j + d_j > 1$, and then that C_j must be smooth at p and also $b_i + d_i < 1$ for all $i \neq j$ with $p \in C_i$. Let that j be 0; thus $b_0 + d_0 > 1$, C_0 is smooth at p, and $b_i + d_i < 1$ if $i \neq 0$ and $p \in C_i$. We say that C_0 is the **critical curve** at p.

Let c be the PLC **threshold** of (S, B, D) at p:

$$c = \max\{\lambda \in \mathbb{Q}_+ \mid (S, B, \lambda D) \text{ is } PLC \text{ at } p\};$$

explicitly, $b_0 + cd_0 = 1$, i.e. $c = \frac{1 - b_0}{d_0}$. Note that $0 < c < 1$.

$M - cD \underset{\mathbb{Q}}{\sim} (1 - ct)M$ is still nef and big on S, and we have:

$$\begin{aligned} K_S + \lceil M - cD \rceil &= K_S + B + M - \lfloor B + cD \rfloor \\ &= K_S + B + M - C_0 - N, \end{aligned}$$

with $p \notin \operatorname{Supp}(N)$. (If $p \in C_i$ and $i \neq 0$ then $b_i + d_i < 1$, and therefore $b_i + cd_i < 1$, too, because $c < 1$; hence $p \notin \operatorname{Supp}(N)$.)

(1.2.1)(a) yields $H^1(S, K_S + B + M - C_0 - N) = 0$, and therefore the restriction map $H^0(S, K_S + B + M - N) \to H^0(C_0, (K_S + B + M - N)|_{C_0})$ is surjective. Hence it suffices to show that $p \notin \operatorname{Bs} |(K_S + B + M - N)|_{C_0}|$.

We have

(1.9.1) $$K_S + B + M - N = K_S + \lceil M - cD \rceil + C_0,$$

and therefore $(K_S + B + M - N)|_{C_0} = K_{C_0} + \lceil M - cD \rceil|_{C_0}$; by (1.2.2)(a), it suffices to show that $\lceil M - cD \rceil \cdot C_0 \geq 2$. In any event $\lceil M - cD \rceil \cdot C_0$ is an integer; we will show that $\lceil M - cD \rceil \cdot C_0 > 1$.

$\lceil M - cD \rceil = (M - cD) + \Delta$, where $\Delta = \lceil M - cD \rceil - (M - cD) = \lceil (M + B) - (B + cD) \rceil - (M - cD) = (M + B) - \lfloor B + cD \rfloor - (M - cD) = (B + cD) - \lfloor B + cD \rfloor = \{B + cD\}$. Δ is an effective divisor which intersects C_0 properly, because C_0 has integer coefficient (namely, 1) in $B + cD$. Moreover, in a neighborhood of p we have $\{B + cD\} = (B + cD) - C_0$, because $B + cD = C_0 + \sum_{i \neq 0} (b_i + cd_i) C_i$, and $0 \leq b_i + cd_i < 1$ for every $i \neq 0$ such that $p \in C_i$. In particular, we have

$$\operatorname{ord}_p(\Delta) = \operatorname{ord}_p(B + cD) - 1 = \mu + c(2 - \mu) - 1.$$

(1.10) $$\begin{aligned} &\lceil M - cD \rceil \cdot C_0 \\ &= (M - cD) \cdot C_0 + \Delta \cdot C_0 \geq (1 - ct) M \cdot C_0 + \operatorname{ord}_p(\Delta) \\ &\geq (1 - ct)\beta_1 + \mu + c(2 - \mu) - 1. \end{aligned}$$

Therefore the inequality $\lceil M - cD \rceil \cdot C_0 > 1$ follows from

(1.10.1) $$(1 - ct)\beta_1 > (1 - c)(2 - \mu).$$

If $\beta_1 \geq 2 - \mu$ then (1.10.1) is trivial, because $t < 1 \Longrightarrow 1 - ct > 1 - c$.

(1.11) When $\mu < 1$ the inequality we assume for β_1 (namely, (1.3.4)) is weaker than $\beta_1 \geq 2 - \mu$. However, in this case the equation $B + cD =$

$C_0 +$ *other terms* yields a nontrivial lower bound for c: $\mu + c(2-\mu) = \mathrm{ord}_p(B+cD) \geq \mathrm{ord}_p(C_0) = 1$, and therefore $c \geq \frac{1-\mu}{2-\mu} > 0$.

The inequality (1.10.1) can also be written as

$$c(2-\mu-t\beta_1) > 2-\mu-\beta_1. \tag{1.11.1}$$

We may assume that $\beta_1 < 2-\mu$ (or else (1.10.1) is already proved). We have $c \geq \frac{1-\mu}{2-\mu}$, $t < \frac{2-\mu}{\beta_2}$ (see (1.6.1)), and $\frac{1-\mu}{\beta_2} \leq 1 - \frac{1}{\beta_1}$ (by (1.3.4)); therefore

$$\begin{aligned} c(2-\mu-t\beta_1) &> \frac{1-\mu}{2-\mu}(2-\mu-\frac{2-\mu}{\beta_2}\beta_1) = (1-\mu-\frac{1-\mu}{\beta_2}\beta_1) \\ &\geq (1-\mu) - (1-\frac{1}{\beta_1})\beta_1 = 2-\mu-\beta_1. \end{aligned}$$

(1.11.1) is proved. This concludes the proof of proposition 3.

2. Separation of points

(2.1) Let (S, B, M) be as in (1.1). Fix two distinct points $p, q \in S$. In this section we give criteria for $|K_S + B + M|$ to separate (p, q).

Note that in each case $|K_S+B+M|$ is free at p and q, by proposition 3, and therefore it suffices to find $s \in H^0(S, K_S + B + M)$ such that $s(p) = 0, s(q) \neq 0,$ *or* vice-versa.

Notation. $\mu_p = \mathrm{ord}_p(B)$, $\mu_q = \mathrm{ord}_q(B)$.

Proposition 4. *$|K_S+B+M|$ separates (p, q) in each of the following cases:*

1. $\mu_p \geq 2$ *and* $\mu_q \geq 2$;
2. $\mu_q \geq 2$; $0 \leq \mu_p < 2$; $M^2 > (\beta_2)^2, M \cdot C \geq \beta_1$ *for every irreducible curve* $C \subset S$ *passing through* p, *where* β_2, β_1 *are positive numbers which satisfy* (1.3.2) *and* (1.3.3) *for* $\mu = \mu_p$;
3. $0 \leq \mu_p < 2$ *and* $0 \leq \mu_q < 2$; $M^2 > (\beta_{2,p})^2 + (\beta_{2,q})^2$, *and*
   - (i) $M \cdot C \geq \beta_{1,p}$ *for every curve* $C \subset S$ *passing through* p,
   - (ii) $M \cdot C \geq \beta_{1,q}$ *for every curve* $C \subset S$ *passing through* q,
   - (iii) $M \cdot C \geq \beta_{1,p} + \beta_{1,q}$ *if* C *passes through* both p and q,

where $\beta_{2,p}, \beta_{1,p}\,; \beta_{2,q}, \beta_{1,q}$ *are positive numbers which satisfy the inequalities*

$$\beta_{2,p} \geq 2 - \mu_p, \quad \beta_{2,q} \geq 2 - \mu_q; \tag{2.1.1}$$

$$\beta_{1,p} \geq \min\left\{(2-\mu_p); \frac{\beta_{2,p}}{\beta_{2,p}-(1-\mu_p)}\right\}, \text{ and similarly for } \beta_{1,q}. \tag{2.1.2}$$

Proof of Proposition 4

(2.2) Let $f : S_1 \to S$ be the blowing-up of S at p and q, with exceptional curves E_p, E_q. As in (1.4), we have:

$$K_{S_1} + \lceil f^*M \rceil = f^*(K_S + B + M) - (\lfloor \mu_p \rfloor - 1)E_p - (\lfloor \mu_q \rfloor - 1)E_q.$$

In particular, if $\mu_p \geq 2$ and $\mu_q \geq 2$ (case 1 of the proposition), we get

$$H^1(S, \mathcal{O}_S(K_S + B + M) \otimes \mathfrak{m}_p{}^{t_p} \otimes \mathfrak{m}_q{}^{t_q}) = 0$$

for positive integers t_p, t_q (compare to (1.5.2)); the conclusion follows as in (1.5).

(2.3) Next assume that $\mu_p < 2, \mu_q \geq 2$, $M^2 > (\beta_2)^2$ with $\beta_2 \geq 2 - \mu_p$, etc. (case 2 of the proposition). Write $\mu = \mu_p$. As in (1.6.1), we can find an effective $\mathbb{Q}$-divisor D on S such that $\mathrm{ord}_p(D) = 2 - \mu$ and $D \underset{\mathbb{Q}}{\sim} tM$ for some $t \in \mathbb{Q}\,, 0 < t < \frac{2-\mu}{\beta_2}$.

If (S, B, D) is PLC at p, the argument of (1.8) yields a vanishing

$$H^1(S_1, f^*(K_S + B + M) - E_p - N_0) = 0 \tag{2.3.1}$$

where N_0 is an effective divisor supported away from E_p. Note that in this case $N_0 \geq E_q$, because $\mu_q \geq 2$. Indeed, (2.3.1) is obtained by applying (1.2.1)(b) to

$$\begin{aligned} K_{S_1} &+ \lceil f^*(M - D) \rceil \\ &= f^*(K_S + B + M) - E_p - t_q E_q - \sum \lfloor b_i + d_i \rfloor D_i \\ &= f^*(K_S + B + M) - E_p - t_q E_q - \sum{}' D_i - N_1, \end{aligned} \tag{2.3.2}$$

where $\sum' D_i$ and N_1 are as in (1.8.1) and $t_q = \lfloor \mu_q + \mathrm{ord}_q(D) \rfloor - 1$ is an integer, $t_q \geq 1$; then $N_0 = N_1 + t_q E_q \geq E_q$.

The vanishing (2.3.1) implies the surjectivity of the restriction map

$$H^0(S_1, f^*(K_S+B+M)-N_0) \longrightarrow H_0(E_p, (f^*(K_S+B+M)-N_0)|_{E_p}) \cong \mathbb{C}$$

(note that $f^*(K_S+B+M)|_{E_p}$ is trivial, and so is $N_0|_{E_p}$ because $N_0 \cap E_p = \emptyset$).

Hence we can find $\Gamma \in |f^*(K_S+B+M)-N_0|$ such that $\Gamma \cap E_p = \emptyset$. As $\Gamma+N_0 \in |f^*(K_S+B+M)|$, we have $\Gamma+N_0 = f^*\Lambda$ for some $\Lambda \in |K_S+B+M|$. Moreover, $p \notin \operatorname{Supp}(\Lambda)$, because $f^*\Lambda \cap E_p = \emptyset$, but $q \in \operatorname{Supp}(\Lambda)$, because $f^*\Lambda = \Gamma + N_0 \geq E_q$. Thus $|K_S+B+M|$ separates (p,q) in this case.

(2.4) Now assume that (S,B,D) is not PLC at p. Let c, C_0 be the PLC threshold and the critical curve at p, as in §1, (1.9)–(1.11). Let $\phi : S_2 \longrightarrow S$ be the blowing-up of S at q (only), with exceptional curve F_q. Let $C_0' \subset S_2$ be the proper transform of C_0 in S_2. Let $p' = \phi^{-1}(p)$. We have:

$$K_{S_2} + \lceil \phi^*(M-cD) \rceil = \phi^*(K_S+B+M) - C_0' - N_0,$$

where $p' \notin \operatorname{Supp}(N_0)$, as in (1.9), and $N_0 \geq F_q$, as in (2.3).

The argument in (1.9)–(1.11) shows that there exists $\Gamma \in |\phi^*(K_S+B+M)-N_0|$ with $p' \notin \operatorname{Supp}(\Gamma)$. Now the proof can be completed as in the last part of (2.3).

(2.5) Finally, consider the case $\mu_p < 2$ and $\mu_q < 2$, with $M^2 > (\beta_{2,p})^2 + (\beta_{2,q})^2$, etc. (case 3 of the proposition).

As in (1.6.1), we can find $G \in |nM|$ with $\operatorname{ord}_p(G) > \beta_{2,p} n$ and $\operatorname{ord}_q(G) > \beta_{2,q} n$. Let $r = \max\left\{\frac{2-\mu_p}{\operatorname{ord}_p(G)}, \frac{2-\mu_q}{\operatorname{ord}_q(G)}\right\}$, and $D = rG$. Then $\operatorname{ord}_p(D) \geq 2-\mu_p$ and $\operatorname{ord}_q(G) \geq 2-\mu_q$, and at least one of the last two inequalities is an equality. Without loss of generality we may assume that $\operatorname{ord}_p(D) = 2-\mu_p$ and $m_q \stackrel{\text{def}}{=} \operatorname{ord}_q(D) \geq 2-\mu_q$. We have $D \underset{\mathbb{Q}}{\sim} tM$, with

$$0 < t = rn = \frac{2-\mu_p}{\operatorname{ord}_p(G)} n < \frac{2-\mu_p}{\beta_{2,p}} \leq 1; \tag{2.5.1}$$

also, $m_q = \operatorname{ord}_q(D) = r \cdot \operatorname{ord}_q(G) > r \cdot (\beta_{2,q} n) = t\beta_{2,q}$, and therefore

$$t < \frac{m_q}{\beta_{2,q}} \tag{2.5.2}$$

(this is the analogue of (2.5.1) at q).

If (S, B, D) is PLC at p, then (1.2.1)(b) yields

$$H^1(S_1, f^*(K_S + B + M) - E_p - N_0) = 0, \tag{2.5.3}$$

with $N_0 \cap E_p = \emptyset, N_0 \geq E_q$ (the computation in (2.3.2) applies unchanged in this situation). In this case we conclude as in (2.3).

(2.6) Now assume that (S, B, D) is not PLC at p. Let c, C_0 be the PLC threshold and the critical curve at p. (1.2.1)(a) yields

$$H^1(S_1, f^*(K_S + B + M) - D_0 - N_0) = 0, \qquad N_0 \cap E_p = \emptyset. \tag{2.6.1}$$

If $N_0 \cap E_q \neq \emptyset$, we use (2.6.1) to find $\Gamma \in |f^*(K_S + B + M) - N_0|$ which does not pass through $\tilde{p} = D_0 \cap E_p$; the proof is the same as in (1.9)–(1.11). Then the conclusion follows as in (2.3).

Assume that $N_0 \cap E_q = \emptyset$. We discuss separately the subcases $q \in C_0$ and $q \notin C_0$. If $q \in C_0$, we separate (p, q) on C_0. If $q \notin C_0$, we reverse the roles of p and q.

(2.7) First consider the subcase $q \in C_0$. The vanishing (2.6.1) implies

$$H^1(S, K_S + B + M - C_0 - N) = 0, \tag{2.7.1}$$

with $N = f_* N_0$, $\mathrm{Supp}(N) \cap \{p, q\} = \emptyset$. Consequently, the restriction map

$$H^0(S, K_S + B + M - N) \longrightarrow H^0(C_0, (K_S + B + M - N)|_{C_0})$$

is surjective, and it suffices to show that $|(K_S + B + M - N)|_{C_0}|$ separates (p, q) on C_0. As in (1.9.1), we have

$$(K_S + B + M - N)|_{C_0} = K_{C_0} + \lceil M - cD \rceil|_{C_0};$$

by (1.2.2)(b) it is enough to show that $\lceil M - cD \rceil \cdot C_0 > 2$ (and consequently ≥ 3).

We proceed as in §1: $\lceil M - cD \rceil = (M - cD) + \Delta$, with $\Delta = \{B + cD\}$; Δ and C_0 intersect properly, and $\mathrm{ord}_p(\Delta) = \mu_p + c(2 - \mu_p) - 1, \mathrm{ord}_q(\Delta) = \mu_q + cm_q - 1$ (note that $N_0 \cap E_q = \emptyset \Longrightarrow$ the only component with coefficient ≥ 1 of $B + cD$ through q is C_0, and moreover C_0 must be smooth at q). Therefore

$$\begin{aligned}
&\lceil M - cD \rceil \cdot C_0 \\
&= (M - cD) \cdot C_0 + \Delta \cdot C_0 \\
&\geq (1 - ct) M \cdot C_0 + \mathrm{ord}_p(\Delta) + \mathrm{ord}_q(\Delta) \\
&\geq (1 - ct)(\beta_{1,p} + \beta_{1,q}) + (\mu_p + c(2 - \mu_p) - 1) + (\mu_q + cm_q - 1)
\end{aligned}$$

($M \cdot C_0 \geq \beta_{1,p} + \beta_{1,q}$, because this time C_0 passes through both p and q.)

Hence $\lceil M - cD \rceil \cdot C_0 > 2$ follows from

$$(2.7.2)\quad (1-ct)(\beta_{1,p}+\beta_{1,q}) + (\mu_p + c(2-\mu_p) - 1) + (\mu_q + cm_q - 1) > 2,$$

which in turn follows from the following two inequalities:

$$(2.7.3)\qquad (1-ct)\beta_{1,p} + (\mu_p + c(2-\mu_p) - 1) > 1 \quad \text{and}$$

$$(2.7.4)\qquad (1-ct)\beta_{1,q} + (\mu_q + cm_q - 1) > 1.$$

(2.7.3) is proved like (1.10.1) in §1: if $\beta_{1,p} \geq 2 - \mu_p$, then $t < 1 \Longrightarrow (1-ct)\beta_{1,p} > (1-c)(2-\mu_p) \Longrightarrow$ (2.7.3). If $\beta_{1,p} < 2-\mu_p$ (which can happen only if $\mu_p < 1$), then we have $c \geq \frac{1-\mu_p}{2-\mu_p}$ as in (1.11), $t < \frac{2-\mu_p}{\beta_{2,p}}$ by (2.5.1), and $\frac{1-\mu_p}{\beta_{2,p}} \leq 1 - \frac{1}{\beta_{1,p}}$ by (2.1.2), and therefore (2.7.3) follows as in (1.11).

(2.7.4) is proved similarly. First, since $m_q = \mathrm{ord}_q(D) \geq 2 - \mu_q$, the inequality is true when $\beta_{1,q} \geq 2-\mu_q$, as in the proof of (2.7.3) above. When $\beta_{1,q} < 2-\mu_q$ we must have $\mu_q < 1$; then $B + cD \geq C_0 \Longrightarrow \mu_q + cm_q \geq 1 \Longrightarrow c \geq \frac{1-\mu_q}{m_q}$, $t < \frac{m_q}{\beta_{2,q}}$ by (2.5.2), and $\frac{1-\mu_q}{\beta_{2,q}} \leq 1 - \frac{1}{\beta_{1,q}}$ by (2.1.2); consequently

$$\begin{aligned} c(m_q - t\beta_{1,q}) &> \frac{1-\mu_q}{m_q}\left(m_q - \frac{m_q}{\beta_{2,q}}\beta_{1,q}\right) \\ &= (1-\mu_q) - \frac{1-\mu_q}{\beta_{2,q}}\beta_{1,q} \geq 2 - \mu_q - \beta_{1,q}, \end{aligned}$$

which yields (2.7.4).

Thus (2.7.2) is proved; this concludes the proof when $q \in C_0$.

(2.8) To complete the proof of the proposition in case 3, consider the remaining subcase, $q \notin C_0$. In this subcase separation of (p, q) is obtained by reversing the roles of p and q. Namely, let $D' = \alpha D$, for the positive rational number α such that $\mathrm{ord}_q(D') = 2-\mu_q$; that is, $\alpha = \frac{2-\mu_q}{\mathrm{ord}_q(D)} = \frac{2-\mu_q}{m_q}$. Note that $D' \underset{\mathbb{Q}}{\sim} t'M$, where $t' = \alpha t < \frac{2-\mu_q}{m_q} \cdot \frac{m_q}{\beta_{2,q}}$ (by (2.5.2)), i.e.

$$(2.8.1)\qquad 0 < t' < \frac{2-\mu_q}{\beta_{2,q}} \leq 1.$$

Let c' be the *PLC* threshold for (S, B, D') at q; note that $c'\alpha > c$ ($c'\alpha$ is the *PLC* threshold of (S, B, D) at q, and therefore $c < c'\alpha$ follows from $N_0 \cap E_q = \emptyset$ in (2.6.1)). This, in turn, implies $B + c'D' = B + c'\alpha D \geq C_0$.

If (S, B, D') is PLC at q (i.e. if $c' = 1$), then (1.2.1)(b) yields

$$(2.5.3') \qquad H^1(S_1, f^*(K_S + B + M) - E_q - N_0') = 0, \qquad N_0' \cap E_q = \emptyset$$

(Compare to (2.5.3)).

If (S, B, D') is not PLC at q (i.e. if $c' < 1$), and C_0' is the critical curve at q, then (1.2.1.)(a) yields

$$(2.6.1') \qquad H^1(S_1, f^*(K_S + B + M) - D_0' - N_0') = 0, \qquad N_0' \cap E_q = \emptyset$$

(Compare to (2.6.1), noting that now p and q are interchanged.)

In both cases, the arguments in (1.8) and, respectively, (1.9)–(1.11) show that there exists $\Lambda \in |K_S + B + M - N'|$ with $q \notin \mathrm{Supp}(\Lambda)$, where $N' = f_* N_0'$ is an effective divisor with $q \notin \mathrm{Supp}(N')$. Now, however, $N' \geq C_0$ (because $B + c'D' \geq C_0$, as noted earlier, and $q \notin C_0 \implies C_0$ is not discarded even when the vanishing theorem is used in the form (1.2.1)(b)); thus $\Gamma + N' \in |K_S + B + M|$ passes through p but not through q.

This completes the proof of proposition 4.

3. Separation of tangent directions

(3.1) Let (S, B, M) be as in §1. Fix a point $p \in S$. In this section we give criteria for $|K_S + B + M|$ to separate directions at p.

The statements (and proofs) are somewhat similar to those in §2. The main difference is in the part of the proof corresponding to the discussion in (2.8). So far in our proofs we worked with $M - cD$, where c was always the PLC threshold at some point or another; this made the arguments relatively transparent. In (2.8), when we passed from $c = PLC$ threshold at p to $c'\alpha = PLC$ threshold at q, the relevant fact was that $q \notin C_0$, where C_0 was the critical curve at p, and therefore C_0 did not affect the local computations around q. In separating tangent directions, the analogue is a curve C_0 through p, such that $\vec{v} \notin T_p(C_0)$ for some fixed $\vec{v} \in T_p(S), \vec{v} \neq \vec{0}$. Then we will have to increase c to some larger value c', but clearly in that case $(S, B, c'D)$ will no longer be PLC at p. While this complicates the computations, the geometric idea is still the same: find a divisor $\Gamma \in |K_S + B + M - C_0 - N|$, $p \notin \mathrm{Supp}(N)$, such that Γ does not pass through p; then $\Gamma + C_0 + N$ has only one component through p, namely, C_0, and $\vec{v} \notin T_p(C_0)$ – therefore $\Gamma + C_0 + N$ passes through p and is not tangent to $\vec{v}$, as required.

Another technical problem, which did not arise before, is that in some cases the "minimizing" curve C_0 may be singular at p. (This possibility is directly related to the need, in some cases, to increase c beyond the PLC threshold at p.) In those cases we separate the tangent directions on C_0, using (1.2.2)(b) (note that C_0 singular at $p \implies T_p(S) = T_p(C_0)$); the vanishing (1.2.1) is then used to lift from C_0 to S.

(3.2) Let S be a smooth surface, as before; let p denote a point on S, and fix $\vec{v} \in T_p(S), \vec{v} \neq \vec{0}$. Let Z denote the zero-dimensional subscheme of length 2 of S, corresponding to $(p, \vec{v})$; in local coordinates (x, y) at p such that $\vec{v}$ is tangent to $(y = 0)$, Z is defined by the ideal $\mathcal{I}_Z = (x^2, y) \cdot \mathcal{O}_S$.

Let $f : S_1 \to S$ be the blowing-up of S at p, with exceptional curve E_p, and let $V \in E_p$ correspond to (the direction of) $\vec{v}$. Let $g : S_2 \to S_1$ be the blowing-up of S_1 at V, with exceptional curve $F_{\vec{v}}$, and let $F_p = g^{-1}E_p$. Let $h = f \circ g$. Write

$$h^*B = h^{-1}B + \mu_p F_p + \mu_{\vec{v}} F_{\vec{v}}; \tag{3.2.1}$$

$\mu_p = \operatorname{ord}_p(B)$, while (3.2.1) is the definition of $\mu_{\vec{v}}$.

More generally, if G is any effective $\mathbb{Q}$-divisor on S, denote the order of h^*G along $F_{\vec{v}}$ by $o_{\vec{v}}(G)$; $o_{\vec{v}}(G) = \operatorname{ord}_p(G) + \operatorname{ord}_V(f^{-1}G)$. For convenience, let $o_V(G) \stackrel{\text{def}}{=} \operatorname{ord}_V(f^{-1}G)$, and let $\mu_V = o_V(B)$.

Note that, in general, $o_{\vec{v}} = \operatorname{ord}_p + o_V$ and $o_V \leq \operatorname{ord}_p$; in particular:

$$\mu_p \leq \mu_{\vec{v}} \leq 2\mu_p. \tag{3.2.2}$$

(3.3) Consider again (S, B, M) as in §1, and fix $p, \vec{v}$ as in (3.2). Since the proofs will now be more complex, we will state the criteria for separating $\vec{v}$ at p one by one, in increasing order of difficulty.

The first (and easiest) case is:

Proposition 5. (Case 1) *If $\mu_p \geq 3$ or $\mu_{\vec{v}} \geq 4$, then $|K_S + B + M|$ separates $\vec{v}$ at p. (M must still be nef and big.)*

Proof. Recall that the conclusion means that the restriction map

$$H^0(S, K_S + B + M) \to H^0(Z, K_S + B + M|_Z) \cong \mathcal{O}_Z$$

is surjective.

If $\mu_p \geq 3$, we use the vanishing theorem in the form (1.2.1)(a) for

$$\begin{aligned} K_{S_1} + \lceil f^*M \rceil &= f^*(K_S + B + M) + E_p - \lfloor f^*B \rfloor \\ &= f^*(K_S + B + M) - tE_p, \end{aligned}$$

where $t = \lfloor \mu_p \rfloor - 1 \geq 2$, as in (1.4)–(1.5); then $H^0(S, K_S+B+M) \to \mathcal{O}_S/\mathfrak{m}_p{}^t$ is surjective, and since $t \geq 2$, we have $\mathfrak{m}_p{}^t \subset \mathcal{I}_Z$, i.e. $\mathcal{O}_S/\mathfrak{m}_p{}^t \to \mathcal{O}_Z$ is also surjective. (See also Remark 1.5.3.)

If $\mu_{\vec{v}} \geq 4$, the argument is similar, starting on S_2:

$$\begin{aligned} K_{S_2} + \lceil h^*M \rceil &= h^*(K_S + B + M) + F_p + 2F_{\vec{v}} - \lfloor h^*B \rfloor \\ &= h^*(K_S + B + M) - t_pF_p - t_{\vec{v}}F_{\vec{v}}, \end{aligned}$$

where $t_{\vec{v}} = \lfloor \mu_{\vec{v}} \rfloor - 2 \geq 2$, and $t_p = \lfloor \mu_p \rfloor - 1 \geq 1$ (indeed, by (3.2.2), $\mu_p \geq \frac{1}{2}\mu_{\vec{v}} \geq 2$.) As in the previous case, we get a vanishing $H^1(S, \mathcal{O}_S(K_S + B + M) \otimes \mathcal{I}) = 0$ for $\mathcal{I} = h_*\mathcal{O}_{S_2}(-t_pF_p - t_{\vec{v}}F_{\vec{v}})$; $\mathrm{Supp}(\mathcal{O}_S/\mathcal{I}) = \{p\}$ and $\mathcal{I} \subset \mathcal{I}_Z$, so the conclusion follows as before. □

(3.4) Now assume that $\mu_p < 3$ and $\mu_{\vec{v}} < 4$.

First consider the case $2 \leq \mu_p < 3$. Then $2 \leq \mu_{\vec{v}} < 4$, and therefore $0 < (4 - \mu_{\vec{v}}) \leq 2$.

Proposition 5. (Case 2) *Let $2 \leq \mu_p < 3$ and $2 \leq \mu_{\vec{v}} < 4$. Assume that $M^2 > (4 - \mu_{\vec{v}})^2$, $M \cdot C \geq \frac{1}{2}(4 - \mu_{\vec{v}})$ for every curve $C \subset S$ through p, and $M \cdot C \geq (4 - \mu_{\vec{v}})$ for every curve C containing Z – i.e., such that $p \in C$ and $\vec{v} \in T_p(C)$. Then $|K_S + B + M|$ separates $\vec{v}$ at p.*

Proof.

(3.5) Claim. We can find an effective $\mathbb{Q}$-divisor D on S such that $o_{\vec{v}}(D) = 4 - \mu_{\vec{v}}$ and $D \underset{\mathbb{Q}}{\sim} tM$ for some $t \in \mathbb{Q}, 0 < t < 1$. (See (3.2) for the definition of $o_{\vec{v}}(D)$.)

Proof of (3.5). Choose $a > (4 - \mu_{\vec{v}})$ such that $M^2 > a^2$. Then $(h^*M - aF_{\vec{v}})^2 = M^2 - a^2 > 0$ and $(h^*M - aF_{\vec{v}}) \cdot h^*M = M^2 > 0$; therefore $h^*M - aF_{\vec{v}} \in N(S_2)^+$, the positive cone of S_2, and in particular it is big. (See, for example, [KM, (1.1)].) Therefore $\exists T$, effective $\mathbb{Q}$-divisor on S_2, such that $T \underset{\mathbb{Q}}{\sim} h^*M - aF_{\vec{v}}$. Put $D_1 = h_*(T + aF_{\vec{v}})$; then $D_1 \underset{\mathbb{Q}}{\sim} h_*(h^*M) = M$. Also, $h^*D_1 = T + aF_{\vec{v}}$ (their difference has support contained in $F_p \cup F_{\vec{v}}$;

on the other hand, $T+aF_{\vec{v}} \underset{\mathbb{Q}}{\sim} h^*M \Longrightarrow (h^*D_1-(T+aF_{\vec{v}}))\cdot F_p = (h^*D_1-(T+aF_{\vec{v}}))\cdot F_{\vec{v}} = 0$, and $h^*D_1 = T+aF_{\vec{v}}$ follows from the negative definiteness of the intersection form on $h^{-1}(p) = F_p \cup F_{\vec{v}}$). We have $D_1 \underset{\mathbb{Q}}{\sim} M$ and $o_{\vec{v}}(D_1) \geq a > 4-\mu_{\vec{v}}$. Take $D = tD_1$, $t = \frac{4-\mu_{\vec{v}}}{o_{\vec{v}}(D_1)} < 1$. ∎

Remark. The statement of (3.5) is similar to that of (1.6.1), and indeed, we could have proved it as in §1. However, the proof we gave here is easier to generalize, especially on *singular* surfaces.

(3.6) We return to the proof of proposition 5, case 2. Choose D as in (3.5). Write $B = \sum b_iC_i, D = \sum d_iC_i; D_i = f^{-1}C_i, T_i = g^{-1}D_i = h^{-1}C_i; h^*B = h^{-1}B + \mu_pF_p + \mu_{\vec{v}}F_{\vec{v}}, h^*D = h^{-1}D + m_pF_p + (4-\mu_{\vec{v}})F_{\vec{v}}$, where $m_p = \mathrm{ord}_p(D)$. We have $K_{S_2} = h^*K_S + F_p + 2F_{\vec{v}}$.

If $b_i + d_i \leq 1$ for every C_i through p, then

(3.6.1)
$$\begin{aligned} &K_{S_2} + \lceil h^*(M-D)\rceil \\ &\quad = h^*(K_S+B+M) + F_p + 2F_{\vec{v}} - \lfloor h^*(B+D)\rfloor \\ &\quad = h^*(K_S+B+M) - t_pF_p - 2F_{\vec{v}} - {\sum}' T_i - N_2, \end{aligned}$$

where ${\sum}' T_i$ extends over all i with $b_i + d_i = 1$ and $p \in C_i$ (if any), N_2 is an effective divisor on S_2 such that $\mathrm{Supp}(N_2) \cap h^{-1}(p) = \emptyset$, and $t_p = \lfloor \mu_p + m_p \rfloor - 1 \geq 1$ (because $\mu_p \geq 2$ by hypothesis). Then we conclude as in (3.3) (case 1 of the proposition), using the vanishing (1.2.1)(b) to dispose of ${\sum}' T_i$ (if it is not zero).

(3.7) Now assume that $b_i + d_i > 1$ for at least one C_i through p. Let

(3.7.1)
$$c \overset{\text{def}}{=} \min\left\{\frac{3-\mu_p}{m_p}; \frac{1-b_i}{d_i} : b_i + d_i > 1 \text{ and } p \in C_i\right\}.$$

If $c = \frac{3-\mu_p}{m_p}$, we finish again as in case 1, using (1.2.1)(b) for

$$K_{S_1} + \lceil f^*(M-cD)\rceil = f^*(K_S+B+M) - 2E_p - {\sum}' D_i - N_1$$

on S_1, where ${\sum}' D_i$ extends over all i such that $b_i + cd_i = 1$ and $p \in C_i$ (if any), and $\mathrm{Supp}(N_1) \cap E_p = \emptyset$.

(3.8) If $c = \frac{1-b_0}{d_0} < \frac{3-\mu_p}{m_p}$ for some C_0 through p, then

$$\sum (b_i + cd_i) \cdot \operatorname{mult}_p(C_i) = \mu_p + cm_p < 3;$$

therefore $\operatorname{mult}_p(C_0) \leq 2$, and moreover, if $\operatorname{mult}_p(C_0) = 2$, then $b_i + cd_i < 1$ for all C_i through p with $i \neq 0$. Also, $\mu_{\vec{v}} + c(4 - \mu_{\vec{v}}) < 4$ (since $c < 1$), and therefore $B + cD \geq C_0 \Longrightarrow o_{\vec{v}}(C_0) \leq 3$.

(3.9) If C_0 is singular at p and $\vec{v} \notin TC_p(C_0)$ (the *tangent cone* to C_0 at p), then $o_{\vec{v}}(C_0) = 2$. We have $Z \subset C_0$, and

$$\begin{aligned} K_S + \lceil M - cD \rceil &= (K_S + B + M) - \lfloor B + cD \rfloor \\ &= (K_S + B + M) - C_0 - N, \end{aligned}$$

with $p \notin \operatorname{Supp}(N)$. Using (1.2.1)(a), as in §1, it suffices to show that $((K_S + B + M) - N)|_{C_0}$ separates $\vec{v}$ at p on C_0; that, in turn, will follow from (1.2.2)(b), *if* we can show that $\lceil M - cD \rceil \cdot C_0 > 2$.

As before, write $\lceil M - cD \rceil = (M - cD) + \Delta$; $\Delta = \{B + cD\}$ and C_0 intersect properly, and $\Delta = B + cD - C_0$ in an open neighborhood of p.

We have: $\operatorname{ord}_p(\Delta) = \mu_p + cm_p - 2$, and therefore $\Delta \cdot C_0 \geq 2(\mu_p + cm_p - 2)$. However, we get a better estimate if we consider orders along $F_{\vec{v}}$, as follows: $o_{\vec{v}}(\Delta) = \mu_{\vec{v}} + c(4 - \mu_{\vec{v}}) - 2$, because $o_{\vec{v}}(C_0) = 2$; $\operatorname{ord}_p(\Delta) \geq \frac{1}{2} o_{\vec{v}}(\Delta)$, and therefore

$$\Delta \cdot C_0 \geq \frac{1}{2} o_{\vec{v}}(\Delta) \cdot 2 = \mu_{\vec{v}} + c(4 - \mu_{\vec{v}}) - 2.$$

Finally,

$$\begin{aligned} (3.9.1) \quad & \lceil M - cD \rceil \cdot C_0 \\ &= (M - cD) \cdot C_0 + \Delta \cdot C_0 = (1 - ct) M \cdot C_0 + \Delta \cdot C_0 \\ &\geq (1 - ct)(4 - \mu_{\vec{v}}) + \mu_{\vec{v}} + c(4 - \mu_{\vec{v}}) - 2 \\ &> 2 \qquad \text{(because } t < 1\text{)}, \end{aligned}$$

as required.

(3.10) If C_0 is singular at p and $\vec{v} \in TC_p(C_0)$, then $o_{\vec{v}}(C_0) = 3$ (≥ 3 is clear, and ≤ 3 was shown in (3.8)).

Working as in (3.9), we can show that

$$(3.10.1) \quad \lceil M - cD \rceil \cdot C_0 \geq (1 - ct)(4 - \mu_{\vec{v}}) + \mu_{\vec{v}} + c(4 - \mu_{\vec{v}}) - 3 > 1$$

(now $o_{\vec{v}}(\Delta) = o_{\vec{v}}(B + cD - C_0) = \mu_{\vec{v}} + c(4 - \mu_{\vec{v}}) - 3$); thus in this case we cannot use (1.2.2)(b) as in (3.9). We will modify the argument as follows:

Start with $f^*(M - cD)$ on S_1; the vanishing theorem yields

$$(3.10.2) \quad H^1(S_1, f^*(K_S + B + M) - E_p - D_0 - N_1) = 0, \qquad N_1 \cap E_p = \emptyset$$

(the coefficient of E_p is -1 because $2 \leq \mu_p + cm_p < 3$; the first inequality follows from $\mu_p \geq 2$, and the second was shown in (3.8)).

$\vec{v} \in TC_p(C_0) \implies V \in D_0$ (recall that $V \in E_p$ corresponds to $\vec{v} \in T_p(S)$). (3.10.2) implies the surjectivity of the restriction map

$$(3.10.3) \qquad H^0(S_1, f^*(K_S + B + M) - E_p - N_1) \\ \to H^0(D_0, f^*(K_S + B + M) - E_p - N_1|_{D_0}).$$

We will show that $\exists \tilde{\Gamma} \in |f^*(K_S + B + M) - E_p - N_1|_{D_0}|$ such that $V \notin \mathrm{Supp}(\tilde{\Gamma})$. Then we can lift $\tilde{\Gamma}$ to $\Gamma \in |f^*(K_S + B + M) - E_p - N_1|$, since (3.10.3) is surjective. $\Gamma + E_p + N_1 \in |f^*(K_S + B + M)|$ has the form $f^*\Lambda$ for some $\Lambda \in |K_S + B + M|$. Finally, $p \in \mathrm{Supp}(\Lambda)$, because $f^*\Lambda \geq E_p$, but $\vec{v} \notin T_p(\Lambda)$, because $V \notin \mathrm{Supp}(f^*\Lambda - E_p)$; this shows that $|K_S + B + M|$ separates $\vec{v}$ at p on S.

To prove the existence of $\tilde{\Gamma}$, note that $(f^*(K_S+B+M)-E_p-N_1)|_{D_0} = K_{D_0} + \lceil f^*(M - cD)\rceil|_{D_0}$; we will show that $\lceil f^*(M - cD)\rceil \cdot D_0 > 1$ – then (1.2.2)(a) implies the existence of $\tilde{\Gamma}$.

As in (1.9), we can write $\lceil f^*(M - cD)\rceil = f^*(M - cD) + \Delta_1$, where $\Delta_1 = \{f^*(B + cD)\}$ and D_0 intersect properly, and $\Delta_1 = f^*(B + cD) - 2E_p - D_0 = f^*(B + cD - C_0)$ in a neighborhood of E_p (the coefficient of E_p in $f^*(B + cD)$ is $\mu_p + cm_p$, and $2 \leq \mu_p + cm_p < 3$). We have:

$$\begin{aligned}(3.10.4) \quad & \lceil f^*(M - cD)\rceil \cdot D_0 \\ &= f^*(M - cD) \cdot D_0 + \Delta_1 \cdot D_0 \\ &\geq (M - cD) \cdot C_0 + \mathrm{ord}_p(B + cD - C_0) \cdot \mathrm{mult}_p(C_0) \\ &\geq (1 - ct)(4 - \mu_{\vec{v}}) + \mu_{\vec{v}} + c(4 - \mu_{\vec{v}}) - 3 \\ &> 1\end{aligned}$$

as in (3.10.1).

(3.11) Now consider the case: C_0 smooth at p and tangent to $\vec{v}$, and $b_i + cd_i < 1$ for all $i \neq 0$ with $p \in C_i$.

Write $\lceil M - cD \rceil = (M - cD) + \Delta$, where Δ and C_0 intersect properly; then $\Delta \cdot C_0 = h^* \Delta \cdot T_0$ (projection formula: recall that $T_0 = h^{-1} C_0) \geq o_{\vec{v}}(\Delta)$, because $F_{\vec{v}} \cdot T_0 = 1$; since $\Delta = \{B + cD\} = B + cD - C_0$ in a neighborhood of p, we have $o_{\vec{v}}(\Delta) = \mu_{\vec{v}} + c(4 - \mu_{\vec{v}}) - 2$, and therefore

$$\lceil M - cD \rceil \cdot C_0 \geq (1 - ct)(4 - \mu_{\vec{v}}) + \mu_{\vec{v}} + c(4 - \mu_{\vec{v}}) - 2 > 2,$$

exactly as in (3.9.1). Thus $K_{C_0} + \lceil M - cD \rceil|_{C_0}$ separates $\vec{v}$ on C_0; we conclude as in (3.9).

(3.12) If C_0 is smooth at p and tangent to $\vec{v}$, and moreover $b_i + cd_i = 1$ for at least one $i \neq 0$ with $p \in C_i$, then: such an i is unique, say $i = 1$, and C_1 must be smooth at p and not tangent to $\vec{v}$; indeed, $B + cD \geq C_0 + C_1$, while $\operatorname{ord}_p(B + cD) < 3$ and $o_{\vec{v}}(B + cD) < 4$.

In this case reverse the roles of C_0 and C_1: thus C_0 will be smooth at p and not tangent to $\vec{v}$. This situation is covered below, in (3.13).

(3.13) Finally, assume that C_0 is smooth at p and not tangent to $\vec{v}$. In this case we work with $M - c'D$ for some $c' \geq c$, namely:

$$c' \stackrel{\text{def}}{=} \min \left\{ 1; \frac{3 - \mu_p}{m_p}; \frac{2 - b_0}{d_0}; \frac{1 - b_i}{d_i} \text{ with } i \neq 0, p \in C_i \text{ and } b_i + d_i \geq 1 \right\}.$$

In all cases, $M - c'D \underset{\mathbb{Q}}{\sim} (1 - c't)M$ is still nef and big; using the vanishing $H^1(S, K_S + \lceil M - c'D \rceil) = 0$, or the corresponding vanishing on S_1 or S_2, we will show that $\exists \Lambda \in |K_S + B + M - C_0 - N|, p \notin \operatorname{Supp}(N)$, such that $p \notin \operatorname{Supp}(\Lambda)$. Then $\Lambda + C_0 + N \in |K_S + B + M|$ has the unique component C_0 through p not tangent to $\vec{v}$, as required.

It remains to prove the existence of Λ.

(3.13.1) If $c' = \frac{3 - \mu_p}{m_p}$, then (1.2.1)(b) yields

$$H^1(S_1, f^*(K_S + B + M) - 2E_p - D_0 - N_1) = 0, \qquad N_1 \cap E_p = \emptyset;$$

thus $H^1(S_1, f^*(K_S + B + M - C_0 - N) - E_p) = 0$, where $N = f_* N_1$, and the existence of Λ follows.

When $c' = 1$ the proof is similar, starting on S_2, as in the proof of case 1 of the proposition.

(3.13.2) If $c' = \frac{1 - b_1}{d_1} < \frac{3 - \mu_p}{m_p}$ for another curve C_1 through p with $b_1 + d_1 > 1$, then C_1 must be smooth at p (because $B + c'D \geq C_0 + C_1$, and $\mathrm{ord}_p(B+c'D)=\mu_p+c'm_p<3$). We may have $c'=c$ (e.g., in the case discussed in (3.12)), or $c'>c$. In any event, $b_i+c'd_i<1$ for all curves C_i through p, $i\neq 0,1$. (1.2.1)(a) yields:

$$H^1(S,K_S+B+M-C_0-C_1-N)=0, \qquad p\notin \mathrm{Supp}(N). \tag{3.13.3}$$

We claim that $p\notin \mathrm{Bs}\,|K_S+B+M-C_0-N|_{C_1}|$, which in turn follows from (1.2.2)(a) once we show that $\lceil M-c'D\rceil\cdot C_1>1$. Then we use (3.13.3) to lift from C_1 to S, proving the existence of Λ as stated.

$\lceil M-c'D\rceil=(M-c'D)+\Delta'$, with $\Delta'=\{B+c'D\}=B+c'D-C_0-C_1$ in a neighborhood of p, and Δ',C_1 intersect properly.

If $\vec{v}\notin T_p(C_1)$, then $M\cdot C_1\geq\frac{1}{2}(4-\mu_{\vec{v}})$ by hypothesis, and $\mathrm{ord}_p(\Delta')\geq\frac{1}{2}o_{\vec{v}}(\Delta')=\frac{1}{2}(\mu_{\vec{v}}+c'(4-\mu_{\vec{v}})-2)$; therefore

$$\lceil M-c'D\rceil\cdot C_1\geq\frac{1}{2}(1-c't)(4-\mu_{\vec{v}})+\frac{1}{2}(\mu_{\vec{v}}+c'(4-\mu_{\vec{v}})-2)>1,$$

as required (compare to (3.9.1)). The proof is the same when $c'=\frac{2-b_0}{d_0}$, i.e. $C_1=C_0$; in that case $p\notin\mathrm{Bs}\,|K_S+B+M-C_0-N|_{C_0}|$.

If $\vec{v}\in T_p(C_1)$, then $M\cdot C_1\geq 4-\mu_{\vec{v}}$ and $\Delta'\cdot C_1\geq o_{\vec{v}}(\Delta')=\mu_{\vec{v}}+c'(4-\mu_{\vec{v}})-3$; all told, we have

$$\lceil M-c'D\rceil\cdot C_1\geq(1-c't)(4-\mu_{\vec{v}})+\mu_{\vec{v}}+c'(4-\mu_{\vec{v}})-3>1,$$

as claimed.

This concludes the proof of proposition 5, case 2.

(3.14) Finally, consider the case $0\leq\mu_p<2$ (and therefore $0\leq\mu_V<2$ and $0\leq\mu_{\vec{v}}=\mu_p+\mu_V<4$).

PROPOSITION 5. (Case 3) *Assume that $0\leq\mu_p<2$. Assume, moreover, that $M^2>(\beta_{2,p})^2+(\beta_{2,V})^2$ and*

(i) *$M\cdot C\geq\beta_1$ for every curve $C\subset S$ passing through p,*

(ii) *$M\cdot C\geq 2\beta_1$ for every curve C containing Z (i.e., passing through p and with $\vec{v}\in T_p(C)$),*

where $\beta_{2,p},\beta_{2,V},\beta_1$ are positive numbers which satisfy:

$$\beta_{2,p}\geq 2-\mu_p, \qquad \beta_{2,V}\geq 2-\mu_V; \tag{3.14.1}$$

$$\beta_1\geq\min\left\{\frac{1}{2}(4-\mu_{\vec{v}});\frac{\beta_{2,p}+\beta_{2,V}}{\beta_{2,p}+\beta_{2,V}-(2-\mu_{\vec{v}})}\right\} \tag{3.14.2}$$

Proof. The proof is very similar, in many respects, to that of case 2. We indicate the main steps of the proof, and we provide explicit computations in a few cases, to show what kind of alterations are needed.

(3.15) Claim. We can find D, an effective $\mathbb{Q}$-divisor on S, such that $o_{\vec{v}}(D) = 4 - \mu_{\vec{v}}$ and $D \underset{\mathbb{Q}}{\sim} tM$ for some $t \in \mathbb{Q}\,, t > 0$, satisfying

$$t < \frac{4-\mu_{\vec{v}}}{\beta_{2,p}+\beta_{2,V}} \tag{3.15.1}$$

– and therefore, in particular, $t < 1$.

Proof of (3.15). Choose $a > \beta_{2,p}, b > \beta_{2,V}$, such that $M^2 > a^2+b^2$. We have $(aF_p+(a+b)F_{\vec{v}})^2 = -(a^2+b^2)$, and therefore $h^*M-(aF_p+(a+b)F_{\vec{v}})$ is big, as in the proof of (3.5). Thus we can find $D_1 \underset{\mathbb{Q}}{\sim} M$ on S, $D_1 \geq 0$, such that $h^*D_1 \geq aF_p + (a+b)F_{\vec{v}}$. Then take $D = tD_1$, with

$$t = \frac{4-\mu_{\vec{v}}}{o_{\vec{v}}(D_1)} \leq \frac{4-\mu_{\vec{v}}}{a+b} < \frac{4-\mu_{\vec{v}}}{\beta_{2,p}+\beta_{2,V}}.$$

∎

(3.16) If $D = \sum d_iC_i$, as before, and $b_i + d_i \leq 1$ for every C_i through p, we conclude as in (3.6).

If $b_i + d_i > 1$ for at least one C_i through p, then define

$$c = \min\left\{\frac{3-\mu_p}{m_p}; \frac{1-b_i}{d_i} : b_i+d_i > 1 \text{ and } p \in C_i\right\}. \tag{3.16.1}$$

If $c = \frac{3-\mu_p}{m_p}$, we finish as in (3.7).

If $c = \frac{1-b_0}{d_0} < \frac{3-\mu_p}{m_p}$ for some C_0 through p, then $\mathrm{mult}_p(C_0) \leq 2$ and $o_{\vec{v}}(C_0) \leq 3$; if C_0 is singular at p, then it is the only C_i through p with $b_i + cd_i \geq 1$, and we proceed as in (3.9) or (3.10), according to whether $\vec{v} \in TC_p(C_0)$ or not. Only the proof of $\lceil M - cD\rceil \cdot C_0 > 2$ (if $\vec{v} \notin TC_p(C_0)$) or > 1 (if $\vec{v} \in TC_p(C_0)$) needs adjustment.

Assume first that $\vec{v} \notin TC_p(C_0)$ (with C_0 singular at p). Then $\lceil M - cD\rceil = (M-cD)+\Delta, \Delta = \{B+cD\} = B + cD - C_0$ in a neighborhood of p, and $o_{\vec{v}}(\Delta) = \mu_{\vec{v}} + c(4-\mu_{\vec{v}}) - 2$; $\mathrm{ord}_p(\Delta) \geq \frac{1}{2}o_{\vec{v}}(\Delta)$ and $\mathrm{mult}_p(C_0) = 2$, so that

$$\lceil M - cD\rceil \cdot C_0 \geq (1-ct)(2\beta_1) + \mu_{\vec{v}} + c(4-\mu_{\vec{v}}) - 2. \tag{3.16.2}$$

If $\beta_1 \geq \frac{1}{2}(4-\mu_{\vec{v}})$, then $\lceil M-cD\rceil \cdot C_0 > 2$ follows from (3.16.2) and $t<1$. In particular, this is true if $\mu_{\vec{v}} \geq 2$. If $\mu_{\vec{v}} < 2$, the hypothesis is weaker than $\beta_1 \geq \frac{1}{2}(4-\mu_{\vec{v}})$, namely:

$$\beta_1 \geq \frac{\beta_{2,p}+\beta_{2,V}}{\beta_{2,p}+\beta_{2,V}-(2-\mu_{\vec{v}})}. \tag{3.16.3}$$

Assume also that $\beta_1 < \frac{1}{2}(4-\mu_{\vec{v}})$ (otherwise we are done). Then $\lceil M-cD\rceil \cdot C_0 > 2$ follows from (3.16.2), (3.16.3), (3.15.1), and $c \geq \frac{2-\mu_{\vec{v}}}{4-\mu_{\vec{v}}}$, exactly as in (1.11).

Now consider the case $\vec{v} \in TC_p(C_0)$ (with C_0 still singular at p). Using the strategy of (3.10), all we need to prove is $\lceil M-cD\rceil \cdot C_0 > 1$, which follows from

$$\lceil M-cD\rceil \cdot C_0 \geq (1-ct)(2\beta_1)+\mu_{\vec{v}}+c(4-\mu_{\vec{v}})-3 \tag{3.16.4}$$

(same computation as in (3.10) – see (3.10.4)). Using (3.16.4), the inequality $\lceil M-cD\rceil \cdot C_0 > 1$ is proved exactly as in the previous paragraph.

(3.17) If C_0 is smooth at p, $\vec{v} \in T_p(C_0)$, and $b_i+cd_i<1$ for every C_i through p with $i \neq 0$, then the proof goes as in (3.11); the inequality we need in this case, $\lceil M-cD\rceil \cdot C_0 > 2$, is proved as above.

As in the proof of case 2 of the proposition, if C_0 is smooth at p and tangent to $\vec{v}$, and $b_1+cd_1=1$ for one more curve C_1 through p, then C_1 is unique with these properties, and is smooth at p and $\vec{v} \notin T_p(C_1)$. Switching C_0 and C_1, we are in the situation discussed below. (Compare to (3.12).)

(3.18) Finally, assume that C_0 is smooth at p and $\vec{v} \notin T_p(C_0)$. Define

$$c' = \min\left\{1; \frac{3-\mu_p}{m_p}; \frac{2-b_0}{d_0}; \frac{1-b_i}{d_i} : i \neq 0, b_i+d_i>1 \text{ and } p \in C_i\right\}.$$

Consider, for example, the case $c' = \frac{2-b_0}{d_0} < 1$ and $< \frac{3-\mu_p}{m_p}$. In this case, we show that $p \notin \operatorname{Bs}|K_S+B+M-C_0-N|$, for some effective divisor N supported away from p. Using the vanishing

$$H^1(S, K_S+\lceil M-c'D\rceil) = H^1(S, K_S+B+M-2C_0-N) = 0,$$

it suffices to show that $p \notin \operatorname{Bs}|K_S+B+M-C_0-N|_{C_0}|$; this, in turn, will follow from (1.2.2)(a) and $\lceil M-c'D\rceil \cdot C_0 > 1$.

Now C_0 passes through p but is not tangent to $\vec{v}$, and therefore we have only $M \cdot C_0 \geq \beta_1$ (rather than $2\beta_1$). $\lceil M - c'D \rceil = (M - c'D) + \Delta'$, with $\Delta' = B + c'D - 2C_0$ in a neighborhood of p, and therefore $\mathrm{ord}_p(\Delta') \geq \frac{1}{2}(\mu_{\vec{v}} + c'(4 - \mu_{\vec{v}}) - 2)$; it suffices to show that

$$(1 - c't)\beta_1 + \frac{1}{2}(\mu_{\vec{v}} + c'(4 - \mu_{\vec{v}}) - 2) > 1.$$

An inequality equivalent to this one was already proved in (3.16).

The proof in the remaining cases is a similar adaptation of the arguments in (3.13).

4. Example

Fix an integer n, $n \geq 1$. Let S be the n^{th} Hirzebruch surface, i.e. the geometrically ruled rational surface $\mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is the rank 2 vector bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$ on $\mathbb{P}^1$. Let $\pi : S \to \mathbb{P}^1$ be the ruling of S, and let F denote a fiber of π. S contains a unique irreducible curve G with negative self-intersection, $G^2 = -n$. $\mathrm{Pic}(S) \cong \mathbb{Z} \oplus \mathbb{Z}$, with generators F and G; $F^2 = 0, F \cdot G = 1$. $K_S \sim -2G - (n+2)F$. If C is any irreducible curve on S, then $C = G$, $C \sim F$, or $C \sim aG + bF$ with $a, b \in \mathbb{Z}, a \geq 1$ and $b \geq na$. All these properties are proved, for example, in [Har1, Ch.V, §2].

Let $H_m = G + mF$. We will use the Reider-type results for $\mathbb{Q}$-divisors to prove the following facts:

Claim. (1) (See [Bv, Ch.IV, Ex.1].) *$|H_n|$ is base-point-free, and defines a morphism $\phi_n : S \to \mathbb{P}^{n+1}$. Moreover, ϕ_n is an isomorphism on $S \setminus G$, and $\bar{S} = \phi_n(S) \subset \mathbb{P}^{n+1}$ is a (projective) cone with vertex $x = \phi_n(G)$. (Thus $\bar{S}$ is the cone over a normal rational curve contained in a hyperplane $\mathbb{P}^n \subset \mathbb{P}^{n+1}$, because $G \cong \mathbb{P}^1$ and $G^2 = -n$. ϕ_n is the blowing-up of $\bar{S}$ at x.)*

(2) *$|H_m|$ is very ample for $m \geq n+1$, defining an embedding $\phi_m : S \to \mathbb{P}^{2m-n+1}$.* (See [Bv, Ch.IV, Ex.2] for other properties of $|H_m|$.)

Certainly, these facts can be proved in many different ways. For instance, for (1): if $G \subset S$ is a smooth rational curve with negative self-intersection on *any* smooth surface S, then there is a projective contraction $\phi : S \to \bar{S}$, which is an isomorphism on $S \setminus G$ and contracts G to a normal point x. (This is a direct generalization of the "easy" part of Castelnuovo's criterion – the "hard" part being the regularity of $\bar{S}$ at x when $G^2 = -1$.) The proof can be adapted to the situation of the Claim. Alternatively, most of the Claim is proved in [Har1, Ch.V, Theorem 2.17].

The methods used in these proofs are somewhat specialized (the "normal contraction" approach depends on $\mathrm{Pic}(G) \cong \mathbb{Z}$; the proof in [Har1] is typical for ruled surfaces). From this point of view, Reider's theorem, which is based only on intersection numbers, is much more general. However, as we will see, Reider's theorem doesn't apply in the situation of the Claim. The proof we give below shows that there are instances where the scope of Reider's original results can be broadened by allowing $\mathbb{Q}$-divisors into the picture.

Proof of the Claim

(1) Write $H_n = K_S + L$, thus defining $L = H_n - K_S = 3G + (2n+2)F$. Then $L \cdot G = 3(-n) + (2n+2) = -n+2$; thus L is not nef for $n \geq 3$, and therefore Reider's criterion does not apply. However, write L as $B + M$, with $B = (1-\epsilon)G$ and $M = (2+\epsilon)G + (2n+2)F$, $\epsilon \in (0,1)$. Then

$$M \cdot F = (2+\epsilon), \quad M \cdot G = (2 - \epsilon n), \quad M^2 = (2+\epsilon)(2n+4-\epsilon n).$$

In particular, for $\epsilon \to 0$, we have $M \cdot F \to 2, M \cdot G \to 2, M^2 \to 2(2n+4) \geq 12$. Fix $\epsilon > 0, \epsilon \ll 1$, such that $M^2 > 9, M \cdot F \geq \frac{3}{2}$, and $M \cdot G \geq \frac{3}{2}$. Since *any* irreducible curve $C \subset S$ is either $C = G$, or $C \sim F$, or $C \sim aG + bF$ with $a \geq 1$ and $b \geq na$, we automatically have $M \cdot C \geq \frac{3}{2}$ for all such C. (We will use this observation again later: if $M \cdot G \geq 0$ and $C \neq G$ is an irreducible curve, then $M \cdot C \geq M \cdot F$.) Therefore $|H_n|$ is base-point-free by Proposition 3, part 2, with $\beta_2 = 3$ and $\beta_1 = \frac{3}{2} = \frac{\beta_2}{\beta_2 - 1}$.

Thus $|H_n|$ defines a morphism $\phi_n : S \to \mathbb{P}^\nu$, $\nu = \dim |H_n|$. We compute ν. By Riemann–Roch, we have:

$$\chi(S, H_n) = \frac{H_n \cdot (H_n - K_S)}{2} + \chi(S, \mathcal{O}_S) = n + 2.$$

We get $\nu = h^0(S, H_n) - 1 = n+1$, as stated in the Claim, *if* we can show that $h^i(S, H_n) = 0$ for $i \geq 1$. If we write $H_n = K_S + L$, as before, Kodaira's vanishing theorem does not apply, because L is not ample (it is not even nef). If we write $L = B + M$ as above, though, we get $h^i(S, H_n) = 0$ for $i \geq 1$, by (1.2.1)(a).

Next we show that ϕ_n is an isomorphism on $S \setminus G$. Consider two distinct points $p, q \in F \setminus G$. Write $L = B' + M'$, with $B' = (1-\epsilon)G + (1-\alpha)F$, $M' = (2+\epsilon)G + (2n+1+\alpha)F$, $\epsilon, \alpha \in (0,1)$. (Note that we may use *any* decomposition of L of the form $B + M$, as long as $\lceil M \rceil = L$.) We have:

$$M' \cdot F = (2+\epsilon), \quad M' \cdot G = (1 + \alpha - \epsilon n),$$

$$(M')^2 = (2+\epsilon)(2n+2+2\alpha-\epsilon n).$$

In particular, for $\epsilon, \alpha \to 0$, M' is nef and big and $M' \to 2(2n+2) \geq 8$. Let $\mu \overset{\text{def}}{=} \mu_p = \mu_q = 1-\alpha$. Choose $\beta_2 = \beta_{2,p} = \beta_{2,q} = \frac{3}{2}$ (say); then $\beta_2 \geq 2-\mu = 1+\alpha$ and $(M')^2 > 2(\beta_2)^2$ for $\epsilon, \alpha \ll 1$.

Fix $\epsilon \ll 1$, and then choose $\alpha \ll \epsilon$ such that $1+\frac{\epsilon}{2} \geq \frac{\beta_2}{\beta_2-(1-\mu)} = \frac{\beta_2}{\beta_2-\alpha}$; this can be done, because $\frac{\beta_2}{\beta_2-\alpha} \to 1$ for $\alpha \to 0$. Then $M' \cdot F = 2+\epsilon = 2\beta_1$, with $\beta_1 = 1+\frac{\epsilon}{2}$ — and therefore $M' \cdot C \geq 2\beta_1$ for every irreducible curve C through p or q. Hence proposition 4, part 3, applies (with $\beta_{1,p} = \beta_{1,q} = \beta_1$): $|H_n|$ separates (p,q).

If $p, q \in S \setminus G$ are distinct points on another irreducible curve $\bar{F} \sim F$, the proof is similar — take $B' = (1-\epsilon)G + (1-\alpha)\bar{F}$. (We say $\bar{F} \sim F$ instead of "fiber of $\pi : S \to \mathbb{P}^1$", to emphasize that the proof uses numerical arguments only.) Finally, if no such curve passes through both p and q, the proof is even easier.

Separation of tangent directions on $S \setminus G$ is proved exactly the same way; note that $\mu_p(B') = \mu_V(B') = 1-\alpha$ if $B' = (1-\epsilon)G + (1-\alpha)F$, $p \in F \setminus G$, and $\vec{v} \in T_p(F) \setminus \{\vec{0}\}$.

$H_n \cdot G = 0$ and $H_n \cdot F = 1$; therefore ϕ_n contracts G to a point $x \in \bar{S} = \phi_n(S) \subset \mathbb{P}^{n+1}$, and $\phi_n(\bar{F})$ is a straight line in $\mathbb{P}^{n+1}$ for every $\bar{F} \sim F$.

(2) As in part (1) of the Claim, we can show that $|H_m|$ is base-point-free for $m \geq n+1$, and defines a morphism $\phi_m : S \to \mathbb{P}^{2m-n+1}$ which is an isomorphism on $S \setminus G$. For $m \geq n+1$, we must show that $|H_m|$ separates p, q even when p (or q, or both) is on G, and also that $|H_m|$ separates tangent directions at every point $p \in G$.

Let $\{p\} = F \cap G$ and $\vec{v} \in T_p(G) \setminus \{\vec{0}\}$. We will show that $|H_{n+1}|$ separates $\vec{v}$ at p; the other properties have similar proofs.

Write $H_{n+1} = K_S + L$, $L = 3G + (2n+3)F$. Write $L = B + M$, $B = (1-\epsilon)G$, $M = (2+\epsilon)G + (2n+3)F$, $\epsilon \in (0,1)$. We have:

$$M \cdot F = (2+\epsilon), \quad M \cdot G = (3-\epsilon n), \quad M^2 = (2+\epsilon)(2n+6-\epsilon n).$$

For $\epsilon \to 0$ we have $M \cdot F \to 2$, $M \cdot G \to 3$, and $M^2 \to 2(2n+6) \geq 16$; in particular M is nef and big. (Note that L itself is not nef, if $n \geq 4$; indeed, $L \cdot G = 3-n$.) We have $M \cdot C \geq 2+\epsilon$ for every irreducible curve $C \subset S$ (assuming that $\epsilon \ll 1$); also, if $\vec{v} \in T_p(C)$, then $M \cdot C \geq 3-\epsilon n$, because in that case $C \sim aG + bF$ with $a \geq 1$ (proof: if $C \neq G$, then $C \cdot G \geq 2$, because $\vec{v} \in T_pC \cap T_pG$; therefore $C \not\sim F$.)

We have $\mu_p = \mu_V = 1 - \epsilon$, and $\mu_{\vec{v}} = 2(1-\epsilon)$. Choose $\beta_2 = \beta_{2,p} = \beta_{2,V} = 2$ (say), so that $M^2 > 2(\beta_2)^2$, $\beta_{2,p} \geq 2 - \mu_p$, and $\beta_{2,V} \geq 2 - \mu_V$. Put $\beta_1 = \frac{2\beta_2}{2\beta_2 - (2-\mu_{\vec{v}})} = \frac{\beta_2}{\beta_2 - \epsilon}$. For $\epsilon \ll 1$, we have:

$$M \cdot C = 2 + \epsilon \geq \beta_1 \quad \text{for all curves } C \subset S,$$
$$M \cdot C = 3 - \epsilon n \geq 2\beta_1 \quad \text{for all } C \text{ containing } (p, \vec{v}).$$

(Note that $\beta_1 = \frac{\beta_2}{\beta_2 - \epsilon} \to 1$ as $\epsilon \to 0$, so these inequalities are verified for all $\epsilon \ll 1$.) Now use proposition 5, case 3. ∎

By inspecting the proof of the Claim, we can see that the only assumptions we used were that $\mathrm{Pic}(S) = \mathbb{Z}G \oplus \mathbb{Z}F$, $G^2 = -n$, $F^2 = 0$, $G \cdot F = 1$, and $K_S = -2G - (n+2)F$ (if the other hypotheses are satisfied, the last condition is equivalent to: G and F are smooth rational curves); this suggests the following

Exercise. A surface S with these properties is isomorphic to the n^{th} Hirzebruch surface.

Hint. There are several ways to see this. One, of course, is to use part (1) of the Claim: after all, we have shown that S is the blowing-up of the cone over the normal rational curve of degree n.

Another solution is to show that $|F|$ is base-point-free and $\dim |F| = 1$, as in the proof of part (1) of the Claim; thus $\phi = \phi_{|F|}$ realizes S as a geometrically ruled surface over $\mathbb{P}^1$, as required. (S is minimal, because $C^2 \geq 0$ for every irreducible curve $C \neq G$; this follows easily from the hypotheses.)

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