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ONE DIMENSIONAL PERTURBATION OF INVARIANT SUBSPACES IN THE HARDY SPACE OVER THE BIDISK II

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Abstract. This paper is a continuation of the previous paper [9]. Let M_1 be an invariant subspace of H^2 over the bidisk. Then there exists a nonzero f_0 in M_1 such that $M_2 := M_1 \ominus \mathbb{C} \cdot f_0$ is also an invariant subspace. A relationship is given the ranks of the cross commutators $[R_z^*, R_w]$ on M_1 and M_2. We also give a relationship of the ranks of the cross commutators $[S_w, S_z^*]$ on $H^2 \ominus M_1$ and $H^2 \ominus M_2$.

1. Introduction

Let $H^2 = H^2(\mathbb{D}^2)$ be the Hardy space over the bidisk $\mathbb{D}^2$ with two variables z and w. Let T_z and T_w be the multiplication operators on H^2 by z and w, respectively. A nonzero closed subspace M of H^2 is said to be invariant if $T_zM \subset M$ and $T_wM \subset M$. We write $R_z^M = T_z|_M$ and $R_w^M = T_w|_M$. Let $N = H^2 \ominus M$. Then $T_z^*N \subset N$ and $T_w^*N \subset N$, where T_z^*, T_w^* are adjoint operators of T_z, T_w, so N is called a backward shift invariant subspace of H^2. We denote by S_z^N, S_w^N the compression operators of T_z, T_w on N, that is, $S_z^N = P_NT_z|_N$ and $S_w^N = P_NT_w|_N$, where P_N is the orthogonal projection from H^2 onto N. We note that $R_z^{M*} = P_MT_z^*|_M$ and $S_z^{N*} = T_z^*|_N$.

In [12], Mandrekar showed that $[R_w^{M*}, R_z^M] := R_w^{M*}R_z^M - R_z^MR_w^{M*} = 0$ if and only if $M = \varphi H^2$ for an inner function φ (see also [1, 2, 4, 8, 13]). In [10], Nakazi, Seto and the first author proved that $[S_w^N, S_z^{N*}] = 0$ if and only if $M = \varphi(z)H^2 + \psi(w)H^2$, where $\varphi(z), \psi(w)$ are either one variable inner functions or 0 (see also [3, 5, 6, 7, 11]). So it is considered that the cross commutators $[R_w^{M*}, R_z^M]$ on M and $[S_w^N, S_z^{N*}]$ on N are important operators to study the structure of invariant subspaces H^2. We denote by $\operatorname{rank} T$ the rank of the operator T, that is, $\operatorname{rank} T$ is the dimension of the range of T.

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Let M_1 be an invariant subspace of H^2. Then there is $f_0 \in M_1$ with $\|f_0\| = 1$ such that $M_2 := M_1 \ominus \mathbb{C} \cdot f_0$ is an invariant subspace. To study the structure of invariant subspaces of H^2, one of the basic questions is what kind of changes of properties occur under the one dimensional perturbation. Let $N_j = H^2 \ominus M_j$ for $j = 1, 2$. In the previous paper [9], we described the spaces

$$M_2 \ominus (zM_2 + wM_2) \quad \text{and} \quad \big\{h \in N_2 : zh \in M_2, wh \in M_2\big\}$$

using the words of f_0,

$$M_1 \ominus (zM_1 + wM_1) \quad \text{and} \quad \big\{h \in N_1 : zh \in M_1, wh \in M_1\big\},$$

respectively and studied some related topics, and see the references given in [9] for the study of invariant subspaces of H^2. In this paper, we shall concentrate on the study of the relationship of the ranks of the cross commutators on M_1, M_2 and on N_1, N_2, respectively.

In Section 2, we shall show that

$$\operatorname{rank}\,[R_w^{M_1 *}, R_z^{M_1}] - 1 \leq \operatorname{rank}\,[R_w^{M_2 *}, R_z^{M_2}] \leq \operatorname{rank}\,[R_w^{M_1 *}, R_z^{M_1}] + 1.$$

Since M_2 is one dimensional perturbation of M_1, this is an expectable fact.

In Section 3, we shall show that

$$\operatorname{rank}\,[S_w^{N_1}, S_z^{N_1 *}] - 1 \leq \operatorname{rank}\,[S_w^{N_2}, S_z^{N_2 *}] \leq \operatorname{rank}\,[S_w^{N_1}, S_z^{N_1 *}] + 3.$$

The authors think that this is a remarkable fact.

We shall give examples of M_1 and $f_0 \in M_1$ which satisfy the cases in the above inequalities.

2. Invariant subspaces

For an invariant subspace M of H^2, it is not difficult to see that

$$[R_w^{M*}, R_z^M]M = P_{wM} z(M \ominus wM). \tag{2.1}$$

We note that

$$\operatorname{rank}\,[R_w^{M*}, R_z^M] = \operatorname{rank}\,[R_z^{M*}, R_w^M]. \tag{2.2}$$

Let M_1 be an invariant subspace of H^2 and $f_0 \in M_1$ with $\|f_0\| = 1$ such that $M_2 := M_1 \ominus \mathbb{C} \cdot f_0$ is an invariant subspace. The following is given in Lemmas 3.2 and 4.2 in [9].

Lemma 2.1. *If $f_0 \in M_1 \ominus wM_1$, then*

$$M_2 \ominus wM_2 = \big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0\big) \oplus \mathbb{C} \cdot wf_0.$$

Suppose that $f_0 \notin M_1 \ominus wM_1$. Since $M_1 = M_2 \oplus \mathbb{C} \cdot f_0$ and $R_w^{M_1*} f_0 \in \mathbb{C} \cdot f_0$, there is a nonzero $\beta \in \mathbb{D}$ such that $f_0 = P_{M_1 \ominus wM_1} f_0 + \beta w f_0$. The following is given in Lemmas 5.1 and 5.2 in [9].

Lemma 2.2. *Suppose that $f_0 \notin M_1 \ominus wM_1$ and $f_0 \notin M_1 \ominus zM_1$. Then we have the following.*

(i) *Either $P_{M_1 \ominus wM_1} f_0 \notin M_1 \ominus (zM_1 + wM_1)$ or $P_{M_1 \ominus zM_1} f_0 \notin M_1 \ominus (zM_1 + wM_1)$.*

(ii) *If $P_{M_1 \ominus wM_1} f_0 \notin M_1 \ominus (zM_1 + wM_1)$, then*

$$M_2 \ominus wM_2 = \big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot P_{M_1 \ominus wM_1} f_0\big) \oplus \mathbb{C} \cdot g_0,$$

where

$$g_0 = f_0 - \frac{1}{1 - |\beta|^2} P_{M_1 \ominus wM_1} f_0.$$

.

Theorem 2.1.

$$\operatorname{rank}[R_w^{M_1*}, R_z^{M_1}] - 1 \le \operatorname{rank}[R_w^{M_2*}, R_z^{M_2}] \le \operatorname{rank}[R_w^{M_1*}, R_z^{M_1}] + 1.$$

Proof. Step 1. Suppose that $f_0 \in M_1 \ominus wM_1$. By Lemma 2.1, we have

$$M_2 \ominus wM_2 = \big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0\big) \oplus \mathbb{C} \cdot w f_0. \tag{2.3}$$

Then

$$wM_2 = wM_1 \ominus \mathbb{C} \cdot w f_0. \tag{2.4}$$

We have

$$\begin{aligned}
\operatorname{rank}[R_w^{M_2*}, R_z^{M_2}] &= \dim P_{wM_2} z(M_2 \ominus wM_2) \qquad \text{by (2.1)} \\
&\le \dim P_{wM_1} z(M_2 \ominus wM_2) \qquad \text{because of } M_2 \subset M_1 \\
&\le \dim P_{wM_1} z\big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0\big) + 1 \qquad \text{by (2.3)} \\
&\le \dim P_{wM_1} z(M_1 \ominus wM_1) + 1 \\
&= \operatorname{rank}[R_w^{M_1*}, R_z^{M_1}] + 1 \qquad \text{by (2.1)}.
\end{aligned}$$

Let

$$A = \big\{h \in (M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0 : zh \perp w f_0\big\}.$$

By (2.4), $P_{wM_1} zh = P_{wM_2} zh$ for every $h \in A$. Then we have

$$\begin{aligned}
&\operatorname{rank}[R_w^{M_2*}, R_z^{M_2}] = \dim P_{wM_2} z(M_2 \ominus wM_2) \\
&= \dim P_{wM_2} z\big(\big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0\big) \oplus \mathbb{C} \cdot w f_0\big) \quad \text{by Lemma 2.1} \\
&= \dim \big(P_{wM_2} z\big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0\big) + \mathbb{C} \cdot P_{wM_2} z w f_0\big) \\
&\ge \dim \big(P_{wM_2} zA + \mathbb{C} \cdot P_{wM_2} z w f_0\big).
\end{aligned}$$

For every $h \in A$, we have

$$\begin{aligned}\big\langle P_{wM_2}zh, P_{wM_2}zwf_0\big\rangle &= \langle P_{wM_2}zh, zwf_0\rangle = \langle P_{wM_1}zh, zwf_0\rangle \\ &= \langle zh, P_{wM_1}zwf_0\rangle = \langle zh, zwf_0\rangle = \langle h, wf_0\rangle = 0.\end{aligned}$$

Since $P_{wM_2}zwf_0 = wP_{M_2}zf_0 \neq 0$, we have

$$\operatorname{rank}[R_w^{M_2*}, R_z^{M_2}] \geq \dim P_{wM_2}zA + 1 = \dim P_{wM_1}zA + 1.$$

By the definition of A, there is $h_1 \in (M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0$ (may be zero) such that

$$A = (M_1 \ominus wM_1) \ominus (\mathbb{C} \cdot f_0 \oplus \mathbb{C} \cdot h_1).$$

Hence

$$\begin{aligned}\operatorname{rank}[R_w^{M_2*}, R_z^{M_2}] &\geq \dim P_{wM_1}zA + 1 \\ &\geq \dim P_{wM_1}z(M_1 \ominus wM_1) - 2 + 1 \\ &= \operatorname{rank}[R_w^{M_1*}, R_z^{M_1}] - 1.\end{aligned}$$

Step 2. Suppose that $f_0 \notin M_1 \ominus wM_1$. If $f_0 \in M_1 \ominus zM_1$, then by Step 1 (exchanging variables z and w) we have

$$\operatorname{rank}[R_z^{M_1*}, R_w^{M_1}] - 1 \leq \operatorname{rank}[R_z^{M_2*}, R_w^{M_2}] \leq \operatorname{rank}[R_z^{M_1*}, R_w^{M_1}] + 1.$$

Hence by (2.2), we get the assertion. So, we may assume that $f_0 \notin M_1 \ominus zM_1$. By Lemma 2.2 (i), either $\eta_0 := P_{M_1 \ominus wM_1} f_0 \notin M_1 \ominus (zM_1 + wM_1)$ or $P_{M_1 \ominus zM_1} f_0 \notin M_1 \ominus (zM_1 + wM_1)$. So, further we may assume that $\eta_0 \notin M_1 \ominus (zM_1 + wM_1)$. For the latter case, we may prove it similarly. By Lemma 2.2 (ii), we have

$$M_2 \ominus wM_2 = \big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot \eta_0\big) \oplus \mathbb{C} \cdot g_0, \tag{2.5}$$

where

$$g_0 = f_0 - \frac{1}{1 - |\beta|^2}\eta_0.$$

In the same way as the first paragraph of Step 1, we have

$$\operatorname{rank}[R_w^{M_2*}, R_z^{M_2}] \leq \operatorname{rank}[R_w^{M_1*}, R_z^{M_1}] + 1.$$

We have

$$wM_2 = w(M_1 \ominus \mathbb{C} \cdot f_0) = wM_1 \ominus \mathbb{C} \cdot wf_0.$$

Since $f_0 = \eta_0 + \beta wf_0$,

$$wM_2 = wM_1 \ominus \mathbb{C} \cdot (f_0 - \eta_0). \tag{2.6}$$

We have

$$
\begin{aligned}
&\operatorname{rank}[R_w^{M_2*}, R_z^{M_2}] = \dim P_{wM_2} z(M_2 \ominus wM_2) \\
&= \dim \big(P_{wM_2} z\big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot \eta_0\big) + \mathbb{C} \cdot P_{wM_2} z g_0\big) \qquad \text{by (2.5)} \\
&\geq \dim P_{wM_2} z\big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot \eta_0\big) \\
&\geq \dim P_{wM_1} z\big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot \eta_0\big) - 1 \qquad \text{by (2.6)} \\
&\geq \dim P_{wM_1} z(M_1 \ominus wM_1) - 2 \\
&= \operatorname{rank}[R_w^{M_1*}, R_z^{M_1}] - 2.
\end{aligned}
$$

By this fact, if $\operatorname{rank}[R_w^{M_1*}, R_z^{M_1}] = \infty$, then we get the assertion. So, we may assume that

$$k := \operatorname{rank}[R_w^{M_1*}, R_z^{M_1}] < \infty.$$

To show that

$$\operatorname{rank}[R_w^{M_2*}, R_z^{M_2}] \geq \operatorname{rank}[R_w^{M_1*}, R_z^{M_1}] - 1,$$

assume that

$$\operatorname{rank}[R_w^{M_2*}, R_z^{M_2}] = \operatorname{rank}[R_w^{M_1*}, R_z^{M_1}] - 2. \tag{2.7}$$

We shall lead a contradiction. By the above inequalities, we have

$$P_{wM_2} z g_0 \in P_{wM_2} z\big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot \eta_0\big), \tag{2.8}$$

$$\dim P_{wM_2} z\big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot \eta_0\big) = \dim P_{wM_1} z\big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot \eta_0\big) - 1 \tag{2.9}$$

and

$$\dim P_{wM_1} z\big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot \eta_0\big) = \dim P_{wM_1} z(M_1 \ominus wM_1) - 1. \tag{2.10}$$

By (2.10), there are $f_1, f_2, \cdots, f_{k-1}$ in $(M_1 \ominus wM_1) \ominus \mathbb{C} \cdot \eta_0$ such that

$$\big\{P_{wM_1} z f_1, P_{wM_1} z f_2, \cdots, P_{wM_1} z f_{k-1}\big\}$$

is a basis of $P_{wM_1} z\big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot \eta_0\big)$.

First, suppose that $P_{wM_1} z f_j \perp f_0 - \eta_0$ for every $1 \leq j \leq k-1$. Then by (2.6), we have $P_{wM_1} z f_j = P_{wM_2} z f_j$ for every $1 \leq j \leq k-1$. Hence

$$
\begin{aligned}
&\operatorname{rank}[R_w^{M_2*}, R_z^{M_2}] \geq \dim P_{wM_2} z\big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot \eta_0\big) \\
&\geq \dim P_{wM_2} z \sum_{j=1}^{k-1} \mathbb{C} \cdot f_j = \dim P_{wM_1} z \sum_{j=1}^{k-1} \mathbb{C} \cdot f_j \\
&= k - 1 = \operatorname{rank}[R_w^{M_1*}, R_z^{M_1}] - 1.
\end{aligned}
$$

This contradicts (2.7).

Next, suppose that $P_{wM_1} z f_j \not\perp f_0 - \eta_0$ for some $1 \leq j \leq k-1$. We may assume that $P_{wM_1} z f_1 \not\perp f_0 - \eta_0$ and $P_{wM_1} z f_j \perp f_0 - \eta_0$ for every $2 \leq j \leq k-1$. Then $P_{wM_1} z f_j = P_{wM_2} z f_j$ for every $2 \leq j \leq k-1$. We divide the proof into two cases.

Case 1. Suppose that

$$P_{wM_2} z f_1 \notin \sum_{j=2}^{k-1} \mathbb{C} \cdot P_{wM_2} z f_j = \sum_{j=2}^{k-1} \mathbb{C} \cdot P_{wM_1} z f_j.$$

Then

$$\begin{aligned}
&\operatorname{rank}\,[R_w^{M_2 *}, R_z^{M_2}] \geq \dim P_{wM_2} z\big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot \eta_0\big) \\
&\geq \dim \sum_{j=1}^{k-1} \mathbb{C} \cdot P_{wM_2} z f_j \\
&= \dim \Big(\mathbb{C} \cdot P_{wM_2} z f_1 + \sum_{j=2}^{k-1} \mathbb{C} \cdot P_{wM_2} z f_j\Big) \\
&= \dim \sum_{j=2}^{k-1} \mathbb{C} \cdot P_{wM_2} z f_j + 1 = \dim \sum_{j=2}^{k-1} \mathbb{C} \cdot P_{wM_1} z f_j + 1 \\
&= k - 2 + 1 = k - 1.
\end{aligned}$$

This contradicts (2.7).

Case 2. Suppose that

$$P_{wM_2} z f_1 \in \sum_{j=2}^{k-1} \mathbb{C} \cdot P_{wM_2} z f_j = \sum_{j=2}^{k-1} \mathbb{C} \cdot P_{wM_1} z f_j.$$

Then

$$P_{wM_2} z f_1 = \sum_{j=2}^{k-1} c_j P_{wM_2} z f_j$$

for some $c_j \in \mathbb{C}$, $2 \leq j \leq k-1$. Replacing f_1 by $f_1 - \sum_{j=2}^{k-1} c_j f_j$, we may assume that $P_{wM_2} z f_1 = 0$. Since $P_{wM_1} z f_1 \neq 0$, by (2.6) we have

$$\mathbb{C} \cdot P_{wM_1} z f_1 = \mathbb{C} \cdot (f_0 - \eta_0). \tag{2.11}$$

In this case, we note that (2.7) holds by (2.8), (2.9) and (2.10).

For every $h \in M_1 \ominus wM_1$, since $f_0 - \eta_0 \in wM_1$ we have

$$\begin{aligned}
0 &= \langle f_0 - \eta_0, h \rangle = \langle z(f_0 - \eta_0), zh \rangle \\
&= \big\langle P_{wM_1} z(f_0 - \eta_0), zh \big\rangle = \big\langle P_{wM_1} z(f_0 - \eta_0), P_{wM_1} zh \big\rangle.
\end{aligned}$$

Then

$$P_{wM_1} z(f_0 - \eta_0) \perp P_{wM_1} z(M_1 \ominus wM_1).$$

By (2.11), $f_0-\eta_0 \in P_{wM_1}z(M_1\ominus wM_1)$. Then $P_{wM_1}z(f_0-\eta_0)\perp f_0-\eta_0$, so by (2.6) we have

$$P_{wM_1}z(f_0-\eta_0)=P_{wM_2}z(f_0-\eta_0).$$

Hence

$$P_{wM_2}z(f_0-\eta_0)\perp P_{wM_1}z(M_1\ominus wM_1),$$

so that we get

$$P_{wM_2}z(f_0-\eta_0)\perp P_{wM_2}z(M_1\ominus wM_1). \tag{2.12}$$

Therefore

$$\begin{aligned}
0&=\big\langle P_{wM_2}z(f_0-\eta_0),P_{wM_2}zg_0\big\rangle \qquad \text{by (2.8)}\\
&=\Big\langle P_{wM_2}z(f_0-\eta_0),P_{wM_2}z\Big(f_0-\frac{1}{1-|\beta|^2}\eta_0\Big)\Big\rangle\\
&=\Big\langle P_{wM_2}z(f_0-\eta_0),P_{wM_2}z\Big(f_0-\eta_0-\frac{|\beta|^2}{1-|\beta|^2}\eta_0\Big)\Big\rangle\\
&=\big\|P_{wM_2}z(f_0-\eta_0)\big\|^2 \qquad \text{by (2.12)}.
\end{aligned}$$

This shows that $z(f_0-\eta_0)\perp wM_2$. Since $f_0-\eta_0\in wM_1$, we have $z(f_0-\eta_0)\in wM_1$ and by (2.6) we have $z(f_0-\eta_0)=c(f_0-\eta_0)$ for some $c\in\mathbb{C}$. Thus we get $f_0=\eta_0$. Since $\eta_0\in M_1\ominus wM_1$, $f_0\in M_1\ominus wM_1$, and this contradicts the starting assumption. $\square$

Example 2.1. Let

$$M_1=z^3H^2+z^2wH^2+w^2H^2.$$

Then

$$\begin{aligned}
&\operatorname{rank}[R_w^{M_1*},R_z^{M_1}]=\dim P_{wM_1}z(M_1\ominus wM_1)\\
&=\dim P_{wM_1}z\big(z^3H^2+\mathbb{C}\cdot z^2w+\mathbb{C}\cdot zw^2+\mathbb{C}\cdot w^2\big)\\
&=\dim(\mathbb{C}\cdot z^3w+\mathbb{C}\cdot z^2w^2)=2.
\end{aligned}$$

We shall take a nonzero f_0 in M_1 such that $M_2=M_1\ominus\mathbb{C}\cdot f_0$ is an invariant subspace and $\operatorname{rank}[R_w^{M_2*},R_z^{M_2}]=1,2,3$, respectively.

(i) Let $f_0=z^2w\in M_1$. Then $M_2=z^3H^2+w^2H^2$ and

$$\operatorname{rank}[R_w^{M_2*},R_z^{M_2}]=\dim\mathbb{C}\cdot P_{wM_2}z^3w^2=1.$$

(ii) Let $f_0=z^3\in M_1$. Then $M_2=z^4H^2+z^2wH^2+w^2H^2$ and

$$\operatorname{rank}[R_w^{M_2*},R_z^{M_2}]=\dim(\mathbb{C}\cdot z^3w+\mathbb{C}\cdot z^2w^2)=2.$$

(iii) Let $f_0=w^2\in M_1$. Then

$$M_2=z^3H^2+z^2wH^2+zw^2H^2+w^3H^2$$

and

$$\operatorname{rank}[R_w^{M_2*}, R_z^{M_2}] = \dim\left(\mathbb{C}\cdot z^3w + \mathbb{C}\cdot z^2w^2 + \mathbb{C}\cdot zw^3\right) = 3.$$

□

3. Backward shift invariant subspaces

Let M be an invariant subspace of H^2 and $N = H^2 \ominus M$.

Lemma 3.1. *We have the following.*

(i) $[S_w^N, S_z^{N*}] = P_N T_z^* P_M T_w|_N$.
(ii) $\operatorname{rank}[S_w^N, S_z^{N*}] = \dim P_N T_z^* P_M wN$.

Proof. (i) We have

$$\begin{aligned}
[S_w^N, S_z^{N*}] &= S_w^N S_z^{N*} - S_z^{N*} S_w^N \\
&= P_N T_w P_N T_z^*|_N - P_N T_z^* P_N T_w|_N \\
&= P_N T_z^* T_w|_N - P_N T_z^* P_N T_w|_N \\
&= P_N T_z^*(I - P_N) T_w|_N = P_N T_z^* P_M T_w|_N.
\end{aligned}$$

(ii) follows from (i). □

Let M_1 be an invariant subspace of H^2 and $f_0 \in M_1$ with $\|f_0\| = 1$ such that $M_2 := M_1 \ominus \mathbb{C}\cdot f_0$ is an invariant subspace. Let $N_j = H^2 \ominus M_j$ for $j = 1, 2$. We have $M_1 = M_2 \oplus \mathbb{C}\cdot f_0$ and $N_2 = N_1 \oplus \mathbb{C}\cdot f_0$.

Theorem 3.1.

$$\operatorname{rank}[S_w^{N_1}, S_z^{N_1*}] - 1 \le \operatorname{rank}[S_w^{N_2}, S_z^{N_2*}] \le \operatorname{rank}[S_w^{N_1}, S_z^{N_1*}] + 3.$$

Proof. We have

$$\begin{aligned}
&\operatorname{rank}[S_w^{N_2}, S_z^{N_2*}] = \dim P_{N_2} T_z^* P_{M_2} wN_2 \qquad \text{by Lemma 3.1 (ii)} \\
&\ge \dim P_{N_1} T_z^* P_{M_2} wN_1 \qquad \text{because } N_1 \subset N_2 \\
&\ge \dim P_{N_1} T_z^* P_{M_1} wN_1 - 1 \qquad \text{because } M_1 = M_2 \oplus \mathbb{C}\cdot f_0. \\
&= \operatorname{rank}[S_w^{N_1}, S_z^{N_1*}] - 1.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\operatorname{rank}[S_w^{N_2}, S_z^{N_2*}] = \dim P_{N_2} T_z^* P_{M_2} wN_2 \\
&\le \dim P_{N_1} T_z^* P_{M_2} wN_2 + 1 \qquad \text{because } N_2 = N_1 \oplus \mathbb{C}\cdot f_0 \\
&\le \dim P_{N_1} T_z^* P_{M_2} wN_1 + 2 \qquad \text{because } N_2 = N_1 \oplus \mathbb{C}\cdot f_0 \\
&\le \dim P_{N_1} T_z^* P_{M_1} wN_1 + 3 \qquad \text{because } M_1 = M_2 \oplus \mathbb{C}\cdot f_0. \\
&= \operatorname{rank}[S_w^{N_1}, S_z^{N_1*}] + 3.
\end{aligned}$$

$\square$

Example 3.1. Let

$$M_1 = z^4H^2 + z^3w^2H^2 + z^2w^4H^2 + w^5H^2$$

and $N_1 = H^2 \ominus M_1$. Then

$$\begin{aligned}&\operatorname{rank}[S_w^{N_1}, S_z^{N_1*}] = \dim P_{N_1}T_z^*P_{M_1}wN_1\\ &= \dim(\mathbb{C}\cdot z^2w^2 + \mathbb{C}\cdot zw^4) = 2.\end{aligned}$$

We shall take a nonzero f_0 in M_1 such that $M_2 = M_1 \ominus \mathbb{C}\cdot f_0$ is an invariant subspace and $\operatorname{rank}[S_w^{N_2}, S_z^{N_2*}] = 1, 2, 3, 4, 5$, respectively, where $N_2 = H^2 \ominus M_2$. Note that $\operatorname{rank}[R_w^{M_1*}, R_z^{M_1}] = 3$.

(i) Let $f_0 = z^2w^4 \in M_1$. Then

$$M_2 = z^4H^2 + z^3w^2H^2 + w^5H^2$$

and

$$\operatorname{rank}[S_w^{N_2}, S_z^{N_2*}] = \dim\mathbb{C}\cdot z^2w^2 = 1.$$

Note that $\operatorname{rank}[R_w^{M_2*}, R_z^{M_2}] = 2$.

(ii) Let $f_0 = z^3w^2 \in M_1$. Then

$$M_2 = z^4H^2 + z^3w^3H^2 + z^2w^4H^2 + w^5H^2$$

and

$$\operatorname{rank}[S_w^{N_2}, S_z^{N_2*}] = \dim(\mathbb{C}\cdot z^2w^3 + \mathbb{C}\cdot zw^4) = 2.$$

Note that $\operatorname{rank}[R_w^{M_2*}, R_z^{M_2}] = 3$.

(iii) Let $f_0 = z^4 \in M_1$. Then

$$M_2 = z^5H^2 + z^4wH^2 + z^3w^2H^2 + z^2w^4H^2 + w^5H^2$$

and

$$\operatorname{rank}[S_w^{N_2}, S_z^{N_2*}] = \dim\left(\mathbb{C}\cdot z^3w + \mathbb{C}\cdot z^2w^2 + \mathbb{C}\cdot zw^4\right) = 3.$$

Note that $\operatorname{rank}[R_w^{M_2*}, R_z^{M_2}] = 4$.

(iv) Let $f_0 = z^3w^2 - w^5 \in M_1$. Then

$$\begin{aligned}M_2 = z^4H^2 + z^3w^3H^2 + z^2w^4H^2 + zw^5H^2 + w^6H^2&\\ + \mathbb{C}\cdot(z^3w^2 + w^5).&\end{aligned}$$

We have

$$
\begin{aligned}
&\operatorname{rank}[S_w^{N_2}, S_z^{N_2*}] = \dim P_{N_2} T_z^* P_{M_2} w N_2 \\
&= \dim P_{N_2} T_z^* P_{M_2} w \big(\mathbb{C}\cdot z^3 w + \mathbb{C}\cdot (z^3 w^2 - w^5) + \mathbb{C}\cdot z^2 w^3 \\
&\qquad\qquad + \mathbb{C}\cdot z w^4 + \mathbb{C}\cdot w^4\big) \\
&= \dim P_{N_2} T_z^* \big(\mathbb{C}\cdot (z^3 w^2 + w^5) + \mathbb{C}\cdot (z^3 w^3 - w^6) \\
&\qquad\qquad + \mathbb{C}\cdot z^2 w^4 + \mathbb{C}\cdot z w^5\big) \\
&= \dim P_{N_2} \big(\mathbb{C}\cdot z^2 w^2 + \mathbb{C}\cdot z^2 w^3 + \mathbb{C}\cdot z w^4 + \mathbb{C}\cdot w^5\big) \\
&= \dim \big(\mathbb{C}\cdot z^2 w^2 + \mathbb{C}\cdot z^2 w^3 + \mathbb{C}\cdot z w^4 + \mathbb{C}\cdot (z^3 w^2 - w^5)\big) \\
&= 4.
\end{aligned}
$$

Note that $\operatorname{rank}[R_w^{M_2*}, R_z^{M_2}] = 4$.

(v) Let $f_0 = z^4 - w^5 \in M_1$. Then

$$
\begin{aligned}
M_2 = z^5 H^2 + z^4 w H^2 + z^3 w^2 H^2 + z^2 w^4 H^2 \\
+ z w^5 H^2 + w^6 H^2 + \mathbb{C}\cdot (z^4 + w^5).
\end{aligned}
$$

We have

$$
\begin{aligned}
&\operatorname{rank}[S_w^{N_2}, S_z^{N_2*}] = \dim P_{N_2} T_z^* P_{M_2} w N_2 \\
&= \dim P_{N_2} T_z^* P_{M_2} w \big(\mathbb{C}\cdot (z^4 - w^5) + \mathbb{C}\cdot z^3 w + \mathbb{C}\cdot z^2 w^3 \\
&\qquad\qquad + \mathbb{C}\cdot z w^4 + \mathbb{C}\cdot w^4\big) \\
&= \dim P_{N_2} T_z^* \big(\mathbb{C}\cdot (z^4 w - w^6) + \mathbb{C}\cdot z^3 w^2 + \mathbb{C}\cdot z^2 w^4 \\
&\qquad\qquad + \mathbb{C}\cdot z w^5 + \mathbb{C}\cdot (z^4 + w^5)\big) \\
&= \dim P_{N_2} \big(\mathbb{C}\cdot z^3 w + \mathbb{C}\cdot z^2 w^2 + \mathbb{C}\cdot z w^4 + \mathbb{C}\cdot w^5 + \mathbb{C}\cdot z^3\big) \\
&= \dim \big(\mathbb{C}\cdot z^3 w + \mathbb{C}\cdot z^2 w^2 + \mathbb{C}\cdot z w^4 + \mathbb{C}\cdot (z^4 - w^5) + \mathbb{C}\cdot z^3\big) \\
&= 5.
\end{aligned}
$$

Note that $\operatorname{rank}[R_w^{M_2*}, R_z^{M_2}] = 4$. □

Remark 3.1. We shall give $\operatorname{rank}[S_w^{N_j}, S_z^{N_j*}]$, $j = 1, 2$, for Example 2.1. We have $\operatorname{rank}[S_w^{N_1}, S_z^{N_1*}] = 1$.

(i) $\operatorname{rank}[S_w^{N_2}, S_z^{N_2*}] = 0$.
(ii) $\operatorname{rank}[S_w^{N_2}, S_z^{N_2*}] = 1$.
(iii) $\operatorname{rank}[S_w^{N_2}, S_z^{N_2*}] = 2$.

□

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