Nihonkai Math. J.
Vol.23(2012), 93–113

THE DUNKL-WILLIAMS CONSTANT OF SYMMETRIC OCTAGONAL NORMS ON $\mathbb{R}^2$

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Abstract. Recently, we constructed a new calculation method for the Dunkl-Williams constant $DW(X)$ of a normed linear space X. In this paper, we determine the Dunkl-Williams constant of symmetric octagonal norms on $\mathbb{R}^2$ by using our method.

1. Introduction

A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be absolute if $\|(a,b)\| = \|(|a|,|b|)\|$ for all $(a,b) \in \mathbb{R}^2$, and normalized if $\|(1,0)\| = \|(0,1)\| = 1$. The set of all absolute normalized norms on $\mathbb{R}^2$ is denoted by AN_2. Bonsall and Duncan [4] showed the following characterization of absolute normalized norms on $\mathbb{R}^2$. Namely, the set AN_2 of all absolute normalized norms on $\mathbb{R}^2$ is in a one-to-one correspondence with the set Ψ_2 of all convex functions ψ on $[0,1]$ satisfying $\max\{1-t,t\} \le \psi(t) \le 1$ for all $t \in [0,1]$ (cf. [24]). The correspondence is given by the equation $\psi(t) = \|(1-t,t)\|$ for all $t \in [0,1]$. Note that the norm $\|\cdot\|_\psi$ associated with the function $\psi \in \Psi_2$ is given by

$$\|(a,b)\|_\psi = \begin{cases} (|a|+|b|)\psi\left(\dfrac{|b|}{|a|+|b|}\right) & \text{if } (a,b) \neq (0,0), \\ 0 & \text{if } (a,b) = (0,0). \end{cases}$$

2010 *Mathematics Subject Classification.* 46B20.
Key words and phrases. Dunkl-Williams constant, absolute normalized norm, norming functional.
The second author is supported in part by Grants-in-Aid for Scientific Research (No. 23540189), Japan Society for Promotion of Science.

For each $\beta \in (1/2, 1)$, let $\psi_\beta(t) = \max\{1-t, t, \beta\}$. Then, $\psi_\beta \in \Psi_2$, and the norm $\|\cdot\|_\beta$ associated with ψ_β is given by

$$\begin{aligned}\|(a,b)\|_\beta &= \max\{|a|, |b|, \beta(|a|+|b|)\} \\ &= \begin{cases} |a| & \left(|b| \leq \dfrac{1-\beta}{\beta}|a|\right), \\ \beta(|a|+|b|) & \left(\dfrac{1-\beta}{\beta}|a| \leq |b| \leq \dfrac{\beta}{1-\beta}|a|\right), \\ |b| & \left(\dfrac{\beta}{1-\beta}|a| \leq |b|\right). \end{cases}\end{aligned}$$

Remark that the unit sphere of $(\mathbb{R}^2, \|\cdot\|_\beta)$ is an octagon, and that the norm $\|\cdot\|_\beta$ is symmetric, that is, $\|(a,b)\|_\beta = \|(b,a)\|_\beta$ for all $(a,b) \in \mathbb{R}^2$. Hence, in this paper, the norm $\|\cdot\|_\beta$ is said to be a symmetric octagonal norm on $\mathbb{R}^2$.

Throughout this paper, the term "normed linear space" always means a real normed linear space which has two or more dimension. Let X be a normed linear space. In 1964, Dunkl and Williams [8] showed that the inequality

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{4\|x-y\|}{\|x\|+\|y\|} \tag{1}$$

holds for all $x, y \in X \setminus \{0\}$, and that if X admits an inner product, the stronger inequality

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x-y\|}{\|x\|+\|y\|} \tag{2}$$

holds for all $x, y \in X \setminus \{0\}$. These inequalities are so called the Dunkl-Williams inequality. There are many results related to this inequality (cf. [1, 5, 6, 7, 16, 17, 21, 22, 23, 25, 26], and so on).

In [8], it was also proved that for any $\varepsilon > 0$ there exist $x, y \in (\mathbb{R}^2, \|\cdot\|_1)$ such that

$$\left\| \frac{x}{\|x\|_1} - \frac{y}{\|y\|_1} \right\|_1 > (4-\varepsilon)\frac{\|x-y\|_1}{\|x\|_1+\|y\|_1}.$$

This means that the constant 4 is the best possible choice for the Dunkl-Williams inequality in the space $(\mathbb{R}^2, \|\cdot\|_1)$. A bit later, Kirk and Smiley [15] completed this result by showing that inequality (2) characterizes inner product spaces.

Thus, the best possible choice for the Dunkl-Williams inequality measures "how much" the space is close (or far) to be an inner product space. Motivated by this fact, Jiménez-Melado et al. [14] defined the Dunkl-Williams constant $DW(X)$ of a normed linear space X as the best constant for the Dunkl-Williams inequality, that is,

$$DW(X) = \sup\left\{ \frac{\|x\|+\|y\|}{\|x-y\|} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| : x, y \in X \setminus \{0\},\ x \neq y \right\}.$$

We collect some basic properties of the Dunkl-Williams constant. Let X be a normed linear space. Then, the following hold:

(i) $2 \leq DW(X) \leq 4$.

(ii) X is an inner product space if and only if $DW(X) = 2$.

(iii) X is uniformly non-square if and only if $DW(X) < 4$ (cf. [2, 14]).

However, the Dunkl-Williams constant is very hard to calculate. It is not known for almost all normed linear spaces. We cannot compute $DW(X)$ even if $X = (\mathbb{R}^2, \|\cdot\|_p)$. In [20], it was shown that $DW(\ell_2\text{-}\ell_\infty) = 2\sqrt{2}$, where $\ell_2\text{-}\ell_\infty$ is the space $\mathbb{R}^2$ endowed with the norm $\|\cdot\|_{2,\infty}$ defined by

$$\|(a,b)\|_{2,\infty} = \begin{cases} (|a|^2 + |b|^2)^{1/2} & \text{if } ab \geq 0, \\ \max\{|a|, |b|\} & \text{if } ab \leq 0, \end{cases}$$

for all $(a, b) \in \mathbb{R}^2$. This is the only nontrivial example that the Dunkl-Williams constant was precisely determined.

In this paper, we determine the Dunkl-Williams constant of the space $\mathbb{R}^2$ endowed with a symmetric octagonal norm $\|\cdot\|_\beta$ by using a calculation method which was constructed in [20].

2. Calculation method

In this section, we describe a calculation method used in this paper. Let X be a normed linear space, and let B_X and S_X denote the unit ball and the unit sphere of X, respectively. When we make use of the calculation method, the notion of Birkhoff orthogonality plays an important role. We recall that $x \in X$ is said to be Birkhoff orthogonal to $y \in X$, denoted by $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$. Obviously, Birkhoff orthogonality is always homogeneous, that is, $x \perp_B y$ implies $\alpha x \perp_B \beta y$ for all $\alpha, \beta \in \mathbb{R}$. More details about Birkhoff orthogonality can be found in Birkhoff [3], Day [9, 10] and James [11, 12, 13].

To construct a calculation method, we introduced some notations in [20]. Suppose that X is a normed linear space. For each $x \in S_X$, let $V(x)$ be a subset of X defined by $V(x) = \{y \in X : x \perp_B y\}$. For each $x \in S_X$ and each $y \in V(x)$, we define $\Gamma(x, y)$ and $m(x, y)$ by

$$\Gamma(x, y) = \left\{ \frac{\lambda + \mu}{2} : \lambda \leq 0 \leq \mu,\ \|x + \lambda y\| = \|x + \mu y\| \right\}$$

and $m(x, y) = \sup\{\|x + \gamma y\| : \gamma \in \Gamma(x, y)\}$, respectively. Furthermore, let

$$M(x) = \sup\{m(x, y) : y \in V(x)\}.$$

Using these notions, we obtained a calculation method for the Dunkl-Williams constant.

Theorem 2.1 ([20])**.** Let X be a normed linear space. Then,

$$DW(X) = 2\sup\{M(x) : x \in S_X\}.$$

For two-dimensional spaces, Theorem 2.1 has the following improvement.

Theorem 2.2 ([20])**.** Let X be a two-dimensional normed linear space. Then,

$$DW(X) = 2\sup\{M(x) : x \in \mathrm{ext}(B_X)\},$$

where $\mathrm{ext}(B_X)$ denotes the set of all extreme points of B_X.

For each nonzero element x of a normed linear space X, an element f of X^* is said to be a norming functional of x if $\|f\| = 1$ and $f(x) = \|x\|$. Let $D(X, x)$ denote the set of all norming functionals of x. The following is an important characterization of Birkhoff orthogonality.

Lemma 2.3 (James, 1947 [12])**.** Let X be a normed linear space, and let x and y be two elements of X. Then, $x \perp_B y$ if and only if there exists a norming functional f of x such that $f(y) = 0$.

From this result, one can easily have that $V(x) = \bigcup\{\ker f : f \in D(X, x)\}$ for each unit vector x in a normed linear space X.

When we put the method into practice, the following results are needed.

Lemma 2.4. Let X be a normed linear space, and let $x \in S_X$. Then, the following hold:

(i) $m(x, 0) = 1$.
(ii) $m(x, \alpha y) = m(x, y)$ for all $y \in V(x)$ and all $\alpha \in \mathbb{R} \setminus \{0\}$.

Proposition 2.5. Let X and Y be normed linear spaces, and let T be an isometric isomorphism from X onto Y. Then, the following hold:

(i) $m(Tx, Ty) = m(x, y)$ for all $x \in S_X$ and all $y \in V(x)$.
(ii) $M(Tx) = M(x)$ for all $x \in S_X$.

Lemma 2.6. Let X be a normed linear space. Suppose that $x \in S_X$, and that $y \in V(x)$. Then, $m(x, y) = \max\{\|x + \alpha y\|, \|x + \beta y\|\}$, where $\alpha = \inf \Gamma(x, y)$ and $\beta = \sup \Gamma(x, y)$.

Lemma 2.7. Let X be a normed linear space, and let $x \in S_X$. Suppose that D is a dense subset of $V(x)$. Then, $M(x) = \{m(x, y) : y \in D\}$.

All of these results can be found in [20].

3. The Dunkl-Williams constant of $(\mathbb{R}^2, \|\cdot\|_\beta)$

The following is the main theorem in this paper.

Theorem 3.1. Let $\beta \in (1/2, 1)$. Then, the following hold:

(i) If $\beta \in (1/2, 1/\sqrt{2}]$, then

$$DW((\mathbb{R}^2, \|\cdot\|_\beta)) = \frac{2}{\beta^2}\left((1-\beta)^2 + \beta^2\right).$$

(ii) If $\beta \in [1/\sqrt{2}, 1)$, then

$$DW((\mathbb{R}^2, \|\cdot\|_\beta)) = 4\left((1-\beta)^2 + \beta^2\right).$$

Once it has been proved that (i) holds, one can show (ii) easily. Indeed, for each $\beta \in (1/2, 1)$, it is easy to check that $(\mathbb{R}^2, \|\cdot\|_\beta)$ is isometrically isomorphic to $(\mathbb{R}^2, \|\cdot\|_{1/2\beta})$ under the identification

$$(\mathbb{R}^2, \|\cdot\|_\beta) \ni (x_1, x_2) \longleftrightarrow \beta(x_1 + x_2, x_1 - x_2) \in (\mathbb{R}^2, \|\cdot\|_{1/2\beta})$$

since $\max\{|x_1 + x_2|, |x_1 - x_2|\} = |x_1| + |x_2|$ for all $x_1, x_2 \in \mathbb{R}$. If $\beta \in [1/\sqrt{2}, 1)$, then $1/2\beta \in (1/2, 1/\sqrt{2}]$ and hence

$$\begin{aligned} DW((\mathbb{R}^2, \|\cdot\|_\beta)) &= DW((\mathbb{R}^2, \|\cdot\|_{1/2\beta})) \\ &= \frac{2}{(1/2\beta)^2}\left((1-(1/2\beta))^2 + (1/2\beta)^2\right) \\ &= 4\left((1-\beta)^2 + \beta^2\right) \end{aligned}$$

by Theorem 3.1 (i).

Thus, to prove Theorem 3.1, the case of $\beta \in (1/2, 1/\sqrt{2}]$ is essential. Henceforth, we assume that $\beta \in (1/2, 1/\sqrt{2}]$ unless otherwise stated. Put $X_\beta = (\mathbb{R}^2, \|\cdot\|_\beta)$ and $k_\beta = (1-\beta)/\beta$ for short. We remark that $\sqrt{2} - 1 \le k_\beta < 1$ since $1/2 < \beta \le 1/\sqrt{2}$, and that $\beta = 1/(1 + k_\beta)$.

We start the proof of Theorem 3.1 with the following lemma.

Lemma 3.2. $DW(X_\beta) = 2M((1, k_\beta))$.

Proof. It is easy to see that $\mathrm{ext}(B_{X_\beta})$ is the set of all vertices of the octagon S_{X_β}, that is,

$$\mathrm{ext}(B_{X_\beta}) = \{(\varepsilon_1, \varepsilon_2 k_\beta) : |\varepsilon_1| = |\varepsilon_2| = 1\} \cup \{(\varepsilon_1 k_\beta, \varepsilon_2) : |\varepsilon_1| = |\varepsilon_2| = 1\}.$$

Since $\|\cdot\|_\beta$ is a symmetric absolute normalized norm on $\mathbb{R}^2$, both of the maps $(x_1, x_2) \mapsto (x_1, -x_2)$ and $(x_1, x_2) \mapsto (x_2, x_1)$ are isometric isomorphism from X_β onto itself. Hence, we have

$$M((\varepsilon_1, \varepsilon_2 k_\beta)) = M((\varepsilon_1 k_\beta, \varepsilon_2)) = M((1, k_\beta))$$

by Proposition 2.5, which and Theorem 2.2 together imply that

$$\begin{aligned} DW(X_\beta) &= 2\sup\{M(x) : x \in \mathrm{ext}(B_{X_\beta})\} \\ &= 2M((1, k_\beta)). \end{aligned}$$

This completes the proof. $\square$

Put $x_\beta = (1, k_\beta)$. Next, we determine the set $V(x_\beta)$. To do this, we make use of the following lemma found in [4] (cf. [19]).

Lemma 3.3 (Bonsall-Duncan, 1973 [4]; Mitani-Saito-Suzuki, 2003 [19])**.** Let $\psi \in \Psi_2$ and let $x(t) = (1-t, t)/\psi(t)$ for each $t \in [0, 1]$. Then,

$$\begin{aligned} &D((\mathbb{R}^2, \|\cdot\|_\psi), x(t)) \\ &= \begin{cases} \{(1, c(1+a)) : a \in [-1, \psi'_R(0)],\ |c| = 1\} & (t = 0), \\ \{(\psi(t) - at, \psi(t) + a(1-t)) : a \in [\psi'_L(t), \psi'_R(t)]\} & (0 < t < 1), \\ \{(c(1-a), 1) : a \in [\psi'_L(1), 1],\ |c| = 1\} & (t = 1), \end{cases} \end{aligned}$$

where $\psi'_L(t)$ and $\psi'_R(t)$ are, respectively, the left-hand and right-hand derivative of ψ at $t \in [0, 1]$.

Using this result, we have the following lemma.

Lemma 3.4. $V(x_\beta) = \{\alpha(1 + a, -1 + k_\beta a) : a \in [-1, 0],\ \alpha \in \mathbb{R}\}$.

Proof. First, we note that $x_\beta = (\beta, 1-\beta)/\psi_\beta(1-\beta)$. Since $(\psi_\beta)'_L(1-\beta) = -1$ and $(\psi_\beta)'_R(1-\beta) = 0$, we have

$$D(X_\beta, x_\beta) = \{(\beta - a(1-\beta), \beta + a\beta) : a \in [-1, 0]\}.$$

Thus,

$$\begin{aligned} V(x_\beta) &= \bigcup\{\ker f : f \in D(X_\beta, x_\beta)\} \\ &= \{\alpha(\beta + a\beta, -\beta + a(1-\beta)) : a \in [-1, 0],\ \alpha \in \mathbb{R}\} \\ &= \{\alpha(1 + a, -1 + k_\beta a) : a \in [-1, 0],\ \alpha \in \mathbb{R}\}. \end{aligned}$$

The proof is complete. $\square$

To reduce the amount of calculation, we make use of Lemmas 2.4 and 2.7.

Lemma 3.5. $M(x_\beta) = \sup\{m(x_\beta, (1, -t)) : t \in (1, \infty) \setminus \{1/k_\beta, (1 + k_\beta)/(1 - k_\beta)\}\}$.

Proof. It is clear that $\{\alpha(1 + a, -1 + k_\beta a) : a \in (-1, 0),\ \alpha \in \mathbb{R}\}$ is a dense subset of $V(x_\beta)$ by the preceding lemma. On the other hand,

$$\begin{aligned} &\{\alpha(1 + a, -1 + k_\beta a) : a \in (-1, 0),\ \alpha \in \mathbb{R}\} \\ &= \left\{\alpha\left(1, \frac{-1 + k_\beta a}{1 + a}\right) : a \in (-1, 0),\ \alpha \in \mathbb{R}\right\}. \end{aligned}$$

Since the function $a \mapsto (-1+k_\beta a)/(1+a)$ is continuous and increasing, it maps $(-1,0)$ onto $(-\infty,-1)$. Thus, one has that

$$\left\{\alpha\left(1, \frac{-1+k_\beta a}{1+a}\right) : a \in (-1,0),\ \alpha \in \mathbb{R}\right\} = \{\alpha(1,-t) : t \in (1,\infty),\ \alpha \in \mathbb{R}\}.$$

From this, it follows that $\{\alpha(1,-t) : t \in (1,\infty) \setminus \{1/k_\beta, (1+k_\beta)/(1-k_\beta)\},\ \alpha \in \mathbb{R}\}$ is also a dense subset of $V(x_\beta)$. Thus, by Lemma 2.7, we obtain

$$M(x_\beta) = \sup\{m(x_\beta, \alpha(1,-t)) : t \in (1,\infty) \setminus \{1/k_\beta, (1+k_\beta)/(1-k_\beta)\},\ \alpha \in \mathbb{R}\}.$$

Finally, applying Lemma 2.4, we have the lemma. □

For each $t \in \mathbb{R}$, put $y_t = (1,-t)$. Next, we give the formula of $\|x_\beta + \lambda y_t\|_\beta$ for all $t \in (1,\infty) \setminus \{1/k_\beta\}$ and all $\lambda \in \mathbb{R}$.

Lemma 3.6. Let $t \in (1,\infty) \setminus \{1/k_\beta\}$, and let

$$a_t = \frac{2k_\beta}{t-k_\beta}, \quad b_t = \frac{k_\beta^2-1}{1+k_\beta t} \quad \text{and} \quad c_t = \frac{1+k_\beta^2}{k_\beta t - 1}.$$

Then, the following hold:

(i) If $t \in (1, 1/k_\beta)$, then $c_t < b_t < 0 < a_t$ and

$$\|x_\beta + \lambda y_t\|_\beta = \begin{cases} \dfrac{k_\beta - 1 - (1+t)\lambda}{1+k_\beta} & (\lambda \le c_t), \\ k_\beta - t\lambda & (c_t \le \lambda \le b_t), \\ \dfrac{1+k_\beta+(1-t)\lambda}{1+k_\beta} & (b_t \le \lambda \le 0), \\ 1+\lambda & (0 \le \lambda \le a_t), \\ \dfrac{1-k_\beta+(1+t)\lambda}{1+k_\beta} & (a_t \le \lambda). \end{cases}$$

(ii) If $t \in (1/k_\beta, \infty)$, then $b_t < 0 < a_t < c_t$ and

$$\|x_\beta + \lambda y_t\|_\beta = \begin{cases} k_\beta - t\lambda & (\lambda \le b_t), \\ \dfrac{1+k_\beta+(1-t)\lambda}{1+k_\beta} & (b_t \le \lambda \le 0), \\ 1+\lambda & (0 \le \lambda \le a_t), \\ \dfrac{1-k_\beta+(1+t)\lambda}{1+k_\beta} & (a_t \le \lambda \le c_t), \\ t\lambda - k_\beta & (c_t \le \lambda). \end{cases}$$

Proof. First, we note that

$$-1 < b_t < 0 < k_\beta/t < a_t$$

for all $t \in (1, \infty) \setminus \{1/k_\beta\}$. If $t \in (1, 1/k_\beta)$, then one can easily have

$$c_t < -1 < b_t < 0 < k_\beta/t < a_t.$$

If $t \in (1/k_\beta, \infty)$, then we obtain

$$-1 < b_t < 0 < k_\beta/t < a_t < c_t$$

since

$$c_t - a_t = \frac{(1-k_\beta^2)(k_\beta + t)}{(t-k_\beta)(k_\beta t - 1)} > 0.$$

Now, it follows from the definition of $\|\cdot\|_\beta$ that

$$\|x_\beta + \lambda y_t\|_\beta = \begin{cases} |1+\lambda| & (|k_\beta - t\lambda| \le k_\beta|1+\lambda|), \\ \dfrac{|1+\lambda| + |k_\beta - t\lambda|}{1+k_\beta} & (k_\beta|1+\lambda| \le |k_\beta - t\lambda| \le k_\beta^{-1}|1+\lambda|), \\ |k_\beta - t\lambda| & (k_\beta^{-1}|1+\lambda| \le |k_\beta - t\lambda|). \end{cases}$$

On the other hand, we have

$$(k_\beta - t\lambda)^2 - k_\beta^2(1+\lambda)^2 = (t+k_\beta)(t-k_\beta)(\lambda - a_t)\lambda, \text{ and}$$
$$k_\beta^{-2}(1+\lambda)^2 - (k_\beta - t\lambda)^2 = k_\beta^{-2}(1+tk_\beta)(1-tk_\beta)(\lambda - b_t)(\lambda - c_t).$$

From these facts, one can obtain the lemma. □

The following lemma is needed in the sequel.

Lemma 3.7. Let $t \in (1, \infty)$. Then, the function $\lambda \mapsto \|x_\beta + \lambda y_t\|_\beta$ is strictly decreasing on $(-\infty, 0]$, and is strictly increasing on $[0, \infty)$.

Proof. We first note that $y_t \in V(x_\beta)$, that is, $x \perp_B y_t$. Since the function $\lambda \mapsto \|x_\beta + \lambda y_t\|_\beta$ is convex, it is enough to show that $\|x_\beta + \lambda_0 y_t\|_\beta = \min\{\|x_\beta + \lambda y_t\|_\beta : \lambda \in \mathbb{R}\} = 1$ if and only if $\lambda_0 = 0$. To this end, we suppose that $\|x_\beta + \lambda_0 y_t\|_\beta = 1$. Then,

$$\max\{|1+\lambda_0|, |k_\beta - t\lambda_0|, \beta(|1+\lambda_0| + |k_\beta - t\lambda_0|)\} = \|x_\beta + \lambda_0 y_t\|_\beta = 1.$$

Since $|1+\lambda_0| \le 1$, we have $\lambda_0 \le 0$, whence

$$k_\beta - t\lambda_0 = |k_\beta - t\lambda_0| \le 1.$$

It follows from $0 < k_\beta < 1$ and $t > 1$ that

$$\lambda_0 \ge \frac{k_\beta - 1}{t} > k_\beta - 1 > -1,$$

which implies that

$$
\begin{aligned}
1 &\geq \beta(|1+\lambda_0| + |k_\beta - t\lambda_0|) \\
&= \beta((1+\lambda_0) + (k_\beta - t\lambda_0)) \\
&= 1 - \beta(t-1)\lambda_0.
\end{aligned}
$$

Thus, we also have $\lambda_0 \geq 0$. This completes the proof. $\square$

We clarify the relationship among $\|x_\beta + a_t y_t\|_\beta$, $\|x_\beta + b_t y_t\|_\beta$, and $\|x_\beta + c_t y_t\|_\beta$. We note that

$$
\frac{1+k_\beta}{1-k_\beta} = \frac{1}{2\beta-1} \geq \frac{\beta}{1-\beta} = \frac{1}{k_\beta}
$$

since $\beta \in (1/2, 1/\sqrt{2}]$.

Lemma 3.8. Let $t \in (1,\infty) \setminus \{1/k_\beta, (1+k_\beta)/(1-k_\beta)\}$. Then, the following hold:

(i) If $t \in (1, 1/k_\beta)$, then $\|x_\beta + b_t y_t\|_\beta < \|x_\beta + a_t y_t\|_\beta < \|x_\beta + c_t y_t\|_\beta$.

(ii) If $t \in (1/k_\beta, (1+k_\beta)/(1-k_\beta))$, then $\|x_\beta + b_t y_t\|_\beta < \|x_\beta + a_t y_t\|_\beta < \|x_\beta + c_t y_t\|_\beta$.

(iii) If $t \in ((1+k_\beta)/(1-k_\beta), \infty)$, then $\|x_\beta + a_t y_t\|_\beta < \|x_\beta + b_t y_t\|_\beta < \|x_\beta + c_t y_t\|_\beta$.

Proof. By Lemma 3.6 (i) and (ii), we have

$$
\|x_\beta + a_t y_t\|_\beta = 1 + a_t \quad \text{and} \quad \|x_\beta + b_t y_t\|_\beta = \frac{1 + k_\beta + (1-t)b_t}{1+k_\beta},
$$

which implies that

$$
\|x_\beta + a_t y_t\|_\beta - \|x_\beta + b_t y_t\|_\beta = \frac{(1-k_\beta)(k_\beta + t)}{(t-k_\beta)(1+k_\beta t)} \left(\frac{1+k_\beta}{1-k_\beta} - t \right).
$$

Thus, $\|x_\beta + a_t y_t\|_\beta > \|x_\beta + b_t y_t\|_\beta$ if $t < (1+k_\beta)/(1-k_\beta)$, and $\|x_\beta + a_t y_t\|_\beta < \|x_\beta + b_t y_t\|_\beta$ if $t > (1+k_\beta)/(1-k_\beta)$.

Suppose that $t \in (1, 1/k_\beta)$. Then, as mentioned above, $\|x_\beta + b_t y_t\|_\beta < \|x_\beta + a_t y_t\|_\beta$. Moreover, by Lemma 3.6 (i), we have

$$
\|x_\beta + a_t y_t\|_\beta = \frac{1 - k_\beta + (1+t)a_t}{1+k_\beta} \quad \text{and} \quad \|x_\beta + c_t y_t\|_\beta = \frac{k_\beta - 1 - (1+t)c_t}{1+k_\beta},
$$

and so

$$
\begin{aligned}
&\|x_\beta + c_t y_t\|_\beta - \|x_\beta + a_t y_t\|_\beta \\
&= \frac{1}{1+k_\beta} \left(2(k_\beta - 1) + (1+t) \left(\frac{1+k_\beta^2}{1-k_\beta t} - \frac{2k_\beta}{t-k_\beta} \right) \right).
\end{aligned}
$$

On the other hand, since $1 - k_\beta t < 1 - k_\beta < t - k_\beta$, we obtain

$$
\frac{1+k_\beta^2}{1-k_\beta t} - \frac{2k_\beta}{t-k_\beta} > \frac{1+k_\beta^2}{1-k_\beta} - \frac{2k_\beta}{1-k_\beta} = 1 - k_\beta,
$$

which implies that

$$\|x_\beta + c_t y_t\|_\beta - \|x_\beta + a_t y_t\|_\beta > \frac{(t-1)(1-k_\beta)}{1+k_\beta} > 0.$$

This shows (i).

Next, we suppose that $t \in (1/k_\beta, (1+k_\beta)/(1-k_\beta))$. Then, we have $\|x_\beta + b_t y_t\|_\beta < \|x_\beta + a_t y_t\|_\beta$. Furthermore, we obtain $0 < a_t < c_t$ by Lemma 3.6 (ii). Thus, Lemma 3.7 assures that $\|x_\beta + a_t y_t\|_\beta < \|x_\beta + c_t y_t\|_\beta$.

Finally, we assume that $t \in ((1+k_\beta)/(1-k_\beta), \infty)$. Then, we have $\|x_\beta + a_t y_t\|_\beta < \|x_\beta + b_t y_t\|_\beta$ as mentioned in the first paragraph. Moreover, since

$$\|x_\beta + b_t y_t\|_\beta = k_\beta - t b_t \quad \text{and} \quad \|x_\beta + c_t y_t\|_\beta = t c_t - k_\beta,$$

it follows that

$$\|x_\beta + c_t y_t\|_\beta - \|x_\beta + b_t y_t\|_\beta = \frac{2(k_\beta + t)}{k_\beta^2 t^2 - 1} > 0.$$

Thus, one has that $\|x_\beta + b_t y_t\|_\beta < \|x_\beta + c_t y_t\|_\beta$. This proves (iii). $\square$

Let $t \in (1, \infty)$. Then, the intermediate value theorem guarantees that the function $\lambda \mapsto \|x_\beta + \lambda y_t\|_\beta$ maps $(-\infty, 0]$ onto $[1, \infty)$ and $[0, \infty)$ onto $[1, \infty)$. Thus, for any $\mu \in [0, \infty)$, there exists a $\lambda \in (-\infty, 0]$ such that $\|x_\beta + \lambda y_t\|_\beta = \|x_\beta + \mu y_t\|_\beta$. Furthermore, by Lemma 3.7, this gives a one-to-one correspondence between $[0, \infty)$ and $(-\infty, 0]$. Now, let p_t, q_t, r_t be real numbers such that $p_t < 0 < q_t$, $c_t r_t < 0$, $\|x_\beta + a_t y_t\|_\beta = \|x_\beta + p_t y_t\|_\beta$, $\|x_\beta + b_t y_t\|_\beta = \|x_\beta + q_t y_t\|_\beta$, and $\|x_\beta + c_t y_t\|_\beta = \|x_\beta + r_t y_t\|_\beta$. Then, we have the following lemma.

Lemma 3.9. Let $t \in (1, \infty) \setminus \{1/k_\beta, (1+k_\beta)/(1-k_\beta)\}$. Then, the following hold:

(i) If $t \in (1, 1/k_\beta)$, then $c_t < p_t < b_t < 0 < q_t < a_t < r_t$ and

$$p_t = \frac{k_\beta - 1 - a_t}{t}, \quad q_t = \frac{(1-t)b_t}{1+k_\beta} \quad \text{and} \quad r_t = \frac{2(k_\beta - 1)}{t+1} - c_t.$$

(ii) If $t \in (1/k_\beta, (1+k_\beta)/(1-k_\beta))$, then $r_t < p_t < b_t < 0 < q_t < a_t < c_t$ and

$$p_t = \frac{k_\beta - 1 - a_t}{t}, \quad q_t = \frac{(1-t)b_t}{1+k_\beta} \quad \text{and} \quad r_t = \frac{2k_\beta}{t} - c_t.$$

(iii) If $t \in ((1+k_\beta)/(1-k_\beta), \infty)$, then $r_t < b_t < p_t < 0 < a_t < q_t < c_t$ and

$$p_t = \frac{(1+k_\beta)a_t}{1-t}, \quad q_t = \frac{2k_\beta + (1-t)b_t}{t+1} \quad \text{and} \quad r_t = \frac{2k_\beta}{t} - c_t.$$

Proof. Suppose that $t \in (1, 1/k_\beta)$. Then, $c_t < b_t < 0 < a_t$ by Lemma 3.6. Using Lemma 3.8, we have the following diagram:

$$\begin{array}{cccccc}
+: & \|x_\beta + q_t y_t\|_\beta & < & \|x_\beta + a_t y_t\|_\beta & < & \|x_\beta + r_t y_t\|_\beta \\
 & \| & & \| & & \| \\
-: & \|x_\beta + b_t y_t\|_\beta & < & \|x_\beta + p_t y_t\|_\beta & < & \|x_\beta + c_t y_t\|_\beta
\end{array}$$

Thus, by Lemma 3.7, it follows that $c_t < p_t < b_t < 0 < q_t < a_t < r_t$. Then, we have

$$k_\beta - tp_t = \|x_\beta + p_t y_t\|_\beta = \|x_\beta + a_t y_t\|_\beta = 1 + a_t,$$

$$1 + q_t = \|x_\beta + q_t y_t\|_\beta = \|x_\beta + b_t y_t\|_\beta = \frac{1 + k_\beta + (1-t)b_t}{1 + k_\beta}, \text{ and}$$

$$\frac{1 - k_\beta + (1+t)r_t}{1 + k_\beta} = \|x_\beta + r_t y_t\|_\beta = \|x_\beta + c_t y_t\|_\beta = \frac{k_\beta - 1 - (1+t)c_t}{1 + k_\beta}.$$

This shows (i).

Similarly, one can prove (ii) and (iii). □

Next, we consider the set $\Gamma(x_\beta, y_t)$. As was mentioned in the paragraph preceding Lemma 3.9, for each $\mu \in [0, \infty)$ there exists a unique $\lambda_\mu \in (-\infty, 0]$ such that $\|x_\beta + \lambda_\mu y_t\|_\beta = \|x_\beta + \mu y_t\|_\beta$. Then, it follows that

$$\Gamma(x_\beta, y_t) = \left\{ \frac{\lambda_\mu + \mu}{2} : \mu \in [0, \infty) \right\}.$$

Remark that

$$1 < \frac{k_\beta(1 + k_\beta)}{3k_\beta - 1} = \frac{1 - \beta}{\beta(3 - 4\beta)} < \frac{\beta}{1 - \beta} = \frac{1}{k_\beta}$$

since $\beta \in (1/2, 1/\sqrt{2}]$.

Lemma 3.10. Let $t \in (1, 1/k_\beta)$. Then,

$$\Gamma(x_\beta, y_t) = \begin{cases} \left[\dfrac{c_t + r_t}{2}, 0\right] & \left(1 < t \le \dfrac{k_\beta(1+k_\beta)}{3k_\beta - 1}\right), \\ \left[\dfrac{c_t + r_t}{2}, \dfrac{a_t + p_t}{2}\right] & \left(\dfrac{k_\beta(1+k_\beta)}{3k_\beta - 1} \le t < \dfrac{1}{k_\beta}\right). \end{cases}$$

Proof. By Lemma 3.9 (i), we have $c_t < p_t < b_t < 0 < q_t < a_t < r_t$. Suppose that $0 \le \mu \le q_t$. Then, Lemma 3.7 guarantees that $b_t \le \lambda_\mu \le 0$, and so

$$\frac{1 + k_\beta + (1-t)\lambda_\mu}{1 + k_\beta} = \|x_\beta + \lambda_\mu y_t\|_\beta = \|x_\beta + \mu y_t\|_\beta = 1 + \mu.$$

Hence, we have

$$\lambda_\mu = \frac{(1 + k_\beta)\mu}{1 - t},$$

which implies that

$$\frac{\lambda_\mu + \mu}{2} = \frac{(t - 2 - k_\beta)\mu}{2(t-1)}.$$

Since $t \in (1, 1/k_\beta)$, we have $t - 2 - k_\beta < 0$. Indeed, it follows from $k_\beta \ge \sqrt{2} - 1$ that

$$2 + k_\beta - t > 2 + k_\beta - \frac{1}{k_\beta} = \frac{1}{k_\beta}(k_\beta^2 + 2k_\beta - 1) \ge 0.$$

Thus, the function $\mu \mapsto (\lambda_\mu + \mu)/2$ is decreasing on $[0, q_t]$, and therefore

$$\left\{ \frac{\lambda_\mu + \mu}{2} : \mu \in [0, q_t] \right\} = \left[\frac{b_t + q_t}{2}, 0 \right].$$

Next, we suppose that $q_t \le \mu \le a_t$. Then, we have $p_t \le \lambda_\mu \le b_t$, and so

$$k_\beta - t\lambda_\mu = \|x_\beta + \lambda_\mu y_t\|_\beta = \|x_\beta + \mu y_t\|_\beta = 1 + \mu.$$

From this, we obtain

$$\lambda_\mu = \frac{k_\beta - 1 - \mu}{t}$$

and

$$\frac{\lambda_\mu + \mu}{2} = \frac{k_\beta - 1 + (t-1)\mu}{2t}.$$

This shows that the function $\mu \mapsto (\lambda_\mu + \mu)/2$ is increasing on $[q_t, a_t]$, which implies that

$$\left\{ \frac{\lambda_\mu + \mu}{2} : \mu \in [q_t, a_t] \right\} = \left[\frac{b_t + q_t}{2}, \frac{a_t + p_t}{2} \right].$$

In the case of $a_t \le \mu \le r_t$, we have $c_t \le \lambda_\mu \le p_t$. Then, we obtain

$$k_\beta - t\lambda_\mu = \|x_\beta + \lambda_\mu y_t\|_\beta = \|x_\beta + \mu y_t\|_\beta = \frac{1 - k_\beta + (1+t)\mu}{1 + k_\beta}.$$

It follows that

$$\lambda_\mu = \frac{k_\beta^2 + 2k_\beta - 1 - (1+t)\mu}{t(1 + k_\beta)}$$

and

$$\frac{\lambda_\mu + \mu}{2} = \frac{k_\beta^2 + 2k_\beta - 1 + (k_\beta t - 1)\mu}{2t(1 + k_\beta)}.$$

Since $t \in (1, 1/k_\beta)$, the function $\mu \mapsto (\lambda_\mu + \mu)/2$ is decreasing on $[a_t, r_t]$, and hence

$$\left\{ \frac{\lambda_\mu + \mu}{2} : \mu \in [a_t, r_t] \right\} = \left[\frac{c_t + r_t}{2}, \frac{a_t + p_t}{2} \right].$$

Finally, we assume that $r_t \le \mu$. Then, it follows from $\lambda_\mu \le c_t$ that

$$\frac{k_\beta - 1 - (1+t)\lambda_\mu}{1 + k_\beta} = \|x_\beta + \lambda_\mu y_t\|_\beta = \|x_\beta + \mu y_t\|_\beta = \frac{1 - k_\beta + (1+t)\mu}{1 + k_\beta}.$$

So we have

$$\lambda_\mu = \frac{2(k_\beta - 1)}{1 + t} - \mu,$$

which implies that

$$\frac{\lambda_\mu + \mu}{2} = \frac{k_\beta - 1}{1 + t} = \frac{c_t + r_t}{2}.$$

Now, since the function $\mu \mapsto (\lambda_\mu + \mu)/2$ is continuous, one has that

$$\begin{aligned}
&\Gamma(x_\beta, y_t)\\
&= \left\{ \frac{\lambda_\mu + \mu}{2} : \mu \in [0, \infty) \right\}\\
&= \left[\frac{b_t + q_t}{2}, 0 \right] \cup \left[\frac{b_t + q_t}{2}, \frac{a_t + p_t}{2} \right] \cup \left[\frac{c_t + r_t}{2}, \frac{a_t + p_t}{2} \right]\\
&= \left[\min\left\{ \frac{b_t + q_t}{2}, \frac{c_t + r_t}{2} \right\}, \max\left\{ 0, \frac{a_t + p_t}{2} \right\} \right].
\end{aligned}$$

However, since

$$\frac{b_t + q_t}{2} - \frac{c_t + r_t}{2} = \frac{(t-1)(k_\beta + t)(1 - k_\beta)}{2(1+t)(1+k_\beta t)} > 0$$

and

$$\frac{a_t + p_t}{2} = \frac{3k_\beta - 1}{2t(t - k_\beta)} \left(t - \frac{k_\beta(1 + k_\beta)}{3k_\beta - 1} \right),$$

we have the lemma. □

We remark that

$$\frac{1}{k_\beta} \leq 2 + k_\beta \leq \frac{1 + k_\beta}{1 - k_\beta}$$

since $k_\beta \geq \sqrt{2} - 1$.

Lemma 3.11. Let $t \in (1/k_\beta, (1 + k_\beta)/(1 - k_\beta))$. Then,

$$\Gamma(x_\beta, y_t) = \begin{cases} \left[\frac{b_t + q_t}{2}, \frac{c_t + r_t}{2} \right] & \left(\frac{1}{k_\beta} < t \leq 2 + k_\beta \right), \\ \left[0, \frac{c_t + r_t}{2} \right] & \left(2 + k_\beta \leq t < \frac{1 + k_\beta}{1 - k_\beta} \right). \end{cases}$$

Proof. In the case of $t \in (1/k_\beta, (1 + k_\beta)/(1 - k_\beta))$, we have $r_t < p_t < b_t < 0 < q_t < a_t < c_t$ by Lemma 3.9 (ii). Suppose that $0 \leq \mu \leq q_t$. Then, we have $b_t \leq \lambda_\mu \leq 0$, and so

$$\frac{1 + k_\beta + (1 - t)\lambda_\mu}{1 + k_\beta} = \|x_\beta + \lambda_\mu y_t\|_\beta = \|x_\beta + \mu y_t\|_\beta = 1 + \mu.$$

As in the proof of the preceding lemma, we obtain

$$\frac{\lambda_\mu + \mu}{2} = \frac{(t - 2 - k_\beta)\mu}{2(t - 1)},$$

which implies that $\mu \mapsto (\lambda_\mu + \mu)/2$ is decreasing on $[0, q_t]$ if $t \le 2 + k_\beta$, and is increasing if $t \ge 2 + k_\beta$. Hence, we have

$$\left\{ \frac{\lambda_\mu + \mu}{2} : \mu \in [0, q_t] \right\} = \begin{cases} \left[\frac{b_t + q_t}{2}, 0 \right] & \left(\frac{1}{k_\beta} < t \le 2 + k_\beta \right), \\ \left[0, \frac{b_t + q_t}{2} \right] & \left(2 + k_\beta \le t < \frac{1 + k_\beta}{1 - k_\beta} \right). \end{cases}$$

Assume that $q_t \le \mu \le a_t$. Then, we have $p_t \le \mu \le b_t$ and

$$k_\beta - t\lambda_\mu = \|x_\beta + \lambda_\mu y_t\|_\beta = \|x_\beta + \mu y_t\|_\beta = 1 + \mu,$$

which implies that

$$\frac{\lambda_\mu + \mu}{2} = \frac{k_\beta - 1 + (t-1)\mu}{2t}.$$

Since the function $\mu \mapsto (\lambda_\mu + \mu)/2$ is increasing on $[q_t, a_t]$, which implies that

$$\left\{ \frac{\lambda_\mu + \mu}{2} : \mu \in [q_t, a_t] \right\} = \left[\frac{b_t + q_t}{2}, \frac{a_t + p_t}{2} \right].$$

We suppose that $a_t \le \mu \le c_t$. In this case, we obtain

$$k_\beta - t\lambda_\mu = \|x_\beta + \lambda_\mu y_t\|_\beta = \|x_\beta + \mu y_t\|_\beta = \frac{1 - k_\beta + (1+t)\mu}{1 + k_\beta}$$

since $r_t \le \lambda_\mu \le p_t$. It follows that

$$\frac{\lambda_\mu + \mu}{2} = \frac{k_\beta^2 + 2k_\beta - 1 + (k_\beta t - 1)\mu}{2t(1 + k_\beta)}.$$

Since $t \in (1/k_\beta, (1 + k_\beta)/(1 - k_\beta))$, the function $\mu \mapsto (\lambda_\mu + \mu)/2$ is increasing on $[a_t, c_t]$, and hence

$$\left\{ \frac{\lambda_\mu + \mu}{2} : \mu \in [a_t, c_t] \right\} = \left[\frac{a_t + p_t}{2}, \frac{c_t + r_t}{2} \right].$$

In the case of $c_t \le \mu$, it follows that $\lambda_\mu \le r_t$, and that

$$k_\beta - t\lambda_\mu = \|x_\beta + \lambda_\mu y_t\|_\beta = \|x_\beta + \mu y_t\|_\beta = t\mu - k_\beta.$$

Then, we obtain

$$\lambda_\mu = \frac{2k_\beta}{t} - \mu$$

and

$$\frac{\lambda_\mu + \mu}{2} = \frac{k_\beta}{t} = \frac{c_t + r_t}{2}.$$

Finally, if $1/k_\beta < t \le 2 + k_\beta$, then

$$
\begin{aligned}
&\Gamma(x_\beta, y_t) \\
&= \left\{ \frac{\lambda_\mu + \mu}{2} : \mu \in [0, \infty) \right\} \\
&= \left[\frac{b_t + q_t}{2}, 0 \right] \cup \left[\frac{b_t + q_t}{2}, \frac{a_t + p_t}{2} \right] \cup \left[\frac{a_t + p_t}{2}, \frac{c_t + r_t}{2} \right] \\
&= \left[\frac{b_t + q_t}{2}, \frac{c_t + r_t}{2} \right]
\end{aligned}
$$

since $(c_t + r_t)/2 > 0$. On the other hand, if $2 + k_\beta \le t < (1 + k_\beta)/(1 - k_\beta)$, then

$$
\begin{aligned}
&\Gamma(x_\beta, y_t) \\
&= \left\{ \frac{\lambda_\mu + \mu}{2} : \mu \in [0, \infty) \right\} \\
&= \left[0, \frac{b_t + q_t}{2} \right] \cup \left[\frac{b_t + q_t}{2}, \frac{a_t + p_t}{2} \right] \cup \left[\frac{a_t + p_t}{2}, \frac{c_t + r_t}{2} \right] \\
&= \left[0, \frac{c_t + r_t}{2} \right].
\end{aligned}
$$

This completes the proof. $\square$

Lemma 3.12. Let $t \in ((1 + k_\beta)/(1 - k_\beta), \infty)$. Then,

$$\Gamma(x_\beta, y_t) = \left[0, \frac{c_t + r_t}{2} \right].$$

Proof. First, we note that $r_t < b_t < p_t < 0 < a_t < q_t < c_t$ by Lemma 3.9 (iii). In the case of $0 \le \mu \le a_t$, we have $p_t \le \lambda \le 0$, and hence

$$\frac{\lambda_\mu + \mu}{2} = \frac{(t - 2 - k_\beta)\mu}{2(t - 1)}.$$

Then, the function $\mu \mapsto (\lambda_\mu + \mu)/2$ is increasing on $[0, a_t]$, which implies that

$$\left\{ \frac{\lambda_\mu + \mu}{2} : \mu \in [0, a_t] \right\} = \left[0, \frac{a_t + p_t}{2} \right].$$

If $a_t \le \mu \le q_t$, then $b_t \le \lambda_\mu \le p_t$, and so we obtain

$$\frac{1 + k_\beta + (1 - t)\lambda_\mu}{1 + k_\beta} = \|x_\beta + \lambda_\mu y_t\|_\beta = \|x_\beta + \mu y_t\|_\beta = \frac{1 - k_\beta + (1 + t)\mu}{1 + k_\beta}.$$

It follows from

$$\lambda_\mu = \frac{(1 + t)\mu - 2k_\beta}{1 - t}$$

that

$$\frac{\lambda_\mu + \mu}{2} = \frac{k_\beta - \mu}{t - 1}.$$

This shows that the function $\mu \mapsto (\lambda_\mu + \mu)/2$ is decreasing on $[a_t, q_t]$, and therefore

$$\left\{ \frac{\lambda_\mu + \mu}{2} : \mu \in [a_t, q_t] \right\} = \left[\frac{b_t + q_t}{2}, \frac{a_t + p_t}{2} \right].$$

Next, we assume that $q_t \leq \mu \leq c_t$. Then, we obtain $r_t \leq \lambda_\mu \leq b_t$ and

$$\frac{\lambda_\mu + \mu}{2} = \frac{k_\beta^2 + 2k_\beta - 1 + (k_\beta t - 1)\mu}{2t(1 + k_\beta)}.$$

As in the proof of Lemma 3.11, we have

$$\left\{ \frac{\lambda_\mu + \mu}{2} : \mu \in [q_t, c_t] \right\} = \left[\frac{b_t + q_t}{2}, \frac{c_t + r_t}{2} \right].$$

Let $c_t \leq \mu$. Then, it follows that $\lambda_\mu \leq r_t$, and then

$$\frac{\lambda_\mu + \mu}{2} = \frac{k_\beta}{t} = \frac{c_t + r_t}{2}.$$

Thus, one has that

$$\begin{aligned}
&\Gamma(x_\beta, y_t) \\
&= \left\{ \frac{\lambda_\mu + \mu}{2} : \mu \in [0, \infty) \right\} \\
&= \left[0, \frac{a_t + p_t}{2} \right] \cup \left[\frac{b_t + q_t}{2}, \frac{a_t + p_t}{2} \right] \cup \left[\frac{b_t + q_t}{2}, \frac{c_t + r_t}{2} \right] \\
&= \left[\min \left\{ 0, \frac{b_t + q_t}{2} \right\}, \max \left\{ \frac{a_t + p_t}{2}, \frac{c_t + r_t}{2} \right\} \right].
\end{aligned}$$

On the other hand, we have

$$\frac{b_t + q_t}{2} = \frac{k_\beta^2 + k_\beta^2 t + k_\beta - 1}{(1 + t)(1 + k_\beta t)} > 0.$$

Indeed, since $\beta \leq 1/\sqrt{2}$ and $t > 1/k_\beta$, it follows that

$$k_\beta^2 + k_\beta^2 t + k_\beta - 1 > k_\beta^2 + 2k_\beta - 1 \geq 0.$$

Finally, since

$$\frac{c_t + r_t}{2} - \frac{a_t + p_t}{2} = \frac{k_\beta(k_\beta + t)}{t(t - 1)(t - k_\beta)} > 0,$$

we have the lemma. □

Now, we prove the main theorem.

Proof of Theorem 3.1. Putting

$$\begin{aligned}
&M_1 = \sup\{m(x_\beta, y_t) : t \in (1, 1/k_\beta)\} \text{ and} \\
&M_2 = \sup\{m(x_\beta, y_t) : t \in (1/k_\beta, \infty) \setminus \{(1 + k_\beta)/(1 - k_\beta)\}\},
\end{aligned}$$

we have

$$M(x_\beta) = \max\{M_1, M_2\}$$

by Lemma 3.5. First, we suppose that $t \in (1, 1/k_\beta)$. Then, we obtain $b_t < (c_t + r_t)/2 < 0$. Indeed, one has $(c_t + r_t)/2 = (k_\beta - 1)/(1 + t) < 0$ and

$$\frac{c_t + r_t}{2} - b_t = \frac{(1 - k_\beta)(k_\beta + t)}{(1 + t)(1 + k_\beta t)} > 0.$$

Hence, we have

$$\left\| x_\beta + \frac{c_t + r_t}{2} y_t \right\|_\beta = 1 + \frac{(1 - k_\beta)(t - 1)}{(1 + k_\beta)(t + 1)}.$$

From the fact the function $t \mapsto (t - 1)/(t + 1)$ is strictly increasing on $(1, \infty)$, it follows that

$$\frac{(1 - k_\beta)(t - 1)}{(1 + k_\beta)(t + 1)} < \frac{(1 - k_\beta)(k_\beta^{-1} - 1)}{(1 + k_\beta)(k_\beta^{-1} + 1)} = \frac{(1 - k_\beta)^2}{(1 + k_\beta)^2},$$

which in turn implies

$$\begin{aligned} \left\| x_\beta + \frac{c_t + r_t}{2} y_t \right\|_\beta &< 1 + \frac{(1 - k_\beta)^2}{(1 + k_\beta)^2} \\ &< 1 + k_\beta^2 \end{aligned}$$

since $k_\beta > (1 - k_\beta)/(1 + k_\beta)$. Thus, for each $t \in (1, k_\beta(1 + k_\beta)/(3k_\beta - 1)]$, we have

$$m(x_\beta, y_t) = \max\left\{ \left\| x_\beta + \frac{c_t + r_t}{2} y_t \right\|_\beta, \|x_\beta\|_\beta \right\} < 1 + k_\beta^2$$

by Lemma 2.6.

Let $t \in [k_\beta(1+k_\beta)/(3k_\beta-1), 1/k_\beta)$. Then, as in the proof of Lemma 3.10, we have $0 \leq (a_t+p_t)/2 < a_t$. It follows that

$$\begin{aligned}
\left\| x_\beta + \frac{a_t+p_t}{2} y_t \right\|_\beta &= 1 + \frac{1}{2}\left(1-\frac{1}{t}\right)a_t - \frac{1-k_\beta}{2t} \\
&< 1 + \frac{1}{2}\left(1-\frac{1}{k_\beta^{-1}}\right)a_t - \frac{1-k_\beta}{2k_\beta^{-1}} \\
&= 1 + \frac{1-k_\beta}{2}a_t - \frac{k_\beta(1-k_\beta)}{2} \\
&= 1 + \frac{k_\beta(1-k_\beta)}{2}\left(\frac{2}{t-k_\beta}-1\right) \\
&\leq 1 + \frac{k_\beta(1-k_\beta)}{2}\left(\frac{2}{k_\beta(1+k_\beta)(3k_\beta-1)^{-1}-k_\beta}-1\right) \\
&= 1 + \frac{k_\beta^2+2k_\beta-1}{2} \\
&< 1 + k_\beta^2.
\end{aligned}$$

This shows that

$$\begin{aligned}
&m(x_\beta, y_t) \\
&= \max\left\{ \left\| x_\beta + \frac{c_t+r_t}{2} y_t \right\|_\beta, \left\| x_\beta + \frac{a_t+p_t}{2} y_t \right\|_\beta \right\} \\
&< 1 + k_\beta^2.
\end{aligned}$$

Therefore, we obtain $M_1 \leq 1 + k_\beta^2$.

Next, we suppose that $t \in (1/k_\beta, \infty) \setminus \{(1+k_\beta)/(1-k_\beta)\}$. Since

$$a_t - \frac{c_t+r_t}{2} = \frac{k_\beta(k_\beta+t)}{t(t-k_\beta)} > 0,$$

we have $0 < (c_t+r_t)/2 < a_t$. Then, it follows that

$$\left\| x_\beta + \frac{c_t+r_t}{2} y_t \right\|_\beta = 1 + \frac{k_\beta}{t} < 1 + k_\beta^2.$$

This proves that if $t \geq 2 + k_\beta$, then

$$m(x_\beta, y_t) = \max\left\{ \|x_\beta\|_\beta, \left\| x_\beta + \frac{c_t+r_t}{2} y_t \right\|_\beta \right\} < 1 + k_\beta^2$$

by Lemma 2.6.

In the case of $1/k_\beta < t \leq 2 + k_\beta$, we have $b_t < (b_t+q_t)/2 \leq 0$ since $q_t > 0$ and

$$\frac{b_t+q_t}{2} = \frac{(2+k_\beta-t)b_t}{2(1+k_\beta)} \leq 0.$$

Then, it follows that

$$\left\| x_\beta + \frac{b_t + q_t}{2} y_t \right\|_\beta = 1 + \frac{(1-k_\beta)(t-1)}{2(1+k_\beta)} \cdot \frac{2+k_\beta - t}{1+k_\beta t}.$$

On the other hand, since

$$\begin{aligned} k_\beta - \frac{(1-k_\beta)(t-1)}{2(1+k_\beta)} &\geq k_\beta - \frac{(1-k_\beta)((2+k_\beta)-1)}{2(1+k_\beta)} \\ &= \frac{3k_\beta - 1}{2} > 0 \end{aligned}$$

and

$$\begin{aligned} k_\beta - \frac{2+k_\beta-t}{1+k_\beta t} &= \frac{(1+k_\beta^2)t - 2}{1+k_\beta t} \\ &> \frac{(1+k_\beta^2)k_\beta^{-1} - 2}{1+k_\beta t} \\ &= \frac{k_\beta + k_\beta^{-1} - 2}{1+k_\beta t} > 0, \end{aligned}$$

we obtain

$$\left\| x_\beta + \frac{b_t + q_t}{2} y_t \right\|_\beta < 1 + k_\beta^2,$$

which implies that

$$\begin{aligned} &m(x_\beta, y_t) \\ &= \max\left\{ \left\| x_\beta + \frac{b_t + q_t}{2} y_t \right\|_\beta, \left\| x_\beta + \frac{c_t + r_t}{2} y_t \right\|_\beta \right\} \\ &< 1 + k_\beta^2. \end{aligned}$$

Hence, we have $M_2 \leq 1 + k_\beta^2$.

Finally, since

$$M_2 \geq m(x_\beta, y_t) \geq 1 + \frac{k_\beta}{t}$$

for each $t \in (1/k_\beta, \infty) \setminus \{(1+k_\beta)/(1-k_\beta)\}$, it follows that $M_2 \geq 1 + k_\beta^2$. This shows $M_2 = 1 + k_\beta^2$. Thus, by Lemma 3.2, one has that

$$DW(X_\beta) = 2M(x_\beta) = 2M_2 = 2(1+k_\beta^2) = \frac{2}{\beta^2}\left((1-\beta)^2 + \beta^2\right).$$

The proof is complete. $\square$

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Received September 18, 2012