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OSCILLATION AND ASYMPTOTIC PROPERTIES OF SOLUTIONS OF THIRD ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The third order functional differential equation

$$\left(r_2(t)\left(r_1(t)y'\right)'\right)' + q(t)F\left(y(g(t))\right) = f(t)$$

is studied for its oscillatory and nonoscillatory nature. Conditions have been found to ensure that all solutions are either oscillatory or they approach zero as $t \to \infty$. Moreover, the conditions for asymptotic behavior for the special case of the above equation has been found.

1. Introduction.

In this paper we are interested in the oscillatory behavior of the equation

$$(1) \qquad (r_2(t)(r_1(t)y')')' + q(t)F\left(y(g(t))\right) = f(t) \quad \text{where}$$

$r_1, r_2, q, g, f : [t_0, \infty) \to R, F : \mathbf{R} \to \mathbf{R}$ are continuous, $r_1 > 0, r_2 > 0, r_2' \leq 0$ for $t \in [t_0, \infty), q(t) \geq 0$ and not identically zero for any ray of the form $[t^*, \infty)$ for some $t^* \geq t_0, g(t) \to \infty$ as $t \to \infty$, and $yF(y) > 0$ for $y \neq 0$. S.R. Grace and B.S. Lalli [3] studied the oscillatory behavior of the second order equation

$$(a(t)\dot{x}(t))^{\cdot} + q(t)f\left(x(g(t))\right) = e(t).$$

Thus in this paper we extend the study to the third order and take a more generalized form of the equation.

We consider only solutions of equation (1) which are defined for large t. The oscillatory solution of (1) is considered in the usual sense, i.e., a solution of equation (1) is called oscillatory if it has no last zero, otherwise it is called nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory. It is almost oscillatory if every solution $y(t)$ of equation (1) is either oscillatory or $\lim_{t \to \infty} y(t) = 0$.

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2. Main results.

Theorem 1. *Let*

$$F'(y) \geq k > 0 \text{ for } y \neq 0. \tag{3}$$

and assume that there exist a function $\phi : [t_0, \infty) \to \mathbf{R}$ and differentiable functions

$$\delta, \sigma : [t_0, \infty) \to (0, \infty) \text{ such that}$$

$$(r_2(t)(r_1(t)\phi')')' = f(t), \phi(t) \to 0, \phi'(t) \to 0 \text{ and } \phi''(t) \to 0 \text{ as } t \to \infty, \tag{4}$$

$$\sigma(t) \leq \min\{t, g(t)\}, \sigma'(t) > 0 \text{ and } \sigma(t) \to \infty \text{ as } t \to \infty, \tag{5}$$

$$\delta'(t) < 0, \delta''(t) > 0 \text{ and } \delta'(t) \leq \frac{r_2'(t)}{2r_2(t)}\delta(t) \text{ for all } t \geq t_0. \tag{6}$$

If

$$\int^{\infty} \left\{ \delta(t)q(t) - \frac{r_1(\sigma(t))\, r_2(t)\delta'^2(t)}{4\lambda B\sigma(t)\sigma'(t)\delta(t)} \right\} dt = \infty, 0 < \lambda < 1, B > 0, \tag{7}$$

$$\int^{\infty} \frac{1}{r_1(t)\delta(t)} \int_{t_0}^{t} \int_{t_0}^{u} \frac{\delta(v)q(\nu)}{r_2(\nu)} d\nu du dt = \infty \text{ and } \int^{\infty} \frac{t}{r_1(t)\delta(t)}\, dt < \infty \tag{8}$$

then every solution of equation (1) is either oscillatory or $\lim_{t\to\infty} y(t) = 0$.

Proof. Let $y(t)$ be a nonoscillatory solution of equation (1). We may assume without loss of generality that $y(t) > 0$ for $t \geq t_0$ then there exists a $t_1 \geq t_0$ such that $y(\sigma(t)) > 0$ for $t \geq t_1$. Consider the function

$$y(t) = x(t) + \phi(t) \text{ for } t \geq t_1, \text{ then from equation (1) we get} \tag{9}$$

$$(r_2(t)(r_1(t)x')')' = -q(t)F(y(g(t))) \text{ for } t \geq t_1. \tag{10}$$

It is clear that $-(r_2(t)(r_1(t)x')')'$ is eventually positive for $t \geq t_1$. Hence $x(t)$ is monotone and one-signed, $x'(t)$ and $x''(t)$ are also monotone and one-signed for sufficiently large t. If $x(t) < 0$ for $t \geq t_1$ then $y(t) < \phi(t)$ for $t \geq t_1$ but this contradicts the assumption that $y(t) > 0$ and so we must have

$$x(t) > 0 \quad \text{for} \quad t \geq t_1. \tag{11}$$

__Claim 1.__

$$(r_1(t)x'(t))' > 0 \quad \text{for} \quad t \geq t_1. \tag{12}$$

From equation (10) we have $(r_2(r_1x')')' \leq 0$ or

$$(r_1(t)x'(t))'' \leq -\frac{r_2'(t)}{r_2(t)}(r_1(t)x'(t))' \tag{13}$$

If $(r_1(t)x')' \leq 0$ then $(r_1(t)x'(t))'' \leq 0$, i.e., $r_1(t)x'(t)$ is decreasing and concave down, and hence $r_1(t)x'(t)$ is eventually negative. Therefore $x(t)$ is eventually negative which contradicts (11).

__Claim 2.__

Now we claim that

$$x'(t) < 0 \text{ for } t \geq t_1. \tag{14}$$

If $x'(t) \geq 0$ for $t \geq t_1$ then using equation (9) we have

$$y(g(t)) = x(g(t)) + \phi(g(t)) \quad \text{for} \quad t \geq t_1. \tag{15}$$

Since $x(t)$ is increasing and positive and $\phi(t) \to 0$ as $t \to \infty$ then there exists $t_\lambda \geq t_1$ sufficiently large such that

$$y(g(t)) \geq \lambda x(g(t)) \quad \text{for} \quad t \geq t_\lambda. \tag{16}$$

Using (3) and (5) in (16) we get

$$F(y(g(t))) \geq F(\lambda x(g(t))) \quad \text{for} \quad t \geq t_\lambda. \tag{17}$$

Now define

$$w(t) = \frac{r_2(t)(r_1(t)x'(t))'}{F(\lambda x(\sigma(t)))}\delta(t) \text{ then for every } t \geq t_\lambda \text{ we have} \tag{18}$$

$$w'(t) = -q(t)\frac{F(y(g(t)))}{F(\lambda x(\sigma))}\delta(t) + w\frac{\delta'}{\delta} - \lambda\sigma' x'(\sigma)\frac{F'(\lambda x(\sigma))}{F(\lambda x(\sigma))}w. \tag{19}$$

Since $x' \geq 0$ and $(r_1 x')' > 0$ then by Kiguradze's lemma [4] we have

$$r_1(t)x'(t) \geq B_1 t(r_1(t)x'(t))' \text{ for some constant } B_1 > 0. \text{ Also we have} \tag{20}$$

$$(r_1(t)x'(t))' \leq (r_1(\sigma(t))x'(\sigma(t)))' \text{ for } t \geq t_\lambda. \tag{21}$$

Using (17), (20) and (21) in (19) we get

$$\begin{aligned} w'(t) &\leq -q(t)\delta(t) + \frac{\delta'(t)}{\delta(t)}w(t) - \lambda B\frac{\sigma'(t)\sigma(t)}{r_2(t)r_1(\sigma(t))}\frac{r_2(t)(r_1(t)x'(t))'}{F(\lambda x(\sigma(t)))}w(t), B = kB_1 \\ &\leq -q(t)\delta(t) + \frac{\delta'}{\delta}w(t) - \lambda B\frac{\sigma'\sigma}{r_2(t)r_1(\sigma)\delta(t)}w^2(t) \end{aligned}$$

Now if we complete the square on the right and then simplify we get

$$\omega'(t) \leq -\left\{q(t)\delta(t) - \frac{r_1(\sigma)r_2(t)\delta'^2(t)}{4\lambda B\sigma\sigma'\delta(t)}\right\}. \tag{23}$$

Integrating (23) from t_λ to t we get

$$\int_{t_\lambda}^{t}\left\{q(s)\delta(s) - \frac{r_1(\sigma(s))r_2(s)(\delta'(s))^2}{4\lambda B\sigma'(s)\sigma(s)\delta(s)}\right\}ds \leq w(t_\lambda) - w(t) \leq w(t_\lambda) < \infty \tag{24}$$

which contradicts (7) hence $x'(t) < 0$.

From (11) and (14) there exists a constant $c \geq 0$ such that $\lim_{t\to\infty} x(t) = c$. We will prove that $c = 0$.

Assume $c > 0$ and from (9) we have $\lim_{t\to\infty} y(g(t)) = \lim_{t\to\infty} x(g(t)) = c$. Hence there exists a $t_2 \geq t_1$ such that

$$y(g(t)) \geq \frac{c}{2} \text{ for } t \geq t_2. \tag{25}$$

Define

$$G(t) = r_1(t)x'(t)\delta(t) \text{ for } t \geq t_2. \tag{26}$$

Differentiating we get

$$G'(t) = (r_1(t)x'(t))'\delta(t) + (r_1(t)x'(t))\delta'(t). \tag{27}$$

Multiplying (27) by $r_2(t)$, differentiating and simplifying we get

$$G''(t) = -\frac{1}{r_2(t)}q(t)F(y(g(t)))\delta(t) + (r_1(t)x'(t))\delta''(t) + (r_1x')'(2\delta'(t) - \frac{r_2'(t)}{r_2(t)}\delta(t)) \tag{28}$$

Using (6), (14) and (25) we get

$$G''(t) \leq -\frac{q(t)\delta(t)}{r_2(t)}F\left(\frac{c}{2}\right) \text{ for } t \geq t_2. \tag{29}$$

Now integrating two times from t_2 to t, we get

$$G(t) \leq G(t_2) + G'(t_2)(t - t_2) - F\left(\frac{c}{2}\right)\int_{t_2}^{t}\int_{t_2}^{s}\frac{\delta(u)}{r_2(u)}q(u)du\, ds$$

$$\leq G'(t_2)t - F\left(\frac{c}{2}\right)\int_{t_2}^{t}\int_{t_2}^{s}\frac{\delta(u)}{r_2(u)}q(u)du\, ds. \tag{30}$$

From (26) we get

$$x'(t) \leq G'(t_2)\frac{t}{r_1(t)\delta(t)} - \frac{F\left(\frac{c}{2}\right)}{r_1(t)\delta(t)}\int_{t_2}^{t}\int_{t_2}^{s}\frac{\delta(u)q(u)}{r_2(u)}du\, ds. \tag{31}$$

Integrating we get

$$x(t) \leq x(t_2) + G'(t_2)\int_{t_2}^{t}\frac{s}{r_1(s)\delta(s)}ds - F\left(\frac{c}{2}\right)\int_{t_2}^{t}\frac{1}{r_1(s)\delta(s)}\int_{t_2}^{s}\int_{t_2}^{u}\frac{q(v)\delta(v)}{r_2(v)}dv\, du\, ds. \tag{32}$$

Now taking the limit as $t \to \infty$ and using (8) we get a contradiction to (11). This completes the proof.

Remark.

Note that condition (3) is quite restrictive since it can't be satisfied by a function of the type $F(x) = x^\gamma, \gamma > 1$ where γ is a quotient of odd positive integers. However, we relax that condition in this theorem.

Theorem 2. *Suppose that*

$$F'(y) \geq 0 \text{ for } y \neq 0 \tag{33}$$

and the conditions (4), (5), (6) and (8) hold. If

$$\int^{\infty} q(t)\delta(t)dt = \infty \tag{34}$$

then the conclusion of theorem 1 holds.

Proof. Let $y(t)$ be a nonoscillatory solution of equation (1), say $y(t) > 0$ and $y(\sigma(t)) > 0$ for every $t \geq t_1 \geq t_0$. As in the proof of Theorem 1 we obtain (19) and by using (33) we get

$$w'(t) \leq -q(t)\frac{F(y(g(t)))}{F(\lambda x(\sigma(t)))}\delta(t) + w(t)\frac{\delta'(t)}{\delta(t)}, \tag{35}$$

Using (6) and (17) we get

$$w'(t) \leq -q(t)\delta(t), \text{ and then integrating from } t_1 \text{ to } t \tag{36}$$

and using (34) we get a contradiction to the fact that $w(t) > 0$ for $t \geq t_1$. Thus we have (14) and the rest of the proof is the same as in Theorem 1.

Now we relax a condition on σ and state the following result.

Theorem 3. *Let conditions (4), (6) and (33) hold and suppose that*

$$\sigma(t) \leq \min\{t, g(t)\}, \sigma'(t) \geq 0 \text{ and } \sigma(t) \to \infty \text{ as } t \to \infty. \tag{37}$$

If

$$\int^{\infty} \delta(\sigma(s))q(s)ds = \infty, \tag{38}$$

$$\int^{\infty} \frac{1}{r_1(s)\delta(\sigma(s))} \int_{t_0}^{t} \int_{t_0}^{s} \frac{\delta(\sigma(u))}{r_2(u)} q(u)du\, ds\, dt = \infty \text{ and } \int^{\infty} \frac{t}{r_1(t)\delta(\sigma)}dt < \infty \tag{39}$$

then the conclusion of Theorem 1 holds.

Proof. The proof is similar to that of Theorem 2 except that we let the function w and G be defined as

$$w(t) = \frac{r_2(t)(r_1(t)x'(t))'}{F(\lambda x(\sigma(t)))}\delta(\sigma(t)) \text{ and } G(t) = r_1(t)x'(t)\delta(\sigma(t)).$$

Dahiya and Singh [2] studied the asymptotic nature of

(40) $\qquad y''' + a(t)y(t - \tau(t)) = f(t),$ under the two conditions

$\int^{\infty} t^2|f|dt < \infty$ and $\int^{\infty} t^2|a(t)|dt, \infty$. Our next result is for the asymptotic nature of the equation

(41) $\qquad y''' + q(t)F(y(\sigma(t))) = f(t)$ which is a special case of equation (1)

when $r_1 = r_2 = 1$. Equation (41) is more general than equation (40) and the required conditions are more relaxed than that of (40).

Theorem 4. *If*

$$\int^{\infty} |f(t)|dt < \infty, \tag{42}$$

$$\int^{\infty} t^2 q(t)dt < \infty \text{ and} \tag{43}$$

$$|F(y)| \le y \tag{44}$$

then equation (41) has nonoscillatory solutions asymptotic to $a_0 + a_1 t + a_2 t^2$, where $a_2 \neq 0$.

Proof. Integrating equation (41) from t_0 to t to get

$$y''(t) = y''(t_0) - \int_{t_0}^{t} q(s)F(y(\sigma(s)))ds + \int_{t_0}^{t} f(s)ds. \tag{45}$$

Integrating again we get

$$(46)\qquad y'(t) = y'(t_0) + y''(t_0)(t-t_0) - \int_{t_0}^{t}\int_{t_0}^{s} q(r)F(y(\sigma(r)))dr\ ds + \int_{t_0}^{t}\int_{t_0}^{s} f(r)dr\ ds.$$

By interchanging the integral signs in (46) we get

$$(47)\qquad \begin{aligned} y'(t) &= y'(t_0) + y''(t_0)(t-t_0) - \int_{t_0}^{t}(t-r)q(r)F dr + \int_{t_0}^{t}(t-r)f(r)dr \\ &= c_0 + c_1 t - \int_{t_0}^{t}(t-r)q(r)F dr + \int_{t_0}^{t}(t-r)f(r)dr, \text{ where } c_0 \end{aligned}$$

and c_1 are appropriate constants.

Integrating (47) from t_0 to $\sigma(t)$, where $\sigma(t) > t_0$ for large t, we get

$$(48)\qquad \begin{aligned} y(\sigma(t)) = y(t_0) + c_0(\sigma(t)-t_0) + \frac{c_1}{2}(\sigma^2(t)-t_0^2) - \int_{t_0}^{\sigma}\int_{t_0}^{s}(s-r)q(r)F dr\ ds \\ + \int_{t_0}^{\sigma}\int_{t_0}^{s}(s-r)f(r)dr\ ds. \end{aligned}$$

Hence

$$(49)\qquad |y(\sigma(t))| \le c_2 + c_3 t + c_4 t^2 + \frac{1}{2}\int_{t_0}^{t}(t-s)^2 q(s)|y(\sigma(s))|ds + \frac{1}{2}\int_{t_0}^{t}(t-s)^2|f(s)|ds,$$

where c_2, c_3 and c_4 are appropriate constants and $0 < \sigma(t) - t_0 < t$ for large t. From (49) we have $|y(\sigma(t))| \le (c_2+c_3+c_4)t^2 + t^2\int_{t_0}^{t} q(s)|y(\sigma)|ds + t^2\int_{t_0}^{t}|f(s)|ds$, where $t > 1$ large. Hence

$$\frac{|y(\sigma(t))|}{t^2} \le c_5 + \int_{t_0}^{t} s^2 q(s)\frac{y(\sigma(s))}{s^2}ds + L, \text{ where } c_5 = c_2 + c_3 + c_4$$

and $\int_{t_0}^{t}|f(s)|ds \le L$. By using (42), we get

$$\frac{|y(\sigma)|}{t_2} \le c + \int_{t_0}^{t} s^2 q(s)\frac{|y(s)|}{s^2}ds, \text{ where } c = c_5 + L$$

Applying Gronwall's inequality [5, p. 107] we get

$$\frac{|y(\sigma(t))|}{t^2} \leq k \exp\left(\int_{t_0}^{t} s^2 q(s)ds\right) \leq k_0, \text{ where } k_0 \text{ is a positive constant.} \tag{50}$$

From the first integral of equation (47), it follows that

$$\begin{aligned} \left|\int_{t_0}^{t}(t-s)q(s)F(y(\sigma(s)))ds\right| &\leq t\int_{t_0}^{t} s^2 q(s)\frac{|y(\sigma)|}{s^2}ds \\ &\leq k_0 t\int_{t_0}^{t} s^2 q(s)ds \text{ by using (50),} \\ &\leq k_0' t \text{ by using (43) where} \end{aligned} \tag{51}$$

k_0' is a positive constant.

Similarly the second integral of (47) gives

$$\left|\int_{t_0}^{t}(t-s)f(s)ds\right| \leq t\int_{t_0}^{t} |f(s)|\, ds \leq Lt. \tag{52}$$

Using (51) and (52) in (47) we get

$$\begin{aligned} &y'(t) \to c_0 + c_1' t \text{ as } t \to \infty, \text{i.e.,} \\ &y(t) \to a_0 + a_1 t + a_2 t^2 \text{ as } t \to \infty \text{ where } a_0, a_1 \text{ and } a_2 \end{aligned}$$

are appropriate constants and $a_2 \neq 0$.

Example 1. Consider the equation

$$\begin{aligned} &(e^t y')'' + e^{t-\pi} y(t-\pi) = -\cos t, \text{ where } r_1(t) = e^t, r_2(t) = 1, \\ &g(t) = t - \pi, q(t) = e^{t-\pi}, F(y) = y, \phi(t) = \frac{1}{2}e^{-t}(\sin t - \cos t) \text{ and} \\ &\delta(t) = e^{-t/2}. \end{aligned} \tag{53}$$

All conditions of theorem 1 are satisfied, and the conclusion holds. It is clear that $y(t) = e^{-t}\cos t$ is a solution of equation (53).

Example 2. Consider the equation

$$(54)\qquad \begin{aligned} &(e^{-t}(e^t y')')' + e^{4t} y^{\frac{5}{3}}(3t) = e^{-t}, \text{ where } r_1(t) = e^t, r_2(t) = e^{-t}, \\ &g(t) = 3t, q(t) = e^{4t}, F(y) = y^{\frac{5}{3}}, \phi(t) = te^{-t} + e^{-t} \text{ and } \delta(t) = e^{-t/2}. \end{aligned}$$

All conditions of Theorem 2 are satisfied, and the conclusion holds. It is clear that $y(t) = e^{-t}$ is a solution of equation (54).

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