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On two–parameter discrete time optimal starting–stopping problems

Teruo Tanaka

Abstract

We discuss the finiteness of an optimal stopping point and an ϵ–optimal stopping point for the discrete time two–parameter optimal stopping problem.

We also formulate the two–parameter optimal starting–stopping problem for the discrete time case. Further, several problems with the constraints on a starting time and a stopping time are studied, and several nested Bellman equations, named by Sun [11], are investigated.

Keywords : two–parameter stochastic process * starting–stopping problem * strategy * tactic * nested Bellman equation

1 Introduction

In this paper we study a stochastic control problem with not only a stopping time but also a starting time for two–parameter stochastic processes, which may be termed the two–parameter optimal starting–stopping problem.

The two–parameter optimal stopping problem has been studied by several authors. Haggstrom [3], Krengel and Sucheston [4], Lawler and Vanderbei [5] and Mandelbaum and Vanderbei [7] formulated the optimal stopping problems for the stochastic processes indexed by a partially ordered set and solved them through the dynamic programming approach. That is to say, by introducing the Snell envelopes, which are well–known in the one–parameter optimal stopping problems, and the concept of a stopping point, a strategy and a tactic, they gave a construction of an optimal solution. Mandelbaum [6] formulated the optimal stopping problem for the multi–parameter stochastic processes and developed not only the well–known dynamic programming approach but also the theory of the dynamic allocation index.

In particular, Mandelbaum and Vanderbei [7] showed that if a partially ordered set is a two–dimensional non–negative lattice, the two–parameter optimal stopping problem is equivalent to the control problem of finding the tactic maximizing the given expected reward function under the conditionally independent property (F3) in section 2.

Further Haggstrom [3] gave one sufficient condition for a reward process to guarantee that an optimal stopping point be finite. In contrast to the one–parameter optimal stopping problem, the finiteness of an optimal stopping point and an ϵ–optimal stopping point for the two–parameter optimal stopping problem remains open.

The first aim in this paper is to argue them.

Recently Sun [11] introduced the optimal starting–stopping problem for the one–parameter continuous time stochastic process described by a stochastic differential equation and gave a characterization of the optimal value function in terms of the nested variational inequality. The discussion of the one–parameter starting–stopping problem for the general stochastic process in both discrete time and continuous time case is given in Tanaka [13].

The second aim in this paper is to formulate optimal starting–stopping problems for two–parameter processes both with and without time constraint conditions and to discuss their optimal values. In the discrete time case, two–parameter optimal stopping problems with time constraints have been studied by Arenas [1] and Mandelbaum and Vanderbei [7].

In section 2 we give notations and definition, make preparations for optimal stopping problems for two–parameter processes and show some properties of an optimal stopping rules and the finiteness of an optimal stopping point and an ϵ–optimal stopping point under a certain assumption. In section 3 and 4 we give a precise formulation of two–parameter starting–stopping problem in the discrete time case. Further, we investigate several version of our problem with constraints on a starting time and a stopping time. In section 5 we study the monotone condition for the two–parameter stopping problem.

2 Preliminaries

Throughout this paper we consider the stochastic processes indexed by $\mathbf{N}^2$. Let $\mathbf{T} = \mathbf{N}^2$. The index set $\mathbf{T}$ is extended to its one–point compactification $\mathbf{T} \cup \{\infty\}$ endowed with the following partial order : for all $z = (s,t), z' = (s',t') \in \mathbf{T}$,

$$\begin{array}{ll} z \leq z' & \text{if and only if} \quad s \leq s', t \leq t', \\ z < z' & \text{if and only if} \quad s < s', t < t', \\ z \leq \infty & \text{for all } z \in \mathbf{T}. \end{array}$$

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with the complete two–parameter filtration $\{\mathcal{F}_z, z \in \mathbf{T}\}$ satisfying the conditions

(F1) if $z \leq z'$, then $\mathcal{F}_z \subset \mathcal{F}_{z'}$,

(F2) $\mathcal{F} = \sigma(\cup_z \mathcal{F}_z)$,

(F3) $\mathcal{F}_w$ and $\mathcal{F}_z$ are conditionally independent given $\mathcal{F}_{w\wedge z}$ for all $z, w \in \mathbf{T}$, where the notation $w \wedge z$ denotes the coordinatewise minimum.

A stopping point is the random variable T taking values in $\mathbf{T} \cup \{\infty\}$ such that for all $z \in \mathbf{T}$, $\{T \leq z\} \in \mathcal{F}_z$.

A strategy starting at $z \in \mathbf{T}$ is the family of stopping points $\{\sigma_t, t \geq 0\}$ satisfying the conditions :

$\sigma_0 = z$,

$\sigma_{t+1} = \sigma_t + e_1$ or $\sigma_t + e_2$,

σ_{t+1} is measurable with respect to $\mathcal{F}_{\sigma_t}$,

where $e_1 = (1,0)$ and $e_2 = (0,1)$.

A tactic starting at z is the pair (σ_t, τ) of a strategy at z and a stopping time τ with respect to $\mathcal{F}_{\sigma_t}$, that is, $\{\tau \leq t\} \in \mathcal{F}_{\sigma_t}$ $\forall t \geq 0$, where $\mathcal{F}_{\sigma_t} = \{A \in \mathcal{F} | A \cap \{\sigma_t \leq z\} \in \mathcal{F}_z, \forall z\}$.

Let $\{X(z), z \in \mathbf{T}\}$ be a real-valued $\mathcal{F}_z$-adapted two-parameter stochastic process on $(\Omega, \mathcal{F}, P)$ such that $E[\sup_{z\in\mathbf{T}} X(z)^+] < \infty$. We use the convention that $X(\infty) = \limsup_{z\to\infty} X(z)$.

Let Σ_z denote the set of all the tactics starting at z with $P(\tau < \infty) = 1$ and $E[X(\sigma_\tau)^-] < \infty$, and $\bar{\Sigma}_z$ the set of all tactics starting at z with $E[X(\sigma_\tau)^-] < \infty$.

Denote by C_z the set of all stopping points T such that $P(z \leq T < \infty) = 1$ and $E[X(T)^-] < \infty$, $\bar{C}_z$ the set of all stopping points such that $P(z \leq T) = 1$ and $E[X(T)^-] < \infty$.

The two-parameter optimal stopping problem is to find $T^* \in C_0(\text{resp.}\bar{C}_0)$ such that

$$E[X(T^*)] = \sup_{T\in C_0(\text{resp.}\bar{C}_0)} E[X(T)],$$

which is equivalent to find $(\sigma_t^*, \tau^*) \in \Sigma_0(\text{resp.}\bar{\Sigma}_0)$ such that

$$E[X(\sigma^*_{\tau^*})] = \sup_{(\sigma_t,\tau)\in\Sigma_0(\text{resp.}\bar{\Sigma}_0)} E[X(\sigma_\tau)],$$

because of the assumption (F3) (see Lemma 2.1).

We define the Snell envelopes by, for $z \in \mathbf{T}$,

$$\begin{aligned} f(z) &= \operatorname*{ess\,sup}_{T\in C_z} E[X(T)|\mathcal{F}_z], \\ \bar{f}(z) &= \operatorname*{ess\,sup}_{T\in \bar{C}_z} E[X(T)|\mathcal{F}_z]. \end{aligned}$$

Lemma 2.1 *We have, for $z \in \mathbf{T}$,*

$$f(z) = \operatorname*{ess\ sup}_{(\sigma_t,\tau)\in\Sigma_z} E[X(\sigma_\tau)|\mathcal{F}_z],$$
$$\bar{f}(z) = \operatorname*{ess\ sup}_{(\sigma_t,\tau)\in\bar{\Sigma}_z} E[X(\sigma_\tau)|\mathcal{F}_z].$$

Lemma 2.1 is immediately obtained by the assumption (F3).

At first we state the fundamental results which are able to obtain by the classical methods.

Theorem 2.1 $\{f(z)\}$ *satisfies the following properties :*

(i) $\{f(z)\}$ *satisfies the stochastic dynamic programming equation*

$$f(z) = \max\{X(z), \max_{i=1,2} E[f(z+e_i)|\mathcal{F}_z]\}$$

(ii) $E[f(z)] = \sup_{(\sigma_t,\tau)\in\Sigma_z} E[X(\sigma_\tau)]$

(iii) $\{f(z)\}$ *is the smallest supermartingale dominating* $\{X(z)\}$

(iv) $\limsup_{z\to\infty} X(z) \le \limsup_{z\to\infty} f(z)$

(v) *for all* $S \in C_0$, $f(S) = \operatorname{ess\,sup}_{T\in C_S} E[X(T)|\mathcal{F}_S]$, *where* $C_S = \{T \in C_0 | S \le T\}$.

If C_0, C_S *and* $f(S)$ *are changed by* $\bar{C}_0$, $\bar{C}_S$ *and* $\bar{f}(S)$, *respectively, then the last assertion* (v) *also holds true.*

Definition 2.1 *We call the tactics with the following relations f-admissible*

$$E[f(\sigma_{t+1})|\mathcal{F}_{\sigma_t}] = f(\sigma_t) \qquad on \quad \{t < \tau\}, \tag{1}$$
$$\tau = \inf\{t : X(\sigma_t) = f(\sigma_t)\}. \tag{2}$$

We also call tactics with the relations (1) *and* (2) *in which the Snell envelope* f *is changed by* $\bar{f}$ $\bar{f}$*-admissible.*

Proposition 2.1 (i)*Let the tactic* (σ_t, τ) *be f-admissible such that* $P(\tau < \infty) = 1$. *Then* (σ_t, τ) *is optimal in the class* Σ_0.

(ii)*Let the tactic* (σ_t, τ) *be $\bar{f}$-admissible. If* $\limsup_{z\to\infty} X(z) = \limsup_{z\to\infty} \bar{f}(z)$, *then* (σ_t, τ) *is optimal in the class* $\bar{\Sigma}_0$.

Proposition 2.2 *If for any w,* $\sup_{z\ge w} E[\sup_{p\ge w} X(p)|\mathcal{F}_z] \le \sup_{p\ge w} X(p)$, *then*

$$f(z) = \bar{f}(z) \qquad \forall z \qquad \text{a.s.}.$$

Proof. Since $f(z) \leq \bar{f}(z)$, it is sufficient to prove that $E[\bar{f}(z)] = E[f(z)]$. By the same line as in Theorem 2.1

$$\begin{aligned} E[\bar{f}(z)] &= \sup_{(\sigma_t,\tau)\in\bar{\Sigma}_z} E[X(\sigma_\tau)] \\ \bar{f}(z) &= \max\{X(z), \max_{i=1,2} E[\bar{f}(z+e_i)|\mathcal{F}_z]\} \end{aligned}$$

Therefore we can define an $\bar{f}$–admissible such that admissible tactic (ξ_t, η) by means of the above dynamic programming equation. By our assumption, $\limsup_z \bar{f}(z) \leq X(\infty)$. By Proposition 2.1 (ii), then we have

$$\begin{aligned} E[\bar{f}(z)] &= E[X(\xi_\eta)] \\ &\leq \sup_{\tau\leq\infty} E[X(\xi_\tau)] \\ &= \sup_{\tau<\infty} E[X(\xi_\tau)] \\ &\leq \sup_{\{\sigma_t\}} \sup_{\tau<\infty} E[X(\sigma_\tau)] \\ &= E[f(z)]. \end{aligned}$$

□

PROPOSITION 2.3 *We assume that* $\limsup_{z\to\infty} X(z) = \limsup_{z\to\infty} \bar{f}(z)$ *and* $X(z)$ *converges to* $-\infty$ *as* $z \to \infty$ *with respect to the one–point compactification topology. Let* (σ_t, τ) *be an* $\bar{f}$*–admissible tactic. Then* $\sigma_\tau < \infty$ *a.s..*

Proof. Suppose that $P(\tau = \infty) > 0$. By Proposition 2.1 and Proposition 2.2, we have

$$\begin{aligned} \sup_{\Sigma_0} E[X(\xi_\eta)] &= E[X(\sigma_\tau)] \\ &= \int_{\{\tau<\infty\}} X(\sigma_\tau)dP + \int_{\{\tau=\infty\}} X(\infty)dP \\ &= \int_{\{\tau<\infty\}} X(\sigma_\tau)dP + \int_{\{\tau=\infty\}} \limsup X(z)dP \\ &\leq -\infty. \end{aligned} \tag{3}$$

On the other hand,

$$-\infty < E[X(0)] \leq \sup_{\Sigma_0} E[X(\xi_\eta)].$$

This contradicts (3), therefore we have $P(\tau < \infty) = 1$. □

Next we discuss the ϵ–optimality. Let ϵ be a positive number.

The stopping point T^* is called ϵ–optimal if it satisfies

$$E[X(T^*)] \geq \sup_{T \in C_0(\text{resp}.\bar{C}_0)} E[X(T)] - \epsilon. \tag{4}$$

Since we assume the condition (F3), the problem (4) is equivalent to find $(\sigma_t^*, \tau_\epsilon^*) \in \Sigma_0(\text{resp}.\bar{\Sigma}_0)$ such that

$$E[X(\sigma_{\tau^*}^*)] \geq \sup_{(\sigma_t,\tau)\in\Sigma_0(\text{resp}.\bar{\Sigma}_0)} E[X(\sigma_\tau)] - \epsilon.$$

Theorem 2.2 *We assume that* $\limsup_{z\to\infty} X(z) = \limsup_{z\to\infty} \bar{f}(z)$. *Let* $(\sigma_t, \tau) \in \bar{\Sigma}_0$ *be an* $\bar{f}$*-admissible tactic. We define* τ_ϵ^* *by*

$$\tau_\epsilon^* = \inf\{t : X(\sigma_t) > f(\sigma_t) - \epsilon\}.$$

Then τ_ϵ^* *is* ϵ*–optimal and*

$$P(\tau_\epsilon^* < \infty) = 1.$$

Proof. By the assumption we have

$$\operatorname*{ess\,sup}_{\eta\geq t} E[X(\sigma_\eta)|\mathcal{F}_{\sigma_t}] = \bar{f}(\sigma_t) \qquad \text{on} \quad \{\tau = \infty\}. \tag{5}$$

where η is an $\mathcal{F}_{\sigma_t}$–stopping time. In fact, let $A = \{\operatorname{ess\,sup} E[X(\sigma_\eta)|\mathcal{F}_{\sigma_t}] < \bar{f}(\sigma_t)\}$, suppose that $P(A) > 0$ for some t. Then

$$E[X(\sigma_\tau)|\mathcal{F}_{\sigma_t}] < \bar{f}(\sigma_t) \quad \text{a.s.} \quad \text{on} \quad A \cap \{t < \tau\}.$$

From the fact that $\{\bar{f}(\sigma_{\tau\wedge t})\}$ is a one–parameter martingale, the limit $\lim_{t\to\infty} \bar{f}(\sigma_{\tau\wedge t})$ exists and is finite on $\{\tau = \infty\}$. Moreover we get $\lim_{t\to\infty} f(\sigma_{\tau\wedge t}) = \bar{f}(\sigma_\tau)$, and $E[\bar{f}(\sigma_\tau)|\mathcal{F}_{\sigma_t}] = \bar{f}(\sigma_{\tau\wedge t})$. Therefore we have on $A \cap \{t < \tau\}$

$$\begin{aligned} \bar{f}(\sigma_t) &> E[X(\sigma_\tau)|\mathcal{F}_{\sigma_t}] \\ &= E[\bar{f}(\sigma_\tau)|\mathcal{F}_{\sigma_t}] \\ &= \bar{f}(\sigma_t). \end{aligned}$$

This proves (5).

By the definition of the admissible tactic (σ_t, τ), we get $\tau_\epsilon \leq \tau$ and $\{\tau_\epsilon = \infty\} \subset \{\tau = \infty\}$. Since that $\{\bar{f}(\sigma_{\tau_\epsilon\wedge t})\}$ is a martingale, the limit $\lim_{t\to\infty} \bar{f}(\sigma_{\tau_\epsilon\wedge t})$ exists and is finite on $\{\tau_\epsilon = \infty\}$. Then we have

$$\bar{f}(\sigma_t) \geq X(\sigma_t) + \epsilon \qquad \text{on} \quad \{\tau_\epsilon = \infty\},$$

and then

$$\lim \bar{f}(\sigma_t) \geq \limsup X(\sigma_t) + \epsilon \qquad \text{on} \quad \{\tau_\epsilon = \infty\}. \tag{6}$$

It follows from (5) and (6) that $P(\tau_\epsilon = \infty) = 0$. Finally, by the martingale property of $\{\bar{f}(\sigma_{\tau_\epsilon \wedge t})\}$ and Theorem 2.1 we have

$$\begin{aligned} E[X(\sigma^*_{\tau^*_\epsilon})] &\geq E[\bar{f}(\sigma^*_{\tau_\epsilon})] - \epsilon \\ &= E[\bar{f}(\sigma_0)] - \epsilon \\ &= E[\bar{f}(0)] - \epsilon \\ &= \sup_{(\sigma_t, \tau)} E[X(\sigma_\tau)] - \epsilon. \end{aligned}$$

□

3 Starting–stopping problem

In this section we use the definitions and notation introduced in the previous section.

At first we shall formulate the two–parameter starting–stopping problem.

Let $\{f(z), z \in \mathbf{T}\}$ and $\{g(z), z \in \mathbf{T}\}$ be $\mathcal{F}_z$–adapted integrable stochastic processes such that $E[\sup_z f(z)^+] < \infty$ and $E[\sup_z g(z)^+] < \infty$. $f(\infty)$ (resp. $g(\infty)$) is interpreted as $\limsup f(z)$ (resp. $\limsup g(z)$).

The two–parameter starting–stopping problem is to find the stopping points S^* and T^* with $S^* \leq T^*$ such that

$$E[f(S^*) + g(T^*)] = \sup_{S \leq T} E[f(S) + g(T)].$$

In the starting–stopping problem, it is effective to consider the path in the time space connecting a starting time (point) and a stopping time (point). In the one–parameter case, it is obviously that for all stopping times S and T with $S \leq T$ there exists a path connecting S and T. As for the two–parameter case, the concept of the strategy is essential to the two–parameter optimal stopping problem.

Proposition 3.1 *For all stopping points S and T with $S \leq T$, there exists the triplicate $(\sigma_t, \tau_0, \tau_1)$ of a strategy and two stopping times with respect to $\mathcal{F}_{\sigma_t}$ with $\tau_0 \leq \tau_1$ such that*

$$S = \sigma_{\tau_0} \quad \textit{and} \quad T = \sigma_{\tau_1} \quad \textit{a.s.}.$$

This result is due to Walsh [15].

From this proposition, it should be noted that the following relation holds :

$$\sup_{(\sigma_t, \tau_0, \tau_1)} E[f(\sigma_{\tau_0}) + g(\sigma_{\tau_1})] = \sup_{S \leq T} E[f(S) + g(T)]$$

where $(\sigma_t, \tau_0, \tau_1)$ in the left–hand term is the triplicate of a strategy and two stopping times with respect to $\mathcal{F}_{\sigma_t}$ with $\tau_0 \le \tau_1$ and S and T in the right–hand term is the pair of stopping points with $S \le T$.

Therefore we consider the following criterion : find $(\sigma_t^*, \tau_0^*, \tau_1^*)$ such that

$$E[f(\sigma^*_{\tau_0^*}) + g(\sigma^*_{\tau_1^*})] = \sup_{(\sigma_t, \tau_0, \tau_1)} E[f(\sigma_{\tau_0}) + g(\sigma_{\tau_1})].$$

We define the nested Snell envelope $\{X(z)\}$ and $\{Y(z)\}$ by

$$\begin{aligned} Y(z) &= \operatorname{ess}\sup_{(\sigma_t,\tau)\in\Sigma_z} E[g(\sigma_\tau)|\mathcal{F}_z], \\ X(z) &= \operatorname{ess}\sup_{(\sigma_t,\tau)\in\Sigma_z} E[f(\sigma_\tau) + Y(\sigma_\tau)|\mathcal{F}_z]. \end{aligned}$$

The process $X(z)$ is well–defined. In fact, $Y(z)$ is $\mathcal{F}_z$–adapted and $g(z) \le Y(z) \le \sup_w g(w)^+$. For any $(\sigma_t, \tau) \in \Sigma_z$, $E[Y(\sigma_\tau)^-] \le E[g(\sigma_\tau)^-] < \infty$. Then by the standard argument, we get the following nested equation.

THEOREM 3.1 *We have for any $z \in \mathbf{T}$,*

$$\begin{aligned} Y(z) &= \max\{g(z), \max_{i=1,2} E[Y(z+e_i)|\mathcal{F}_z]\}, \\ X(z) &= \max\{f(z) + Y(z), \max_{i=1,2} E[X(z+e_i)|\mathcal{F}_z]\}. \end{aligned}$$

This theorem gives the construction of an optimal tactic.

THEOREM 3.2 *Let $\{A_z, A_z^1, A_z^2\}$ and $\{B_z, B_z^1, B_z^2\}$ be the partitions of Ω defined by*

$$\begin{aligned} A_z &= \{X(z) = f(z) + Y(z)\}, \\ A_z^1 &= \{X(z) = E[X(z+e_1)|\mathcal{F}_z]\} \setminus A_z, \\ A_z^2 &= \{X(z) = E[X(z+e_2)|\mathcal{F}_z]\} \setminus (A_z \cup A_z^1), \\ B_z &= \{Y(z) = g(z)\}, \\ B_z^1 &= \{Y(z) = E[Y(z+e_1)|\mathcal{F}_z]\} \setminus B_z, \\ B_z^2 &= \{Y(z) = E[Y(z+e_2)|\mathcal{F}_z]\} \setminus (B_z \cup B_z^1). \end{aligned}$$

And let (σ_t, τ) and (ξ_t, η) be tactics defined by

$$\begin{aligned} \sigma_0 &= 0, \\ \sigma_{t+1} &= \sigma_t + e_i \qquad on \quad A^i_{\sigma_t}, \\ \tau &= \inf\{t | X(\sigma_t) = f(\sigma_t) + Y(\sigma_t)\}, \\ \xi_0 &= \sigma_\tau, \\ \xi_{t+1} &= \xi_t + e_i \qquad on \quad B^i_{\xi_t} \cap A_{\xi_0}, \\ \eta &= \inf\{t | Y(\xi_t) = g(\xi_t)\}. \end{aligned}$$

Then the tactic $(\sigma_t^*, \tau_0^*, \tau_1^*)$ *defined by*

$$\begin{aligned} \sigma_t^* &= \begin{cases} \sigma_t & t \leq \tau \\ \xi_{t-\tau} & t \geq \tau, \end{cases} \\ \tau_0^* &= \tau, \\ \tau_1^* &= \tau + \eta \end{aligned}$$

is optimal if $\tau_0^* < \infty$ *and* $\tau_1^* < \infty$.

Theorem 3.2 is obtained by Theorem 2.1 (v) and the following lemma. In order that $\tau_0^* < \infty$ and $\tau_1^* < \infty$, it is sufficient that X and Y satisfy the conditions in Proposition 2.3.

Lemma 3.1 *For any* $(\sigma_t, \tau) \in \Sigma_0$ *and* $(\xi_t, \eta) \in \Sigma_{\sigma_\tau}$, *the pair* $(\hat{\xi}_t, \hat{\tau})$ *defined by*

$$\begin{aligned} \hat{\xi}_t &= \begin{cases} \sigma_t & t \leq \tau \\ \xi_{t-\tau} & t \geq \tau, \end{cases} \\ \hat{\tau} &= \tau + \eta \end{aligned}$$

is a tactic.

Theorem 3.3

$$E[X(0)] = \sup_{(\sigma_t, \tau_0, \tau_1)} E[f(\sigma_{\tau_0}) + g(\sigma_{\tau_1})].$$

Proof. By Theorem 2.1, we get

$$E[X(0)] = \sup_{(\sigma_t, \tau)} E[f(\sigma_\tau) + Y(\sigma_\tau)] = \lim_n E[f(\sigma^n_{\tau^n}) + Y(\sigma^n_{\tau^n})]$$

for some $\{(\sigma_t^n, \tau^n)\} \subset \Sigma_0$, and by Theorem 2.1 (v)

$$E[Y(\sigma^n_{\tau^n})] = \lim_m E[g(\xi^m_{\eta^m})]$$

for some $\{(\xi_t^m, \eta^m)\} \subset \Sigma_{\sigma^n_{\tau^n}}$. Hence we have

$$E[X(0)] = \lim_n \lim_m E[f(\sigma^n_{\tau^n}) + g(\xi^m_{\eta^m})].$$

This completes the proof. □

REMARK 3.1 (i) *We can also discuss the discounted criterion*

$$E[\sum_{t=\tau_0}^{\tau_1-1} a^t f(\sigma_t) + a^{\tau_1} g(\sigma_{\tau_1})]. \tag{7}$$

In the case of the running reward type, it is known that the one-parameter optimal starting problem ends in the usual one-parameter optimal stopping problem (see [11]*). However the concept of the strategy obstructs this transformation of the two-parameter starting problem (the stopping time* $\tau_1 = \infty$ *in* (7)*) to the two-parameter stopping problem. Hence it is significant to study the two-parameter optimal stopping problem containing the starting problem.*

(ii) *We set*

$$\begin{aligned} \bar{Y}(z) &= ess \sup_{(\sigma_t,\tau)\in\bar{\Sigma}_z} E[g(\sigma_\tau)|\mathcal{F}_z], \\ \bar{X}(z) &= ess \sup_{(\sigma_t,\tau)\in\bar{\Sigma}_z} E[f(\sigma_\tau) + \bar{Y}(\sigma_\tau)|\mathcal{F}_z]. \end{aligned}$$

Using the same argument, we can obtain the same results as those of Theorem 3.1, 3.2 *and* 3.3. *In this case, since the tactic defined in Theorem* 3.2 *is optimal in* $\bar{\Sigma}_0$, *it is possible to omit the assumption on the finiteness* $\tau_0^* < \infty$ *and* $\tau_1^* < \infty$.

4 Constrained problems and Markov cases

In this section we state the Snell envelopes for the problems with time constraints, and the nested Bellman equation for bi–Markov cases. Let N be a fixed positive integer. we are interested in the tactics with the following conditions :

(P1)	$\{\sigma_t\}_{t\le N}$,	$\tau \le N$
(P2)	$\{\sigma_t\}_{t\in\mathbf{T}}$,	$N \le \tau$
(P3)	$\{\sigma_t\}_{t\in\mathbf{T}}$,	$\tau_0 \le N \le \tau_1$
(P4)	$\{\sigma_t\}_{t\in\mathbf{T}}$,	$\tau_1 - \tau_0 \ge N$
(P5)	$\{\sigma_t\}_{t\in\mathbf{T}}$,	$\tau_1 - \tau_0 \le N$

Denote by $\Lambda_{(i)}$ the set of all tactics satisfying the condition (Pi).

In the case of (P1) and (P2), our problem is to find a tactic $(\sigma_t^*, \tau^*) \in \Lambda_{(i)}$ such that

$$E[f(\sigma^*_{\tau^*})] = \sup_{(\sigma_t,\tau)\in\Lambda_{(i)}} E[f(\sigma_\tau)].$$

In the case of (P3),(P4) and (P5), our problem is to find a tactic with starting time $(\sigma_t^*, \tau_0^*, \tau_1^*) \in \Lambda_{(i)}$ such that

$$E[f(\sigma_{\tau_0^*}^*) + g(\sigma_{\tau_1^*}^*)] = \sup_{(\sigma_t, \tau_0, \tau_1) \in \Lambda_{(i)}} E[f(\sigma_{\tau_0}) + g(\sigma_{\tau_1})].$$

<u>Case (P1)</u>

Put

$$\begin{aligned} Y(t, N-t) &= f(t, N-t) \quad \text{for} \quad t = 0, 1, \cdots, N, \\ Y(t, N-s-t) &= \max\{f(t, N-s-t), \max_{i=1,2} E[Y((t, N-s-t) + e_i) | \mathcal{F}_{(t,N-s-t)}]\}. \\ & \qquad \text{for} \quad t = 0, 1, \cdots, N-s, \quad s = 0, 1, \cdots, N \end{aligned}$$

Krengel and Sucheston [4] obtained the following theorem.

Theorem 4.1 *We have for* $t = 0, 1, \cdots, N-s, \quad s = 0, 1, \cdots, N,$

$$Y(t, N-s-t) = \operatorname{ess} \sup_{(\sigma_t, \tau) \in \Lambda^1(t, N-s-t)} E[f(\sigma_\tau) | \mathcal{F}_{(t,N-s-t)}],$$

where $\Lambda^1(t, N-s-t) = \{(\sigma_t, \tau) \in \Sigma_{(t,N-s-t)} | \tau \le s\}$.

<u>Case (P2)</u>

The Snell envelopes of this case are

$$\begin{aligned} Y(z) &= \operatorname{ess} \sup_{(\sigma_t, \tau) \in \Sigma_z} E[f(\sigma_\tau) | \mathcal{F}_z], \qquad z \in \mathbf{T} \quad \text{with} \quad |z| \ge N, \\ X(z) &= \operatorname{ess} \sup_{(\sigma_t, \tau) \in \Lambda^2(z)} E[Y(\sigma_\tau) | \mathcal{F}_z], \qquad z \in \mathbf{T} \quad \text{with} \quad |z| \le N \end{aligned}$$

where $\Lambda^2(z) = \{(\sigma_t, \tau) \in \Sigma_z | \tau = N - |z|\}$.

By Theorem 2.1, we have

$$E[X(0)] = \sup_{(\sigma_t, \tau) \in \Lambda_{(2)}} E[f(\sigma_\tau)].$$

Furthermore, an optimal tactic is constructed as follows : Put

$$\sigma_0 = 0, \quad \sigma_{t+1} = \sigma_t + e_i \quad \text{on} \quad A_{\sigma_t}^i \quad \text{for} \quad t = 0, 1, \cdots, N-1$$

$$\xi_0 = \sigma_N, \quad \xi_{t+1} = \xi_t + e_i \quad \text{on} \quad B^i_{\xi_t}, \quad \eta = \inf\{t | Y(\xi_t) = g(\xi_t)\}$$

where $A^i_\bullet$ and $B^i_\bullet$ are the partitions of Ω defined the similar way as Theorem 3.2. Then

$$\sigma^*_t = \begin{cases} \sigma_t & t \le N \\ \xi_{t-N} & t \ge N \end{cases}, \qquad \tau^* = N + \eta$$

is optimal for the case (P2) if $\tau^* < \infty$.

Case (P3)

The Snell envelopes of this case are

$$\begin{aligned} W(z) &= \text{ess} \sup_{(\sigma_t,\tau)\in\Sigma_z} E[g(\sigma_\tau)|\mathcal{F}_z], \\ Y(z) &= \text{ess} \sup_{(\sigma_t,\tau)\in\Lambda^2(z)} E[W(\sigma_\tau)|\mathcal{F}_z], \\ X(z) &= \text{ess} \sup_{(\sigma_t,\tau)\in\Lambda^1(z)} E[f(\sigma_\tau) + Y(\sigma_\tau)|\mathcal{F}_z]. \end{aligned}$$

By Theorem 2.1, we have

$$E[X(0)] = \sup_{(\sigma_t,\tau)\in\Lambda^1(0)} E[f(\sigma_\tau) + Y(\sigma_\tau)],$$

and an optimal tactic of this problem is as follows :

$$\sigma_0 = 0, \quad \sigma_{t+1} = \sigma_t + e_i \quad \text{on} \quad A^i_{\sigma_t}, \quad \tau = \inf\{t | X(\sigma_t) = f(\sigma_t) + Y(\sigma_t)\}.$$

Similarly, we get

$$E[Y(\sigma_\tau)] = \sup_{(\xi_t,\eta)\in\Lambda^2(\sigma_\tau)} E[W(\xi_\eta)]$$

and the tactic (ξ_t, η) defined by

$$\xi_0 = \sigma_\tau, \quad \xi_{t+1} = \xi_t + e_i \quad \text{on} \quad A^i_{\xi_t}, \quad \eta = \inf\{t | Y(\xi_t) = g(\xi_t)\}$$

is optimal. Then the tactic

$$\xi^*_t = \begin{cases} \sigma_t & t \le \tau \\ \xi_{t-\tau} & t \ge \tau \end{cases}, \qquad \eta^* = \tau$$

is optimal for the problem

$$E[X(0)] = \sup_{(\sigma_t,\tau)\in\Lambda^1(0)} E[f(\sigma_\tau) + W(\sigma_\tau)].$$

Finally, put

$$\alpha_0 = \xi_N^*, \quad \alpha_{t+1} = \alpha_t + e_i \quad \text{on} \quad B^i_{\alpha_t}, \quad \beta = \inf\{t | W(\alpha_t) = g(\alpha_t)\}.$$

Then the tactic with starting time

$$\sigma_t^* = \begin{cases} \xi_t^* & t \le N \\ \alpha_{t-N} & t \ge N \end{cases}, \quad \tau_0^* = \tau, \quad \tau_1^* = N + \beta$$

is optimal for the case (P3) if $\tau_0^* < \infty$ and $\tau_1^* < \infty$.

Case (P4)

The Snell envelopes of this case are

$$\begin{aligned} W(z) &= \operatorname*{ess\ sup}_{(\sigma_t,\tau)\in\Sigma_z} E[g(\sigma_\tau)|\mathcal{F}_z], \\ Y(z) &= \operatorname*{ess\ sup}_{(\sigma_t,\tau)\in\Lambda^2(z)} E[W(\sigma_\tau)|\mathcal{F}_z], \\ X(z) &= \operatorname*{ess\ sup}_{(\sigma_t,\tau)\in\Sigma_z} E[f(\sigma_\tau) + Y(\sigma_\tau)|\mathcal{F}_z]. \end{aligned}$$

At first we put

$$\sigma_0 = 0, \quad \sigma_{t+1} = \sigma_t + e_i \quad \text{on} \quad A^i_{\sigma_t}, \quad \tau = \inf\{t | X(\sigma_t) = f(\sigma_t) + Y(\sigma_t)\}.$$

Then (σ_t, τ) is optimal for the problem

$$E[X(0)] = \sup_{(\sigma_t,\tau)\in\Sigma_0} E[f(\sigma_\tau) + Y(\sigma_\tau)].$$

Next we put

$$\xi_0 = \sigma_\tau, \quad \xi_{t+1} = \xi_t + e_i \qquad \text{on} \quad B^i_{\xi_t}.$$

Then the tactic

$$\xi_t^* = \begin{cases} \sigma_t & t \le \tau \\ \xi_{t-\tau} & t \ge \tau \end{cases}, \quad \eta^* = \tau + N$$

is optimal for the problem

$$E[X(0)] = \sup_{(\sigma_t,\tau)\in\Sigma_0} E[f(\sigma_\tau) + W(\sigma_{\tau+N})].$$

Finally, put

$$\alpha_0 = \xi^*_{\eta^*}, \quad \alpha_{t+1} = \alpha_t + e_i \quad \text{on} \quad C^i_{\alpha_t}, \quad \beta = \inf\{t | W(\alpha_t) = g(\alpha_t)\}.$$

Then the tactic with starting time

$$\sigma_t^* = \begin{cases} \xi_t^* & t \leq \eta^* \\ \alpha_{t-\eta^*} & t \geq \eta^* \end{cases}, \quad \tau_0^* = \tau, \quad \tau_1^* = \eta^* + \beta$$

is optimal for the case (P4) if $\tau_0^* < \infty$ and $\tau_1^* < \infty$.

Case $(P5)$

The Snell envelopes of this case are

$$\begin{aligned} Y(z) &= \operatorname{ess} \sup_{(\sigma_t,\tau)\in\Lambda^1(z)} E[g(\sigma_\tau)|\mathcal{F}_z], \\ X(z) &= \operatorname{ess} \sup_{(\sigma_t,\tau)\in\Sigma_z} E[f(\sigma_\tau) + Y(\sigma_\tau)|\mathcal{F}_z]. \end{aligned}$$

We put

$$\sigma_0 = 0, \quad \sigma_{t+1} = \sigma_t + e_i \quad \text{on} \quad A^i_{\sigma_t}, \quad \tau = \inf\{t | X(\sigma_t) = f(\sigma_t) + Y(\sigma_t)\}$$

$$\xi_0 = \sigma_\tau, \quad \xi_{t+1} = \xi_t + e_i \quad \text{on} \quad B^i_{\xi_t}, \quad \eta = \inf\{t | Y(\xi_t) = g(\xi_t)\}.$$

Then the tactic with starting time

$$\sigma_t^* = \begin{cases} \sigma_t & t \leq \tau \\ \xi_{t-\tau} & t \geq \tau \end{cases}, \quad \tau_0^* = \tau, \quad \tau_1^* = \eta + \tau.$$

is optimal for the case (P5) if $\tau_0^* < \infty$ and $\tau_1^* < \infty$.

We end this section by stating the Markov case of our problem.

Let $X^i = (\Omega^i, \mathcal{F}^i, \mathcal{F}^i_t, X^i(t), P^i_x)$ be a time homogeneous Markov chain with the state space E^i, which is assumed to be mutually independent. We define a bi–Markov process introduced in Mazziotto [8] , that is, the family of a two–parameter process taking values in $E = E^1 \times E^2$

$$X(z) = (X^1(s), X^2(t)) \quad z = (s,t) \in \mathbf{T}$$

on the probability space $(\Omega = \Omega^1 \times \Omega^2, \mathcal{F} = \mathcal{F}^1 \otimes \mathcal{F}^2, P_{(x,y)} = P^1_x \otimes P^2_y, (x,y) \in E)$ endowed with the smallest two–parameter filtration $\{\mathcal{F}_z, z \in \mathbf{T}\}$ containing $\{\mathcal{F}^1_s \otimes \mathcal{F}^2_t, (s,t) \in \mathbf{T}\}$.

Let T^i be a transition operator of X^i, then,

$$\begin{aligned} T^1 f(x,y) &= E_{(x,y)}[f(X_{(1,0)})], \\ T^2 f(x,y) &= E_{(x,y)}[f(X_{(0,1)})]. \end{aligned}$$

The Markov version of our problem is to find a tactic $(\sigma_t^*, \tau_0^*, \tau_1^*)$ such that

$$E_{(x,y)}[f(X(\sigma^*_{\tau_0^*})) + g(X(\sigma^*_{\tau_1^*}))] = \sup_{(\sigma_t, \tau_0, \tau_1)} E_{(x,y)}[f(X(\sigma_{\tau_0})) + g(X(\sigma_{\tau_1}))],$$

where the functions f and g are measurable functions on E such that $E_{(x,y)}[\sup_z f(X(z))^+] < \infty$ and $E_{(x,y)}[\sup_z g(X(z))^+] < \infty$. Let $f(X(\infty)) = \limsup f(X(z))$ and $g(X(\infty)) = \limsup g(X(z))$.

We consider the following two-parameter stopping problems :

$$\begin{aligned} S(x,y) &= \sup_{(\sigma_t,\tau)\in\Sigma_0} E_{(x,y)}[g(X(\sigma_\tau))], \\ \bar{S}(x,y) &= \sup_{(\sigma_t,\tau)\in\bar{\Sigma}_0} E_{(x,y)}[g(X(\sigma_\tau))], \\ H(x,y) &= \sup_{(\sigma_t,\tau)\in\Sigma_0} E_{(x,y)}[f(X(\sigma_\tau)) + S(X(\sigma_\tau))], \\ \bar{H}(x,y) &= \sup_{(\sigma_t,\tau)\in\bar{\Sigma}_0} E_{(x,y)}[f(X(\sigma_\tau)) + \bar{S}(X(\sigma_\tau))]. \end{aligned}$$

Then we have the following results, whose proofs are omitted.

PROPOSITION 4.1 *We have for any* $(x,y) \in E$

$$H(x,y) = \sup_{(\sigma_t,\tau_0,\tau_1)} E_{(x,y)}[f(X(\sigma_{\tau_0})) + g(X(\sigma_{\tau_1}))].$$

Moreover if it is satisfies that $\limsup_z S(X(z)) = \limsup_z g(X(z))$ *and* $\limsup_z H(X(z)) = \limsup_z \{f(X(z)) + S(X(z))\}$, *then we have for any* $(x,y) \in E$

$$\begin{aligned} S(x,y) &= \bar{S}(x,y), \\ H(x,y) &= \bar{H}(x,y). \end{aligned}$$

PROPOSITION 4.2 *If f and g are bounded, the functions S and H are the smallest solutions of the nested Bellman equation :*

$$\begin{aligned} S &= \max\{g, \max_{i=1,2} T^i S\}, \\ H &= \max\{f + S, \max_{i=1,2} T^i H\}. \end{aligned}$$

REMARK 4.1 (i) *The Bellman equation corresponding to the two-parameter optimal stopping problem is the following type :*

$$H = \max\{f, \max_{i=1,2} T^i H\}.$$

Hence the nested Bellman equation in Proposition 4.2 is the brand-new type.

(ii) *In the situation where f and g are not necessarily bounded, in order that Proposition 4.2 hold true, we need the boundary condition at ∞. The details are given in Tanaka* [12].

5 Monotone case

In this section we discuss the monotone case for the two–parameter stopping problem studied in section 2 and the starting–stopping problem in section 3. The monotone case for the one–parameter stopping problem is studied by Chow and Robbins [2].

If a two–parameter stochastic process $\{X(z)\}$ is such that

$$A_z \subset A_{z+e_i} \qquad \text{for} \quad z,\ i,$$
$$\cup_z A_z = \Omega$$

where $A_z = \{\max_{i=1,2} E[X(z+e_i)|\mathcal{F}_z] \le X(z)\}$, we say that we are in the monotone case.

We assume that the process $\{X(z)\}$ satisfies the conditions made in section 2. Let N be a positive integer, $\{Y(z), |z| \le N\}$ the Snell envelope for the truncated process $\{X(z), |z| \le N\}$, which is introduced in section 4 Case (P1), and $\{f(z)\}$ the Snell envelope for $\{X(z)\}$, which is introduced in section 2.

Proposition 5.1 *Suppose that we are in the monotone case. We define the random sets by*

$$\begin{aligned} A &= \{z \mid |z| \le N-1,\ \max_{i=1,2} E[X(z+e_i)|\mathcal{F}_z] \le X(z)\}, \\ B &= \{z \mid |z| \le N-1,\ X(z) = Y(z)\}. \end{aligned}$$

Then $A = B$.

Proof. By the same argument as in [2], we have

$$\max_{i=1,2} E[X(z+e_i)|\mathcal{F}_z] \le X(z) \implies \max_{i=1,2} E[Y(z+e_i)|\mathcal{F}_z] \le X(z)$$

which follows $A \subset B$. Conversely, let $z \in B$. From the fact that Y is a supermartingale dominating X, we have

$$E[X(z+e_i)|\mathcal{F}_z] \ \le\ E[Y(z+e_i)|\mathcal{F}_z] \ \le\ Y(z) \ \le\ X(z),$$

which follows $B \subset A$. □

Corollary 5.1 *Suppose that we are in the monotone case and, besides the integrable condition made in section 3, $\{X(z)\}$ is bounded below by an integrable random variable. We define the random sets by*

$$\begin{aligned} A' &= \{z \mid \max_{i=1,2} E[X(z+e_i)|\mathcal{F}_z] \le X(z)\}, \\ B' &= \{z \mid X(z) = f(z)\}. \end{aligned}$$

Then $A' = B'$.

Finally we give the monotone condition for the starting–stopping problem. Throughout the remainder of this paper, we use the notation in section 3.

We assume that

$$\begin{array}{lll} A_z^1 \subset A_{z+e_i}^1 \quad \text{for} \quad z,\ i & \text{and} & A_z^2 \subset A_{z+e_i}^2 \quad \text{for} \quad z,\ i \\ \cup_z A_z^1 = \Omega & & \cup_z A_z^2 = \Omega \end{array}$$

where $A_z^1 = \{\max_{i=1,2} E[g(z+e_i)|\mathcal{F}_z] \leq g(z)\}$ and $A_z^2 = \{\max_{i=1,2} E[f(z+e_i)|\mathcal{F}_z] \leq f(z)\}$. Noting that Y is a supermartingale, then we have

$$\{z \mid \max_{i=1,2} E[g(z+e_i)|\mathcal{F}_z] \leq g(z)\} = \{z \mid g(z) = Y(z)\},$$
$$\{z \mid \max_{i=1,2} E[f(z+e_i)|\mathcal{F}_z] \leq f(z)\} = \{z \mid f(z) + Y(z) = X(z)\}.$$

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Teruo Tanaka
Department of Mathematics
Faculty of Science
Kyushu University
Fukuoka 812, JAPAN

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