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A note on the Grothendieck-Cousin complex on the flag variety in positive characteristic

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The Grothendieck-Cousin complex of a dominant line bundle on the flag variety with respect to the Schubert filtration is made up of the dual Verma modules in characteristic 0, the dual of a Bernstein-Gelfand-Gelfand complex, as observed by G. Kempf [8], J.L. Brylinski and M. Kashiwara [1], and M. Kashiwara [7], but that does not carry over to positive characteristic. The failure seems not as accessible as the author feels it should be. We intend to remedy the situation by reworking Kashiwara [7], § 3.

The difference stems from the one in SL_2. Thus let K be an algebraically closed field, G the K-group SL_2, B a Borel subgroup of G, T a maximal torus of B, B^+ the Borel subgroup of G opposite to B, α the root of B^+, $W = \langle s_\alpha \rangle$ the Weyl group of G, x_0 the point B of the flag variety $X = G/B$, $\mathcal{L}(\lambda)$ the invertible $\mathcal{O}_X$-module on X induced by a 1-dimensional B-module $\lambda \in \mathrm{Hom}(B, GL_1)$, and $\mathrm{Dist}(G)$ the algebra of distributions of G. In characteristic 0, the $\mathrm{Dist}(G)$-modules $H^1_{B^+ s_\alpha x_0}(X, \mathcal{L}(\lambda))$ and $H^0_{B^+ x_0}(X, \mathcal{L}(s_\alpha \cdot \lambda))$ are isomorphic iff $\langle \lambda, \alpha^\vee \rangle \geq -1$, where $\cdot$ is the dot multiplication and $\alpha^\vee$ is the coroot of α. On the other hand, we will find in § 2 that in positive characteristic they are isomorphic iff $\langle \lambda, \alpha^\vee \rangle = -1$. General results are summarized in (3.4).

In what follows K will denote an algebraically closed field of positive characteristic p, G a simply connected semisimple algebraic group over K, B a Borel subgroup of G, and X the flag variety G/B.

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§ 1

In this section we recall some generalities on the Grothendieck-Cousin complex of a G-linearized sheaf on X from Kempf [9] and on the representation theory of algebraic groups from Jantzen [6].

(1.1) We fix an action $\sigma : X \times G \longrightarrow X$ of G on X given by

(1) $$(x, g) \longmapsto g^{-1}x.$$

For a locally free $\mathcal{O}_X$-module $\mathcal{E}$ let $q : \mathbf{V}(\mathcal{E}) \longrightarrow X$ be the vectorial fibration of $\mathcal{E}$ on X [3], (9.4.9). A G-linearization of $\mathcal{E}$ is a G-action $\sigma_{\mathcal{E}} : \mathbf{V}(\mathcal{E}) \times G \longrightarrow \mathbf{V}(\mathcal{E})$ on $\mathbf{V}(\mathcal{E})$ making the fibration q G-equivariant such that $\sigma_{\mathcal{E}}(\_, g)$ induces a K-linear isomorphism from the geometric fibre over x onto the geometric fibre over $g^{-1}x$ for each $g \in G(K)$ and $x \in X(K)$. It induces an $\mathcal{O}_X$-homomorphism

(2) $$\mathcal{E} \longrightarrow \sigma_* p_X{}^* \mathcal{E},$$

where $p_X : X \times G \longrightarrow X$ is the natural map. In particular, the fibration $\mathbb{V}(\mathcal{O}_X) \longrightarrow X$ is isomorphic to

$$(3) \qquad X \times \mathbb{A}^1 \longrightarrow X \quad \text{via} \quad (x, \xi) \longmapsto x$$

and admits a unique G-linearization given by

$$(4) \qquad X \times \mathbb{A}^1 \times G \longrightarrow X \times \mathbb{A}^1 \quad \text{via} \quad (x, \xi, g) \longmapsto (g^{-1}x, \xi).$$

Let H be a closed subgroup of G, F the Frobenius morphism on H, and $H_r = \ker F^r$, $r \in \mathbb{N}$, a closed infinitesimal normal subgroup scheme of H called the r-th Frobenius kernel of H. Let $\mathrm{Dist}(H)$ be the algebra of distributions on H [6], (I.7.7). Then

$$(5) \qquad \mathrm{Dist}(H) = \varinjlim_{r} \mathrm{Dist}(H_r).$$

The G_r-linearization of $\mathcal{E}$ obtained from its G-linearization by restriction induces like (2) a system of compatible $\mathcal{O}_X$-homomorphisms

$$(6) \qquad \begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{E} \otimes_K K[G_r] \\ & \searrow & \downarrow \\ & & \mathcal{E} \otimes_K K[G_{r+1}], \end{array}$$

which defines a structure of $\mathrm{Dist}(G)$-module on $\mathcal{E}$. In particular, $\mathcal{O}_X$ is a $\mathrm{Dist}(G)$-module and the $\mathcal{O}_X$-module structure on $\mathcal{E}$ is compatible with the $\mathrm{Dist}(G)$-actions. We will call such an $\mathcal{O}_X$-module an $\mathcal{O}_X$-$\mathrm{Dist}(G)$-module.

(1.2) Let Z_1, Z_2 be closed subsets of X with $Z_1 \supseteq Z_2$. Define a functor $\Gamma_{Z_1/Z_2}(X, \_)$ from the category of abelian sheaves on X into the category of abelian groups by

$$(1) \qquad \mathcal{F} \longmapsto \Gamma_{Z_1}(X, \mathcal{F})/\Gamma_{Z_2}(X, \mathcal{F}),$$

where $\Gamma_{Z_i}(X, \mathcal{F})$ is the set of global sections of $\mathcal{F}$ with support contained in Z_i. We denote by $H^i_{Z_1/Z_2}(X, \mathcal{F})$ the i-th cohomology group of the complex $\Gamma_{Z_1/Z_2}(X, \mathcal{C}^\bullet(\mathcal{F}))$ for the Godement resolution $\mathcal{F} \longrightarrow \mathcal{C}^\bullet(\mathcal{F})$ of $\mathcal{F}$.

In the notation of (1.1) each cohomology group $H^i_{Z_1/Z_2}(X, \mathcal{E})$ of the G-linearized $\mathcal{O}_X$-module $\mathcal{E}$ inherits the structure of $\Gamma(X, \mathcal{O}_X)$-Dist(G)-module. If Z_1 and Z_2 are both H-invariant, then $H^i_{Z_1/Z_2}(X, \mathcal{E})$ comes from (1.1.2) equipped with a structure of H-module, which in turn makes $H^i_{Z_1/Z_2}(X, \mathcal{E})$ into a Dist(H)-module in a natural way. That, however, coincides with the Dist(H)-module structure obtained from the Dist(G)-module structure on $H^i_{Z_1/Z_2}(X, \mathcal{E})$ by restriction. We will call such a module a Dist(G)-H-module.

A filtration $\{Z\} = (Z_0 = X \supset Z_1 \supset Z_2 \supset \cdots)$ of X by closed subsets gives rise to a complex of $\Gamma(X, \mathcal{O}_X)$-Dist(G)-modules

$$(2) \qquad H^0_{Z_0/Z_1}(X, \mathcal{E}) \longrightarrow H^1_{Z_1/Z_2}(X, \mathcal{E}) \longrightarrow \cdots,$$

called the global Grothendieck-Cousin complex of the G-linearized

$\mathcal{O}_X$-module $\mathcal{E}$ with respect to the filtration $\{Z\}$. In case the filtration is H-invariant the complex (2) is also H-linear.

All the above can be sheafified to yield a complex of H-linearized $\mathcal{O}_X$-Dist(G)-modules

$$(3) \qquad \mathcal{H}^0_{Z_0/Z_1}(\mathcal{E}) \longrightarrow \mathcal{H}^1_{Z_1/Z_2}(\mathcal{E}) \longrightarrow \cdots,$$

called the local Grothendieck-Cousin complex of the G-linearized $\mathcal{O}_X$-module $\mathcal{E}$ with respect to the H-invariant filtration $\{Z\}$. We have

$$(4) \qquad \mathcal{H}^i_{Z_1/Z_2}(\mathcal{E}) \text{ is quasicoherent } \quad \forall i \in \mathbb{N},$$

and in an open subset V of X

$$(5) \quad \Gamma(V, \mathcal{H}^i_{Z_1/Z_2}(\mathcal{E})) \simeq H^i_{Z_1\cap V/Z_2\cap V}(V, \mathcal{E}) \quad \text{as } \Gamma(V, \mathcal{O}_X)\text{-Dist}(G)\text{-modules}.$$

(1.3) Let $X(B) = \mathrm{Hom}(B, GL_1)$. It forms an abelian group under the multiplication which we will write additively. For $\lambda \in X(B)$ we will abuse the notation and denote by the same letter a 1-dimensional B-module affording λ. Define an invertible sheaf $\mathcal{L}(\lambda)$ on X by

$$(1) \quad \begin{aligned} &\Gamma(V, \mathcal{L}(\lambda)) = \\ &\{f \in \mathrm{Mor}(\pi^{-1}V, \lambda) \mid f(gb) = \lambda(b^{-1})f(g) \ \forall g \in G, b \in B\} \quad \forall V \in \mathrm{Top}\, X, \end{aligned}$$

where $\pi : G \longrightarrow X$ is the natural map. Let x_0 be the point B in X and let $G \times^B \lambda$ be the quotient of $G \times \lambda$ by the B-action

$$(2) \qquad (g,\ \xi) \longmapsto (gb,\ b^{-1}\xi), \qquad b \in B.$$

Then the fibration $\mathbb{V}(\mathcal{L}(\lambda)) \longrightarrow X$ is isomorphic to

$$(3) \qquad G \times^{B}(-\lambda) \longrightarrow X \quad \text{via} \quad [g,\ \xi] \longmapsto gx_0,$$

and the invertible sheaf $\mathcal{L}(\lambda)$ admits a unique G-linearization given by the following commutative diagram

$$(4) \qquad \begin{tikzcd}
(G \times^{B}(-\lambda)) \times G \arrow[rr] \arrow[d] & & G \times^{B}(-\lambda) \arrow[d] \\
([g',\ \xi],\ g) \arrow[r, mapsto] \arrow[d, mapsto] & {[g^{-1}g',\ \xi]} \arrow[d, mapsto] & \\
(g'x_0,\ g) \arrow[r, mapsto] & g^{-1}g'x_0 & \\
X \times G \arrow[rr] & & X,
\end{tikzcd}$$

where $[g,\ \xi]$ is the B-orbit through $(g,\ \xi)$.

Let T be a maximal torus of B, R the root system of G relative to T, R^+ the positive system of R such that the roots of B are $-R^+$, Δ the simple system of R^+. We denote by U_α the root subgroup of G associated with the root α. Put $X(B)^+ =$ $\{\lambda \in X(B) \mid \langle\lambda,\ \alpha^\vee\rangle \geq 0 \quad \forall\alpha \in \Delta\}$, where $\alpha^\vee$ is the coroot of α. Let $W = N_G(T)/T$ the Weyl group of G, $s_\alpha \in W$ the reflection associated with $\alpha \in R$, and let $\ell : W \longrightarrow \mathbb{N}$ the length function with respect to the simple reflexions s_α, $\alpha \in \Delta$. Put $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$. Besides the usual action of W on $X(B)$ we define the dot action by

$$(5) \qquad w \cdot \lambda = w(\lambda + \rho) - \rho.$$

Let B^+ be the Borel subgroup of G opposite to B and U^+ its unipotent radical. From the Bruhat decomposition $X = \dot{\bigcup}_{w\in W} B^+wx_0$ we get a filtration of X by closed subsets

$$(6) \qquad Z_i = \bigcup_{\substack{w\in W \\ \ell(w)=i}} \overline{B^+wx_0}, \qquad i \in \mathbb{N},$$

called the Schubert filtration of X. For each $\lambda \in X(B)$ the filtration yields a global (resp. local) Grothendieck-Cousin complex of $\mathcal{L}(\lambda)$ consisting of $\Gamma(X, \mathcal{O}_X)$-Dist(G)-B^+- (resp. B^+-linearized $\mathcal{O}_X$-Dist(G)-) modules. In particular,

$$(7) \qquad H^i_{Z_i/Z_{i+1}}(X, \mathcal{L}(\lambda)) \simeq \coprod_{\substack{w\in W \\ \ell(w)=i}} H^i_{B^+wx_0}(X, \mathcal{L}(\lambda)) \quad \text{as } \Gamma(X, \mathcal{O}_X)\text{-Dist}(G)\text{-}B^+\text{-modules.}$$

We have for each $w \in W$ and $i \in \mathbb{N} \setminus \ell(w)$

$$(8) \qquad H^i_{B^+wx_0}(X, \mathcal{L}(\lambda)) = 0$$

and

$$(9) \qquad \mathcal{H}^i_{B^+wx_0}(\mathcal{L}(\lambda)) = 0.$$

The local (resp. global) Grothendieck-Cousin complex of $\mathcal{L}(\lambda)$ with respect to the Schubert filtration gives a resolution of $\mathcal{L}(\lambda)$ (resp. $\Gamma(X, \mathcal{L}(\lambda))$ in case $\lambda + \rho \in X(B)^{+}$).

Let $P_\alpha = B \cup Bs_\alpha B$, $P_\alpha^{+} = B^{+} \cup B^{+}s_\alpha B^{+}$ be the minimal parabolic subgroups of G associated with $\alpha \in \Delta$, and U_α^{+} the unipotent radical of P_α^{+}. Then there is a short exact sequence of $\Gamma(X, \mathcal{O}_X)$-Dist(G)-U_α^{+}-modules

$$(10)\quad \begin{aligned} &0 \longrightarrow \Gamma(s_\alpha B^{+}x_0, \mathcal{L}(\lambda)) \longrightarrow \\ &\qquad \Gamma(s_\alpha B^{+}x_0 \setminus B^{+}s_\alpha x_0, \mathcal{L}(\lambda)) \longrightarrow H^1_{B^{+}s_\alpha x_0}(X, \mathcal{L}(\lambda)) \longrightarrow 0. \end{aligned}$$

(1.4) Let J be a closed connected subgroup of G and E a J-module. For a K-subspace E' of E

(1) E' forms a J-submodule of E iff it is a Dist(J)-submodule.

Also for another J-module E''

$$(2)\qquad \mathrm{Hom}_J(E, E'') = \mathrm{Hom}_{\mathrm{Dist}(J)}(E, E'').$$

Now let M be a Dist(G)-T-module. It admits a decomposition into the weight subspaces:

$$(3)\qquad M = \coprod_{\lambda \in X(B)} M_\lambda,$$

where $M_\lambda = \{m \in M \mid tm = \lambda(t)m \ \ \forall t \in T\}$. We also have

(4) $$M_\lambda = \{m \in M \mid \mu m = \mu(\lambda)m \quad \forall \mu \in \mathrm{Dist}(T)\}.$$

As G is simply connected semisimple, Dist(G) has a basis that define Kostant's $\mathbb{Z}$-form of the universal enveloping algebra of the semisimple complex Lie algebra corresponding to G (cf.[6], (II.1.12)). Using the commutation formula among the basis elements (cf.[5], (26.3.D)) one checks

(5) $$\nu\mu m = \nu(\lambda+\eta)\mu m \quad \forall \lambda,\ \eta \in X(B),\ m \in M_\lambda,\ \mu \in \mathrm{Dist}(G)_\eta,\ \nu \in \mathrm{Dist}(T),$$

which implies $\mu m \in M_{\lambda+\eta}$ by (4). Hence we see

(6) $$t\mu t^{-1}m = (\mathrm{Ad}(t)\mu)m \quad \forall m \in M,\ \mu \in \mathrm{Dist}(G),\ t \in T.$$

Assume that

(7) $$\dim M_\lambda < \infty \quad \forall \lambda \in X(B).$$

We will deal only with those Dist(G)-T-modules in this note. Put

(8) $$DM = \coprod_{\lambda \in X(B)} \mathrm{Hom}_K(M_\lambda, K).$$

Using (6) we see that DM inherits the structure of Dist(G)-T-module in a natural way (cf.[6], (I.7.11.8)).

(1.5) For each $\lambda \in X(B)$ put

(1) $$\hat{Z}_\infty(\lambda) = \mathrm{Dist}(G) \otimes_{\mathrm{Dist}(B)} \lambda.$$

With Dist(G) hitting from the left by multiplication and with B acting both on Dist(G) and Dist(B) under the adjoint action and on λ as given, $\hat{Z}_\infty(\lambda)$ carries a structure of Dist(G)-B-module. It is Haboush's generalized Verma module of lowest weight λ, lowest with respect to the partial order defined on $X(B)$ by

(2) $$\nu \geq \eta \quad \text{iff} \quad \nu - \eta \in \sum_{\alpha\in R^+} \mathbb{N}\alpha.$$

Let $\mathbb{Z}[X(B)]$ be the group algebra of $X(B)$ over $\mathbb{Z}$ with natural basis $e(\eta)$, $\eta \in X(B)$. For a T-module M we put

(3) $$\mathrm{ch}\, M = \sum_{\eta\in X(B)} \dim M_\eta\, e(\eta)$$

and call it the formal character of M.

We have (cf.[9], Lemma 12.8) for each $\lambda \in X(B)$ and $w \in W$

(4) $$\mathrm{ch}\, H^{\ell(w)}_{B^+wx_0}(X, \mathcal{L}(\lambda)) = \frac{e(w\cdot\lambda)}{\prod_{\alpha\in R^+}(1-e(-\alpha))} = \mathrm{ch}\, D\hat{Z}_\infty(-w\cdot\lambda).$$

One suspects that

(5) $$H^{\ell(w)}_{B^+wx_0}(X, \mathcal{L}(\lambda)) \simeq D\hat{Z}_\infty(-w\cdot\lambda) \quad \text{as Dist}(G)\text{-}T\text{-modules}.$$

It is known to hold in characteristic 0 if $\lambda + \rho \in X(B)^+$ [8], [1],

[7]. Our aim is to study the question in positive characteristic p.

(1.6) Before closing the section let us verify the following lemma used in the proof of [7], Lemma 3.6.6. Let $\alpha \in \Delta$, $X_\alpha = G/P_\alpha$, $\pi_\alpha : X \longrightarrow X_\alpha$ the natural map, and x_α the point P_α in X_α.

Lemma. Let $w \in W$ with $\ell(ws_\alpha) < \ell(w)$. Then for each $i \in \mathbb{N}$ we have the following isomorphisms of Dist(G)-B^+-modules:

(i) $\quad H^i_{B^+wx_\alpha}(X_\alpha, \pi_{\alpha *}\mathcal{H}^0_{ws_\alpha B^+x_0}(\mathcal{L}(\lambda))) \simeq H^i_{B^+ws_\alpha x_0}(X, \mathcal{L}(\lambda))$.

(ii) $H^i_{B^+wx_\alpha}(X_\alpha, \pi_{\alpha *}\mathcal{H}^1_{ws_\alpha B^+s_\alpha x_0}(\mathcal{L}(\lambda))) \simeq H^{i+1}_{B^+wx_0}(X, \mathcal{L}(\lambda))$.

Proof. By (1.4.2) it is enough to establish the isomorphisms as Dist(G)-modules.

(i) We have by [2], Proposition 5.5 a sectral sequence

$$(1)\quad H^i_{B^+wx_\alpha}(X_\alpha, R^j\pi_{\alpha *}\mathcal{H}^0_{ws_\alpha B^+x_0}(\mathcal{L}(\lambda))) \Longrightarrow H^{i+j}_{\pi_\alpha^{-1}(B^+wx_\alpha)}(X, \mathcal{H}^0_{ws_\alpha B^+x_0}(\mathcal{L}(\lambda))).$$

Let V be an affine open subset of X_α. Then another spectral sequence of [2], Proposition 1.3

$$(2)\quad H^i(\pi_\alpha^{-1}V, \mathcal{H}^j_{ws_\alpha B^+x_0}(\mathcal{L}(\lambda))) \Longrightarrow H^{i+j}_{ws_\alpha B^+x_0 \cap \pi_\alpha^{-1}V}(\pi_\alpha^{-1}V, \mathcal{L}(\lambda))$$

degenerates by (1.3.9) into isomorphisms

$$(3)\quad H^j(\pi_\alpha^{-1}V,\ \mathcal{H}^0_{ws_\alpha B^+x_0}(\mathcal{L}(\lambda))) \simeq H^j_{ws_\alpha B^+x_0 \cap \pi_\alpha^{-1}V}(\pi_\alpha^{-1}V,\ \mathcal{L}(\lambda))$$

$\simeq H^j(ws_\alpha B^+x_0 \cap \pi_\alpha^{-1}V,\ \mathcal{L}(\lambda))$ by excision as $ws_\alpha B^+x_0$ is open in X,

which vanishes for $j > 0$ by Serre's vanishing theorem as $ws_\alpha B^+x_0 \cap \pi_\alpha^{-1}V$ is affine [3], (5.3.10). Hence

$$(4)\quad R^j\pi_{\alpha *}\mathcal{H}^0_{ws_\alpha B^+x_0}(\mathcal{L}(\lambda)) = 0 \qquad \forall j > 0$$

and the spectral sequence (1) degenerates into an isomorphism

$$(5)\quad H^i_{B^+wx_\alpha}(X_\alpha,\ \pi_{\alpha *}\mathcal{H}^0_{ws_\alpha B^+x_0}(\mathcal{L}(\lambda))) \simeq H^i_{\pi_\alpha^{-1}(B^+wx_\alpha)}(X,\ \mathcal{H}^0_{ws_\alpha B^+x_0}(\mathcal{L}(\lambda))).$$

Further, the spectral sequence of [9], Lemma 8.5

$$(6)\quad H^i_{\pi_\alpha^{-1}(B^+wx_\alpha)}(X,\ \mathcal{H}^j_{ws_\alpha B^+x_0}(\mathcal{L}(\lambda))) \Longrightarrow H^{i+j}_{ws_\alpha B^+x_0 \cap \pi_\alpha^{-1}(B^+wx_\alpha)}(X,\ \mathcal{L}(\lambda))$$

degenerates by (1.3.9) again into an isomorphism

$$(7)\quad H^i_{\pi_\alpha^{-1}(B^+wx_\alpha)}(X,\ \mathcal{H}^0_{ws_\alpha B^+x_0}(\mathcal{L}(\lambda))) \simeq H^i_{ws_\alpha B^+x_0 \cap \pi_\alpha^{-1}(B^+wx_\alpha)}(X,\ \mathcal{L}(\lambda)).$$

But $ws_\alpha B^+x_0 \cap \pi_\alpha^{-1}(B^+wx_\alpha) = B^+ws_\alpha x_0$ (cf.[7], Lemma 3.2.1), hence putting together (5) and (7) yields (i).

(ii) The same argument as in (i) reduces us to checking

$$(8)\qquad H^j(\pi_\alpha^{-1}V,\ \mathcal{H}^1_{ws_\alpha B^+ s_\alpha x_0}(\mathcal{L}(\lambda))) = 0 \qquad \forall \text{affine open } V \text{ in } X_\alpha \text{ and } j > 0$$

and

$$(9)\qquad ws_\alpha B^+ s_\alpha x_0 \cap \pi_\alpha^{-1}(B^+ wx_\alpha) = B^+ wx_0.$$

The identity (9) is proved in [7], Lemma 3.2.1. To see (8), argue as in (3) to get an isomorphism

$$(10)\qquad \begin{aligned} &H^j(\pi_\alpha^{-1}V,\ \mathcal{H}^1_{ws_\alpha B^+ s_\alpha x_0}(\mathcal{L}(\lambda))) \simeq \\ &\qquad H^{j+1}_{ws_\alpha B^+ s_\alpha x_0 \cap \pi_\alpha^{-1}V}(wB^+ x_0 \cap \pi_\alpha^{-1}V,\ \mathcal{L}(\lambda)). \end{aligned}$$

But $wB^+ x_0 \cap \pi_\alpha^{-1}V$ is affine and $ws_\alpha B^+ s_\alpha x_0 \cap \pi_\alpha^{-1}V$ is defined by a single polynomial in $wB^+ x_0 \cap \pi_\alpha^{-1}V$, hence (10) vanishes for $j > 0$ by [9], Lemma 11.8, as desired.

§ 2

In this section we assume $G = SL_2$ and $\Delta = \{\alpha\}$.

(2.1) We begin by describing the $\mathrm{Dist}(U_\alpha)$-module structure on the invertible sheaf $\mathcal{L}(\lambda)$, $\lambda \in X(B)$. Recall the G-action σ on the variety X. If we identify $s_\alpha B^+ x_0 \times U_\alpha$ with $\mathbb{A}^2$ via

$$(1)\qquad (a,\ b) \longmapsto \left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}x_0 = \begin{pmatrix}0 & 1\\ -1 & -a\end{pmatrix}x_0,\ \begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\right),$$

and write $K[\mathbf{A}^2] = K[x,y]$, then

$$(2)\qquad \mathbf{A}^2_{1+xy} = \sigma|_{s_\alpha B^+ x_0 \times U_\alpha}{}^{-1}(s_\alpha B^+ x_0) \supseteq s_\alpha B^+ x_0 \times U_{\alpha,r} \qquad \forall r \in \mathbf{N},$$

where $U_{\alpha,r}$ is the r-th Frobenius kernel of U_α.

The U_α-linearization of $\mathcal{L}(\lambda)$ restricts to a commutative diagram

$$(3)\qquad \begin{tikzcd} \mathbf{A}^2 \times (-\lambda) \simeq (s_\alpha B^+ B \times^B (-\lambda)) \times U_\alpha \arrow[r] \arrow[d] & G \times^B (-\lambda) \arrow[d] \\ \mathbf{A}^2 \arrow[r] & X \end{tikzcd}$$

and further to

$$(4)\qquad \begin{tikzcd} \mathbf{A}^2_{1+xy} \times (-\lambda) \arrow[r] \arrow[dd] & s_\alpha B^+ B \times^B (-\lambda) \arrow[r, "\sim"] & \mathbf{A}^1 \times (-\lambda) \arrow[dd] \\ (a,b,c) \arrow[r, mapsto] \arrow[d, mapsto] & \left[\begin{pmatrix}b & 1+ab\\ -1 & -a\end{pmatrix},\ c\right] \arrow[r, mapsto] & \left(\frac{a}{1+ab},\ (1+ab)^{\langle\lambda,\alpha^\vee\rangle}c\right) \arrow[d, mapsto] \\ (a,b) \arrow[rr, mapsto] & & a(1+ab)^{-1} \\ \mathbf{A}^2_{1+xy} \arrow[rr] & & \mathbf{A}^1 \end{tikzcd}$$

as $\begin{pmatrix}1 & -b\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & -a\end{pmatrix} =$

$$\begin{pmatrix}b & 1+ab\\ -1 & -a\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & -a(1+ab)^{-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ b(1+ab) & 1\end{pmatrix}\begin{pmatrix}(1+ab)^{-1} & 0\\ 0 & 1+ab\end{pmatrix}.$$

By (2) one can read off from (4) the $U_{\alpha,r}$-linearization of $\mathcal{L}(\lambda)|_{s_\alpha B^+ x_0}$ to find the effect on the global sections to be

$$K[x] \simeq \Gamma(s_\alpha B^+ x_0, \mathscr{L}(\lambda)) \longrightarrow \Gamma(s_\alpha B^+ x_0, \mathscr{L}(\lambda)) \otimes_K K[U_{\alpha,r}] \simeq K[x,y]/(y^{p^r})$$

(5)

$$\text{via} \quad x^n \longmapsto x^n(1+xy)^{\langle \lambda, \alpha^\vee \rangle - n}, \qquad n \in \mathbb{N}.$$

(2.2) Lemma. Let $\lambda \in X(B)$.

(i) $H^0_{B^+ x_0}(X, \mathscr{L}(\lambda)) \simeq D\hat{Z}_\infty(-\lambda)$ as Dist(G)-T-modules.

(ii) For $\nu \in X(B)$ there is a Dist(G)-isomorphism between $H^1_{B^+ s_\alpha x_0}(X, \mathscr{L}(\lambda))$ and $H^0_{B^+ x_0}(X, \mathscr{L}(\nu))$ iff $s_\alpha \cdot \lambda = \lambda = \nu$.

Proof. (i) By (1.4.2) we have only to show that there is a Dist(G)-isomorphism. Put $M = H^0_{B^+ x_0}(X, \mathscr{L}(\lambda))$. By (1.5.4) we know

(1) $$\operatorname{ch} M = \operatorname{ch} D\hat{Z}_\infty(-\lambda).$$

Also

$$\begin{aligned} \operatorname{Hom}_{\mathrm{Dist}(G)}(M, D\hat{Z}_\infty(-\lambda)) &\simeq \operatorname{Hom}_{\mathrm{Dist}(G)}(\hat{Z}_\infty(-\lambda), DM) \\ &\simeq \operatorname{Hom}_{\mathrm{Dist}(B)}(-\lambda, DM) \quad \text{by the Frobenius reciprocity} \\ &\simeq K \quad \text{by (1.4.6) as } -\lambda \text{ is the lowest weight of } DM. \end{aligned} \tag{2}$$

Let φ be a nonzero Dist(G)-homomorphism from M into $D\hat{Z}_\infty(-\lambda)$. By (1) it suffices to show that φ is injective. But $\operatorname{soc}_{\mathrm{Dist}(G)} M$ is a U^+-submodule of M by (1.4.1) and $M^{U^+} = \operatorname{Mor}(U^+, \lambda)^{U^+} = \lambda$, hence $\operatorname{soc}_{\mathrm{Dist}(G)} M$ is simple of highest weight λ. As φ preserves λ, the assertion follows.

(ii) Put $M_0 = H^0_{B^+x_0}(X, \mathcal{L}(\nu))$ and $M_1 = H^1_{B^+s_\alpha x_0}(X, \mathcal{L}(\lambda))$. If there is a Dist$(G)$-isomorphism between M_0 and M_1, then that is a T-isomorphism by (1.4.2), hence

$$(3) \qquad \nu = s_\alpha \cdot \lambda$$

by the character consideration. Also from (i) we must have

$$(4) \qquad M_1^{U^+} = s_\alpha \cdot \lambda.$$

Conversely, if (4) holds, then arguing as in (i) will yield a Dist(G)-isomorphism from M_1 onto $D\hat{Z}_\infty(-s_\alpha \cdot \lambda)$ and the assertion will follow.

Hence we are reduced to showing

$$(5) \qquad M_1^{U^+} = s_\alpha \cdot \lambda \quad \text{iff} \quad s_\alpha \cdot \lambda = \lambda, \text{ i.e., } \langle \lambda, \alpha^\vee \rangle = -1.$$

We have

$$(6) \qquad \begin{aligned} M_1^{U^+} &\simeq \mathrm{Hom}_{U^+}(K, M_1) = \mathrm{Hom}_{\mathrm{Dist}(U^+)}(K, M_1) \qquad \text{by (1.4.2)} \\ &\simeq \mathrm{Ann}_{M_1}(\mathrm{Dist}^+(U^+)), \end{aligned}$$

where the last term is $\{m \in M_1 \mid \mu m = 0 \ \ \forall \mu \in \mathrm{Dist}^+(U^+)\}$ and $\mathrm{Dist}^+(U^+) = \{\mu \in \mathrm{Dist}(U^+) \mid \mu(1) = 0\}$. Further, we can by (1.3.10) write $M_1 \simeq K[x,x^{-1}]/K[x]$ in the notation of (2.1). Put $N = \mathrm{Ann}_{K[x,x^{-1}]/K[x]}(\mathrm{Dist}^+(U_\alpha))$. Then (5) is equivalent to

$$(7) \qquad N = Kx^{-1} + K[x] \qquad \text{iff} \qquad \langle\lambda, \alpha^\vee\rangle = -1.$$

By (2.1.5) the $\mathrm{Dist}(U_{\alpha,r})$-module structure on $K[x,x^{-1}]/K[x]$ has the effect of

$$(8) \qquad x^{-m} \longmapsto x^{-m}(1+xy)^{\langle\lambda,\alpha^\vee\rangle+m}, \qquad m \geq 2.$$

If $\langle\lambda, \alpha^\vee\rangle \leq -2$, then $x^{\langle\lambda,\alpha^\vee\rangle} \in N \setminus 0$.

If $\langle\lambda, \alpha^\vee\rangle \geq 0$, take $k \in \mathbb{N}$ large enough that $m = p^k - \langle\lambda,\alpha^\vee\rangle \geq 2$. Then $x^{-m} \in N \setminus 0$.

Finally, if $\langle\lambda, \alpha^\vee\rangle = -1$, then

$$(9) \qquad \sum_{i=1}^{n} c_i x^{-i} \longmapsto \sum_{j=0}^{n-1} \Big(\sum_{i=1+j}^{n} \binom{i-1}{j} c_i x^{j-i} \Big) y^j, \qquad c_i \in K.$$

If $\sum c_i x^{-i} \in N \setminus 0$, then

$$(10) \qquad \sum_{i=1+j}^{n} \binom{i-1}{j} c_i x^{j-i} = 0 \qquad \forall j \in [1, n-1].$$

Hence if $c_n \neq 0$, we must have $n = 1$, and (7) follows.

(2.3) <u>Remark.</u> If there is an isomorphism in (2.2)(ii), then that is an isomorphism of $\mathrm{Dist}(G)$-B^+-modules by (1.4.2).

§ 3

In this section we follow Kashiwara [7] to study $H^i_{B^+wx_0}(X, \mathcal{L}(\lambda))$.

(3.1) <u>Proposition.</u> Let $\lambda \in X(B)$.

(i) $H^0_{B^+x_0}(X, \mathcal{L}(\lambda)) \simeq D\hat{Z}_\infty(-\lambda)$ as $\mathrm{Dist}(G)$-T-modules.

(ii) For $\alpha \in \Delta$ and $\nu \in X(B)$ there is a $\mathrm{Dist}(G)$-isomorphism between $H^1_{B^+s_\alpha x_0}(X, \mathcal{L}(\lambda))$ and $H^0_{B^+x_0}(X, \mathcal{L}(\nu))$ iff $\lambda = s_\alpha\cdot\lambda = \nu$.

<u>Proof.</u> (i) holds just as (2.2)(i) does.

(ii) Put $M = H^1_{B^+s_\alpha x_0}(X, \mathcal{L}(\lambda))$. As in the proof of (2.2)(ii), one has only to show

(1) $$M^{U^+} = s_\alpha\cdot\lambda \quad \text{iff} \quad \lambda = s_\alpha\cdot\lambda.$$

Recall the parabolic subgroup $P^+_\alpha = B^+ \cup B^+s_\alpha B^+$ and its unipotent radical U^+_α. We have

(2) $$M^{U^+} = (M^{U^+_\alpha})^{U_\alpha} = \mathrm{Ann}_{M^{U^+_\alpha}}(\mathrm{Dist}^+(U_\alpha)).$$

As $s_\alpha B^+x_0 = U^+_\alpha s_\alpha U_\alpha x_0$ and as $B^+s_\alpha x_0 = U^+_\alpha s_\alpha x_0$, we are reduced via (1.3.10) to the SL_2 case, and the assertion follows from (2.2)(ii).

(3.2) Remark. If there is an isomorphism in (3.1)(ii), then that is an isomorphism of $\mathrm{Dist}(G)$-B^{+}-modules by (1.4.2).

(3.3) Recall the natural map $\pi_{\alpha} : X \longrightarrow X_{\alpha} = G/P_{\alpha}$ and the point $x_{\alpha} = P_{\alpha}$ in X_{α}.

Proposition. Let $\lambda \in X(B)$, $\alpha \in \Delta$, and $w \in W$. If $s_{\alpha}\cdot\lambda = \lambda$, then $\pi_{\alpha *}\mathcal{H}^{1}_{wB^{+}s_{\alpha}x_{0}}(\mathcal{L}(\lambda)) \simeq \pi_{\alpha *}\mathcal{H}^{0}_{wB^{+}x_{0}}(\mathcal{L}(\lambda))$ as $\mathcal{O}_{X_{\alpha}}$-$\mathrm{Dist}(G)$-modules.

Proof. As $\mathcal{H}^{1}_{wB^{+}s_{\alpha}x_{0}}(\mathcal{L}(\lambda))$ is quasicoherent by (1.2.4) and as $wB^{+}x_{\alpha}$ is affine open in X_{α} containing $\pi_{\alpha}(wB^{+}s_{\alpha}x_{0}) = \pi_{\alpha}(wB^{+}x_{0})$, it is enough by [3], (1.7.4) to show

$$
(1)\qquad
\begin{aligned}
&\Gamma(wB^{+}x_{\alpha}, \pi_{\alpha *}\mathcal{H}^{1}_{wB^{+}s_{\alpha}x_{0}}(\mathcal{L}(\lambda))) \simeq \\
&\qquad \Gamma(wB^{+}x_{\alpha}, \pi_{\alpha *}\mathcal{H}^{0}_{wB^{+}x_{0}}(\mathcal{L}(\lambda))) \quad \text{as } \Gamma(wB^{+}x_{\alpha}, \mathcal{O}_{X_{\alpha}})\text{-}\mathrm{Dist}(G)\text{-modules.}
\end{aligned}
$$

Assume first $w = 1$ and put $M_{0} = \Gamma(B^{+}x_{\alpha}, \pi_{\alpha *}\mathcal{H}^{0}_{B^{+}x_{0}}(\mathcal{L}(\lambda)))$, $M_{1} = \Gamma(B^{+}x_{\alpha}, \pi_{\alpha *}\mathcal{H}^{1}_{B^{+}s_{\alpha}x_{0}}(\mathcal{L}(\lambda)))$. Then

$$
(2)\qquad M_{0} \simeq \Gamma(B^{+}x_{0}, \mathcal{L}(\lambda)),
$$

and

$$
\begin{aligned}
M_1 &\simeq \Gamma(s_\alpha B^+ x_0, \mathcal{H}^1_{B^+ s_\alpha x_0}(\mathcal{L}(\lambda))) && \text{by excision} \\
\text{(3)} \qquad &\simeq H^1_{B^+ s_\alpha x_0}(s_\alpha B^+ x_0, \mathcal{L}(\lambda)) && \text{by (1.2.5)} \\
&\simeq H^1_{B^+ s_\alpha x_0}(X, \mathcal{L}(\lambda)).
\end{aligned}
$$

By (3.1) we know

(4) $$M_1 \simeq D\hat{Z}_\infty(-\lambda) \simeq M_0 \quad \text{as } \mathrm{Dist}(G)\text{-}T\text{-modules},$$

hence $\mathrm{Hom}_{\mathrm{Dist}(G)}(M_1, M_0) \simeq K$. Let $\varphi \in \mathrm{Hom}_{\mathrm{Dist}(G)}(M_1, M_0) \setminus 0$. We must show that φ is $\Gamma(B^+ x_\alpha, \mathcal{O}_{X_\alpha})$-linear, i.e., the following diagram commutes:

(5)
$$
\begin{tikzcd}
\Gamma(B^+ x_\alpha, \mathcal{O}_{X_\alpha}) \otimes_K M_1 \arrow[rr, hook] \arrow[d, "id \otimes \varphi"'] & & \Gamma(s_\alpha B^+ x_0, \mathcal{O}_X) \otimes_K M_1 \arrow[d, "\text{multiplication}"] \\
\Gamma(B^+ x_\alpha, \mathcal{O}_{X_\alpha}) \otimes_K M_1 \arrow[d, hook] & & M_1 \arrow[d, "\varphi"] \\
\Gamma(B^+ x_0, \mathcal{O}_X) \otimes_K M_0 \arrow[rr, "\text{multiplication}"'] & & M_0 .
\end{tikzcd}
$$

Let φ_1, φ_2 be the two maps in question induced by φ. As φ_1 and φ_2 are both $\mathrm{Dist}(G)$-linear, $\mathrm{im}(\varphi_1 - \varphi_2)$ forms a B^+-submodule of M_0 by (1.4.1), hence

(6) $$(\mathrm{im}(\varphi_1 - \varphi_2))^{U^+} \le M_0^{U^+} = \lambda.$$

But $\varphi_1 - \varphi_2$ is T-linear by (1.4.2) and

(7) $$(\Gamma(B^+s_\alpha, \mathcal{O}_{X_\alpha}) \otimes_K M_1)_\lambda = K \otimes_K \lambda$$

on which $\varphi_1 = \varphi_2$, hence $\mathrm{im}(\varphi_1 - \varphi_2) = 0$, as desired.

For arbitrary $w \in W$ one just twists the above argument by w.

(3.4) Corollary. Let $\lambda \in X(B)$ and $w \in W$.

(i) $H^i_{B^+wx_0}(X, \mathcal{L}(\lambda)) = 0$ unless $i = \ell(w)$.

(ii) $\mathrm{ch}\, H^{\ell(w)}_{B^+wx_0}(X, \mathcal{L}(\lambda)) = \mathrm{ch}\, D\hat{Z}_\infty(-w\cdot\lambda)$.

(iii) $H^0_{B^+x_0}(X, \mathcal{L}(\lambda)) \simeq D\hat{Z}_\infty(-\lambda)$ as $\mathrm{Dist}(G)$-T-modules.

(iv) For $\alpha \in \Delta$ and $\nu \in X(B)$ the following are equivalent:

(a) $H^1_{B^+s_\alpha x_0}(X, \mathcal{L}(\lambda)) \simeq H^0_{B^+x_0}(X, \mathcal{L}(\nu))$ as $\mathrm{Dist}(G)$-modules.

(b) $H^1_{B^+s_\alpha x_0}(X, \mathcal{L}(\lambda)) \simeq H^0_{B^+x_0}(X, \mathcal{L}(\nu))$ as $\mathrm{Dist}(G)$-T-modules.

(c) $H^1_{B^+s_\alpha x_0}(X, \mathcal{L}(\lambda)) \simeq H^0_{B^+x_0}(X, \mathcal{L}(\nu))$ as $\mathrm{Dist}(G)$-B^+-modules.

(d) $s_\alpha\cdot\lambda = \lambda = \nu$.

(v) Let $\Delta_\lambda = \{\alpha \in \Delta \mid s_\alpha\cdot\lambda = \lambda\}$. Then for any $y \in \langle s_\alpha \mid \alpha \in \Delta_\lambda\rangle$
$H^{\ell(y)}_{B^+yx_0}(X, \mathcal{L}(\lambda)) \simeq H^0_{B^+x_0}(X, \mathcal{L}(\lambda))$ as $\mathrm{Dist}(G)$-B^+-modules.

Proof. Only (v) may require an explanation. We have only to establish an isomorphism as $\mathrm{Dist}(G)$-modules by (1.4.2). If $\ell(y) \neq 0$, take s_α with $\alpha \in \Delta_\lambda$ such that $\ell(ys_\alpha) < \ell(y)$. Then

$$H^{\ell(y)}_{B^+yx_0}(X, \mathcal{L}(\lambda)) \simeq H^{\ell(y)-1}_{B^+yx_\alpha}(X_\alpha, \pi_{\alpha*}\mathcal{H}^1_{ys_\alpha B^+s_\alpha x_0}(\mathcal{L}(\lambda))) \quad \text{by (1.6)(ii)}$$

$$\simeq H^{\ell(y)-1}_{B^+yx_\alpha}(X_\alpha, \pi_{\alpha*}\mathcal{H}^0_{ys_\alpha B^+x_0}(\mathcal{L}(\lambda))) \quad \text{by (3.3)}$$

$$\simeq H^{\ell(ys_\alpha)}_{B^+ys_\alpha x_0}(X, \mathcal{L}(\lambda)) \quad \text{by (1.6)(i)},$$

hence the assertion will follow by induction.

(3.5) Remarks. (i) The assertions (3.4)(i) and (ii) hold free of characteristic, and appear in Kempf [9].

(ii) If $s_\alpha \cdot \lambda = \lambda$, $\lambda \in X(B)$, for some $\alpha \in \Delta$, then

$$H^{\bullet}(X, \mathcal{L}(\lambda)) = 0$$

due to H.H. Andersen (cf.[6], (II.5.4)).

(iii) In ch 0, (3.4)(iii), (iv), and (v) are replaced by the statement: if $\lambda + \rho \in X(B)^+$, then for each $w \in W$

$$H^{\ell(w)}_{B^+wx_0}(X, \mathcal{L}(\lambda)) \simeq H^0_{B^+x_0}(X, \mathcal{L}(w\cdot\lambda)) \quad \text{as Dist}(G)\text{-}B^+\text{-modules.}$$

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