Positive linear maps of Banach algebras with an involution

By

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1. Introduction

A linear map $T: A \rightarrow B$ is called a *positive linear map* if $T(A^+) \subset B^+$, where A and B are complex Banach *-algebras, and, A^+ and B^+ are the sets of all finite sums of the form x^*x ($x \in A$ or $x \in B$.) In [7], we investigated some properties of positive linear maps of Banach *-algebras. In this paper, we shall also consider some properties of positive linear maps of complex *-Banach algebras with an identity (namely, Banach *-algebras with an isometric involution and an identity of norm one)

Let A be a complex *-Banach algebra with an identity e_A. By $\|x\|$, we denote the norm of $x \in A$. Moreover, we denote the well known pseud-norms on A as follows:

$$\|x\|_{1,A} = \sup\{|f(x)|; f \text{ is positive linear functional on } A \text{ such that } f(e_A) \leqq 1\},$$
$$\|x\|_{2,A} = \sup\{(f(x^*x))^{\frac{1}{2}}; f \text{ is positive linear functional on } A \text{ such that } f(e_A) \leqq 1\}.$$

Then we have $\|x\|_{1,A} \leqq \|x\|_{2,A} \leqq \|x\|$. If A is a C*-algebra, we have $\|x\|_{1,A} = \|x\|_{2,A} = \|x\|$ for every hermitian element x of A. Moreover $\{x \in A; \|x\|_{1,A} = 0\}$ and $\{x \in A; \|x\|_{2,A} = 0\}$ coincide with the *-radical $R^{(*)}{}_A$ of A. We recall that, if A has an identity, any positive linear map is self-adjoint (namely, $T(x^*) = (T(x))^*$). The notations given in [7] will be quoted without notice.

2. Operator norm of positive linear map

In [7], we discussed the continuity of positive linear maps of Banach *-algebras. In this section, we consider the operator norm of positive linear map of *-Banach algebras with an identity.

We need the following definition.

DEFINITION 2. 1. *Let A and B be a *-Banach algebra and a C*-algebra respectively, and T be a positive linear map of A into B. Then T is said to satisfy the stronger form of generalized Schwarz inequality provided $T(x^*)T(x) \leqq \|T\| T(x^*x)$ for every $x \in A$.*

If $T(x)$ is of the form $V^*\rho(x)V$ for every $x \in A$, where ρ is a *-representation of A on a complex Hilbert space K, and H is a complex Hilbert space on which B acts, and V is a bounded linear operator of H into K, then T satisfies the stronger form of generalized Schwarz inequality. Indeed, let e_A be the identity element of A, then $\|Te_A\| \leqq \|T\|$, $\|(\rho(e_A)V)^* (\rho(e_A) V\| \leqq \|T\|$. Then, we have $\|(\rho(e_A)V) (\rho(e_A)V)^* \| \leqq \|T\|$. Thus, we have $(\rho(e_A)V) (\rho(e_A)V)^* \leqq \|T\|\cdot I$, where I is the identity operator on K. Then, we have

$$\begin{aligned} T(x^*)T(x) &= V^*\rho(x)^*VV^*\rho(x)V \\ &= V^*\rho(x)^*(\rho(e_A)V)(\rho(e_A)V)^*\rho(x)V \\ &\leqq V^*\rho(x)^*\|T\|\cdot I\rho(x)V \\ &= \|T\|\,V^*\rho(x^*x)V = \|T\|\,T(x^*x). \end{aligned}$$

PROPOSITION 2. 2. *Let A and B be complex *-Banach algebras with an identity e_A and e_B respectively, and T be a positive linear map of A into B. If B is *-semi-simple, then the operator bound of T with respect to the norm $\| \ \|_{1,B}$ coincides with the norm $\|T(e_A)\|_{1,B}$. In particular, if B is a C*-algebra and T satisfies the stronger form of generalized Schwarz inequality, then the operator norm $\|T\|$ of T coincides with $\|T(e_A)\|$.*

PROOF. It is clear that we have, for every $x \in A$,

$$\|Tx\|_{1,B} \leqq \|Te_A\|_{1,B}\|x\|.$$

Since $\|e_A\|=1$, the first part of proposition follows.
Next, suppose B is a C*-algebra, and T satisfies the stronger form of generalized Schwarz inequality. Since $T(H_A) \subset H_B$ (H_A and H_B mean the sets of all hermitian elements of A and B respectively), it follows, for every $x \in H_A$,

$$\|Tx\| = \|Tx\|_{1,B} \leqq \|Te_A\|_{1,B}\|x\| = \|Te_A\|\,\|x\|.$$

Then, for every $x \in A$, we have

$$\|Tx\|^2 = \|(Tx)^*(Tx)\| \leqq \|T\|\,\|Tx^*x\| \leqq \|T\|\,\|Te_A\|\,\|x\|^2.$$

Thus we have $\|T\| \leqq \|Te_A\|$ which implies that $\|T\| = \|Te_A\|$ and completes the proof.

If A and B be C*-algebras, any positive linear map T satisfy the stronger form of Generalized Schwarz inequality for unitary operators. Hence we have $\|T\| = \|Te_A\|$. (see. [4], [5])

3. Extreme positive linear maps

In this section, we investigate the extreme points of a certain convex set consisting of positive linear maps. We define $P_0(A, B)$ as follows:

$$P_0(A, B) = \{T: A \to B: \text{positive linear map such that } \|T\|_0 \leqq 1\},$$

where $\|T\|_0$ is the operator bound with respect to the pseud-norm $\| \ \|_{2,A}$. We shall show that if B is symmetric and semi-simple, any multiplicative positive linear map in $P_0(A, B)$

is the extreme point of $P_0(A, B)$ A useful tool in the proof is the generalized Schwarz inequality due to R. V. Kadison.

We need the following lemmas.

LEMMA 3. 1. *Let A and B be complex *-Banach algebras with an identity e_A and e_B respectively and T be a positive linear map of A into B. Then we have* $T(R^{(*)}{}_A) \subset R^{(*)}{}_B$.

PROOF. For every $x \in A$, we have

$\|T(x)\|_{1,B}$

$= \textbf{sup}\ \{|f(T(x))|\,;f$ is positive linear functional on B such that $f(e_B) \leqq 1\}$

$\leqq \|T(e_A)\|_{1,B} \cdot \textbf{sup}\ \{|g(x)|\,;g$ is positive linear functional on A such that $g(e_A) \leqq 1\}$

$= \|T(e_A)\|_{1,B}\|x\|_{1,A}$. Therefore we have $T(R^{(*)}{}_A) \subset R^{(*)}{}_B$. q. e. d.

In his paper [2], Kadison has proved the following tool in study of positive linear maps.

LEMMA 3. 2. (Generalized Schwarz inequality) *Let A be a C^*-algebra, and T be a linear order-preserving map of A into the algebra of all bounded operators on some Hilbert space such that $\|T\| \leqq 1$. Then we have $T(a^2) \leqq (T(a))^2$ for every $a \in H_A$.*

Now we have the following two lemmas.

LEMMA 3. 3. *Suppose that A is a *-Banach algebra and B is a C^*-algebra. Let T be a positive linear map of A into B suchthat $\|T\|_0 \leqq 1$. Then $T(a^2)-(T(a))^2$ is contained in B^+ for every $a \in H_A$.*

PROOF. Suppose that A is *-semi-simple. Let $\|\ \|_{2,A}$ be the C*-norm of A and $C^*(A)$ be the completed C*-algebra of A with respect to $\|\ \|_{2,A}$, that is, the enveloping C*-algebra of A. Since T is continuous on A with respect to the C*-norm $\|\ \|_{2,A}$, T may be extended to a positive linear map $\widetilde{T}$ of the C*-algebra $C^*(A)$ into the C*-algebra B such that $\|\widetilde{T}\|_0 \leqq 1$. From lemma 3. 2, we have $T(a^2)-(T(a))^2 \in B^+$ for every $a \in H_A$.

Next suppose that A is non *-semi-simple. Let $R^{(*)}{}_A$ be the *-radical of A. Then the quotient *-Banach algebra $A/R^{(*)}{}_A$ is *-semi-simple. Let π be the canonical *-homomorphism of A onto $A/R^{(*)}{}_A$. Since C*-algebra is *-semi-simple, T vanishes on $R^{(*)}{}_A$ from lemma 3. 1. Thus we may define a linear map $\widehat{T}$ of $A/R^{(*)}{}_A$ into B by $\widehat{T}(\pi(x)) = \widehat{T}(x)$ for every $x \in A$. It is clear that T is a positive linear map of $A/R^{(*)}{}_A$ into B such that $\|\widehat{T}\|_0 \leqq 1$. Therefore we have $T(a^2)-(T(a))^2 = \widehat{T}(\pi(a^2))-(\widehat{T}(\pi(a)))^2 \in B^+$ which completes the proof.

LEMMA 3. 4. *Let A and B be complex *-Banach algebras and T be a positive linear map of A into B such that $\|T\|_0 \leqq 1$. If B is symmetric, $T(a^2)-(T(a))^2$ is contained in the norm closure of B^+ for every $a \in H_A$.*

PROOF. Let π be any *-representation of B on a complex Hilbert spae H. Then $\pi \circ T$

is a positive linear map of A into $B(H)$ (the C*-algebra of all bounded linear operators on H) such that $\|\pi\circ T\|_0\leqq 1$. From lemma 3. 2, we have

$$\pi(T(a^2)-(T(a))^2)=(\pi\circ T)(a^2)-((\pi\circ T)(a))^2\in(B(H))^+.$$

Now let f be any positive linear functional on B. We denote the *-representation and the cyclic vector associated to f by π_f and ξ_f respectively. Then we have

$$f(T(a^2)-(T(a))^2)=(\pi_f(T(a^2)-(Ta)^2)\xi_f\mid\xi_f)\geqq 0.$$

Therefore $T(a^2)-(T(a))^2$ has a non-negative real spectrum. This implies that $T(a^2)-(T(a))^2\in H^+{}_B=B^+$ and so completes the proof.

Definition 3. 5. *Let A and B be eomplex *-Banach algebras. By a C*-homomorphism we mean a positive linear map T such that $T(a^2)=(T(a))^2$ whenever a is an element of H_A.* Of course any multiplicative element of P (A, B) is C*-homomorphism.

We have the following

Theorem 3. 6. *Let A and B be complex *-Banach algebras. If B is symmetric and semi-simple, all C*-homomorphisms in $P_0(A, B)$ are extreme points of $P_0(A, B)$.*

Since the proof is almost the same as that of Theorem 3.4 in [7], we omit.

Remark. We can replace the symmetricity and semi-simplicity on B by *-semi-simplicity. Indeed, for any irreducible *-representation π of B on a complex Hilbert space H, $\pi\circ T$ is C*-homomorphism in $P_0(A, B(H))$. From lemma 3.3 and the argument used in the proof of the theorem 3.4 in [7] applying to the map $\pi\circ T$, the desired conclusion follows.

We call that $P_1(A, B)$ is the set of all positive limear maps of A into B which preserve the identity.

In the following, let A and B be C*-algebras with an identity. We denote the conjugate space of A and B by A^*andB^* respectively, and the canonical injection of a Banach space into the second conjugate space by J. We may define a certain convex set similar to $P_1(A, B)$ in $L(B^*, A^*)$ which is the set of all bounded linear maps of B^* into A^*. In the remainder of this section, we obtain some results on the connection between the extreme point in $P_1(A.B)$ and the extremality of its adjoint in the certain convex set.

We define the set $Q_1(B^*, A^*)$ of linear maps of B^* into A^* as follows:

$$Q_1(B^*, A^*) = \{S:B^*\to A^*: \textit{linear, bounded with respect to the functional norm and } S(E_B)\subset E_A\}$$

where E_A and E_B are the sets of all states of A and B respectively. It is clear that $Q_1(B^*, A^*)$ is convex and $T\in P_1(A, B)$ if and only if $T^*\in Q_1(B^*, A^*)$.

Proposition 3. 7. *If T^* is an extreme point in $Q_1(B^*, A^*)$, T is an extreme point of $P_1(A, B)$.*

PROOF. Suppose that there exist T_1, $T_2 \in P_1(A, B)$ such that $T=\frac{1}{2}(T_1+T_2)$. Then $T^*=\frac{1}{2}(T_1^*+T_2^*)$ with T_1^*, $T_2^* \in Q_1(B^*, A^*)$. The extremality of T^* implies $T^*=T_1^* = T_2^*$. Therefore we have $T=T_1=T_2$ which completes the proof.

Next, we consider the converse statement, that is, if $T \in P_1(A, B)$ is an extreme point, is T^* the extreme point of $Q_1(B^*, A^*)$? we shall show that, for any C*-homomorphism $T \in P_1(A, B)$ (of course such a map is an extreme point of $P_1(A, B)$), T^* is an extreme point in $Q_1(B^*, A^*)$.

We need the following lemma.

LEMMA 3. 8. *Suppose that A is a C*-algebra and B is a von Neummn algebra acting on a comlex Hilbert space. Let B_* be the predual* (the set of all ultra-weakly continuous linear functionals on B). *If T is an extreme point in $P_1(A, B)$, the restriction of T^* on B_* is an extreme point of $Q_1(B_*, A^*)$.*

PROOF. It is clear that $T^*|B_*$ (the restriction of T^* on B_*) is contained in $Q_1(B_*, A^*)$. Suppose that there exist $S_1, S_2 \in Q_1(B_*, A^*)$ such that $T^*|B^*=\frac{1}{2}(S_1+S_2)$. Since the conjugate Banach space of the predual B_* is B, we define two linear maps T_1, T_2 of A into B in the following manner:

$$J(T_1(a))=S_1^*(J(a)),\ J(T_2(a))=S_2^*(J(a)) \text{ for every } a \in A.$$

It is clear that T_1, $T_2 \in P_1(A, B)$. For every $f \in B_*$ and $a \in A$, we have

$$S_1(f)(a)=S_1^*(J(a))(f)=J(T_1(a))(f)=f(T_1(a))=T_1^*(f)(a).$$

Therefore, we have $S_1=T_1^*|B_*$. Similarly we have $S_2=T_2^*|B_*$.
Now, since $(T^*|_{B_*})^*=\frac{1}{2}(S_1^*+S_2^*)$, we have, for every $f \in B$ and $a \in A$,

$$(T^*|_{B_*})^*(Ja)(f)=\frac{1}{2}(S_1^*(Ja)(f)+S_2^*(Ja)(f)),$$
$$T^*(f)(a)=\frac{1}{2}(J(J(T_1(a))(f)+J(T_2(a)(f)),$$
$$f(Ta-\frac{1}{2}(T_1a+T_2a))=0.$$

Since f is an arbitrary element of B_*, we have $Ta=\frac{1}{2}(T_1a+T_2a)$. Hence we have $T=\frac{1}{2}(T_1+T_2)$. From the extremality of T, we have $T=T_1=T_2$ and therefore $T^*|_{B_*}=S_1=S_2$ which implies the extremality of $T^*|_{B_*}$ in $Q_1(B_*, A^*)$. The proof is completed.

From the above argument, if B is a finite dimensional C*-algebra, T is extreme if and only if T^* is extreme.

PROPOSITION 3. 9. *Let A and B be C*-algebras and T be a C*-homomorphism in $P_1(A, B)$. Then T^* is an extreme point of $Q_1(B^*, A^*)$.*

PROOF. Let π and C be the universal representation of B and the enveloped von

Neumann algebra of B. Since π is non-degenerate, $\pi(e_B)$ is the identity operator on H (the representation space of π). Thus $\pi\circ T$ is a C*-homomorphism in $P_1(A, C)$ and therefore $T^*\circ\pi^*|C_*=(\pi\circ T)^*|C_*$ is an point of $Q_1(C_*, A^*)$. If $T^*=\frac{1}{2}(S_1+S_2)$ with S_1, $S_2\in Q_1(B^*, A^*)$, we have

$$T^*\circ\pi^*|_{C*}=\frac{1}{2}(S_1\circ\pi^*|_{C*}+S_2\circ\pi^*|_{C*}).$$

From the extremality of $(\pi\circ T)^*{}_{C*}$, we have

$$T^*\circ\pi|_{C*}=S_1\circ\pi^*|_{C*}=S_2\circ\pi^*|_{C*}.$$

Consequently, we have $T^*=S_1=S_2$ which implies the extremality of T* in $Q_1(B^*, A^*)$.

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References

1 J DIXMIER: Les C*-algè bres et leurs représentations, Gauthier-villars, Paris, 1964.
2. R. V. KADISON: *A generalized Schwartz inequality and algebraic invariants for operator algebras*, Ann. of Math., 56 (1952), 494–503.
3. C. E. RICKART: *General theory of Banach algebras*, D. Van Nostrand, New York, 1960.
4. B. RUSSO and H. A. DYE: *A note on unitary operators in C*-algebras*, Duke J. Math. 33 (1966) 413–416.
5. W. F. STEINSPRING: *Positive functions on C*-Algebras*, Proc. Amer. Math. Soc. 6 (1955), 211–216
6 E. STØRMER: *Positive linear maps ofoperator algebras*, Acta Math., 110 (1963), 233–278.
7. S WATANABE: *Note on positive linear maps of Banach algebras with an iinvolution*, Sci. Rep. Niigata Univ., Ser. A, No. 7(1969), 17–21.