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# BARKHAUSEN EFFECT: A STICK–SLIP MOTION IN A RANDOM MEDIUM*

NATALIE GRUNEWALD†

**Abstract.** A one–dimensional model for the Barkhausen effect is considered. This model describes a motion in a random medium. The motion exhibits a stick–slip type behaviour in the limit of small correlation length of the random medium. However, we prove that the velocity of the limiting motion is positive almost everywhere. For this the corresponding Fokker–Planck equation is examined. This equation is degenerated and has a critical singularity as well as no gradient structure. Therefore, the proof relies mainly on choosing the right test functions, which gives natural boundary conditions in the limit.

**Key words.** Barkhausen Effect, Stick-Slip Motion, Random Medium, Singular and Degenerate Fokker-Planck Equation

**AMS subject classifications.** 35K20, 35Q60, 60K40, 60G30

**1. Introduction.** We consider a one–dimensional model for the motion of a point in a random medium. This model was introduced by Bertotti et al. [ABM] in (1990). It describes the non–uniform movement of a magnetic wall in a ferromagnetic sample, the Barkhausen effect. The wall moves because the magnetization of its magnetic domain is favoured in the sample as an external electromagnetic field is applied. The motion is driven by increasing this external field linearly in time, but hindered by magnetostatic effects which increase as the domain grows. On the other hand, the wall is moving in the medium of the other magnetic domains, which have different random magnetizations. Some of them hinder the motion of the wall more than others, whose magnetization is already close to the one of the growing domain. For a detailed description of the model see either [ABM] or [B], chapter 9. Consider the equation for the velocity

$$v(t) \;=\; \dot{x}(t) \;=\; a\,t \;-\; \eta\, x(t) \;-\; A\; H_0\left(\frac{x(t)}{\xi}\right), \tag{1}$$

with the dimensions

$[x] = m, \qquad [t] = s, \qquad [a] = \frac{m}{s^2}, \qquad [\eta] = \frac{1}{s}, \qquad [A] = \frac{m}{s}, \qquad [\xi] = m.$

Here, $at$ represents the driving force, which is linearly increased in time, and $\eta x$ are the magnetostatic effects. $AH_0$ is the heterogeneity coming from the different magnetisations of the magnetic domains. This heterogeneity has correlation length $\xi$ and amplitude $A$. $H_0$ is dimensionless and of order one. Equation (1) describes a gradient flow in the potential

$$\mathcal{V}(t,x) := \frac{\eta}{2}x^2 - atx + A\xi\bar{H}\left(\frac{x}{\xi}\right),$$

where $\bar{H}$ is the antiderivative of $H_0$. This potential is a moving, perturbed parabola. It has local minima in an interval of length $\sim \frac{A}{\eta}$ around the global minimum. In the limit $\frac{A}{\eta} \gg \xi$ the potential develops more and more local minima, in which the motion

*Received December 23, 2004; accepted for publication March 23, 2005.
†Institut für Angewandte Mathematik, University of Bonn, Wegelerstr. 10, 53115 Bonn, Germany (grunewald@iam.uni-bonn.de).

of the gradient flow might get trapped. However, we prove that in the limit $\frac{\eta^2\xi^2}{A^2} \to 0$ the velocity stays positive almost everywhere in space.
Differentiating (1) and nondimensionalizing it via the dimensionless, positive parameters $c = \frac{a\xi}{A^2}$ and $\varepsilon = \frac{\xi^2\eta^2}{A^2}$ ($\ll 1$) gives the dimensionless form

$$\frac{dz}{d\hat{t}} = c - z - \frac{z}{\sqrt{\varepsilon}}\, h\left(\frac{\hat{x}(\hat{t})}{\varepsilon}\right). \tag{2}$$

Here, $z$ is the nondimensionalized velocity and $h$ is the derivative of $H_0$. This equation is not closed because $z$ depends on time whereas $h$ depends on space, as it represents the random medium. Therefore, write $z(\hat{x}) = z(\hat{t}(\hat{x}))$, which is equivalent as long as time and space have a one–to–one correspondence, i.e. as long as $z > 0$, which is satisfied for $z(0) > 0$ and continuous $h$, as can be seen from equation (2). This gives

$$\frac{dz}{d\hat{x}} = \frac{c}{z} - 1 - \frac{1}{\sqrt{\varepsilon}}\, h\left(\frac{\hat{x}}{\varepsilon}\right). \tag{3}$$

In the following write $t$ and $x$ for $\hat{t}$ and $\hat{x}$. The random medium shall be described by an Ornstein–Uhlenbeck process. That is,

$$dh_x = -\, h_x\, dx + dW_x. \tag{4}$$

This process has almost surely continuous sample paths. It also is a stationary process, which is needed as the statistical properties should not depend on space. Equation (4) gives, for $y^\varepsilon(x) := -h(\frac{x}{\varepsilon})$,

$$dy^\varepsilon_x = -\frac{1}{\varepsilon}\, y^\varepsilon_x\; dx + \frac{1}{\sqrt{\varepsilon}}\, dW_x, \tag{5}$$

which has the formal solution

$$y^\varepsilon(x) = \frac{1}{\sqrt{\varepsilon}} \int_0^x e^{-\frac{1}{\varepsilon}(x-s)}\, \frac{dW}{dx}(s)\; ds.$$

Thus, $\frac{1}{\sqrt{\varepsilon}} y_\varepsilon$ converges formally to $\frac{dW}{dx}$ as $\varepsilon$ tends to zero. This gives a formal limit of equation (3):

$$dz^0_x = \left(\frac{c}{z^0_x} - 1\right) dx + dW_x. \tag{6}$$

Equation (6) is defined on $(0, \infty)$ but, as can be seen by using speed-scale analysis (see e.g. [RY]), for $c < \frac{1}{2}$ the boundary $\{z = 0\}$ can be reached. Therefore equation (6) needs a boundary condition at $\{z = 0\}$. Notice, this boundary condition is only needed for $c < \frac{1}{2}$. Thus the singularity is exactly critical.
The main purpose of this paper is to rigorously derive the limit equation (6) in the sense of its Fokker–Planck equation, as well as to determine the right boundary condition from the limiting procedure. There is no formal argument for the right choice of the boundary condition.
To this end, equations (3) and (5) are combined to the two–dimensional process on $\mathbb{R}\times(0, \infty)$

$$\begin{aligned} dy^\varepsilon_x &= -\frac{1}{\varepsilon}\, y^\varepsilon_x\; dx + \frac{1}{\sqrt{\varepsilon}}\, dW_x \\ dz^\varepsilon_x &= \left(\frac{c}{z^\varepsilon_x} - 1 + \frac{1}{\sqrt{\varepsilon}}\, y^\varepsilon_x\right) dx. \end{aligned} \tag{7}$$

This process exists because the Ornstein–Uhlenbeck process exists. It does not need boundary conditions as the boundary of $\mathbb{R}\times(0,\infty)$ cannot be reached. The Fokker–Planck equation for the (integrable, positive) density $\varphi_\varepsilon$ of this process reads in the (formal) strong formulation as follows:

$$\partial_x\varphi_\varepsilon\ +\ \partial_z\left[\left(\frac{c}{z}-1+\frac{y}{\sqrt{\varepsilon}}\right)\varphi_\varepsilon\right]\ -\ \partial_y\left(\frac{y}{\varepsilon}\varphi_\varepsilon\right)-\frac{1}{2\varepsilon}\partial_y^2\varphi_\varepsilon\ =0$$

on $(0,\infty)\times\mathbb{R}\times(0,\infty)$ with initial condition $\varphi_\varepsilon(0,y,z)=\varphi_A(y,z)$ and the boundary condition $(\frac{c}{z}-1+\frac{y}{\sqrt{\varepsilon}})\varphi_\varepsilon=0$ at $z=0$. In the limit $\varepsilon\to 0$ we want to recover the process in (6). Its associated Fokker–Planck equation is in (formal) strong formulation

$$\partial_x\bar{\varphi}_0\ +\ \partial_z\left[\left(\frac{c}{z}-1\right)\bar{\varphi}_0\right]\ -\ \frac{1}{2}\partial_z^2\bar{\varphi}_0\ =\ 0$$

on $(0,\infty)\times(0,\infty)$ with initial condition $\bar{\varphi}_0(0,z)=\int\varphi_A(0,y,z)dy$. We also derive the boundary condition. To prove this we use the corresponding weak formulations of the Fokker–Planck equations which can be derived rigorously, see [G]. It turns out that in the limiting procedure the natural boundary conditions contained in the weak formulation remain valid. This gives reflecting boundary conditions for (6) where needed, i.e. for $c<\frac{1}{2}$.

In the following, let $\mathbb{R}_+$ and $\mathbb{R}^2_+$ denote the open sets $(0,\infty)$ and $\mathbb{R}\times(0,\infty)$ respectively. We denote by $\mathcal{M}_{loc}(\Gamma)$ the set of all nonnegative measures on $\Gamma$ which have finite mass on bounded subsets of $\Gamma$. For $\varphi_\varepsilon\in\mathcal{M}_{loc}(\mathbb{R}_+\times\mathbb{R}^2_+)$ we write $\int\varphi_\varepsilon\,dydzdx$ for the integral with respect to the measure $\varphi_\varepsilon$. Furthermore, let $C_0^k(\Gamma)$ denote the set of functions from $\Gamma$ to $\mathbb{R}$ with compact support in $\Gamma$ which are $k$–times continuously differentiable. Notice that for compact $\Gamma$ a function $f\in C_0^k(\Gamma)$ does not have to vanish at the boundary of $\Gamma$. Define $C_b^k(\Gamma)$ as the set of functions from $\Gamma$ to $\mathbb{R}$ which are $k$–times continuously differentiable, bounded and have bounded derivatives up to order $k$. Finally denote by $C^{k,1}(\Gamma)$ the set of functions from $\Gamma$ to $\mathbb{R}$ which are $k$–times continuously differentiable, Lipschitz continuous and have Lipschitz continuous derivatives up to order $k$.

## 2. Statement of the result.

THEOREM 1. *Let $c>0$ be constant and for all $\varepsilon>0$ let $\varphi_\varepsilon\in\mathcal{M}_{loc}\big((0,\infty)\times\mathbb{R}\times(0,\infty)\big)$ solve*

(8)
$$\begin{aligned}&-\int_0^\infty\int_{\mathbb{R}^2_+}\varphi_\varepsilon\,\partial_x\xi\ dydzdx\ -\ \int_0^\infty\int_{\mathbb{R}^2_+}\left(\frac{c}{z}-1+\frac{y}{\sqrt{\varepsilon}}\right)\varphi_\varepsilon\ \partial_z\xi\ dydzdx\\ &+\frac{1}{\varepsilon}\int_0^\infty\int_{\mathbb{R}^2_+}y\varphi_\varepsilon\,\partial_y\xi\,dydzdx-\frac{1}{2\varepsilon}\int_0^\infty\int_{\mathbb{R}^2_+}\varphi_\varepsilon\,\partial_y^2\xi\,dydzdx\ =\int_{\mathbb{R}^2_+}\varphi_A(y,z)\,\xi(0,y,z)\,dydz\end{aligned}$$

*for all $\xi\in C_b^2\big([0,\infty)\times\mathbb{R}\times[0,\infty)\big)$ with bounded support in $x$-direction. Let the initial condition $\varphi_A\in C_0^\infty(\mathbb{R}^2_+)$ be independent of $\varepsilon$ and nonnegative with $\int_{\mathbb{R}^2_+}\varphi_A\,dzdy=1$. In addition assume that $\int_0^X\int_{\mathbb{R}^2_+}z^2\varphi_\varepsilon(dydzdx)<\infty$ for any $X>0$.*

*Then, for any $X>0$ there exists a subsequence $\{\varphi_{\varepsilon'}\}_{\varepsilon'>0}$ that converges weakly in the sense of measures on $(0,X)\times\mathbb{R}\times(0,\infty)$:*

$$\varphi_{\varepsilon'}\rightharpoonup\varphi_0\qquad \textit{for } \varepsilon'\to 0.$$

*For every such subsequence $\{\varphi_{\varepsilon'}\}_{\varepsilon'>0}$ the limit quantity $\bar{\varphi}_0 := \int_{\mathbb{R}} \varphi_0 dy \in \mathcal{M}_{loc}\big((0,\infty)\times \mathbb{R}_+\big)$ is uniquely determined by:*

$$
\begin{aligned}
&-\int_0^\infty \int_0^\infty \bar{\varphi}_0\, \partial_x \xi\, dz dx \;-\; \int_0^\infty \int_0^\infty \bar{\varphi}_0\, \left(\frac{c}{z}-1\right) \partial_z \xi\, dz dx \\
&-\int_0^\infty \int_0^\infty \bar{\varphi}_0\; \partial_z^2 \xi\, dz dx \;=\; \int_0^\infty \xi(0,z)\; \bar{\varphi}_A(z)\, dz,
\end{aligned} \tag{9}
$$

*with $\bar{\varphi}_A := \int_{\mathbb{R}} \varphi_A dy$ for all $\xi \in C_b^{2,1}\big([0,\infty)\times(0,\infty)\big)$ with bounded support in the $x$-direction.*

Thus, as discussed in Section 1 Theorem 1 gives convergence of the densities of the process in (7) to the densities of the process in (6). It also includes boundary conditions in equation (9) which come from the limiting procedure. These conditions are needed for $c < \frac{1}{2}$ to uniquely determine the solutions. They are natural boundary conditions which is reflecting boundary conditions for the process in (6).
Nevertheless Theorem 1 is only a partial answer to the question whether the motion exhibits a stick–slip type behaviour. It says that the densities of the limit process in space have no mass at $\{z=0\}$. But to observe stick–slip motion time–dependency is needed. Unfortunately it is not yet clear that the densities of the time–dependent limit process have no mass at $\{z=0\}$. The estimates in the proof of Theorem 1 are not sharp enough to conclude this, see Section 4.
Notice also that from the theorem follows existence of a solution to (9) as (8) has solutions. These are the densities of the process in (7). For details see [G].
As the coefficients in the equations (8) and (9) are smooth in the interior, we expect smooth measures $\varphi_\varepsilon$ respectively $\bar{\varphi}_0$ in the interior, in spite of the singularity at the boundary and the degenerate main part of equation (8). But in this paper nothing is proved in that direction.
The assumption on the existence of the second moments in $z$ for finite $X$ is a technical assumption to make things easier at the less interesting boundary $z \to \infty$. It does not seem to be essential, as there is nothing which drives the process to large $z$.
The proof of Theorem 1 has five steps:

1. Properties of the solutions for fixed $\varepsilon > 0$.
2. Weak convergence in the sense of measures.
3. The weak limit equation in the interior.
4. The weak limit equation at the boundary.
5. Uniqueness of the solution.

The parabolic equation of the $\varepsilon$–problem has no gradient structure and is non symmetrizable. This makes it impossible to work with standard $L^2$–theory. Instead, the proof is mainly based on the right choice of test functions. The main observation in this context is that the dual equation of the Fokker–Planck equation

$$A_\varepsilon^* \xi \;=\; -\left(\frac{c}{z}-1\right)\partial_z \xi \;-\; \frac{y}{\sqrt{\varepsilon}}\,\partial_z \xi \;+\; \frac{y}{\varepsilon}\,\partial_y \xi \;-\; \frac{1}{2\varepsilon}\,\partial_y^2 \xi,$$

has a symmetry. For test functions of the form

$$\xi(y,z) = f(z+\sqrt{\varepsilon} y), \quad \text{with} \quad f : \mathbb{R} \longrightarrow \mathbb{R}$$

the middle two terms as well as the $\varepsilon$–dependence of the coefficients cancel:

$$A_\varepsilon^* \left(f(z+\sqrt{\varepsilon}y)\right) \;=\; -\left(\frac{c}{z}-1\right) f'(z+\sqrt{\varepsilon}y) - \frac{1}{2} f''(z+\sqrt{\varepsilon}y).$$

For $\varepsilon = 0$, this is the main part of the dual equation to (9). A similar symmetry is also used by Perthame in [Pt] for an operator without singularity and without boundary effects.

## 3. Proof of Theorem 1.

### 3.1. Properties of the solutions for fixed $\varepsilon > 0$.

LEMMA 2. *Let $\varepsilon > 0$ be fixed and $\varphi_\varepsilon$ like in Theorem 1. Then*

$$-\int_0^\infty \partial_x \xi(x) \int_{\mathbb{R}^2_+} \varphi_\varepsilon \, dydzdx \;=\; \xi(0),$$

*for all $\xi \in C_0^1\big([0,\infty)\big)$. This also implies $\int_0^X \int_{\mathbb{R}^2_+} \varphi_\varepsilon \; dydzdx = X$ , for all $0 < X < \infty$. Furthermore,*

$$\int\limits_0^\infty \varphi_\varepsilon(x,y,z)dz = \frac{1}{\sqrt{\pi(1-e^{-\frac{2x}{\varepsilon}})}} \int\limits_{\mathbb{R}} \exp\left\{-\frac{(y-e^{-\frac{x}{\varepsilon}}y_0)^2}{1-e^{-\frac{2x}{\varepsilon}}}\right\} \left(\int\limits_0^\infty \varphi_A(y_0,z)dz\right) dy_0.$$

*Proof.* For the first part choose $\xi(x)\cdot 1_{(\mathbb{R}^2_+)}$ as a test function in equation (8). Then the weak $x$–derivative of the mass of $\varphi_\varepsilon$ vanishes and $\varphi_A$ has mass one.
The second part follows because the $y$–component in the process decouples. Therefore, $\int_0^\infty \varphi_\varepsilon dz$ obeys the weak Fokker–Planck equation for the Ornstein–Uhlenbeck process. Green's function for this equation is:

$$p_x(y_0,y) \;=\; \frac{1}{\sqrt{\pi(1-e^{-\frac{2x}{\varepsilon}})}} \;\exp\left\{-\frac{(y-e^{-\frac{x}{\varepsilon}}y_0)^2}{1-e^{-\frac{2x}{\varepsilon}}}\right\}.$$

□

**3.2. Weak convergence in the sense of measures.** To show that there exists a converging subsequence of $\{\varphi_\varepsilon\}_{\varepsilon>0}$ we will show that the sequence is tight, i.e. there is no mass concentrating at $\{z=0\}$ as $\varepsilon \to 0$ and no mass is escaping to infinity. This holds as integrals over $\varphi_\varepsilon$ weighted with diverging functions are bounded independently of $\varepsilon$.

LEMMA 3. *Let $\varphi_\varepsilon$ be as in Theorem 1. Then there exists a constant $C$, such that for all $0 < X < \infty$ and all $\varepsilon > 0$ small enough,*

1. $\displaystyle\int_0^X \int_{\mathbb{R}^2_+} e^{\frac{y^2}{2}} \varphi_\varepsilon \, dydzdx \;\le C\,(1+X)$
2. $\displaystyle\int_0^X \int_{\mathbb{R}^2_+} \frac{c}{z}\varphi_\varepsilon \, e^{-2z-1} dydzdx \;\le C\,(1+X)$
3. $\displaystyle\int_0^X \int_{\mathbb{R}^2_+} z\,\varphi_\varepsilon \, dydzdx \;\le C\,(1+X).$

*Proof.* The boundedness of the first integral follows from Lemma 2, as $\int_0^\infty \varphi_\varepsilon dz$ decreases at $|y| \to \infty$ faster than $e^{-\frac{y^2}{2}}$. At $\{x=0\}$ the integral exists because of the compact support of $\varphi_A$.

For the second integral choose $\xi(x) \cdot e^{h_\varepsilon(y,z)} := \xi(x) \cdot e^{-2z-2\sqrt{\varepsilon}y+\varepsilon y^2}$ with $\xi \in C_0^\infty([0,\infty))$ as a test function in equation (8), to be allowed to do so cut it off at large $|y|$. The introduced error disappears for the cutoff boundary tending to infinity. This is because $e^{h_\varepsilon}$ is smaller than $e^{\frac{y^2}{2}}$ for large $|y|$ and the first integral exists. Testing equation (8) with $\xi \cdot e^{h_\varepsilon}$ gives

$$-\int_0^\infty \int_{\mathbb{R}^2_+} \varphi_\varepsilon\, e^{h_\varepsilon}\, \partial_x \xi \; dydzdx \; + 2\int_0^\infty \int_{\mathbb{R}^2_+} \Big(\frac{c}{z} - 1\Big)\, \varphi_\varepsilon\, e^{h_\varepsilon}\, \xi \; dydzdx$$
$$+\int_0^\infty \int_{\mathbb{R}^2_+} \varphi_\varepsilon \Big[2y^2 - 1 - 2\big(\sqrt{\varepsilon}y - 1\big)^2\Big]\, e^{h_\varepsilon}\, \xi \, dydzdx = \int_{\mathbb{R}^2_+} \varphi_A(y,z)\, e^{h_\varepsilon(y,z)} dydz\, \xi(0).$$

Choosing $\xi = \eta_X$ with $\eta_X$ a smooth, monotone cutoff function which is equal to one on $[0, X]$ and zero on $[X + 1, \infty)$ we get

$$\int_0^X \int_{\mathbb{R}^2_+} \frac{c}{z}\, \varphi_\varepsilon\, e^{h_\varepsilon}\, dydzdx \le 2\int_0^\infty \int_{\mathbb{R}^2_+} \frac{c}{z}\, \varphi_\varepsilon\, e^{h_\varepsilon}\, dydz\, \eta_X\, dx$$
$$= \int_0^\infty \int_{\mathbb{R}^2_+} \varphi_\varepsilon\, e^{h_\varepsilon}\, dydz\, \partial_x \eta_X\, dx \; + \int_{\mathbb{R}^2_+} \varphi_A\, e^{h_\varepsilon}\, dydz$$
$$+ \int_0^\infty \int_{\mathbb{R}^2_+} \varphi_\varepsilon \big(-2\,y^2 + 3 + 2(\sqrt{\varepsilon}y - 1)^2\big)\, e^{h_\varepsilon}\, \eta_X\, dydzdx.$$

As $z > 0$ estimate $e^{-2z-2\sqrt{\varepsilon}y+\varepsilon y^2}$ by $e^{-2\sqrt{\varepsilon}y+\varepsilon y^2}$ and notice that $\partial_x \eta \le 0$ to get

$$\int_0^X \int_{\mathbb{R}^2_+} \frac{c}{z}\, \varphi_\varepsilon\, e^{h_\varepsilon}\, dydzdx \; \le \int_{\mathbb{R}^2_+} \varphi_A\, e^{h_\varepsilon}\, dydz \; + \int_0^{X+1} \int_{\mathbb{R}^2_+} \varphi_\varepsilon \; \big(3 + 2(\sqrt{\varepsilon}y - 1)^2\big)\, e^{-2\sqrt{\varepsilon}y+\varepsilon y^2}\, dydzdx$$

which is bounded by $C(1 + X)$ by the existence of the first integral and the compact support of $\varphi_A$. To get the desired estimate notice that $-2\sqrt{\varepsilon}y + \varepsilon y^2 = (\sqrt{\varepsilon}y - 1)^2 - 1 \ge -1$.

For the third integral we choose $\xi(x) \cdot u_\varepsilon(y,z) := \xi(x) \cdot (z + \sqrt{\varepsilon}y)^2$ with $\xi(x) \in C_0^\infty([0,\infty))$ as a test function in equation (8). To be allowed to do so cut it off at the boundaries $|y|, z \to \infty$. The error between the cut off function and the real one vanishes for large cutoff areas, because of the boundedness of the first integral and the existence of the second moment in $z$, which holds by assumption. This gives

$$-\int_0^\infty \int_{\mathbb{R}^2_+} \varphi_\varepsilon \; (z + \sqrt{\varepsilon}y)^2\, \partial_x \xi \; dydzdx + 2\int_0^\infty \int_{\mathbb{R}^2_+} z\, \varphi_\varepsilon\, \xi\, dydzdx$$
$$= \int_{\mathbb{R}^2_+} \varphi_A(y,z)\, (z + \sqrt{\varepsilon}y)^2\, dydz\, \xi(0) \; + \int_0^\infty \int_{\mathbb{R}^2_+} \Big\{\Big(\frac{c}{z} - 1\Big)\, 2\sqrt{\varepsilon}y + 2c + 1\Big\}\, \varphi_\varepsilon\, dydz\, \xi(x)\, dx.$$

We choose again $\xi = \eta_X$. The right–hand–side is bounded by $C(1 + X)$ because of the compact support of $\varphi_A$ and the boundedness of the first integral, as well as the estimate:

$$\left| 2c \int_0^{X+1} \int_{\mathbb{R}_+^2} \frac{\sqrt{\varepsilon} y}{z} \varphi_\varepsilon \, dydz \; \eta_X \, dx \right| \le 2c \int_0^{X+1} \int_{\mathbb{R}_+^2} \frac{\sqrt{\varepsilon}|y|}{z} \varphi_\varepsilon \, dydzdx$$
$$= 2c \left\{ \int_0^{X+1} \int_{\mathbb{R}} \int_0^{z_0} \frac{\sqrt{\varepsilon}|y|}{z} \varphi_\varepsilon \, dydzdx + \int_0^{X+1} \int_{\mathbb{R}} \int_{z_0}^{\infty} \frac{\sqrt{\varepsilon}|y|}{z} \varphi_\varepsilon \, dydzdx \right\}$$
$$\le 2c \left\{ C \int_0^{X+1} \int_{\mathbb{R}} \int_0^{\infty} \left( \frac{1}{z} \varphi_\varepsilon \; e^{-2z - \sqrt{\varepsilon} y + \varepsilon y^2} \; + \sqrt{\varepsilon}|y| \varphi_\varepsilon \right) dzdydx \right\} \le C\,(1+X)$$

which is true from the estimates of the first two integrals as seen above. This gives

$$\left| - \int_0^{X+1} \int_{\mathbb{R}_+^2} \varphi_\varepsilon \, (z + \sqrt{\varepsilon} y)^2 \, \partial_x \eta_X \, dydzdx \; + \; 2 \int_0^{X+1} \int_{\mathbb{R}_+^2} z \, \varphi_\varepsilon \, \eta_X \, dydzdx \right| \le C\,(1+X).$$

As $\partial_x \eta_X \le 0$ and $\eta_X \ge 0$, we have

$$2 \int_0^X \int_{\mathbb{R}_+^2} z \; \varphi_\varepsilon \, dzdydx \le C\,(1+X).$$

□

The existence of a converging subsequence is now a corollary of the theorem of Prohorov:

Corollary 4. *Given the assumptions of Theorem 1, the sequence of measures* $\{\varphi_\varepsilon\}_{\varepsilon>0}$ *is tight on* $(0,X) \times \mathbb{R} \times (0,\infty)$ *for all* $0 < X < \infty$*. This implies that there exists a subsequence* $\{\varphi_{\varepsilon'}\}_{\varepsilon'>0}$ *and a nonnegative measure* $\varphi_0$ *on* $(0,X) \times \mathbb{R} \times (0,\infty)$ *with*

$$\varphi_{\varepsilon'} \rightharpoonup \varphi_0 \quad \textit{in the sense of measures for} \quad \varepsilon' \to 0.$$

*That is*

$$\int_0^X \int_{\mathbb{R}_+^2} f \; d\varphi_{\varepsilon'} \longrightarrow \int_0^X \int_{\mathbb{R}_+^2} f \; d\varphi_0 \qquad \forall\, f \in C_b\big((0,X) \times \mathbb{R}_+^2\big).$$

*Proof.* A sequence of measures $\{\varphi_\varepsilon\}_{\varepsilon>0}$ is called tight on $(0,X) \times \mathbb{R} \times (0,\infty)$, if there exists for all $\delta > 0$ a compact set $K \subset (0,X) \times \mathbb{R}_+^2$ such that

$$\int\limits_{\{(0,X)\times\mathbb{R}_+^2\}\setminus K} \varphi_\varepsilon \, dydzdx \;\le\; \delta \qquad \forall \varepsilon > 0\,.$$

This is true in our situation. It follows from the estimates proved in Lemma 3 and Lemma 2. There is no mass concentrating outside of any compact set $K$ as $\varepsilon \to 0$, because otherwise the integrals in Lemma 3 would diverge in the limit $\varepsilon \to 0$. The weak convergence in the sense of measures for a subsequence follows from the theorem of Prohorov, see e.g. [Bil]. □

In the following denote the subsequence $\{\varphi_{\varepsilon'}\}_{\varepsilon'>0}$ again by $\{\varphi_\varepsilon\}_{\varepsilon>0}$.

Corollary 5. *Let* $\{\varphi_\varepsilon\}_{\varepsilon>0}$ *be any subsequence converging in the sense of Corollary 4 and* $\varphi_0$ *its weak limit. Then*

$$\int_0^X \int_{\mathbb{R}^2_+} 1_{(0,X)\times\mathbb{R}\times(0,\infty)}\ \varphi_0\ dydzdx \ = \ X, \qquad \forall\ X > 0.$$

*Furthermore, for all* $\xi \in C_0^1\big((0,\infty)\big)$,

$$-\int_0^\infty \int_{\mathbb{R}^2_+} \varphi_0\ dydz\ \partial_x \xi\, dx = \ \xi(0).$$

*There is also no mass close to zero. That is, for all* $0 < X < \infty$,

$$\int_0^X \frac{1}{\delta} \int_\delta^{2\delta} \int_{\mathbb{R}} \varphi_0\ dydzdx \underset{\delta\to 0}{\longrightarrow} 0.$$

*Proof.* The first part follows by Lemma 2 from choosing $1_{(0,X)\times\mathbb{R}^2_+}$ as a test function in the limit in Corollary 4. The second equation states that in this case the weak $x$–derivative of the mass vanishes.
The last limit follows from Lemma 3 and Fatou's lemma by

$$\int_0^X \int_{\mathbb{R}^2_+} \frac{1}{z}\,\varphi_0\, dydzdx \ \le \ \lim_{\alpha\to 0} \int_0^X \int_{\mathbb{R}^2_+} \frac{1}{z+\alpha}\,\varphi_0\, dydzdx$$
$$= \lim_{\alpha\to 0}\ \lim_{\varepsilon\to 0} \int_0^X \int_{\mathbb{R}^2_+} \frac{1}{z+\alpha}\ \varphi_\varepsilon\, dydzdx \ \le \ C\,(1+X).$$

The term $\int_0^X \int_{\mathbb{R}} \int_\delta^{2\delta} \frac{1}{\delta}\,\varphi_0\, dzdydx$ is a part of the above integral and thus has to vanish, as the integral, which is the summation over all these parts, exists. ∎

**3.3. The weak limit equation in the interior.** We now identify the equation satisfied by the limit measure. The proof relies on the symmetry discussed in Section 2.

Lemma 6. *Let* $\{\varphi_\varepsilon\}_{\varepsilon>0}$ *be any subsequence converging in the sense of Corollary 4 and* $\varphi_0$ *its weak limit. Then* $\bar\varphi_0 := \int_{\mathbb{R}} \varphi_0\, dy$ *satisfies*

$$-\int_0^\infty\!\!\int_0^\infty \bar\varphi_0\ \partial_x\xi\ dzdx \ - \int_0^\infty\!\!\int_0^\infty \bar\varphi_0 \left(\frac{c}{z} - 1\right) \partial_z \xi\, dzdx$$
$$-\frac{1}{2}\int_0^\infty\!\!\int_0^\infty \bar\varphi_0\ \partial_z^2 \xi\, dzdx = \int_0^\infty \bar\varphi_A(z)\,\xi(0,z)\ dz,$$

*with* $\bar\varphi_A := \int_{\mathbb{R}} \varphi_A\, dy$ *for all* $\xi \in C_0^{2,1}\big([0,\infty)\times(0,\infty)\big)$.

*Proof.* We start by proving that for all $g \in C_0(\mathbb{R}_+)$ and all $\xi \in C_0^{0,1}\big([0,\infty)\times\mathbb{R}_+\big)$,

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^2_+} \varphi_\varepsilon(x,y,z)\, g(z)\, \xi(x, z+\sqrt{\varepsilon}y)\, dydzdx \\ &\qquad \underset{\varepsilon\to 0}{\longrightarrow} \int_0^\infty \int_0^\infty g(z)\,\xi(x,z) \int_{\mathbb{R}} \varphi_0(x,y,z)\, dy\ dzdx.\end{aligned} \tag{10}$$

Note that this is not immediate as $\xi(x, z+\sqrt{\varepsilon}y)$ is not converging strongly in $C_b$. For $\tilde{x} := \max\{x \,|\, x \in supp\,\xi\}$ holds

$$
\begin{aligned}
&\left| \int_0^\infty \int_{\mathbb{R}_+^2} \varphi_\varepsilon(x,y,z)\ g(z)\ \big(\xi(x,z+\sqrt{\varepsilon}y) - \xi(x,z)\big)\, dydzdx \right| \\
&\quad \le L \cdot \int_0^{\tilde{x}} \int_{\mathbb{R}_+^2} \varphi_\varepsilon(x,y,z)\ |g(z)|\, \sqrt{\varepsilon}\ |y|\ dydzdx \\
&\quad \le L \cdot \sqrt{\varepsilon}\ \cdot \max|g| \int_0^{\tilde{x}} \int_{\mathbb{R}_+^2} \varphi_\varepsilon\,(1+y^2)\, dydzdx \le L \cdot \sqrt{\varepsilon}\ \cdot \max|g| \cdot C\,(1+\tilde{x}),
\end{aligned}
$$

for some Lipschitz constant $L$, by Lemma 3. This implies (10) by the triangle inequality as $\{\varphi_\varepsilon\}_{\varepsilon>0}$ converges weakly.
Next, take an arbitrary function $\xi(x,z) \in C_0^{2,1}\big([0,\infty)\times(0,\infty)\big)$. There exists a cutoff function $\eta \in C_0^\infty\big((0,\infty)\big)$ such that $\eta(z)\cdot 1(x) \equiv 1$ on $supp\ \xi$. Using $\eta(z)\cdot\xi(x,z+\sqrt{\varepsilon}y)$ as a test function in equation (8) and writing $\partial_2$ for the derivative in the second argument of $\xi$, gives by (10) in the limit $\varepsilon \to 0$

(11)

$$
\begin{aligned}
&-\int_0^\infty\int_0^\infty \bar{\varphi}_0(x,z)\ \eta(z)\,\partial_x\xi(x,z)\ dzdx - \int_0^\infty\int_0^\infty \Big(\frac{c}{z}-1\Big)\,\bar{\varphi}_0(x,z)\,\partial_z\eta(z)\,\xi(x,z)\ dzdx \\
&-\int_0^\infty\int_0^\infty \Big(\frac{c}{z}-1\Big)\bar{\varphi}_0(x,z)\,\eta(z)\,\partial_2\xi(x,z)\,dzdx - \frac{1}{2}\int_0^\infty\int_0^\infty \bar{\varphi}_0(x,z)\,\eta(z)\,\partial_2^2\xi(x,z)\,dzdx \\
&-\lim_{\varepsilon\to 0}\left\{\frac{1}{\sqrt{\varepsilon}}\int_0^\infty\int_{\mathbb{R}_+^2} \partial_z\eta(z)\,y\,\varphi_\varepsilon(x,y,z)\,\xi(x,z+\sqrt{\varepsilon}y)\,dydzdx\right\} = \int_0^\infty \bar{\varphi}_A(z)\,\xi(0,z)\,dz.
\end{aligned}
$$

The last term on the left–hand–side vanishes, because $\partial_z\eta$ and $\xi$ have disjoint supports, which gives

$$
\begin{aligned}
&\left|\frac{1}{\sqrt{\varepsilon}}\int_0^\infty \int_{\mathbb{R}_+^2} \partial_z\eta(z)\,y\,\varphi_\varepsilon(x,y,z)\ \xi(x,z+\sqrt{\varepsilon}y)\,dydzdx\right| \\
&= \left|\frac{1}{\sqrt{\varepsilon}}\int_0^\infty\int_{\mathbb{R}_+^2} \partial_z\eta(z)\,y\,\varphi_\varepsilon(x,y,z)\big(\xi(x,z+\sqrt{\varepsilon}y) - \xi(x,z) - \sqrt{\varepsilon}y\,\partial_z\xi(x,z)\big)dydzdx\right| \\
&\le \frac{L}{\sqrt{\varepsilon}} \int_0^{\max\{x\in supp\xi\}} \int_{\mathbb{R}_+^2} \partial_z\eta(z)\,|y|\,\varphi_\varepsilon\ \varepsilon\ y^2\,dzdydx\ \le\ \sqrt{\varepsilon}\cdot C\cdot\big(1+\max\{x\in supp\,\xi\}\big)
\end{aligned}
$$

for a constant $L > 0$, as $\xi \in C_0^{2,1}\big([0,\infty)\times(0,\infty)\big)$ and by Lemma 3. The limit equation follows from (11) by noticing again that $\partial_z\eta$ and $\xi$ have disjoint support and $\eta(z)\cdot 1(x) \equiv 1$ on the support of $\xi$. □

**3.4. The weak limit equation at the boundary.** We now prove natural boundary conditions, i.e. we enlarge the space of test functions to functions which do not vanish at the boundary. To do so we need existence of the integral $\int\int \frac{1}{z}\,\bar{\varphi}_0\,dzdx$.

Lemma 7. *Let $\{\varphi_\varepsilon\}_{\varepsilon>0}$ be any subsequence converging in the sense of Corollary 4 and $\varphi_0$ its weak limit. Let $\bar\varphi_0 := \int_{\mathbb{R}} \varphi_0 \, dy$. Then*

$$\begin{aligned}&-\int_0^\infty\int_0^\infty \bar\varphi_0\,\partial_x\xi\,dzdx - \int_0^\infty\int_0^\infty \bar\varphi_0\left(\frac{c}{z}-1\right)\partial_z\xi\,dzdx\\&-\frac12\int_0^\infty\int_0^\infty \bar\varphi_0\,\partial_z^2\xi\,dzdx = \int_0^\infty \bar\varphi_A(z)\;\xi(0,z)dz,\end{aligned}$$

*with $\bar\varphi_A := \int \varphi_A dy$ for all $\xi \in C_b^{2,1}\big([0,\infty)\times(0,\infty)\big)$ with bounded support in the $x$–direction.*

*Proof.* Take a test function $\xi$ as described above, and cut it off at the boundaries using the following two smooth cut off functions. First, $\eta_\delta$ which has support on $(\delta,\infty)$ and is equal to one on $(2\delta,\infty)$, and second, $\eta_R$ with support on $[0,R+1)$ being equal to one on $[0,R)$. This gives

$$\begin{aligned}&-\int_0^\infty\int_0^\infty \bar\varphi_0\,\partial_x\xi\,\eta_\delta(z)\eta_R(z)\,dzdx - \int_0^\infty\int_0^\infty \bar\varphi_0\left(\frac{c}{z}-1\right)\partial_z(\xi\,\eta_\delta\,\eta_R)\,dzdx\\&-\frac12\int_0^\infty\int_0^\infty \bar\varphi_0\,\partial_z^2(\xi\,\eta_\delta\,\eta_R)dzdx = \int_0^\infty \bar\varphi_A\;\xi(0,z)\eta_\delta\,\eta_R\,dz\end{aligned}\tag{12}$$

by Lemma 6. We let $R\to\infty$ and get that the terms containing derivatives of $\eta_R$ vanish as $\bar\varphi_0$ has finite mass by Corollary 5. Therefore, equation (12) gives

$$\begin{aligned}&-\int_0^\infty\int_0^\infty \bar\varphi_0\,\partial_x\xi\,\eta_\delta\,dzdx - \int_0^\infty\int_0^\infty \bar\varphi_0\Big(\frac{c}{z}-1\Big)\partial_z\xi\,\eta_\delta\,dzdx\\&-\frac12\int_0^\infty\int_0^\infty \bar\varphi_0\,\partial_z^2\xi\,\eta_\delta\,dzdx - \int_0^\infty\int_0^\infty \bar\varphi_0\left(\frac{c}{z}-1\right)\xi\,\partial_z\eta_\delta\,dzdx\\&-\int_0^\infty\int_0^\infty \bar\varphi_0\,\partial_z\xi\,\partial_z\eta_\delta\,dzdx - \frac12\int_0^\infty\int_0^\infty \bar\varphi_0\,\xi\,\partial_z^2\eta_\delta\,dzdx = \int_0^\infty \bar\varphi_A\,\xi(0,z)\eta_\delta\,dz.\end{aligned}\tag{13}$$

From Taylor's formula, $\xi(x,z)=\xi_0(x)+zh(x,z)$ with bounded $h\in C\big([0,\infty)\times(0,\infty)\big)$. Equation (13) gives for $\xi_0$ independent of $z$

$$\begin{aligned}&-\int_0^\infty\int_0^\infty \bar\varphi_0\left(\frac{c}{z}-1\right)\xi_0\,\partial_z\eta_\delta\,dzdx - \frac12\int_0^\infty\int_0^\infty \bar\varphi_0\,\xi_0\,\partial_z^2\eta_\delta\,dzdx\\&=\int_0^\infty \bar\varphi_A\,\eta_\delta(z)\,dz\;\xi_0(0) + \int_0^\infty\int_0^\infty \bar\varphi_0\,\partial_x\xi_0(x)\,\eta_\delta(z)\,dzdx.\end{aligned}$$

The right hand side vanishes in the limit $\delta\to0$ as the integrals for $\eta_0\equiv1$ exist by Corollary 5 and vanish also by Corollary 5. All other terms in equation (13) containing derivatives of $\eta_\delta$ vanish for $\delta\to0$ because

$$\begin{aligned}&\int_0^\infty\int_\delta^{2\delta} \bar\varphi_0\left\{(-ch+zh)\,\partial_z\eta_\delta - \frac12\,z\,h\,\partial_z^2\eta_\delta - \partial_z\xi\;\partial_z\eta_\delta\right\}dzdx\\&\le (1+\max|\partial_z\xi|)\int_0^{\tilde x}\int_\delta^{2\delta} \bar\varphi_0\;dzdx\left(\frac{C}{\delta}+C\right)\end{aligned}$$

which goes to zero by Corollary 5. This gives the desired equation as the remaining terms in equation (13) converge by the monotone convergence theorem because they exist for $1\equiv\eta_0\ge\eta_\delta$ by assumption and by Corollary 5. ☐

**3.5. Uniqueness.** LEMMA 8. *Let $\bar{\varphi}_0 \in \mathcal{M}_{loc}((0,\infty)\times(0,\infty))$ and let for some $c>0$ hold*

$$\begin{aligned}
&-\int_0^\infty\int_0^\infty \bar{\varphi}_0\ \partial_x\xi\ dzdx - \int_0^\infty\int_0^\infty \bar{\varphi}_0\left(\frac{c}{z}-1\right)\partial_z\xi\, dzdx \\
&-\frac{1}{2}\int_0^\infty\int_0^\infty \bar{\varphi}_0\ \partial_z^2\xi\, dzdx \ = \int_0^\infty \bar{\varphi}_A(z)\ \xi(0,z)dz,
\end{aligned} \tag{14}$$

*for all $\xi \in C_b^{2,1}\big([0,\infty)\times(0,\infty)\big)$ with bounded support in the $x$–direction. Let the initial condition be given by a nonnegative $\bar{\varphi}_A \in C_0^\infty(\mathbb{R}_+)$.*
*Then $\bar{\varphi}_0$ is uniquely determined.*

*Proof.* Notice that (14) holds for all $\zeta \in \big\{C^2\big([0,\infty)\times(0,\infty)\big)\cap C_b^1\big([0,\infty)\times[0,\infty)\big)\big\}$ with bounded support in the $x$–direction, which satisfy $|\partial_z^2\zeta| < C + \frac{C}{z}$ for some constant $C$ and which have Lipschitz continuous second derivatives in the interior. To see this look at (14) with the test function $\zeta\cdot\eta_\delta$ for some smooth cut off function $\eta_\delta$. In the limit $\delta\to 0$, i. e. $\eta_\delta \to \eta_0 \equiv 1$, all integrals in (14) converge by the monotone convergence theorem as their limits exist by Corollary 5.
Now look at the dual equation of (14). We prove that

$$\left\{\begin{array}{lll}
\partial_x v \ - \ \left(\frac{c}{z}-1\right)\ \partial_z v \ - \ \frac{1}{2}\partial_z^2 v & = 0 & \text{on } (0,\infty)\times(0,\infty)\\
 & v = u & \text{at } \{x=0\}
\end{array}\right. \tag{15}$$

has a solution $v\in C^2\big([0,\infty)\times(0,\infty)\big)\cap C_b^1\big([0,\infty)\times[0,\infty)\big)$ with $|\partial_z^2 v| < \frac{C}{z}+C$ for some constant $C$ and Lipschitz continuous second derivatives in the interior for every $u\in C_0^\infty\big((0,\infty)\big)$.
To see this, look at the smoothed version of the above equation:

$$\left\{\begin{array}{lll}
\partial_x v_\varepsilon \ - \ \left(\frac{c}{z+\varepsilon}-1\right)\ \partial_z v_\varepsilon \ - \ \frac{1}{2}\partial_z^2 v_\varepsilon & = 0 & \text{on } (0,\infty)\times(0,\infty)\\
 & v_\varepsilon = u & \text{at } \{x=0\}\\
 & \partial_z v_\varepsilon = 0 & \text{at } \{z=0\}\\
 & \partial_z v_\varepsilon \to 0 & \text{for } z\to\infty\,.
\end{array}\right. \tag{16}$$

This has a smooth solution for every $\varepsilon>0$, by standard theory of parabolic partial differential equations. With the maximum principle for equation (16) and the equations satisfied by the derivatives of $v_\varepsilon$ we get

1. $|v_\varepsilon| + |\partial_z v_\varepsilon| + |\partial_x v_\varepsilon| \le C$ on $[0,\infty)\times[0,\infty)$
2. $|\partial_z^2 v_\varepsilon| \le C + \frac{C}{z}$ on $[0,\infty)\times(0,\infty)$
3. $|\partial_x\partial_z v_\varepsilon| + |\partial_z^3 v_\varepsilon| + |\partial_x^2 v_\varepsilon| + |\partial_x\partial_z^2 v_\varepsilon| + |\partial_x^2\partial_z v_\varepsilon| + |\partial_x^3 v_\varepsilon| \le C(\delta)$ on $[0,\infty)\times(\delta,\infty)$.

Thus, the theorem of Arzela–Ascoli in the limit $\varepsilon\to 0$ gives the desired solution.
Now look at the primal equation. We prove uniqueness by looking at the difference, $\varphi$, of two possible solutions of (14). We show that $\varphi$ vanishes as

$$\int_0^\infty \varphi(x_0,z)\ u(z)\ dz \ = 0 \qquad \forall\, x_0 \in\ [0,\infty)$$

and for all $u\in C_0^\infty(\mathbb{R}_+)$.

To this end, pick $u \in C_0^\infty(\mathbb{R}_+)$ and $x_0 \in [0,\infty)$ and look at the time–reversed solution of equation (15): $\bar v(x,z) := v(x_0 - x, z)$. It satisfies

$$\begin{cases} \partial_x \bar v \;+\; \left(\frac{c}{z} - 1\right)\; \partial_z \bar v \;+\; \frac{1}{2}\partial_z^2 \bar v \;=0 & \text{on } (-\infty, x_0)\times(0,\infty) \\ \bar v = u & \text{at } \{x = x_0\}. \end{cases}$$

As $\bar v$ is smooth and compactly supported at $x_0$ it can be $C^2$–extended with compact support up to $x_0 + \delta$ for some $\delta$ small enough. By equation (14) we have for all $\xi_\delta \in C_0^\infty\big([0, x_0+\delta)\big)$

$$\begin{aligned} &- \int_0^{x_0+\delta} \partial_x \xi_\delta(x) \int_0^\infty \varphi(x,z)\, \bar v(x,z)\, dz dx \\ &= \int_0^{x_0} \xi_\delta \int_0^\infty \varphi \underbrace{\left[\partial_x \bar v + \left(\frac{c}{z} - 1\right)\partial_z \bar v + \frac{1}{2}\partial_z^2 \bar v\right]}_{=0} dz dx + \int_{x_0}^{x_0+\delta} \xi_\delta \int_0^\infty \varphi \left[\partial_x \bar v + \left(\frac{c}{z} - 1\right)\partial_z \bar v + \frac{1}{2}\partial_z^2 \bar v\right] dz dx. \end{aligned}$$

The right–hand–side vanishes in the limit $\delta \to 0$ as $\varphi$ has finite mass. Therefore the weak $x$–derivative of $\int_0^\infty \varphi(x,z)\,\bar v(x,z) dz$ vanishes on $[0, x_0]$. But $\int_0^\infty \varphi(0,z)\,\bar v(0,z) dz = 0$ as $\varphi = 0$ at $x = 0$. This implies

$$0 \;=\; \int_0^\infty \varphi(x_0,z)\,\bar v(x_0,z)\; dz \;=\; \int_0^\infty \varphi(x_0,z)\, u(z)\; dz.$$

□

This also completes the proof of Theorem 1, which is a combination of the just proven lemmata.

**4. Conclusion.** Theorem 1 proves the convergence of the probability densities of the two–dimensional stochastic process on $\mathbb{R}\times(0,\infty)$

$$\begin{aligned} dy_x^\varepsilon \;&=\; -\,\frac{1}{\varepsilon}\; y_x^\varepsilon\; dx \;+\; \frac{1}{\sqrt{\varepsilon}}\; dW_x \\ dz_x^\varepsilon \;&=\; \left(\frac{c}{z_x^\varepsilon} \;-\; 1 \;+\; \frac{1}{\sqrt{\varepsilon}}\, y_x^\varepsilon\right) dx \end{aligned}$$

to the densities of the one–dimensional process

$$dz_x^0 \;=\; \left(\frac{c}{z_x^0} \;-\; 1\right) dx \;+\; dW_x$$

on $(0,\infty)$ with reflecting boundary conditions.

As both processes are ergodic (see [G]) this gives a good understanding of the limiting behaviour of the sample paths. Thus, almost surely no path of the limit process gets stuck at the boundary.

However, this is only a partial answer to the problem posed at the beginning. There is still the open question of what happens to the process as a process of time and not of space. This cannot directly be derived from the process of space as for zero velocity time may pass while the motion sits on a single point $x$. The probability density of the time–dependent process $\rho_\varepsilon(t,y,z)$ satisfies

$$\rho_\varepsilon(t,y,z) \;=\; \frac{1}{z}\,\varphi_\varepsilon(x,y,z).$$

This makes the time dependent problem harder to answer. It requires estimates on the integral

$$\int_0^X \int_{\mathbb{R}^2_+} \frac{1}{z^{1+\delta}} \varphi_\varepsilon(x,y,z)\, dydzdx$$

for some $\delta > 0$, to get tightness of the sequence $\{\rho_\varepsilon\}_{\varepsilon>0}$. We do not have these estimates yet. Our method to get estimates is inflexible because the $\frac{1}{z}$ term in the integral for the tightness comes from the coefficients in the equation.

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