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THE GROUP OF HAMILTONIAN HOMEOMORPHISMS AND C^0-SYMPLECTIC TOPOLOGY

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The main purpose of this paper is to carry out some of the foundational study of C^0-*Hamiltonian geometry* and C^0-*symplectic topology*. We introduce the notion of *Hamiltonian topology* on the space of Hamiltonian paths and on the group of Hamiltonian diffeomorphisms. We then define the *group*, denoted by $\mathrm{Hameo}(M, \omega)$, consisting of *Hamiltonian homeomorphisms* such that

$$\mathrm{Ham}(M, \omega) \subsetneq \mathrm{Hameo}(M, \omega) \subset \mathrm{Sympeo}(M, \omega),$$

where $\mathrm{Sympeo}(M, \omega)$ is the *group* of *symplectic homeomorphisms*. We prove $\mathrm{Hameo}(M, \omega)$ is a *normal subgroup* of $\mathrm{Sympeo}(M, \omega)$ and contains all the time-one maps of Hamiltonian vector fields of $C^{1,1}$-functions, and $\mathrm{Hameo}(M, \omega)$ is path-connected and so contained in the identity component $\mathrm{Sympeo}_0(M, \omega)$ of $\mathrm{Sympeo}(M, \omega)$.

We also prove that the *mass flow* of any Hamiltonian homeomorphism vanishes. In the case of a closed orientable surface, this implies that $\mathrm{Hameo}(M, \omega)$ is strictly smaller than the identity component of the group of area-preserving homeomorphisms when $M \neq S^2$. For $M = S^2$, we conjecture that $\mathrm{Hameo}(S^2, \omega)$ is still a proper subgroup of $\mathrm{Sympeo}_0(S^2, \omega)$.

Dedicated to Dusa McDuff

1. Introduction

Let (M, ω) be a connected symplectic manifold. *Unless explicit mention is made to the contrary, M will be closed.* See Section 6 for the necessary changes in the non-compact case or in the case with boundary. Denote by $\mathrm{Symp}(M, \omega)$ the group of symplectic diffeomorphisms i.e., the subgroup of $\mathrm{Diff}(M)$ consisting of diffeomorphisms $\phi : M \to M$ such that $\phi^*\omega = \omega$. We equip $\mathrm{Diff}(M)$ with the C^∞-topology. Then $\mathrm{Symp}(M, \omega)$ forms a closed topological subgroup. We call the induced topology on $\mathrm{Symp}(M,\omega)$ the C^∞-topology on $\mathrm{Symp}(M,\omega)$. We denote by $\mathrm{Symp}_0(M,\omega)$ the path-connected component of the identity in $\mathrm{Symp}(M,\omega)$. The celebrated C^0-rigidity theorem by Eliashberg [**3**, **6**] in symplectic topology states

Theorem 1.1 (C^0-symplectic rigidity, [3]). *The subgroup* $\mathrm{Symp}(M,\omega) \subset \mathrm{Diff}(M)$ *is closed in the* C^0*-topology.*

Therefore it is reasonable to define a *symplectic homeomorphism* as any element of

$$\overline{\mathrm{Symp}(M,\omega)} \subset \mathrm{Homeo}(M),$$

where the closure is taken inside the group $\mathrm{Homeo}(M)$ of homeomorphisms of M with respect to the C^0-topology (or compact-open topology). This closure forms a group and is a topological group with respect to the induced C^0-topology. We refer to Section 2 for the precise definition of the C^0-topology on $\mathrm{Homeo}(M)$.

Definition 1.2 (Symplectic homeomorphism group). We denote the above closure equipped with the C^0-topology by

$$\mathrm{Sympeo}(M,\omega) := \overline{\mathrm{Symp}(M,\omega)},$$

and call this group the *symplectic homeomorphism group*.

We provide two justifications for this definition.

Firstly, it is easy to see that any symplectic homeomorphism preserves the *Liouville measure* induced by the Liouville volume form

$$\Omega = \frac{1}{n!}\omega^n,$$

which is an easy consequence of Fatou's lemma in measure theory. In fact, this measure-preserving property follows from a general fact that the set of measure-preserving homeomorphisms is closed in the group of homeomorphisms under the compact-open topology. In particular in two dimensions, $\mathrm{Sympeo}(M,\omega)$ coincides with $\mathrm{Homeo}^\Omega(M)$, where $\mathrm{Homeo}^\Omega(M)$ is the group of homeomorphisms that preserve the Liouville measure. This follows from the fact that any area-preserving homeomorphism can be C^0-approximated by an area-preserving diffeomorphism in two dimensions (see Theorem 5.1). Secondly, it is easy to see from Eliashberg's rigidity that we have

$$\mathrm{Sympeo}(M,\omega) \subsetneq \mathrm{Homeo}^\Omega(M) \tag{1.1}$$

when $\dim M \geq 4$. In this sense, the symplectic homeomorphism group is a good high dimensional *symplectic* generalization of the group of area-preserving homeomorphisms.

There is another smaller subgroup $\mathrm{Ham}(M,\omega) \subset \mathrm{Symp}_0(M,\omega)$, the *Hamiltonian diffeomorphism group*, which plays a prominent role in many problems in the development of symplectic topology, starting implicitly from

Hamiltonian mechanics and more conspicuously from the Arnold conjecture. One of the purposes of the present paper is to give a precise definition of the C^0-counterpart of $\mathrm{Ham}(M,\omega)$. This requires some lengthy discussion on the Hofer geometry of Hamiltonian diffeomorphisms.

The remarkable Hofer norm of Hamiltonian diffeomorphisms introduced in [**8**, **9**] is defined by

$$\|\phi\| = \inf_{H \mapsto \phi} \|H\|, \tag{1.2}$$

where $H \mapsto \phi$ means that $\phi = \phi_H^1$ is the time-one map of Hamilton's equation

$$\dot{x} = X_H(t,x)$$

where $t \mapsto \phi_H^t$ is the flow of the Hamiltonian vector field X_H associated to the Hamiltonian function $H : [0,1] \times M \to \mathbb{R}$. The norm $\|H\|$ is defined by

$$\|H\| = \int_0^1 \operatorname{osc} H_t \, dt = \int_0^1 \left(\max_{x \in M} H_t(x) - \min_{x \in M} H_t(x) \right) dt. \tag{1.3}$$

This is a version of the $L^{(1,\infty)}$-norm on $C^\infty([0,1] \times M, \mathbb{R})$.

Here (M,ω) is a general symplectic manifold, which may be open or closed. We will always assume that X_H is compactly supported in $\mathrm{Int}(M)$ when M is open so that the flow exists for all time and is supported in $\mathrm{Int}(M)$. For the closed case, we will always assume that the Hamiltonians are normalized by

$$\int_M H_t \, d\mu = 0, \quad \text{for all } t \in [0,1],$$

where $d\mu$ is the Liouville measure. We call such Hamiltonian functions *normalized*. In both cases, there is a one–one correspondence between H and the path $\phi_H \colon t \mapsto \phi_H^t$. There is the L^∞-version of the Hofer norm originally adopted by Hofer [**8**] and defined by

$$\|H\|_\infty := \max_{t \in [0,1]} \operatorname{osc} H_t.$$

Although this L^∞-norm would be easier to handle and enough for most of the geometric purposes in the smooth category, we would like to emphasize that it is important to use the $L^{(1,\infty)}$-norm (1.3) for the purpose of working with the C^0-dynamics: one essential point that distinguishes the $L^{(1,\infty)}$-norm from the L^∞-norm is that the important *boundary flattening procedure* is $L^{(1,\infty)}$-continuous but *not* L^∞-continuous. (See Section 3 and Appendix 2 for more precise remarks.) Recall that this flattening procedure is important for the various constructions involving concatenation in symplectic geometry. Because of this, we adopt the $L^{(1,\infty)}$-norm in our exposition from the beginning.

When we do not explicitly mention otherwise, we always assume that all the functions and diffeomorphisms are smooth. In particular, $\mathrm{Ham}(M,\omega)$

is a subgroup of $\mathrm{Symp}_0(M,\omega)$. Banyaga [**1**] proved that this group is a simple group. Recently Ono [**23**] gave a proof of the C^∞-Flux Conjecture which implies that $\mathrm{Ham}(M,\omega)$ is a closed subgroup of $\mathrm{Symp}_0(M,\omega)$ and locally contractible in the C^∞-topology. The question whether $\mathrm{Ham}(M,\omega)$ is C^0-closed in $\mathrm{Symp}_0(M,\omega)$ is sometimes called the C^0-Flux Conjecture.

The above norm $\|H\|$ can be identified with the Finsler length

$$\mathrm{leng}(\phi_H) = \int_0^1 \Big(\max_{x\in M} H(t,(\phi_H^t)(x)) - \min_{x\in M} H(t,(\phi_H^t)(x)) \Big)\, dt \tag{1.4}$$

of the path $\phi_H : t \mapsto \phi_H^t$, where the Banach norm on $T_{\mathrm{id}}\mathrm{Ham}(M,\omega) \cong C^\infty(M)/\mathbb{R}$ is defined by

$$\|h\| = \mathrm{osc}(h) = \max h - \min h$$

for a normalized function $h : M \to \mathbb{R}$.

Definition 1.3. We call a continuous path $\lambda : [0,1] \to \mathrm{Symp}(M,\omega)$ a (smooth) *Hamiltonian path* if it is generated by the flow of $\dot{x} = X_H(t,x)$ with respect to a smooth Hamiltonian $H : [0,1] \times M \to \mathbb{R}$ (see also Definition A.1). We denote by $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega))$ the set of Hamiltonian paths λ and by $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$ the set of Hamiltonian paths λ that satisfy $\lambda(0) = \mathrm{id}$. We also denote by

$$\mathrm{ev}_1 : \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id}) \to \mathrm{Symp}(M,\omega) \tag{1.5}$$

the evaluation map $\mathrm{ev}_1(\lambda) = \lambda(1) = \phi_H^1$.

For readers' convenience, we will give a precise description of the C^∞-topology on $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$ in Appendix 1. By definition, $\mathrm{Ham}(M,\omega)$ is the set of images of ev_1. We will be mainly interested in the Hamiltonian paths lying in the identity component $\mathrm{Symp}_0(M,\omega)$ of $\mathrm{Symp}(M,\omega)$.

Definition 1.4 (Hofer topology). Consider the metric

$$d_H : \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id}) \to \mathbb{R}_{\geq 0},$$

defined by

$$d_H(\lambda,\mu) := \mathrm{leng}(\lambda^{-1} \circ \mu), \tag{1.6}$$

where $\lambda^{-1} \circ \mu$ is the Hamiltonian path $t \in [0,1] \mapsto \lambda(t)^{-1}\mu(t)$. We call the induced topology on $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$ the *Hofer topology*. We define the Hofer topology on $\mathrm{Ham}(M,\omega)$ to be the strongest topology for which the evaluation map (1.5) is continuous.

It is easy to see that this definition of the Hofer topology on $\mathrm{Ham}(M,\omega)$ coincides with the usual one induced by (1.2), which also shows that the Hofer topology is metrizable. Of course nontriviality of this topology is not a trivial matter, which was proven by Hofer [**8**] for $\mathbb{C}^n$, by Polterovich [**25**] for rational symplectic manifolds and by Lalonde and McDuff [**11**] in its complete generality. It is also immediate to check that the Hofer topology is locally path-connected.

The relation between the Hofer topology on $\mathrm{Ham}(M,\omega)$ and the C^∞-topology or the C^0-topology thereon is rather delicate. However, it is known (see [**25**] and Example 4.2) that the Hofer norm function

$$\phi \in \mathrm{Ham}(M,\omega) \to \|\phi\|$$

is *not* continuous with respect to the C^0-topology in general. We refer to [**9**, **28**] for some results for compactly supported Hamiltonian diffeomorphisms on $\mathbb{R}^{2n}$ in this direction.

The main purpose of this paper is to carry out a foundational study of C^0-Hamiltonian geometry. We first give the precise definition of a topology on the space of Hamiltonian paths with respect to which the spectral invariants for Hamiltonian paths constructed in [**17**–**20**] will all be continuous [**20**]. We then define the notion of *Hamiltonian homeomorphisms* and denote the set thereof by $\mathrm{Hameo}(M,\omega)$. We provide many evidences for our thesis that the Hamiltonian topology is the right topology for the study of topological Hamiltonian geometry. In fact, the notion of Hamiltonian topology has been vaguely present in the literature without much emphasis on its significance ([**9**, **10**, **17**, **31**] for some theorems related to this topology). *However, all of the previous works fell short of constructing a "group" of continuous Hamiltonian maps.* A precise formulation of the topology will be essential in our study of the continuity property of spectral invariants and also in our construction of C^0-symplectic analogs corresponding to various C^∞-objects or invariants. We refer readers to [**20**] for the details of this study.

The following is the C^0-analog to the well-known fact that $\mathrm{Ham}(M,\omega)$ is a normal subgroup of $\mathrm{Symp}_0(M,\omega)$.

Theorem 1.5. $\mathrm{Hameo}(M,\omega)$ *forms a normal subgroup of* $\mathrm{Sympeo}(M,\omega)$.

We also prove

Theorem 1.6. $\mathrm{Hameo}(M,\omega)$ *is path-connected and contained in the identity component of* $\mathrm{Sympeo}(M,\omega)$*, i.e., we have*

$$\mathrm{Hameo}(M,\omega) \subset \mathrm{Sympeo}_0(M,\omega).$$

See Theorems 4.4 and 4.5, respectively. In Section 4, we also prove that all Hamiltonian diffeomorphisms generated by $C^{1,1}$-Hamiltonian functions are contained in $\mathrm{Hameo}(M,\omega)$ and give an example of a Hamiltonian homeomorphism that is not even Lipschitz (see Theorem 4.1 and Example 4.2, respectively). We recall the notion of the *mass flow homomorphism* [**4**, **27**, **30**], which is also called *the mean rotation vector* in the literature on area-preserving maps.

We prove (see Theorems 5.2 and 5.5)

Theorem 1.7. *The values of the mass flow homomorphism with respect to the Liouville measure of* ω *are zero on* $\mathrm{Hameo}(M, \omega)$.

As a corollary to Theorems 1.6–1.7, we prove that in dimension two $\mathrm{Hameo}(M, \omega)$ is strictly smaller than the identity component of the group of area-preserving homeomorphisms if $M \neq S^2$. For the case of S^2, we still conjecture

Conjecture 1.8. *Let* $M = S^2$ *with the standard area form* $\omega = \Omega$. *Then* $\mathrm{Hameo}(S^2, \omega)$ *is a proper subgroup of* $\mathrm{Homeo}_0^\Omega(S^2)$.

It is known (see [**22**], [**29**] for a proof) that any area-preserving homeomorphism can be approximated by smooth area-preserving diffeomorphisms. Combined with this smoothing theorem, one consequence of Conjecture 1.8, combined with normality (Theorem 1.5) and path-connectedness (Theorem 1.6), would be the affirmative answer to the following conjecture. The simpleness question of the group of area-preserving homeomorphisms of S^2 has remained open since the work of Fathi [**4**] appeared.

Conjecture 1.9. $\mathrm{Homeo}_0^\Omega(S^2)$, *the identity component of the group of area-preserving homeomorphisms of* S^2, *is not a simple group.*

We refer to Section 5 for further discussions on the relation between $\mathrm{Hameo}(M, \omega)$ and the simpleness question of the area-preserving homeomorphism group of S^2.

In Section 6, we look at the open case and define the corresponding Hamiltonian topology and the C^0-version of compactly supported Hamiltonian diffeomorphisms.

Finally we have two appendices. In Appendix 1, we provide precise descriptions of the C^∞-topology of $\mathrm{Ham}(M, \omega)$ and that of its path space $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M, \omega), \mathrm{id})$. We also give the proof of the fact that C^∞-continuity of a Hamiltonian path implies the continuity with respect to the Hamiltonian topology. In Appendix 2, we recall the proof of the $L^{(1,\infty)}$-approximation lemma from [**17**] in a more precise form for the readers' convenience.

The senior author is greatly indebted to the graduate students of Madison attending his symplectic geometry course in the fall of 2003. He thanks them for their patience listening to his lectures throughout the semester, which were sometimes erratic in some foundational materials concerning the Hamiltonian diffeomorphism group. The present paper partly grew out of the course. He also thanks J. Franks, J. Mather and A. Fathi for a useful communication concerning the smoothing of area-preserving homeomorphisms. Writing of the original version of this paper has been carried out while the senior author was visiting the Korea Institute for Advanced Study in the winter of 2003. He thanks KIAS for its financial support and excellent research atmosphere.

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During the preparation of the revisions of the current paper, Viterbo [**32**] answered affirmatively to Question 3.16 in the C^0 context on closed manifolds, and subsequently the senior author [**22**] refined Viterbo's scheme and proved the uniqueness of continuous Hamiltonians on open manifolds.

Notations

(1) Unless otherwise stated, H always denotes a *normalized smooth* Hamiltonian function $[0,1] \times M \to \mathbb{R}$, and we always denote by $\|\cdot\|$ the $L^{(1,\infty)}$-norm

$$\|H\| = \int_0^1 \left(\max_{x\in M} H(t,x) - \min_{x \in M} H(t,x) \right) dt.$$

We denote by $C_m^\infty([0,1]\times M, \mathbb{R})$ the space of such functions H with the norm $\|\cdot\|$ and by $L_m^{(1,\infty)}([0,1]\times M,\mathbb{R})$ its completion with respect to $\|\cdot\|$.

(2) Our convention is that ϕ_H always denotes a smooth Hamiltonian path $\phi_H : t \mapsto \phi_H^t$, while ϕ or ϕ_H^t denotes a single diffeomorphism. Unless otherwise stated, $\|\phi\|$ always denotes the Hofer norm (1.2) for $\phi \in \mathrm{Ham}(M,\omega)$.

(3) G_0: the identity path-component of any topological group G.

(4) $\mathrm{Homeo}(M)$: the group of homeomorphisms of M with the C^0-topology. We will often abbreviate composition of maps by $\psi\circ\phi = \psi\phi$.

(5) $\mathcal{P}(G)$, $\mathcal{P}(G, \mathrm{id})$: the space of continuous paths $\lambda : [0,1] \to G$ and the space of continuous paths with $\lambda(0) = \mathrm{id}$, respectively.

(6) $\mathrm{Homeo}^\Omega(M)$: the topological subgroup of $\mathrm{Homeo}(M)$ consisting of measure (induced by the volume form Ω) preserving homeomorphisms of M.

(7) $\mathrm{Symp}(M,\omega)$: the group of symplectic diffeomorphisms with the C^∞-topology.

(8) $\mathrm{Sympeo}(M,\omega)$: the C^0-closure of $\mathrm{Symp}(M,\omega)$ in $\mathrm{Homeo}(M)$.

(9) $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$: the space of smooth Hamiltonian paths $\lambda : [0,1] \to \mathrm{Symp}(M,\omega)$ with $\lambda(0) = \mathrm{id}$.

(10) $\mathrm{Ham}(M,\omega) \subset \mathrm{Symp}_0(M,\omega)$: the subgroup of Hamiltonian diffeomorphisms with the C^∞-topology.

(11) $\mathcal{H}\mathrm{am}(M,\omega)$: $\mathrm{Ham}(M,\omega)$ with the (strong) Hamiltonian topology.

(12) $\mathrm{ev}_1 : \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id}) \to \mathrm{Ham}(M,\omega)$ the evaluation map.

(13) $\mathrm{Hameo}(M,\omega)$: the group of (strong) Hamiltonian homeomorphisms with the C^0-topology.
(14) $\mathcal{H}\mathrm{ameo}(M,\omega)$: $\mathrm{Hameo}(M,\omega)$ with the (strong) Hamiltonian topology.

2. Symplectic homeomorphisms and the mass flow homomorphism

Let (M,ω) be as in the introduction. We fix any Riemannian metric and denote by d the induced Riemannian distance function on M. We denote by $\mathrm{Homeo}_0(M)$ the path-connected component of the identity in $\mathrm{Homeo}(M)$, the group of homeomorphisms of M. Denote by $\mathcal{P}(\mathrm{Homeo}(M),\mathrm{id})$ the set of continuous paths $\lambda : [0,1] \to \mathrm{Homeo}(M)$ with $\lambda(0) = \mathrm{id}$. We denote by d_{C^0} the standard C^0-distance of *maps* defined by

$$d_{C^0}(\phi,\psi) = \max_{x\,\in\, M} \left(d(\phi(x),\psi(x))\right).$$

Then for any two *homeomorphisms* ϕ, $\psi \in \mathrm{Homeo}(M)$, we define their C^0-distance

$$\overline{d}(\phi,\psi) = \max\left\{d_{C^0}(\phi,\psi), d_{C^0}(\phi^{-1},\psi^{-1})\right\}. \tag{2.1}$$

With respect to this, $\mathrm{Homeo}(M)$ becomes a complete metric space. We call the topology induced by $\overline{d}$ the C^0-*topology* on $\mathrm{Homeo}(M)$. It is easy to see that this topology coincides with the compact-open topology. In particular, it does not depend on the choice of the particular Riemannian metric. As defined in Definition 1.1 of the introduction, the symplectic homeomorphism group $\mathrm{Sympeo}(M,\omega)$ is the closure of $\mathrm{Symp}(M,\omega)$ in $\mathrm{Homeo}(M)$ with respect to this topology.

Then for given continuous paths λ, $\mu : [0,1] \to \mathrm{Homeo}_0(M)$ with $\lambda(0) = \mu(0) = \mathrm{id}$, we define their C^0-distance by

$$\overline{d}(\lambda,\mu) := \max_{t\,\in\,[0,1]} \overline{d}(\lambda(t),\mu(t)), \tag{2.2}$$

and call the induced metric topology, the C^0-*topology* on $\mathcal{P}(\mathrm{Homeo}(M),\mathrm{id})$.

If ψ_i is a Cauchy sequence in the C^0-topology converging to a homeomorphism $\psi \in \mathrm{Homeo}(M)$, we will write $\lim_{C^0} \psi_i = \psi$. It is easy to see that $\lim_{C^0} \psi_i^{-1} = \psi^{-1}$ and $\lim_{C^0} \psi_i\phi_i = \psi\phi$ for two sequences $\lim_{C^0} \psi_i = \psi$ and $\lim_{C^0} \phi_i = \phi$. The same observations hold for the complete metric (2.2) for continuous paths. More precisely, let λ_i and $\mu_i \in \mathcal{P}(\mathrm{Homeo}(M),\mathrm{id})$ be two Cauchy sequences of continuous paths. Then there exist *continuous* paths $\lambda = \lim_{C^0} \lambda_i \in \mathcal{P}(\mathrm{Homeo}(M),\mathrm{id})$, $\mu = \lim_{C^0} \mu_i \in \mathcal{P}(\mathrm{Homeo}(M),\mathrm{id})$, and we have $\lim_{C^0} \lambda_i\mu_i = \lambda\mu$ and $\lim_{C^0} \lambda_i^{-1} = \lambda^{-1}$. Here $\lambda^{-1} : [0,1] \to \mathrm{Homeo}(M)$ denotes the path $t \mapsto (\lambda(t))^{-1}$. We will use this frequently in Sections 3 and 4.

Recall that the symplectic form ω induces a measure on M by integrating the volume form

$$\Omega = \frac{1}{n!}\omega^n.$$

We will call the induced measure the *Liouville measure* on M. We denote the Liouville measure by $d\mu = d\mu^\omega$.

The following is an immediate consequence of the well-known fact (see [**4**, Corollary 1.6], for example) that for any given finite Borel measure $d\mu$, the group of measure-preserving homeomorphisms is closed in the above compact-open topology.

Proposition 2.1. *Any symplectic homeomorphism* $h \in \mathrm{Sympeo}(M, \omega)$ *preserves the Liouville measure. More precisely,* $\mathrm{Sympeo}(M, \omega)$ *forms a closed subgroup of* $\mathrm{Homeo}^\Omega(M)$*.*

It is easy to derive from Eliashberg's rigidity theorem the properness of the subgroup $\mathrm{Sympeo}(M, \omega) \subset \mathrm{Homeo}^\Omega(M)$ when $\dim M \geq 4$.

Next we briefly review the construction from [**4**] of the *mass flow homomorphism* for measure-preserving homeomorphism. When considered on an orientable surface, it coincides with the symplectic flux (up to Poincaré duality), and it will be used in Section 5 to prove, when $M \neq S^2$, that $\mathrm{Sympeo}_0(M, \omega)$ is strictly bigger than the group $\mathrm{Hameo}(M, \omega)$ of Hamiltonian homeomorphisms which we will introduce in the next section.

Let Ω be a volume form on M and denote by $\mathrm{Homeo}_0^\Omega(M)$ the path-connected component of the identity in the set of measure (induced by Ω) preserving homeomorphisms with respect to the C^0-topology (or compact-open topology). By Proposition 2.1, we have the inclusion

$$\mathrm{Sympeo}(M, \omega) \subset \mathrm{Homeo}^\Omega(M).$$

We will not be studying this inclusion carefully here except in two dimensions.

For any G one of the above groups, we will denote by $\mathcal{P}(G)$ (respectively, $\mathcal{P}(G, \mathrm{id})$), the space of continuous path from $[0, 1]$ into G (respectively with $c(0) = \mathrm{id}$) with the induced C^0-topology. We denote by $c = (h_t) : [0, 1] \to G$ the corresponding path. Since $\mathrm{Homeo}^\Omega(M)$ is locally contractible [**4**], the universal covering space of $\mathrm{Homeo}_0^\Omega(M)$ is represented by homotopy classes of paths $c \in \mathcal{P}(\mathrm{Homeo}_0^\Omega(M), \mathrm{id})$ with fixed end points. We denote by

$$\pi : \widetilde{\mathrm{Homeo}}_0^\Omega(M) \to \mathrm{Homeo}_0^\Omega(M)$$

the universal covering space and by $[c]$ the corresponding elements. To define the mass flow homomorphism

$$\widetilde{\theta} : \widetilde{\mathrm{Homeo}}_0^\Omega(M) \to H_1(M, \mathbb{R}), \tag{2.3}$$

we use the fact that $H_1(M,\mathbb{R}) \cong \mathrm{Hom}([M,S^1],\mathbb{R})$, where $[M,S^1]$ is the set of homotopy classes of maps from M to S^1.

Denote by $C^0(M,S^1)$ the set of continuous maps $M \to S^1$ equipped with the C^0-topology. Note that $C^0(M,S^1)$ naturally forms a group. Identifying S^1 with $\mathbb{R}/\mathbb{Z}$, write the group law on S^1 additively. Given $c = (h_t) \in \mathcal{P}(\mathrm{Homeo}_0^\Omega(M), \mathrm{id})$, we define a continuous group homomorphism

$$\widetilde{\theta}(c) : C^0(M,S^1) \to \mathbb{R}$$

in the following way: let $f : M \to S^1 = \mathbb{R}/\mathbb{Z}$ be continuous. The homotopy $fh_t - f : M \to S^1$ satisfies $fh_0 - f = 0$, hence we can lift it to a homotopy $\overline{fh_t - f} : M \to \mathbb{R}$ such that $\overline{fh_0 - f} = 0$. Then we define

$$\widetilde{\theta}(c)(f) = \int_M \overline{fh_1 - f}\, d\mu,$$

where $d\mu$ is the given measure on M. This induces a homomorphism

$$\widetilde{\theta} : \mathcal{P}(\mathrm{Homeo}_0^\Omega(M), \mathrm{id}) \to \mathrm{Hom}(C^0(M,S^1),\mathbb{R}). \tag{2.4}$$

One can check that for each given $f \in C^0(M,S^1)$, the assignment $c \mapsto \widetilde{\theta}(c)(f)$ is continuous, i.e., the map (2.4) is *weakly continuous*. Furthermore $\widetilde{\theta}(c)(f)$ depends only on the homotopy class of f, $\widetilde{\theta}(c)$ is a homomorphism, $\widetilde{\theta}(c)$ depends only on the equivalence class of c, and $\widetilde{\theta}$ is a homomorphism [**4**]. Therefore, it induces a group homomorphism (2.3). The weak continuity of (2.4) then implies the continuity of the map (2.3).

If we put

$$\Gamma = \widetilde{\theta}\left(\ker\left(\pi\colon \widetilde{\mathrm{Homeo}_0^\Omega}(M) \to \mathrm{Homeo}_0^\Omega(M)\right)\right),$$

we obtain by passing to the quotient a group homomorphism

$$\theta : \mathrm{Homeo}_0^\Omega(M) \to H_1(M,\mathbb{R})/\Gamma, \tag{2.5}$$

which is also called the *mass flow homomorphism*. The group Γ is shown to be discrete because it is contained in $H_1(M,\mathbb{Z})$ (after normalizing Ω so that $\int_M \Omega = 1$) [**4**, Proposition 5.1].

We summarize the above discussion and some fundamental results by Fathi [**4**] restricted to the case where M is a (smooth) manifold. Note that Fathi equips $\mathcal{P}(\mathrm{Homeo}(M), \mathrm{id})$ with the compact-open topology, while we use the C^0-topology (2.2). It is easy to see that the C^0-topology is stronger than the compact-open topology on the path space $\mathcal{P}(\mathrm{Homeo}(M), \mathrm{id})$, and therefore, Fathi's results also apply to our case.

Theorem 2.2 [4]. *Suppose that M is a closed smooth manifold and Ω is a volume form on M. Then*

(1) $\mathrm{Homeo}^\Omega(M)$ *is locally contractible*;

(2) *the map* $\widetilde{\theta}$ *in* (2.4) *is weakly continuous and* θ *in* (2.5) *is continuous, with respect to the* C^0*-topology;*
(3) *the map* $\widetilde{\theta}$ *in* (2.3) *is surjective, and hence so is* θ*;*
(4) $\ker\theta = [\ker\theta, \ker\theta]$ *is perfect, and* $\ker\theta$ *is simple, if* $n \geq 3$.

The following still remains an open problem concerning the structure of the area-preserving homeomorphism groups in two dimensions (note that since $H_1(S^2, \mathbb{R}) = 0$, we have $\ker\theta = \mathrm{Homeo}_0^{\Omega}(S^2)$).

Question 2.3. Is $\ker\theta$ simple when $n = 2$? In particular, is $\mathrm{Homeo}_0^{\Omega}(S^2)$ a simple group?

3. Definition of Hamiltonian topology and the Hamiltonian homeomorphism group

We start by recalling the following proposition proven by the senior author [**17**] in relation to his study of the length minimizing property of geodesics in Hofer's Finsler geometry on $\mathrm{Ham}(M, \omega)$. This result was the starting point of the senior author's research carried out in this paper.

Proposition 3.1 [17, Lemma 5.1]. *Let* ϕ_{G_i} *be a sequence of smooth Hamiltonian paths and* ϕ_G *be another smooth Hamiltonian path such that*

(1) *each* ϕ_{G_i} *is length minimizing in its homotopy class relative to the end points;*
(2) $\mathrm{leng}(\phi_G^{-1}\phi_{G_i}) \to 0$ *as* $i \to \infty$*;*
(3) *the sequence of Hamiltonian paths* ϕ_{G_i} *converges to* ϕ_G *in the* C^0*-topology.*

Then ϕ_G *is length minimizing in its homotopy class relative to the end points.*

In fact, an examination of the proof of Lemma 5.1 in [**17**] shows that the same holds even without (3). This proposition can be translated into the statement that the length minimizing property of Hamiltonian paths in its homotopy class *relative to the end points* is closed under a certain topology on the space of Hamiltonian paths. In this section, we will first introduce the corresponding topology on the space of Hamiltonian paths. Then using this topology, which we call *Hamiltonian topology*, we will construct the group of *Hamiltonian homeomorphisms*.

We first recall the definition of (C^∞-)Hamiltonian diffeomorphisms (see also Section 1): a C^∞-diffeomorphism ϕ of (M, ω) is C^∞*-Hamiltonian* if $\phi = \phi_H^1$ for a C^∞-function $H : [0, 1] \times M \to \mathbb{R}$. Here ϕ_H^1 is again the time-one map of the Hamilton equation

$$\dot{x} = X_H(t, x).$$

We denote the set of Hamiltonian diffeomorphisms by $\mathrm{Ham}(M,\omega)$, and recall that $\mathrm{Ham}(M,\omega) \subset \mathrm{Symp}_0(M,\omega)$. We will always denote by ϕ_H the corresponding Hamiltonian path $\phi_H : t \mapsto \phi_H^t$ generated by the Hamiltonian H and by $H \mapsto \phi$ when $\phi = \phi_H^1$. In the latter case, we also say that the diffeomorphism ϕ is generated by the Hamiltonian H.

We recall that for two Hamiltonian functions H and K, the product Hamiltonian $H\#K$ is given by the formula

$$(H\#K)_t = H_t + K_t \circ (\phi_H^t)^{-1}, \tag{3.1}$$

and generates the path $\phi_H\phi_K : t \mapsto \phi_H^t\phi_K^t$. And the inverse Hamiltonian $\overline{H}$ corresponding to the inverse path $(\phi_H)^{-1} : t \mapsto (\phi_H^t)^{-1}$ is defined by

$$(\overline{H})_t = -H_t \circ \phi_H^t. \tag{3.2}$$

We also recall that the pull-back Hamiltonian ψ^*H,

$$(\psi^*H)_t = H_t \circ \psi, \tag{3.3}$$

generates the path $\psi^{-1}\phi_H\psi : t \mapsto \psi^{-1}\phi_H^t\psi$ for any $\psi \in \mathrm{Symp}(M,\omega)$. In particular, $\mathrm{Ham}(M,\omega)$ is a normal subgroup of $\mathrm{Symp}(M,\omega)$. We will be mainly interested in paths of the form $\phi_H^{-1}\phi_K$. By the above, this path is generated by $\overline{H}\#K$, and

$$(\overline{H}\#K)_t = -H_t \circ \phi_H^t + K_t \circ \phi_H^t = (K_t - H_t) \circ \phi_H^t. \tag{3.4}$$

Furthermore from the definitions of $\|\cdot\|$ and leng (see (1.3) and (1.4), respectively), we have $\|H\| = \mathrm{leng}\,(\phi_H)$. In particular,

$$\mathrm{leng}(\phi_H^{-1}\phi_K) = \left\|\overline{H}\#K\right\| = \|K - H\|. \tag{3.5}$$

The following simple lemma will be useful later for the calculus of the Hofer length function. The proof of this lemma immediately follows from the definitions and is omitted.

Lemma 3.2. *Let H, $K : [0,1] \times M \to \mathbb{R}$ be smooth. Then we have*

(1) $\mathrm{leng}(\phi_H^{-1}\phi_K) = \mathrm{leng}(\phi_K^{-1}\phi_H)$ *or* $\|\overline{H}\#K\| = \|\overline{K}\#H\|$,
(2) $\mathrm{leng}(\phi_H\phi_K) \leq \mathrm{leng}(\phi_H) + \mathrm{leng}(\phi_K)$ *or* $\|H\#K\| \leq \|H\| + \|K\|$,
(3) $\mathrm{leng}(\phi_H) = \mathrm{leng}(\phi_H^{-1})$ *or* $\|H\| = \|\overline{H}\|$.

In relation to Floer homology and the spectral invariants, one often needs to consider the periodic Hamiltonian functions H satisfying

$$H(t+1,x) = H(t,x).$$

For example, the spectral invariants $\rho(\phi_H;a)$ of the Hamiltonian path $\phi_H : t \mapsto \phi_H^t$ are defined in [**18**] first by reparameterizing the path so that it becomes *boundary flat* (see Definition 3.3 below) and so time-periodic in particular, by applying the Floer homology theory to the Hamiltonian generating the reparameterized Hamiltonian path, and then by proving the resulting spectral invariants are independent of such reparameterization. For this purpose, the senior author used the inequality

$$\int_0^1 -\max(H-K)\,dt \le \rho(\phi_H;a) - \rho(\phi_K;a) \le \int_0^1 -\min(H-K)\,dt$$

in an essential way in [**18**, **19**].

The following basic formula for the Hamiltonian generating a reparameterized Hamiltonian path follows immediately from the definition. It is used for the above purpose and again later in this paper. For a given Hamiltonian function $H : \mathbb{R}\times M \to \mathbb{R}$, not necessarily one-periodic, generating the Hamiltonian path $\lambda = \phi_H$, the reparameterized path

$$t \mapsto \phi_H^{\zeta(t)}$$

is generated by the Hamiltonian function H^ζ defined by

$$H^\zeta(t,x) := \zeta'(t)H(\zeta(t),x)$$

for any smooth function $\zeta : \mathbb{R} \to \mathbb{R}$. Here ζ' denotes the derivative of the function ζ. In relation to the reparameterization of Hamiltonian paths, the following definition will be useful.

Definition 3.3. We call a path $\lambda : [0,1] \to \mathrm{Symp}(M,\omega)$ *boundary flat near* 0 (*near* 1) if λ is constant near $t = 0$ ($t = 1$), and we call the path *boundary flat* if it is constant near $t = 0$ and $t = 1$.

Of course this is the same as saying that any generating Hamiltonian H of λ is constant near the end points. We would like to point out that *the set of boundary flat Hamiltonians is closed under the operations of the product* $(H,K) \mapsto H\#K$ *and taking the inverse* $H \mapsto \overline{H}$ (and similarly for paths that are flat near $t = 0$ or $t = 1$).

We will see in the $L^{(1,\infty)}$-approximation lemma (Appendix 2) that by choosing a suitable ζ so that $\zeta' \equiv 0$ near $t = 0, 1$ any Hamiltonian path can be approximated by a boundary flat one in the Hamiltonian topology which we will introduce later. We would like to emphasize that this approximation cannot be done in the L^∞-norm and that there is no such approximation procedure in the L^∞-topology. This would obstruct the smoothing procedure of concatenated Hamiltonian paths, which is the main reason why we adopt the $L^{(1,\infty)}$-norm, in addition to its natural appearance in Floer theory.

Let $\lambda : [0,1] \to \mathrm{Symp}(M,\omega)$ be a smooth path such that

$$\lambda(t) \in \mathrm{Ham}(M,\omega) \subset \mathrm{Symp}(M,\omega).$$

We know that by definition of $\mathrm{Ham}(M,\omega)$, for each given $s \in [0,1]$, there exists a unique normalized Hamiltonian $H^s = \{H_t^s\}_{0\le t\le 1}$ such that $H^s \mapsto \lambda(s)$. One very important property of a C^∞-path (or C^1 path in general) $\lambda : [0,1] \to \mathrm{Ham}(M,\omega)$ is the following result by Banyaga [**1**].

Proposition 3.4 [1, Proposition II.3.3]. *Let* $\lambda : [0,1] \to \mathrm{Symp}(M,\omega)$ *be a smooth path such that* $\lambda(t) \in \mathrm{Ham}(M,\omega) \subset \mathrm{Symp}(M,\omega)$. *Define the vector field* $\dot\lambda$ *by*

$$\dot\lambda(s) := \frac{\partial \lambda}{\partial s} \circ (\lambda(s))^{-1},$$

and consider the closed one-form $\dot\lambda \rfloor \omega$. *Then this one-form is exact for all* $s \in [0,1]$.

In other words, *any smooth path in* $\mathrm{Symp}(M,\omega)$ *whose image lies in* $\mathrm{Ham}(M,\omega)$ *is Hamiltonian* in the sense of Definition 1.3. Note that this statement does not make sense if the path is not at least C^1 in s, i.e., when we consider a *continuous* path in $\mathrm{Homeo}(M)$ whose image lies in $\mathrm{Ham}(M,\omega)$. As far as we know, it is not known whether one can always approximate a continuous path $\lambda : [0,1] \to \mathrm{Ham}(M,\omega) \subset \mathrm{Symp}_0(M,\omega) \hookrightarrow \mathrm{Homeo}(M)$ by a sequence of smooth Hamiltonian paths. More precisely, it is not known in general whether there is a sequence of smooth Hamiltonian functions $H_j : [0,1] \times M \to \mathbb{R}$ such that the Hamiltonian paths $t \mapsto \phi^t_{H_j}$ uniformly converge to λ.

Not only for its definition but also for many results in the study of the geometry of the Hamiltonian diffeomorphism group, a path being Hamiltonian, not just lying in $\mathrm{Ham}(M,\omega)$, is a crucial ingredient. For that reason, it is reasonable to attempt to preserve this property as one develops *topological Hamiltonian geometry*. Our definition of the Hamiltonian topology in the present paper is the outcome of this attempt.

Obviously there is a one–one correspondence between the set of Hamiltonian paths and that of generating (normalized) Hamiltonians in the smooth category. However, this correspondence gets murkier as the regularity of the Hamiltonian gets worse, say when the regularity is less than $C^{1,1}$. Because of this, we introduce the following terminology for our later discussions.

Definition 3.5. We recall that $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$ denotes the set of (smooth) Hamiltonian paths λ defined on $[0,1]$ satisfying $\lambda(0) = \mathrm{id}$ (see Definitions 1.3 and A.1). Let H be the (unique normalized) Hamiltonian generating a given Hamiltonian path λ. We define two maps

$$\mathrm{Tan},\, \mathrm{Dev} : \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id}) \to C^\infty_m([0,1] \times M, \mathbb{R})$$

by the formulas

$$\mathrm{Tan}(\lambda)(t,x) := H(t, (\phi^t_H)(x)),$$
$$\mathrm{Dev}(\lambda)(t,x) := H(t,x),$$

and call them the *tangent map* and the *developing map*. We call the image of the tangent map Tan the *rolled Hamiltonian* of λ (or of H).

The identity (3.2) implies the identity

$$\mathrm{Tan}(\lambda) = -\mathrm{Dev}(\lambda^{-1}) \tag{3.6}$$

for a general (smooth) Hamiltonian path λ.

The tangent map corresponds to the map of the tangent vectors of the path. Assigning the usual generating Hamiltonian H to a Hamiltonian path corresponds to the *developing map* in Lie group theory: one can “develop” any differentiable path on a Lie group to a path in its Lie algebra using the tangent map and then by right translation.[1]

We also consider the evaluation map

$$\mathrm{ev}_1 : \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id}) \to \mathrm{Symp}(M,\omega), \quad \mathrm{ev}_1(\lambda) = \lambda(1),$$

and the obvious composition of maps

$$\iota_{\mathrm{ham}} : \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id}) \hookrightarrow \mathcal{P}(\mathrm{Symp}(M,\omega),\mathrm{id}) \to \mathcal{P}(\mathrm{Homeo}(M),\mathrm{id}).$$

We next state the following proposition. This proposition is a reformulation of [**10**, Theorem 6, Chapter 5], in our general context, which Hofer and Zehnder proved for compactly supported Hamiltonian diffeomorphisms on $\mathbb{R}^{2n}$. In the presence of the general energy-capacity inequality [**11**], their proof can be easily adapted to our general context. For readers’ convenience, we give the details of the proof here.

Proposition 3.6. *Let $\lambda_i = \phi_{H_i} \in \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$ be a sequence of smooth Hamiltonian paths and $\lambda = \phi_H$ be another smooth path such that*

(1) $\|\overline{H} \# H_i\| \to 0$, *and*
(2) $\mathrm{ev}_1(\lambda_i) = \phi^1_{H_i} \to \psi$ *uniformly to a map* $\psi : M \to M$.

Then we must have $\psi = \phi^1_H$.

Proof. We first note that ψ must be continuous since it is a uniform limit of continuous maps $\phi^1_{H_i}$. Suppose the contrary that $\psi \neq \phi^1_H$, i.e., $(\phi^1_H)^{-1}\psi \neq \mathrm{id}$. Then we can find a small closed ball B such that

$$B \cap \left((\phi^1_H)^{-1}\psi\right)(B) = \emptyset.$$

Since B and hence $\left((\phi^1_H)^{-1}\psi\right)(B)$ is compact and $\phi^1_{H_i} \to \psi$ uniformly, we have

$$B \cap \Big((\phi^1_H)^{-1}\phi^1_{H_i}\Big)(B) = \emptyset$$

for all sufficiently large i. By definition of the Hofer displacement energy e (see [**8**] for the definition), we have $e(B) \leq \|(\phi^1_H)^{-1}\phi^1_{H_i}\|$. Now by the

[1] The senior author would like to take this opportunity to thank A. Weinstein for making this remark almost 9 years ago right after he wrote his first papers [**15**, **16**] on the spectral invariants. Weinstein’s remark answered the questions about the group structure $(\#, -)$ on the space of Hamiltonians and much helped the senior author’s understanding of the group structure at that time.

energy-capacity inequality from [**11**], we know $e(B) > 0$ and hence

$$0 < e(B) \leq \left\|(\phi_H^1)^{-1}\phi_{H_i}^1\right\|$$

for all sufficiently large i. On the other hand, we have

$$\left\|(\phi_H^1)^{-1}\phi_{H_i}^1\right\| \leq \left\|\overline{H}\# H_i\right\| \to 0$$

by hypothesis (1). The last two inequalities certainly contradict each other. That completes the proof. □

What this proposition indicates for the practical purpose is that simultaneously imposing both convergence

$$\|\overline{H}\# H_i\| \longrightarrow 0 \quad \text{and}$$
$$\phi_{H_i}^1 \longrightarrow \phi_H^1 \quad \text{in the } C^0\text{-topology}$$

is consistent in that it gives rise to a nontrivial topology.

We remark that the evaluation map ev_1 is *not* continuous if we equip $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$ with the Hofer topology (Definition 1.4) and Ham (M,ω) with the C^0-topology (and, therefore, Proposition 3.6 is not trivial). If it were, for every sequence H_i such that $\|H_i\| \to 0$, we would have $\phi_{H_i}^1 \to \mathrm{id}$. But, for any pair (x, y) of points $x, y \in M$, it is well-known that there is such a sequence with $\phi_{H_i}^1(x) = y$ for all i: This is because *the transport energy of a point from one place to any other place is always zero*, that is,

$$\inf_H \{\|H\| \mid \phi_H^1(x) = y\} = 0.$$

We will now define the Hamiltonian topology. Its definition is directly motivated by the above Propositions 3.1 and 3.6 (see the remarks after these propositions).

Definition 3.7 (Hamiltonian topology).

(1) We define the *Hamiltonian topology* on the set $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$ of Hamiltonian paths by the one generated by the collection of subsets defined by

$$\mathcal{U}(\phi_H, \epsilon_1, \epsilon_2) := \left\{\phi_{H'} \in \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id}) \middle| \|\overline{H}\# H'\| < \epsilon_1,\ \overline{d}(\phi_H, \phi_{H'}) < \epsilon_2\right\} \tag{3.7}$$

for each choice of $\epsilon_1, \epsilon_2 > 0$ and $\phi_H \in \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$.

(2) We define the *Hamiltonian topology* on $\mathrm{Ham}(M,\omega)$ to be the strongest topology such that the evaluation map

$$\mathrm{ev}_1 : \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id}) \to \mathrm{Ham}(M)$$

is continuous. We denote the resulting topological space by $\mathcal{H}\mathrm{am}(M,\omega)$.

We will call continuous maps with respect to the Hamiltonian topology *Hamiltonian continuous.*

We refer readers to Section 6 for the corresponding definition of Hamiltonian topology either for the non-compact case or the case of manifolds with boundary.

We should now make several remarks concerning our choice of the above definition of the Hamiltonian topology. The combination of the Hofer topology and the C^0-topology in (3.7) will be essential in our study of C^0-analogs to various objects in Hamiltonian geometry and symplectic topology in this paper and in [**20**]. Such a phenomenon was first indicated by Eliashberg [**3**] and partly demonstrated by Viterbo [**31**] and Hofer [**8**, **9**].

We have the following interpretation of the Hamiltonian topology, which will be used later.

By definition, we have the natural continuous maps

(3.8)

$$\iota_{\rm ham}\colon \mathcal{P}^{\rm ham}(\mathrm{Symp}(M,\omega),\mathrm{id}) \to \mathcal{P}(\mathrm{Symp}(M,\omega),\mathrm{id}) \to \mathcal{P}(\mathrm{Homeo}(M),\mathrm{id}),$$
$$\mathrm{Dev}\colon \mathcal{P}^{\rm ham}(\mathrm{Symp}(M,\omega),\mathrm{id}) \to C^\infty_m([0,1]\times M,\mathbb{R}) \to L^{(1,\infty)}_m([0,1]\times M,\mathbb{R}).$$

We call the product map

$$(\iota_{\rm ham},\mathrm{Dev})\colon \mathcal{P}^{\rm ham}(\mathrm{Symp}(M,\omega),\mathrm{id}) \to \mathcal{P}(\mathrm{Symp}(M,\omega),\mathrm{id})\times C^\infty_m([0,1]\times M,\mathbb{R})$$

the *unfolding map*. The Hamiltonian topology on $\mathcal{P}^{\rm ham}(\mathrm{Symp}(M,\omega),\mathrm{id})$ is nothing but the weakest topology for which this unfolding map is continuous.

Here are several other comments.

Remark 3.8.

(1) The way how we define a topology on $\mathrm{Ham}(M,\omega)$ starting from one on the path space $\mathcal{P}^{\rm ham}(\mathrm{Symp}(M,\omega),\mathrm{id})$ is natural since the group $\mathrm{Ham}(M,\omega)$ itself is defined that way. We will repeatedly use this strategy in this paper.

(2) Note that the Hamiltonian topology on $Ham(M,\omega)$ is nothing but the one induced by the evaluation map ev_1.

(3) We also note that the collection of sets (3.7) is symmetric with respect to H and H', i.e., $\phi_{H'}\in\mathcal{U}(\phi_H,\epsilon_1,\epsilon_2) \iff \phi_H\in\mathcal{U}(\phi_{H'},\epsilon_1,\epsilon_2)$.

(4) It is easy to see that for fixed $\phi_H\in\mathcal{P}^{\rm ham}(\mathrm{Symp}(M,\omega),\mathrm{id})$, the open sets (3.7) form a neighborhood basis of the Hamiltonian topology at ϕ_H.

(5) Because of the simple identity

$$(\overline{H}\#H')(t,x) = (H'-H)(t,\phi^t_H(x)),$$

one can write the length in either of the following two ways:

$$\mathrm{leng}(\phi_H^{-1}\phi_{H'}) = \|\overline{H}\#H'\| = \|H'-H\|,$$

if H and H' are smooth (or more generally $C^{1,1}$). In this paper, we will mostly use the first one that manifests the group structure better.

(6) Note that the above identity does not make sense in general even for C^1-functions because their Hamiltonian vector field would be only C^0 and so their flow ϕ_H^t may not exist. Understanding what is going on in such a case touches the heart of C^0-Hamiltonian geometry and dynamics. We will pursue the dynamical issue in [**20**] and focus on the geometry in this paper.

It turns out that $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$ is metrizable. We now define the following natural metric on $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$ which combines the Hofer metric and the C^0-metric appropriately.

Definition 3.9. We define a metric on $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$ by

$$d_{\mathrm{ham}}(\phi_H,\phi_{H'}) = \|\overline{H}\#H'\| + \overline{d}(\phi_H,\phi_{H'}).$$

Proposition 3.10. *The Hamiltonian topology on* $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$ *is equivalent to the metric topology induced by* d_{ham}.

Proof. This is an exercise in using the definitions. Let $\mathcal{U}$ be open in the Hamiltonian topology, and let $\phi_H \in \mathcal{U}$. By Remark 3.8(4), there are $\epsilon_1,\epsilon_2>0$ such that $\mathcal{U}(\phi_H,\epsilon_1,\epsilon_2)\subset\mathcal{U}$. Define $\epsilon=\min(\epsilon_1,\epsilon_2)$. Let

$$\mathcal{U}_\epsilon(\phi_H)=\{\phi_{H'}\in\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id}) \mid d_{\mathrm{ham}}(\phi_H,\phi_{H'})<\epsilon\}$$

be the metric ball of radius ϵ centered at ϕ_H. By our choice for ϵ and by Definitions 3.7(1) and 3.9, we have $\mathcal{U}_\epsilon(\phi_H)\subset\mathcal{U}(\phi_H,\epsilon_1,\epsilon_2)\subset\mathcal{U}$. This holds for any $\phi_H\in\mathcal{U}$, so $\mathcal{U}$ is open in the metric topology.

Conversely, suppose $\mathcal{V}$ is open in the metric topology, and $\phi_H\in\mathcal{V}$. Then $\mathcal{U}_\epsilon(\phi_H)\subset\mathcal{V}$ for some $\epsilon>0$, and $\mathcal{U}(\phi_H,\frac{\epsilon}{2},\frac{\epsilon}{2})\subset\mathcal{U}_\epsilon(\phi_H)\subset\mathcal{V}$. So $\mathcal{V}$ is open in the Hamiltonian topology. □

Proposition 3.11. *The left translations of the group* $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$ *are continuous, i.e., for each* $\lambda\in\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$, *the bijection*

$$L_\lambda:\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})\to\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id}),\quad L_\lambda(\mu)=\lambda\mu,$$

is continuous, and, therefore, a homeomorphism, with respect to the Hamiltonian topology on $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$. *In particular, the sets of the form*

$$\phi_H\left(\mathcal{U}(\mathrm{id},\epsilon_1,\epsilon_2)\right),\quad \epsilon_1,\,\epsilon_2>0, \tag{3.9}$$

form a neighborhood basis at ϕ_H *in* $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$.

Proof. Let $\lambda=\phi_H$. We have to show that $L_\lambda^{-1}(\mathcal{U}(\phi_K,\epsilon_1,\epsilon_2))$ is open for any choice of $\mu=\phi_K$ and $\epsilon_1,\,\epsilon_2>0$. Let $\phi_L\in L_\lambda^{-1}(\mathcal{U}(\phi_K,\epsilon_1,\epsilon_2))$, i.e.,

$$\phi_H\phi_L\in\mathcal{U}(\phi_K,\epsilon_1,\epsilon_2). \tag{3.10}$$

We need to find some $\epsilon_1',\,\epsilon_2'>0$ such that

$$\mathcal{U}(\phi_L,\epsilon_1',\epsilon_2')\subset L_\lambda^{-1}(\mathcal{U}(\phi_K,\epsilon_1,\epsilon_2)),$$

or equivalently, such that

$$L_\lambda(\mathcal{U}(\phi_L,\epsilon_1',\epsilon_2'))=\phi_H(\mathcal{U}(\phi_L,\epsilon_1',\epsilon_2'))\subset\mathcal{U}(\phi_K,\epsilon_1,\epsilon_2). \tag{3.11}$$

For the part of $\overline{d}$, we define

$$\bar\epsilon_2=\epsilon_2-\overline{d}(\phi_H\phi_L,\phi_K)>0 \tag{3.12}$$

by (3.10). By compactness of M, the smooth map $[0,1]\times M\to M, (t,x)\mapsto \phi_H^t(x)$ is in particular uniformly continuous with respect to the standard metric on $[0,1]$ and the metric d on M. Therefore, there exists $0<\epsilon_2'<\bar\epsilon_2$ such that

$$d(x,y)<\epsilon_2'\Longrightarrow d(\phi_H^t(x),\phi_H^t(y))<\bar\epsilon_2$$

for all $x,y\in M$ and all $t\in[0,1]$. Hence if $\overline{d}(\phi_L,\phi_{L'})<\epsilon_2'$, then

$$\begin{aligned}\overline{d}(\phi_H\phi_L,\phi_H\phi_{L'})&=\max\{d_{C^0}(\phi_H\phi_L,\phi_H\phi_{L'}),d_{C^0}(\phi_L^{-1}\phi_H^{-1},\phi_{L'}^{-1}\phi_H^{-1})\}\\&=\max\left\{\max_{(t,x)}d\left(\phi_H^t\phi_L^t(x),\phi_H^t\phi_{L'}^t(x)\right),d_{C^0}(\phi_L^{-1},\phi_{L'}^{-1})\right\}\\&<\max\{\bar\epsilon_2,\epsilon_2'\}=\bar\epsilon_2.\end{aligned}$$

We now estimate

$$\begin{aligned}\overline{d}(\phi_H\phi_{L'},\phi_K)&\le\overline{d}(\phi_H\phi_{L'},\phi_H\phi_L)+\overline{d}(\phi_H\phi_L,\phi_K)\\&<\bar\epsilon_2+\overline{d}(\phi_H\phi_L,\phi_K)=\epsilon_2\end{aligned} \tag{3.13}$$

by (3.12), as long as $\overline{d}(\phi_L,\phi_{L'})<\epsilon_2'$.

On the other hand for the part of $\|\cdot\|$, choose $\epsilon_1'=\epsilon_1-\|H\#L-K\|$, which again is positive by (3.10). It is immediate to check from the definitions that $\|H\#L'-H\#L\|=\|L'-L\|$. Then whenever L' satisfies $\|L'-L\|<\epsilon_1'$, we have by the triangle inequality

$$\|H\#L'-K\|\le\|H\#L'-H\#L\|+\|H\#L-K\|=\|L'-L\|+\|H\#L-K\|<\epsilon_1.$$

That completes the proof of the first statement. Since the inverse of L_λ is the left translation $L_{\lambda^{-1}}$, left translations are in fact homeomorphisms. The last statement is obvious from this and Remark 3.8(4). This finishes the proof. □

As we will see below, $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$ in fact forms a topological group. (Note that we have not yet proved multiplication is continuous in the Hamiltonian topology.) This will follow as a corollary to the fact that its completion $\overline{\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})}$ considered below forms a topological group as well. But we prefer to give an elementary proof of Proposition 3.11 and the following corollaries using only the definitions, and then to complete the discussion of $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$ and $\mathcal{H}\mathrm{am}(M,\omega)$, before dealing with the more complicated arguments involved when considering said completion.

Proposition 3.11 immediately gives rise to the following corollaries.

Corollary 3.12. *The map* $\mathrm{ev}_1 : \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id}) \to \mathcal{H}\mathrm{am}(M,\omega)$ *is an open map with respect to the Hamiltonian topology on* $\mathrm{Ham}(M,\omega)$. *In particular, the following hold:*

(1) *For fixed* $\phi \in \mathrm{Ham}(M,\omega)$ *and* $H \mapsto \phi$, *the sets of the form*

$$\mathrm{ev}_1\Big(\mathcal{U}(\phi_H,\epsilon_1,\epsilon_2)\Big), \quad \epsilon_1,\, \epsilon_2 > 0,$$

form a neighborhood basis at ϕ *in the Hamiltonian topology.*

(2) *For fixed* $\phi \in \mathrm{Ham}(M,\omega)$ *and* $H \mapsto \phi$, *the sets of the form*

$$\phi\Big(\mathrm{ev}_1\Big(\mathcal{U}(\mathrm{id},\epsilon_1,\epsilon_2)\Big)\Big) = \mathrm{ev}_1\Big(\phi_H\Big(\mathcal{U}(\mathrm{id},\epsilon_1,\epsilon_2)\Big)\Big), \quad \epsilon_1,\, \epsilon_2 > 0,$$

also form a neighborhood basis at ϕ *in the Hamiltonian topology.*

Proof. Let $\mathcal{U} \subset \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$ be open in the Hamiltonian topology. We have to show that $\mathrm{ev}_1(\mathcal{U}) \subset \mathcal{H}\mathrm{am}(M,\omega)$ is open with respect to the Hamiltonian topology on $\mathrm{Ham}(M,\omega)$. But by definition of the Hamiltonian topology, $\mathrm{ev}_1(\mathcal{U})$ is open if and only if

$$\mathrm{ev}_1^{-1}\left(\mathrm{ev}_1(\mathcal{U})\right) = \bigcup_\lambda \{\lambda(\mathcal{U}) \mid \lambda \in \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id}), \lambda(0) = \lambda(1) = \mathrm{id}\}$$

is open. But the latter is the union of open sets by Proposition 3.11 and hence itself open. That proves the first part.

Openness and continuity of ev_1 with respect to the Hamiltonian topology together with Remark 3.8(4) now implies (1).

For (2), note that since $\mathrm{Ham}(M,\omega)$ is a group it also acts on itself via left translations. The left translations of $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$ and $\mathrm{Ham}(M,\omega)$ commute with ev_1 in the sense that if $\phi \in \mathrm{Ham}(M,\omega)$ and $H \mapsto \phi$ is any Hamiltonian, then $\mathrm{ev}_1(\phi_H \phi_{H'}) = \phi(\mathrm{ev}_1(\phi_{H'}))$ for any $\phi_{H'} \in \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$. This together with openness and continuity of ev_1 and the last statement of Proposition 3.11 implies (2). □

The following is one indication of good properties of the Hamiltonian topology.

Theorem 3.13. $\mathcal{H}\mathrm{am}(M,\omega)$ *is path-connected and locally path-connected.*

Proof. We first prove that $\mathcal{H}\mathrm{am}(M,\omega)$ is locally path-connected at the identity. Consider the following open neighborhood of the identity element in $\mathcal{H}\mathrm{am}(M,\omega)$

$$\mathcal{U} = \mathrm{ev}_1\Big(\mathcal{U}(\mathrm{id},\epsilon_1,\epsilon_2)\Big)$$

for any $\epsilon_1,\, \epsilon_2 > 0$. Note that by Corollary 3.12, these sets form a neighborhood basis at the identity. So it suffices to prove that $\mathcal{U}$ is path-connected.

Let $\phi_0 \in \mathcal{U}$. We will prove that ϕ_0 can be connected by a continuous path to the identity inside $\mathcal{U}$. Since $\phi_0 \in \mathcal{U}$ there exists $H \mapsto \phi_0$ such that

$$\|H\| < \epsilon_1, \quad \overline{d}(\phi_H, \mathrm{id}) = \sup_{t \in [0,1]} \overline{d}(\phi_H^t, \mathrm{id}) < \epsilon_2.$$

Let H^s be the Hamiltonian generating $t \mapsto \phi_{H^s}^t = \phi_H^{st}$ defined by $H^s(t,x) = sH(st,x)$. We have

$$\overline{d}(\phi_{H^s}, \mathrm{id}) = \sup_{t \in [0,1]} \overline{d}(\phi_{H^s}^t, \mathrm{id}) = \sup_{t \in [0,s]} \overline{d}(\phi_H^t, \mathrm{id}) \leq \sup_{t \in [0,1]} \overline{d}(\phi_H^t, \mathrm{id}) < \epsilon_2.$$

Also note that by substituting $\tau = st$, we get $\|H^s\| \leq \|H\|$. Combining the two, we derive that $\phi_{H^s} \in \mathcal{U}(\mathrm{id}, \epsilon_1, \epsilon_2)$ and hence $\phi_H^s = \phi_{H^s}^1 \in \mathcal{U}$ for all $s \in [0,1]$. Hence the path $\lambda = \phi_H : t \mapsto \phi_H^t$ has its image contained in $\mathcal{U}$, and connects the identity and ϕ_0. Continuity follows from Corollary A.3. So $\mathcal{U}$ is path-connected.

Now let $\phi \in \mathrm{Ham}(M,\omega)$. By Corollary 3.12, the sets $\phi\mathcal{U}$, where $\mathcal{U}$ as above, form a neighborhood basis at ϕ. That they are path-connected follows from their definition and path-connectedness of $\mathcal{U}$. This proves local path-connectedness of $\mathcal{H}\mathrm{am}(M,\omega)$. Path-connectedness of $\mathcal{H}\mathrm{am}(M,\omega)$ follows from its definition (see the remark after Definition A.1) and Corollary A.3. That proves the theorem. □

One crucial advantage of the Hamiltonian topology over the Hofer topology is that it enables one to extend the evaluation map

$$\mathrm{ev}_1 : \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega), \mathrm{id}) \to \mathrm{Ham}(M,\omega)$$

to the completion of $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega), \mathrm{id})$ with respect to the corresponding metric topology. Recall that the evaluation map is *not* continuous if one equips $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega), \mathrm{id})$ with the Hofer topology and $\mathrm{Ham}(M,\omega)$ with the C^0-topology (see the remark after Proposition 3.6). It is also an interesting problem to understand the completion of $\mathrm{Ham}(M,\omega)$ with respect to the Hofer topology, but this is much harder to study, partly because a general element in the completion would not be a continuous map.

We now define the notion of *topological Hamiltonian path*, *topological Hamiltonian function* and *Hamiltonian homeomorphism*. Let (ϕ_i, λ_i, H_i) be a sequence of triples, where $\phi_i \in \mathrm{Ham}(M,\omega)$ are Hamiltonian diffeomorphisms and $H_i \in C_m^\infty([0,1] \times M, \mathbb{R})$ are normalized Hamiltonian functions, such that H_i generates the Hamiltonian path $\lambda_i = \phi_{H_i} : t \mapsto \phi_{H_i}^t$ and $\phi_i = \phi_{H_i}^1 = \lambda_i(1)$. Suppose the sequence is Cauchy in the Hamiltonian topology,

$$\overline{d}\left(\phi_{H_i}, \phi_{H_j}\right) \to 0, \quad \text{as } i,\, j \to \infty,$$

and

$$\|H_i - H_j\| \to 0, \quad \text{as } i,\, j \to \infty.$$

In particular, H_i converges to an $L^{(1,\infty)}$-function $H \in L_m^{(1,\infty)}([0,1] \times M, \mathbb{R})$, and λ_i converges to a continuous path $\lambda \in \mathcal{P}(\mathrm{Homeo}(M), \mathrm{id})$, with

$$\lambda(1) = \lim_{C^0} \phi_i =: h \in \mathrm{Homeo}(M).$$

We call the continuous path λ a *topological Hamiltonian path*, the function H a *topological Hamiltonian function* and the map h a *Hamiltonian homeomorphism.*

More precisely, recall the unfolding map

$$\begin{aligned}(\iota_{\mathrm{ham}}, \mathrm{Dev}) : \mathcal{P}^{\mathrm{ham}}&(\mathrm{Symp}(M,\omega), \mathrm{id}) \\ &\to \mathcal{P}(\mathrm{Symp}(M,\omega), \mathrm{id}) \times C_m^\infty([0,1] \times M, \mathbb{R}) \\ &\to \mathcal{P}(\mathrm{Homeo}(M), \mathrm{id}) \times L_m^{(1,\infty)}([0,1] \times M, \mathbb{R}),\end{aligned}$$

which was defined by $\lambda = \phi_H \mapsto (\lambda, H)$. We denote by $\mathcal{Q}$ the image of $(\iota_{\mathrm{ham}}, \mathrm{Dev})$ equipped with the subspace topology. More precisely, the topology on $\mathcal{Q}$ is induced by the product metric given by the C^0-metric $\overline{d}$ on $\mathcal{P}(\mathrm{Homeo}(M), \mathrm{id})$ and the $L^{(1,\infty)}$-metric on $L_m^{(1,\infty)}([0,1] \times M, \mathbb{R})$. We will refer to this topology on $\mathcal{Q}$ also as the Hamiltonian topology. This will be explained further in Remark 3.17(2) below.

Note that Definition 3.9 implies that both ι_{ham} and Dev are Lipschitz continuous (with $L \leq 1$) with respect to d_{ham} on $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega), \mathrm{id})$, and the C^0-metric $\overline{d}$ on $\mathcal{P}(\mathrm{Homeo}(M), \mathrm{id})$ and the $L^{(1,\infty)}$-metric on $L_m^{(1,\infty)}([0,1] \times M, \mathbb{R})$, respectively. These maps induce natural (Lipschitz continuous) projections from $\mathcal{Q}$ onto the first and second factor, denoted by

$$\begin{aligned}\iota_{\mathrm{ham}}^Q &: \mathcal{Q} \to \mathcal{P}(\mathrm{Symp}(M,\omega), \mathrm{id}) \to \mathcal{P}(\mathrm{Homeo}(M), \mathrm{id}), \\ \mathrm{Dev}^Q &: \mathcal{Q} \to C_m^\infty([0,1] \times M, \mathbb{R}) \to L_m^{(1,\infty)}([0,1] \times M, \mathbb{R}).\end{aligned} \tag{3.14}$$

The map ev_1 is also seen to be Lipschitz continuous (also with $L \leq 1$) with respect to d_{ham} on $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega), \mathrm{id})$ and the C^0-topology on $\mathrm{Ham}(M,\omega) \subset \mathrm{Homeo}(M)$, and hence induces the natural (Lipschitz continuous) map

$$\mathrm{ev}_1^Q : \mathcal{Q} \to \mathrm{Ham}(M,\omega) \subset \mathrm{Homeo}(M), \quad (\lambda, H) \mapsto \lambda(1).$$

We denote by $\overline{\mathcal{Q}}$ the closure of $\mathcal{Q}$ in $\mathcal{P}(\mathrm{Homeo}(M), \mathrm{id}) \times L_m^{(1,\infty)}([0,1] \times M, \mathbb{R})$ with respect to the product metric. By Lipschitz continuity of the above maps, all three maps naturally extend to continuous maps defined on $\overline{\mathcal{Q}}$.

Definition 3.14 (Hamiltonian homeomorphisms). We denote by

$$\overline{\mathrm{ev}}_1^Q : \overline{\mathcal{Q}} \to \mathrm{Homeo}(M), \quad (\lambda, H) \mapsto \lambda(1) \tag{3.15}$$

the natural continuous extension of the evaluation map ev_1^Q. We denote by

$$\mathrm{Hameo}(M,\omega) \subset \mathrm{Homeo}(M)$$

the image of $\overline{\mathcal{Q}}$ under the map $\overline{\mathrm{ev}}_1^Q$ and call any element thereof a *Hamiltonian homeomorphism*, i.e., $h \in \mathrm{Hameo}(M,\omega)$ if and only if there exists a Cauchy sequence (ϕ_{H_i}, H_i) in $\mathcal{Q}$ in the Hamiltonian topology with $h = \lim_{C^0} \phi^1_{H_i}$. We equip $\mathrm{Hameo}(M,\omega)$ with the subspace topology induced from $\mathrm{Homeo}(M)$, i.e., with the C^0-topology. We define the *Hamiltonian topology* on the set $\mathrm{Hameo}(M,\omega)$ to be the strongest topology such that the map $\overline{\mathrm{ev}}_1^Q$ is continuous. We denote by $\mathcal{H}\mathrm{ameo}(M,\omega)$ the resulting topological space. By definition the map

$$\overline{\mathrm{ev}}_1^Q : \overline{\mathcal{Q}} \to \mathcal{H}\mathrm{ameo}(M,\omega) \tag{3.16}$$

is surjective, continuous, and the following diagram commutes

$$\begin{array}{ccc} \mathcal{Q} & \longrightarrow & \mathcal{H}\mathrm{am}(M,\omega) \\ \downarrow & & \downarrow \\ \overline{\mathcal{Q}} & \longrightarrow & \mathcal{H}\mathrm{ameo}(M,\omega), \end{array} \tag{3.17}$$

where the vertical maps are the natural inclusions, and the horizontal maps are the maps induced by the evaluation map.

Definition 3.15 (Topological Hamiltonian paths). We denote by

$$\bar{\iota}^Q_{\mathrm{ham}} : \overline{\mathcal{Q}} \to \mathcal{P}(\mathrm{Homeo}(M), \mathrm{id}), \quad (\lambda, H) \mapsto \lambda,$$

the natural continuous extension of ι^Q_{ham}. By definition of $\mathrm{Sympeo}(M,\omega)$ it follows that the map is factorized into

$$\bar{\iota}^Q_{\mathrm{ham}} : \overline{\mathcal{Q}} \to \mathcal{P}(\mathrm{Sympeo}(M,\omega), \mathrm{id}) \hookrightarrow \mathcal{P}(\mathrm{Homeo}(M), \mathrm{id}).$$

We denote by

$$\mathcal{P}^{\mathrm{ham}}(\mathrm{Sympeo}(M,\omega), \mathrm{id}) \subset \mathcal{P}(\mathrm{Sympeo}(M,\omega), \mathrm{id})$$

the image of the map $\bar{\iota}^Q_{\mathrm{ham}}$ equipped with the subspace topology, i.e., the C^0-topology. We call any element $\lambda \in \mathcal{P}^{\mathrm{ham}}(\mathrm{Sympeo}(M,\omega), \mathrm{id})$ a *topological Hamiltonian path*.

More specifically, a continuous path $\lambda \in \mathcal{P}(\mathrm{Homeo}(M), \mathrm{id})$ is a topological Hamiltonian path if and only if there exists a Cauchy sequence $(\phi_{H_i}, H_i) \in \mathcal{Q}$ in the Hamiltonian topology such that $\lim_{C^0} \phi_{H_i} = \lambda$.

Now we ask the following uniqueness question on the "$L^{(1,\infty)}$-Hamiltonian" concerning the one-oneness of the map $\bar{\iota}^Q_{\mathrm{ham}}$.

Question 3.16. Consider the Cauchy sequences (ϕ_{H_i}, H_i) and $(\phi_{H'_i}, H'_i)$ in the Hamiltonian topology such that $(\phi^t_{H_i})^{-1}(\phi^t_{H'_i}) \to \mathrm{id}$ as $i \to \infty$ uniformly over $[0,1] \times M$. Does this imply $\left\|\overline{H}_i \# H'_i\right\| \to 0$ as $i \to \infty$?

The C^0-(or L^∞-)version of this question has been answered affirmatively by Viterbo [**32**] on closed manifolds, and then subsequently by the senior author [**22**] on open manifolds during the preparation of the current revision of the paper. We refer readers to [**20**, **22**] for the generalization of this uniqueness result in the Lagrangian context and for several other consequences of this uniqueness result.

Here are several remarks.

Remark 3.17.

(1) Similarly, we can define the continuous extension $\overline{\mathrm{Dev}^Q}$ of Dev^Q. The image of this map is by definition the set of topological Hamiltonian functions. These will be studied in a sequel [**20**].
(2) Of course, as topological spaces $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id}) \cong \mathcal{Q}$ via the unfolding map. But it is often more convenient to consider the completion of $\mathcal{Q}$ in $\mathcal{P}(\mathrm{Homeo}(M),\mathrm{id})\times L^{(1,\infty)}_m([0,1]\times M,\mathbb{R})$ rather than the abstract completion $\overline{\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})}$ of $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$, and then dealing with equivalence classes of Cauchy sequences representing elements in $\overline{\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})}$. As topological spaces, $\overline{\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})}$ and $\overline{\mathcal{Q}}$ are homeomorphic by the natural extension of the unfolding map. All statements about $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ can be translated to $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$ and $\overline{\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})}$ by composing all maps with the unfolding map or its inverse, and vice versa.
(3) The way how we define $\mathrm{Hameo}(M,\omega)$ starting from the completion of the path space $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$ is natural since $\mathrm{Ham}(M,\omega)$ itself is defined in a similar way (recall Remark 3.8(1)).

Next recall $\mathrm{Dev}(\phi_H)(t,x) = H(t,x)$ and $\mathrm{Tan}(\phi_H)(t,x) = H(t,(\phi^t_H)(x))$. For convenience, we will often write $H \circ \phi_H$ to denote

$$(H \circ \phi_H)(t,x) = H(t,\phi^t_H(x)) = \mathrm{Tan}(\phi_H)(t,x).$$

Note that from the definitions we immediately get the useful identity

$$\mathrm{leng}\left(\phi_H(\phi_{H'})^{-1}\right) = \|H\#\overline{H'}\| = \|\,\mathrm{Tan}(\phi_H) - \mathrm{Tan}(\phi_{H'})\|. \tag{3.18}$$

Continuity of the maps Dev and Dev^Q is obvious from their definition, but not so that of Tan and Tan^Q. In this regard, we state the following lemma.

Lemma 3.18. *The map*

$$\mathrm{Tan} : \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id}) \to C^\infty_m([0,1]\times M,\mathbb{R})$$

is continuous with respect to the Hamiltonian topology on $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),$ $\mathrm{id})$ *and the* $L^{(1,\infty)}$*-topology on* $C^\infty_m([0,1]\times M,\mathbb{R})$*. The same holds for the map*

$$\mathrm{Tan}^Q : \mathcal{Q} \to C^\infty_m([0,1]\times M,\mathbb{R}), \quad (\lambda,H) \mapsto H\circ\lambda.$$

Proof. Let $\lambda = \phi_H$ be given. Consider another Hamiltonian path $\lambda' = \phi_{H'}$. We have

$$\begin{aligned} \|\operatorname{Tan}(\phi_{H'}) - \operatorname{Tan}(\phi_H)\| &= \|H' \circ \phi_{H'} - H \circ \phi_H\| \\ &\leq \|H' \circ \phi_{H'} - H \circ \phi_{H'}\| + \|H \circ \phi_{H'} - H \circ \phi_H\| \\ &\leq \|H' - H\| + 2L d_{C^0}(\phi_{H'}, \phi_H), \end{aligned} \tag{3.19}$$

where L is a Lipschitz constant that depends only on the smooth function H. It follows from this inequality that Tan is continuous at every $\lambda \in \mathcal{P}^{\text{ham}}(\text{Symp}(M, \omega), \text{id})$ and hence the proof. The proof for Tan^Q is of course the same. □

Since the constant L in (3.19) depends on the Hamiltonian function H, the map Tan is unlikely to be *uniformly continuous*. The constant L cannot be controlled in the Hamiltonian topology, e.g., when we consider a Cauchy sequence (ϕ_{H_i}, H_i) representing a topological Hamiltonian path. This was the source of many erroneous statements and proofs in the previous senior author's own versions of the current paper, many of which are corrected by the junior author in the current version. The crucial lemma to deal with this difficulty is the Reparameterization Lemma 3.21 below.

Very often in the study of the geometry of Hamiltonian diffeomorphisms, one needs to reparameterize a given Hamiltonian path in a way that the reparameterization is close enough to the given parameterization, e.g., in the smoothing process of the concatenation of two paths. We now provide the correct topology describing the closeness of such parameterizations.

Definition 3.19. We call the norm

$$\|f\|_{\text{ham}} := \|f\|_{C^0} + \|f'\|_{L^1}$$

of a (smooth) function $f : [0,1] \to \mathbb{R}$ the *hamiltonian norm* of the function f. Here f' denotes the derivative of the function f. We say that two smooth functions $\zeta_1, \zeta_2 : [0,1] \to [0,1]$ are *hamiltonian-close* to each other if the norm

$$\begin{aligned} \|\zeta_1 - \zeta_2\|_{\text{ham}} &:= \|\zeta_1 - \zeta_2\|_{C^0} + \|\zeta_1' - \zeta_2'\|_{L^1} \\ &= \max_{t \in [0,1]} |\zeta_1(t) - \zeta_2(t)| + \int_0^1 |\zeta_1'(t) - \zeta_2'(t)| \, dt \end{aligned}$$

is small.

Recall that for a given Hamiltonian function H generating the Hamiltonian path ϕ_H, the reparameterized path $t \mapsto \phi_H^{\zeta(t)}$ is generated by the Hamiltonian function H^ζ defined by $H^\zeta(t,x) = \zeta'(t) H(\zeta(t), x)$, where ζ' again denotes the derivative of the reparameterization function $\zeta : [0,1] \to [0,1]$.

Lemma 3.20. *Let $H : [0,1] \times M \to \mathbb{R}$ be a normalized smooth Hamiltonian function, and let ζ_1, $\zeta_2 : [0,1] \to [0,1]$ be two smooth reparameterization* *functions. Then*

$$\|H^{\zeta_1} - H^{\zeta_2}\| \leq C\|\zeta_1 - \zeta_2\|_{\text{ham}}, \tag{3.20}$$

where $C \leq 2\max(\|H\|_{C^0}, L)$ *is a constant that depends only on the* C^0*-norm*

$$\|H\|_{C^0} = \max_{(t,x)} |H(t,x)| < \infty$$

of H *and a Lipschitz constant* (*in the time variable*) L *for* H.

We refer to Appendix 2 for the proof of Lemma 3.20. But note that Lemma 3.20 does *not* hold if we replace the hamiltonian norm by the C^0-norm of $\zeta_1 - \zeta_2$ in (3.20).

We now state the following useful lemma.

Lemma 3.21 (Reparameterization lemma). *Suppose* $H_i : [0,1] \times M \to \mathbb{R}$ *is a Cauchy sequence of smooth functions in the* $L^{(1,\infty)}$*-topology, i.e.,*

$$\|H_i - H_j\| \to 0 \quad \text{as } i, j \to \infty,$$

$\zeta_1, \zeta_2 : [0,1] \to [0,1]$ *are smooth reparameterization functions on* $[0,1]$ *and* λ, $\mu \in \mathcal{P}(\text{Homeo}(M), \text{id})$ *are continuous paths. Let* $\epsilon > 0$ *be given.*

(1) *Then there exist* $\delta = \delta(\{H_i\}) > 0$ *and* $i_0 = i_0(\{H_i\}) > 0$ *such that*

$$\|H_i^{\zeta_1} - H_i^{\zeta_2}\| < \epsilon$$

for all $i \geq i_0$, *if* ζ_1, ζ_2 *satisfy*

$$\|\zeta_1 - \zeta_2\|_{\text{ham}} < \delta.$$

(2) *There exist* $\delta' = \delta'(\{H_i\}) > 0$ *and* $i'_0 = i'_0(\{H_i\}) > 0$ *such that*

$$\|H_i \circ \lambda - H_i \circ \mu\| < \epsilon$$

for all $i \geq i'_0$, *if* λ, μ *satisfy*

$$d_{C^0}(\lambda, \mu) < \delta'.$$

Proof. (1) We can find i_0 sufficiently large such that

$$\|H_i - H_{i_0}\| < \frac{\epsilon}{3} \quad \text{for all } i \geq i_0.$$

Choose $0 < \delta < \frac{\epsilon}{3C}$, where C is as in Lemma 3.20 with H replaced by H_{i_0}. Then

$$\|H_{i_0}^{\zeta_1} - H_{i_0}^{\zeta_2}\| < \frac{\epsilon}{3} \quad \text{when } \|\zeta_1 - \zeta_2\|_{\text{ham}} < \delta.$$

Therefore,

$$\begin{aligned}
\|H_i^{\zeta_1} - H_i^{\zeta_2}\| &\leq \|H_i^{\zeta_1} - H_{i_0}^{\zeta_1}\| + \|H_{i_0}^{\zeta_1} - H_{i_0}^{\zeta_2}\| + \|H_{i_0}^{\zeta_2} - H_i^{\zeta_2}\| \\
&= \|H_i - H_{i_0}\| + \|H_{i_0}^{\zeta_1} - H_{i_0}^{\zeta_2}\| + \|H_{i_0} - H_i\| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,
\end{aligned}$$

when $\|\zeta_1 - \zeta_2\|_{\text{ham}} < \delta$, $i \geq i_0$. That proves (1).

For (2), again choose $i_0' = i_0$ sufficiently large such that

$$\|H_i - H_{i_0}\| < \frac{\epsilon}{3} \quad \text{for all} \quad i \geq i_0.$$

By uniform continuity of H_{i_0}, there exists $\delta' > 0$ such that

$$\|H_{i_0} \circ \lambda - H_{i_0} \circ \mu\|_{C^0} < \frac{\epsilon}{6}$$

when $d_{C^0}(\lambda, \mu) < \delta$. This implies

$$\|H_{i_0} \circ \lambda - H_{i_0} \circ \mu\| < \frac{\epsilon}{3}$$

when $d_{C^0}(\lambda, \mu) < \delta$. Now apply the triangle inequality as above. □

Note that H_i converges to an $L^{(1,\infty)}$-function H, but that we cannot replace H_{i_0} by H in the above proof since H is not even continuous in general.

Proposition 3.22. *There exist continuous maps* $\overline{\mathrm{Tan}^Q}$ *and* $\overline{\mathrm{Dev}^Q}$*, which we again call the tangent map and the developing map, respectively*

$$\overline{\mathrm{Tan}^Q}, \overline{\mathrm{Dev}^Q} : \overline{\mathcal{Q}} \to L_m^{(1,\infty)}([0,1] \times M), \tag{3.21}$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{Q} & \longrightarrow & C_m^\infty([0,1] \times M, \mathbb{R}) \\ \downarrow & & \downarrow \\ \overline{\mathcal{Q}} & \longrightarrow & L_m^{(1,\infty)}([0,1] \times M, \mathbb{R}), \end{array} \tag{3.22}$$

where the vertical maps are the natural inclusions, and the horizontal maps are the tangent and developing maps.

Proof. Since $\overline{\mathrm{Dev}^Q}$ has been already checked before, we will consider only $\overline{\mathrm{Tan}^Q}$. For $\overline{\mathrm{Tan}^Q}$, recall that for any sequence H_i,

$$\begin{aligned} \|\mathrm{Tan}(\phi_{H_i}) - \mathrm{Tan}(\phi_{H_j})\| &= \|H_i \circ \phi_{H_i} - H_j \circ \phi_{H_j}\| \\ &\leq \|H_i \circ \phi_{H_i} - H_j \circ \phi_{H_i}\| + \|H_j \circ \phi_{H_i} - H_j \circ \phi_{H_j}\| \\ &= \|H_i - H_j\| + \|H_j \circ \phi_{H_i} - H_j \circ \phi_{H_j}\|. \end{aligned}$$

Now if (ϕ_{H_i}, H_i) is a Cauchy sequence in the Hamiltonian topology, then the first term converges to zero by definition, and the second term converges to zero by Lemma 3.21(2). So $\mathrm{Tan}(\phi_{H_i})$ converges to an element in $L_m^{(1,\infty)}([0,1] \times M, \mathbb{R})$.

If $(\lambda, H) \in \overline{\mathcal{Q}}$, there exists such a Cauchy sequence (ϕ_{H_i}, H_i) converging to (λ, H) in the Hamiltonian topology. By definition, we set

$$\overline{\mathrm{Tan}^Q}(\lambda, H) = \lim_{i \to \infty} \mathrm{Tan}(\phi_{H_i}).$$

The above discussion shows that the right hand side does not depend on the choice of H_i and so is well defined. It also coincides with the composition $H \circ \lambda$, which is already well defined as an $L^{(1,\infty)}$-function.

To prove continuity, suppose $(\lambda, H) \in \overline{\mathcal{Q}}$ is given, and let $\epsilon > 0$ be given as well. Let $(\lambda', H') \in \overline{\mathcal{Q}}$ be another element. By definition there are sequences (ϕ_{H_i}, H_i) and $(\phi_{H_i'}, H_i')$ converging to (λ, H) and (λ', H'), respectively. We have

$$\begin{aligned}\| \operatorname{Tan}(\phi_{H_i}) - \operatorname{Tan}(\phi_{H_i'})\| &= \|H_i \circ \phi_{H_i} - H_i' \circ \phi_{H_i'}\| \\ &\leq \|H_i \circ \phi_{H_i} - H_i \circ \phi_{H_i'}\| + \|H_i \circ \phi_{H_i'} - H_i' \circ \phi_{H_i'}\| \\ &= \|H_i \circ \phi_{H_i} - H_i \circ \phi_{H_i'}\| + \|H_i - H_i'\|.\end{aligned}$$

By Lemma 3.21, we can find $0 < \delta < \frac{\epsilon}{2}$ and i_0 only depending on the sequence H_i such that if $\|H_i - H_i'\| < \delta$ and $d_{C^0}(\phi_{H_i}, \phi_{H_i'}) < \delta$ for sufficiently large i, say $i \geq N$, then

$$\| \operatorname{Tan}(\phi_{H_i}) - \operatorname{Tan}(\phi_{H_i'})\| \leq \|H_i \circ \phi_{H_i} - H_i \circ \phi_{H_i'}\| + \|H_i - H_i'\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $i \geq \max\{i_0, N\}$. By taking the limit as $i \to \infty$, this implies

$$\|\overline{\operatorname{Tan}^Q}(\lambda, H) - \overline{\operatorname{Tan}^Q}(\lambda', H')\| < \epsilon \quad \text{when } \overline{d}(\lambda, \mu) + \|H - H'\| < \delta,$$

proving that $\overline{\operatorname{Tan}^Q}$ is continuous at (λ, H). □

The images of $\overline{\operatorname{Tan}^Q}$ and $\overline{\operatorname{Dev}^Q}$ contain $C^\infty_m([0,1] \times M, \mathbb{R})$. This is because for any given $F \in C^\infty_m([0,1] \times M, \mathbb{R})$, we have the formula

$$F = \operatorname{Dev}(\phi_F) = -\operatorname{Tan}(\phi_F^{-1}) \tag{3.23}$$

by (3.6). In fact we will see in Theorem 4.1 that $\operatorname{Im} \overline{\operatorname{Dev}^Q}$ and $\operatorname{Im} \overline{\operatorname{Tan}^Q}$ both contain $C^{1,1}([0,1] \times M, \mathbb{R})$. We do not know whether the images of the maps

$$\overline{\operatorname{Tan}^Q}, \overline{\operatorname{Dev}^Q} : \overline{\mathcal{Q}} \to L^{(1,\infty)}_m([0,1] \times M, \mathbb{R})$$

contain the whole $C^0_m([0,1] \times M, \mathbb{R})$. Some of these questions will be studied in [**20**].

The power of our definition of the Hamiltonian topology using the sets (3.7) manifests itself in the proof of the following theorem.

Theorem 3.23. *The set $\overline{\mathcal{Q}}$ forms a topological group.*

Proof. We first have to show that composition and inverses on $\overline{\mathcal{Q}}$ are defined. The other group properties will follow immediately. We then show that composition and inverse operation are continuous.

Let (λ, H) and $(\mu, F) \in \overline{\mathcal{Q}}$. By definition, there are sequences (ϕ_{H_i}, H_i) and (ϕ_{F_i}, F_i) converging to (λ, H) and (μ, F), respectively in the Hamiltonian topology. In particular,

(1) both satisfy

$$\|H - H_i\|, \quad \|F - F_i\| \to 0 \text{ as } i \to \infty, \tag{3.24}$$

(2) and

$$\overline{d}(\lambda, \phi_{H_i}) \to 0, \quad \overline{d}(\mu, \phi_{F_i}) \to 0 \text{ as } i \to \infty. \tag{3.25}$$

We know by our earlier remark about $\overline{d}$ that

$$\overline{d}(\lambda\mu, \phi_{H_i}\phi_{F_i}) \to 0 \text{ as } i \to \infty. \tag{3.26}$$

Moreover, we recall

$$H_i \# F_i = H_i + F_i \circ (\phi_{H_i})^{-1},$$

and this Hamiltonian generates $\phi_{H_i}\phi_{F_i}$. By assumption, we have $\|H_i - H\| \to 0$. On the other hand, we derive

$$\begin{aligned} \|F_i \circ \phi_{H_i}^{-1} - F \circ \lambda^{-1}\| &\leq \|F_i \circ \phi_{H_i}^{-1} - F_i \circ \lambda^{-1}\| + \|F_i \circ \lambda^{-1} - F \circ \lambda^{-1}\| \\ &= \|F_i \circ \phi_{H_i}^{-1} - F_i \circ \lambda^{-1}\| + \|F_i - F\|. \end{aligned}$$

Here the first term converges to zero by Lemma 3.21 and the second does by assumption. We therefore have

$$H_i \# F_i \to H + F \circ \lambda^{-1} \tag{3.27}$$

in the $L^{(1,\infty)}$-topology as $i \to \infty$ under the assumptions (3.24) and (3.25).

Therefore, if we define the $L^{(1,\infty)}$-function $H \# F$ by

$$H \# F := H + F \circ \lambda^{-1},$$

(3.26) and (3.27) imply that the pair $(\lambda\mu, H\#F)$ is the limit of the sequence

$$(\phi_{H_i \# F_i}, H_i \# F_i),$$

and so lies in $\overline{Q}$ again. And the above proof also shows that this limit does not depend on the choices of H_i, F_i but depends only on (λ, H) and (μ, F).

Now we define the product of (λ, H) and (μ, F) by

$$(\lambda, H) \circ (\mu, F) := (\lambda\mu, H\#F). \tag{3.28}$$

When restricted to $\mathcal{Q}$, this obviously agrees with the usual definition of composition.

For the inverse, let (λ, H) be as above. We know that

$$\overline{d}\left(\lambda^{-1}, (\phi_{H_i})^{-1}\right) \to 0 \text{ as } i \to \infty. \tag{3.29}$$

Moreover, by the same proof as for the product, we see that

$$\lim_{i\to\infty} \overline{H_i} = -H \circ \lambda. \tag{3.30}$$

(One can also prove this by recalling $\overline{H_i} = -\overline{\mathrm{Tan}^Q}(\phi_{H_i})$ and then using the continuity of $\overline{\mathrm{Tan}^Q}$ from Proposition 3.22.) We now define

$$\overline{H} := -H \circ \lambda, \tag{3.31}$$

which also coincides with the limit (3.30) for any sequence H_i satisfying $\|H - H_i\| \to 0$ and $\overline{d}(\lambda, \phi_{H_i}) \to 0$. Then we define the inverse by

$$(\lambda, H)^{-1} := (\lambda^{-1}, \overline{H}). \tag{3.32}$$

When restricted to $\mathcal{Q}$, this again agrees with the usual definition of the inverse.

This proves that $\overline{\mathcal{Q}}$ forms a group under $\circ$, and it is straightforward to check that all group axioms are satisfied.

We now have to show that the group operations in $\overline{\mathcal{Q}}$ are continuous, i.e., that the maps

$$\begin{array}{rcl} \overline{\mathcal{Q}} \times \overline{\mathcal{Q}} \to \overline{\mathcal{Q}}, & ((\lambda, H), (\mu, F)) & \mapsto (\lambda\mu, H\#F), \\ \overline{\mathcal{Q}} \quad \to \overline{\mathcal{Q}}, & (\lambda, H) & \mapsto (\lambda^{-1}, \overline{H}) \end{array}$$

are continuous with respect to the metric $\overline{d} + \|\cdot\|$.

For the composition, suppose we have two sequences (λ_i, H_i') and $(\mu_i, F_i') \in \overline{\mathcal{Q}}$ converging to (λ, H) and (μ, F) in the metric $\overline{d} + \|\cdot\|$ on $\overline{\mathcal{Q}}$, respectively. We have to show that

$$\overline{d}(\lambda\mu, \lambda_i\mu_i) \to 0 \text{ as } i \to \infty,$$

and

$$\|H_i'\#F_i' - H\#F\| \to 0 \quad \text{as } i \to \infty.$$

The C^0-convergence is again immediate. For the $\|\cdot\|$-convergence, we compute

$$\begin{aligned} \|H_i'\#F_i' - H\#F\| &= \|H_i' + F_i' \circ \lambda_i^{-1} - H - F \circ \lambda^{-1}\| \\ &\le \|H_i' - H\| + \|F_i' \circ \lambda_i^{-1} - F \circ \lambda^{-1}\| \\ &\le \|H_i' - H\| + \|F_i' \circ \lambda_i^{-1} - F \circ \lambda_i^{-1}\| \\ &\quad + \|F \circ \lambda_i^{-1} - F \circ \lambda^{-1}\| \\ &= \|H_i' - H\| + \|F_i' - F\| + \|F \circ \lambda_i^{-1} - F \circ \lambda^{-1}\|. \end{aligned}$$

The first two terms converge to zero by assumption. For the third term, we derive, with F_i smooth Hamiltonians as above,

$$\begin{aligned} \|F \circ \lambda_i^{-1} - F \circ \lambda^{-1}\| &\le \|F \circ \lambda_i^{-1} - F_i \circ \lambda_i^{-1}\| + \|F_i \circ \lambda_i^{-1} - F_i \circ \lambda^{-1}\| \\ &\quad + \|F_i \circ \lambda^{-1} - F \circ \lambda^{-1}\| \\ &= \|F - F_i\| + \|F_i \circ \lambda_i^{-1} - F_i \circ \lambda^{-1}\| + \|F_i - F\|. \end{aligned}$$

The first and the third terms converge to zero by assumption and the second term by assumption and Lemma 3.21. That proves continuity of composition.

For the inverse, $\overline{d}(\lambda^{-1}, \lambda_i^{-1}) \to 0$. Moreover, it is immediate to check that as in the smooth case (3.18) we have

$$\|\overline{H_i} - \overline{H}\| = \|\overline{\mathrm{Tan}^Q}(\lambda) - \overline{\mathrm{Tan}^Q}(\lambda_i)\| \to 0$$

by continuity of $\overline{\mathrm{Tan}^Q}$. This completes the proof. □

Corollary 3.24. *The set* $\mathcal{Q} \subset \overline{\mathcal{Q}}$ *forms a topological subgroup.*

Proof. $\mathcal{Q}$ is a topological subspace of $\overline{\mathcal{Q}}$ by definition of the latter, and the proof of Theorem 3.23 implies that $\mathcal{Q}$ is a subgroup. □

Corollary 3.25. *The evaluation map*

$$\overline{\mathrm{ev}}_1^Q : \overline{\mathcal{Q}} \to \mathcal{H}\mathrm{ameo}(M, \omega)$$

is an open map. The set $\mathcal{H}\mathrm{ameo}(M, \omega)$ *forms a topological group under composition. In particular,* $\mathrm{Hameo}(M, \omega) \subset \mathrm{Homeo}(M)$ *forms a topological subgroup of* $\mathrm{Homeo}(M)$.

Proof. Theorem 3.23 in particular implies that left multiplication by an element in $\overline{\mathcal{Q}}$ is a continuous map $\overline{\mathcal{Q}} \to \overline{\mathcal{Q}}$. By definition, the topology on $\mathcal{H}\mathrm{ameo}(M, \omega)$ is the strongest topology on the set $\mathrm{Hameo}(M, \omega)$ such that the above evaluation map $\overline{\mathrm{ev}}_1^Q$ is continuous. The proof of openness of $\overline{\mathrm{ev}}_1^Q$ is now the same as the one for ev_1 in Corollary 3.12.

The surjective map

$$\overline{\mathrm{ev}}_1^Q : \overline{\mathcal{Q}} \to \mathcal{H}\mathrm{ameo}(M, \omega)$$

induces a group structure on $\mathcal{H}\mathrm{ameo}(M, \omega)$ in the obvious way. In fact, composition in this group is just the usual composition of maps. The map $\overline{\mathrm{ev}}_1^Q$ becomes a homomorphism of (abstract) groups, which is open, continuous and surjective. From this it is straightforward to check that $\mathcal{H}\mathrm{ameo}(M, \omega)$ indeed forms a topological group.

Since as sets $\mathrm{Hameo}(M, \omega)$ coincides with $\mathcal{H}\mathrm{ameo}(M, \omega)$, $\mathrm{Hameo}(M, \omega)$ forms a group as well. It is immediate that $\mathrm{Hameo}(M, \omega)$ with this group structure forms a topological subgroup of $\mathrm{Homeo}(M)$. □

We now define the notion of topological Hamiltonian fiber bundles.

Definition 3.26 (Topological Hamiltonian bundle). We call a topological fiber bundle $P \to B$ with fiber (M, ω) a topological Hamiltonian bundle if its structure group can be reduced to the group $\mathrm{Hameo}(M, \omega)$. More precisely, $P \to B$ is a topological Hamiltonian bundle if it allows a trivializing chart $\{(U_\alpha, \Phi_\alpha)\}$ such that its transition maps are contained in $\mathrm{Hameo}(M, \omega)$.

Recall that in the smooth case, this definition coincides with that of a symplectic fiber bundle that carries a fiber-compatible *closed* 2-form when either the fiber or the base is simply connected [**7**, **12**, **13**]. It seems to be a very interesting problem to formulate the corresponding C^0-analog to the latter. We hope to study this issue among others elsewhere.

Remark 3.27 (Weak Hamiltonian topology). We can define the notion of *weak Hamiltonian topology* similarly to (strong) Hamiltonian topology. In the sets (3.7), we just replace the C^0-distance of the whole paths by the C^0-distance of the time-one maps only. So in the weak Hamiltonian topology, we do not have any control over the C^0-convergence of the whole paths other than the time-one maps. Although this seems natural in light of Proposition 3.6, it turns out that the weak Hamiltonian topology does not behave as nicely as the strong Hamiltonian topology. For example, it is unlikely that the map Tan is continuous with respect to the weak Hamiltonian topology, and that the sets $\overline{\mathcal{Q}_{\mathrm{w}}}$ and therefore $\mathrm{Hameo}^{\mathrm{w}}(M,\omega)$ defined in the same way as in the strong case form groups. One can easily verify that Remark 3.8, Proposition 3.10, Proposition 3.11, Corollary 3.12 and Theorem 4.1 still hold, respectively, in the weak case, while in Theorem 3.13 only path-connectedness, but not local path-connectedness, still holds. It seems unlikely that the analog to Theorem 4.5 below holds as well. The strong Hamiltonian topology is obviously stronger than the weak one, but it is an open question whether they are indeed different in general.

4. Basic properties of the group of Hamiltonian homeomorphisms

In this section, we extract some basic properties of the group $\mathrm{Hameo}(M,\omega)$ that immediately arise from its definition. We first note that

$$\mathrm{Ham}(M,\omega) \subset \mathrm{Hameo}(M,\omega) \subset \mathrm{Sympeo}(M,\omega) \tag{4.1}$$

from their definitions. The following theorem proves that $\mathrm{Hameo}(M,\omega)$ contains all expected C^k-Hamiltonian diffeomorphisms with $k \geq 2$.

Theorem 4.1. *The group* $\mathrm{Hameo}(M,\omega)$ *contains all* $C^{1,1}$*-Hamiltonian diffeomorphisms. More precisely, if* ϕ *is the time-one map of Hamilton's equation* $\dot{x} = X_H(t,x)$ *for a* C^1*-function* $H : [0,1] \times M \to \mathbb{R}$ *such that*

(1) $\|H_t\|_{C^{1,1}} \leq C$*, where* $C > 0$ *is independent of* $t \in [0,1]$*, and*
(2) *the map* $(t,x) \mapsto dH_t(x)$*,* $[0,1] \times M \to T^*M$ *is continuous,*

then $\phi \in \mathrm{Hameo}(M,\omega)$*.*

Proof. Note that any such $C^{1,1}$-function can be approximated by a sequence of smooth functions $H_i : [0,1] \times M \to \mathbb{R}$ so that

$$\|H - H_i\| \to 0, \tag{4.2}$$

where $\|\cdot\|$ denotes the $L^{(1,\infty)}$-norm as before. On the other hand, the vector fields $X_{H_i}(t,x)$ converge to $X_H(t,x)$ in $C^{0,1}(TM)$ uniformly over $t \in [0,1]$. Therefore, the flow $\phi^t_{H_i} \to \phi^t_H$ and so $\phi^1_{H_i} \to \phi^1_H$ in the C^0-topology by the standard existence and continuity theorem of ODE for Lipschitz vector fields. In particular, this C^0-convergence together with (4.2) implies that the sequence (ϕ_{H_i}, H_i) is a Cauchy sequence in $\mathcal{Q}$ with

$$\lim_{C^0} \phi^1_{H_i} = \phi^1_H = \phi. \tag{4.3}$$

Therefore, $\phi \in \mathrm{Hameo}(M, \omega)$. □

The following provides an example of an area-preserving homeomorphism on a surface that is not C^1, but still a Hamiltonian homeomorphism. Therefore, we have the following *proper* inclusion relation

$$\mathrm{Ham}(M, \omega) \subsetneq \mathrm{Hameo}(M, \omega) \subset \mathrm{Sympeo}(M, \omega).$$

Example 4.2. We will construct an area-preserving homeomorphism on the unit disc D^2 that is the identity near the boundary ∂D^2 and continuous but not differentiable. By extending the homeomorphism by the identity on $\Sigma = D^2 \cup (\Sigma \setminus D^2)$ to the outside of the disc, we can construct a similar example on a general surface Σ (for example, by choosing D inside the domain of a Darboux chart). Similarly one can construct such an example in higher dimensions. Furthermore a slight modification of an example like this combined with Polterovich's theorem on S^2 [**25**] provides a sequence ϕ_i of Hamiltonian diffeomorphisms on S^2 such that $\phi_i \to \mathrm{id}$ uniformly but $\|\phi_i\| \to \infty$, which demonstrates that the Hofer norm function $\phi \mapsto \|\phi\|$ is not continuous in the C^0-topology on $\mathrm{Ham}(M, \omega)$.

Let (r, θ) be polar coordinates on D^2. Then the standard area form is given by

$$\Omega = r\, dr \wedge d\theta.$$

Consider maps $D^2 \to D^2$ of the form

$$\phi_\rho : (r, \theta) \mapsto (r, \theta + \rho(r)),$$

where $\rho : (0, 1] \to [0, \infty)$ is a smooth function that satisfies for some small $\epsilon > 0$

(1) $\rho' < 0$ on $(0, 1-\epsilon)$, $\rho \equiv 0$ on $[1-\epsilon, 1]$; and
(2) $\lim_{r\to 0^+} r\rho'(r) = -\infty$.

It follows that ϕ_ρ is smooth except at the origin at which ϕ_ρ is continuous but not differentiable. Obviously the map $\phi_{-\rho}$ is the inverse of ϕ_ρ, which shows that it is a homeomorphism. Furthermore, we have

$$\phi_\rho^*(r\, dr \wedge d\theta) = r\, dr \wedge d\theta \quad \text{on} \quad D^2 \setminus \{0\},$$

which implies that ϕ_ρ is area-preserving.

Now it remains to show that if we choose ρ suitably, ϕ_ρ becomes a Hamiltonian homeomorphism. We will in fact consider *time-independent* Hamiltonians for this purpose. Consider the isotopy

$$t \in [0, 1] \mapsto \phi_{t\rho} \in \mathrm{Homeo}^\Omega(D^2).$$

A straightforward calculation shows that a corresponding Hamiltonian is given by the time-independent function

$$H_\rho(r,\theta) = -\int_1^r s\rho(s)\,ds.$$

The $L^{(1,\infty)}$-norm of H_ρ becomes

$$\int_0^1 s\rho(s)\,ds.$$

Choose any ρ so that the integral becomes finite, e.g., $\rho(r) = \frac{1}{\sqrt{r}}$ near $r = 0$. Now we choose any smoothing sequence ρ_n of ρ by regularizing ρ at 0, and consider the corresponding Hamiltonians H_{ρ_n} and their time one-maps ϕ_{ρ_n}. Then it follows that $(\phi_{H_{\rho_n}}, H_{\rho_n})$ is a Cauchy sequence in the Hamiltonian topology and $\phi_{\rho_n} \to \phi_\rho$ in the C^0-topology. So ϕ_ρ is a Hamiltonian homeomorphism that is neither differentiable nor Lipschitz at 0.

The following question seems to be one of fundamental importance (See Conjectures 5.3 and 5.4 later).

Question 4.3. In Example 4.2, consider ρ such that

$$\int_{0^+}^1 s\rho(s)\,ds = +\infty.$$

Is the homeomorphism ϕ_ρ still contained in $\mathrm{Hameo}(M,\omega)$?

The following theorem is the C^0-version of the well-known fact that $\mathrm{Ham}(M,\omega)$ is a normal subgroup of $\mathrm{Symp}_0(M,\omega)$.

Theorem 4.4. $\mathrm{Hameo}(M,\omega)$ *is a normal subgroup of* $\mathrm{Sympeo}(M,\omega)$.

Proof. We have to show

$$\psi h \psi^{-1} \in \mathrm{Hameo}(M,\omega)$$

for any $h \in \mathrm{Hameo}(M,\omega)$ and $\psi \in \mathrm{Sympeo}(M,\omega)$. By definition, there are Cauchy sequences $(\phi_{H_i}, H_i) \in \mathcal{Q}$ and $\psi_i \in \mathrm{Symp}(M,\omega)$ such that

$$h = \lim_{C^0} \phi^1_{H_i} \quad \text{and} \quad \lim_{C^0} \psi_i = \psi.$$

Let $\phi_i = \phi^1_{H_i}$. Recall from (3.3) that $\psi_i^{-1}\phi_i\psi_i$ is generated by $H_i \circ \psi_i$ for all i. It, therefore, suffices to prove that $(\psi_i^{-1}\phi_i\psi_i, H_i \circ \psi_i)$ is a Cauchy sequence in $\mathcal{Q}$ and $\lim_{C^0} \psi_i^{-1}\phi_i\psi_i = \psi^{-1}h\psi$. The C^0-convergence of the paths and time-one maps is obvious. Hence it remains to prove that $H_i \circ \psi_i$ is a Cauchy sequence in the $L^{(1,\infty)}$-topology,

$$\|H_i \circ \psi_i - H_j \circ \psi_j\| \to 0 \text{ as } i,\, j \to \infty. \tag{4.4}$$

But

$$\|H_i \circ \psi_i - H_j \circ \psi_j\| \le \|H_i \circ \psi_i - H_j \circ \psi_i\| + \|H_j \circ \psi_i - H_j \circ \psi_j\| \to 0.$$

Here the first term goes to zero as $\|H_i \circ \psi_i - H_j \circ \psi_i\| = \|H_i - H_j\| \to 0$ by assumption, and the second does by assumption and by Lemma 3.21(2) (by viewing the ψ_i as constant paths). That finishes the proof. $\square$

The following is an important property of $\text{Hameo}(M,\omega)$, which demonstrates that it is the "correct" C^0-counterpart of $\text{Ham}(M,\omega)$.

Theorem 4.5. *$\mathcal{H}\text{ameo}(M,\omega)$ is path-connected and locally path-connected. Consequently,* $\text{Hameo}(M,\omega)$ *is path-connected and we have*

$$\text{Hameo}(M,\omega) \subset \text{Sympeo}_0(M,\omega) \subset \text{Sympeo}(M,\omega) \cap \text{Homeo}_0^\Omega(M).$$

Proof. Let $h \in \mathcal{H}\text{ameo}(M,\omega)$. For the path-connectedness of $\mathcal{H}\text{ameo}(M,\omega)$, it suffices to prove that h can be connected to the identity by a Hamiltonian continuous path $\ell : [0,1] \to \mathcal{H}\text{ameo}(M,\omega)$ such that $\ell(0) = \text{id}$ and $\ell(1) = h$.

By definition, there exists a sequence $(\phi_{H_i}, H_i) \in \mathcal{Q}$ converging to an element $(\lambda, H) \in \overline{\mathcal{Q}}$, and $h = \overline{\text{ev}}_1^Q(\lambda, H) = \lambda(1) = \lim_{C^0} \phi_{H_i}^1$. As in Theorem 3.13, consider the Hamiltonians H_i^s generating the Hamiltonian paths $t \mapsto \phi_{H_i^s}^t = \phi_{H_i}^{st}$ for all $s \in [0,1]$ and all i. By the same arguments as in Theorem 3.13, we have

$$\overline{d}(\phi_{H_i^s}, \phi_{H_{i'}^s}) \leq \overline{d}(\phi_{H_i}, \phi_{H_{i'}}) \to 0 \text{ as } i,\, i' \to \infty,$$

and

$$\|H_i^s - H_{i'}^s\| \leq \|H_i - H_{i'}\| \to 0 \text{ as } i,\, i' \to \infty.$$

So $(\phi_{H_i^s}, H_i^s)$ is a Cauchy sequence in the Hamiltonian topology. Denote by $(\lambda^s, H^s) \in \overline{\mathcal{Q}}$ its limit, and note that λ^s is nothing but the path $t \mapsto \lambda(st)$. By the above, $\ell(s) = \overline{\text{ev}}_1^Q(\lambda^s, H^s) = \lambda(s) \in \mathcal{H}\text{ameo}(M,\omega)$ for all $s \in [0,1]$, and $\ell(0) = \text{id}$, $\ell(1) = h$. It remains to show that ℓ is continuous with respect to the Hamiltonian topology on $\mathcal{H}\text{ameo}(M,\omega)$.

Now ℓ factors through

$$[0,1] \to \overline{\mathcal{Q}} \to \mathcal{H}\text{ameo}(M,\omega), \quad s \mapsto (\lambda^s, H^s) \mapsto \overline{\text{ev}}_1^Q(\lambda^s, H^s) = \ell(s).$$

By definition of the topology on $\mathcal{H}\text{ameo}(M,\omega)$, it suffices to show that the first map is continuous, that is, that $s \mapsto (\lambda^s, H^s)$ is continuous with respect to the standard metric on $[0,1]$ and the product metric $\overline{d} + \|\cdot\|$ on $\overline{\mathcal{Q}}$. But

$$\begin{aligned}\overline{d}\Big((\lambda^s, H^s), (\lambda^{s'}, H^{s'})\Big) &= \|H^s - H^{s'}\| + \overline{d}(\lambda^s, \lambda^{s'}) \\ &= \lim_{i\to\infty} \|H_i^s - H_i^{s'}\| + \max_{t \in [0,1]} \overline{d}(\lambda(st), \lambda(s't)).\end{aligned}$$

Let $\epsilon > 0$. Note that if we consider the functions $\zeta_1(t) = ts$ and $\zeta_2(t) = ts'$, we see that

$$\|\zeta_1 - \zeta_2\|_{\text{ham}} = 2|s - s'|.$$

Therefore, it follows from Lemma 3.21 that we can find $\delta > 0$ and i_0 sufficiently large such that

$$\|H_i^s - H_i^{s'}\| < \frac{\epsilon}{2},$$

when $|s - s'| < \delta$ and $i \geq i_0$, and therefore

$$\lim_{i\to\infty} \|H_i^s - H_i^{s'}\| < \frac{\epsilon}{2},$$

when $|s - s'| < \delta$. For the second term, use continuity of λ and λ^{-1} to see that by making δ smaller if necessary,

$$\overline{d}(\lambda(st), \lambda(s't)) < \frac{\epsilon}{2},$$

when $|st - s't| \leq |s - s'| < \delta$. That proves continuity of ℓ, and hence completes the proof of path-connectedness of $\mathcal{H}$ameo(M, ω).

For the proof of local path-connectedness, we can, using Corollary 3.25, combine the above proof with the ideas in the proof of Theorem 3.13. Since the proof is essentially the same, we leave the details to the reader.

Now as sets, Hameo(M, ω) coincides with $\mathcal{H}$ameo(M, ω). Note that the path ℓ constructed above is a topological Hamiltonian path. Since a topological Hamiltonian path is in particular a continuous path with respect to the C^0-topology, this implies path-connectedness of Hameo(M, ω). The other statements about Hameo(M, ω) follow from this immediately. That completes the proof. □

It follows immediately from the $L^{(1,\infty)}$-approximation lemma (Appendix 2) that given any Cauchy sequence in $\mathcal{Q}$, we may assume that each path in the sequence is boundary flat. This implies that the concatenation of two topological Hamiltonian path is again a topological Hamiltonian path. So in fact we have proved that Hameo(M, ω) is path-connected by *topological Hamiltonian path.*

Question 4.6. Is Hameo(M, ω) locally path-connected?

Recall from (4.1) that we have Hameo$(M, \omega) \subset$ Sympeo(M, ω). But note that a priori it is not clear whether Hameo(M, ω) is different from Sympeo(M, ω). In fact, if one naively takes just the C^0-closure of Ham(M, ω), then it can end up becoming the whole Sympeo(M, ω). We refer to [**2**] for a nice observation that this is really the case for Ham$^c(\mathbb{R}^{2n})$. We refer to Section 6 for further discussion on this phenomenon.

In the next section, we will study the case $\dim M = 2$. Here we want to state the following theorem which is an immediate application of Arnold's conjecture.

Theorem 4.7. *Let (M, ω) be a closed symplectic manifold. Then any C^0-limit of Hamiltonian diffeomorphism has a fixed point. In particular, any Hamiltonian homeomorphism has a fixed point.*

Proof. Let $h = \lim_{C^0} \phi_i$ for a sequence $\phi_i \in \mathrm{Ham}(M,\omega)$. We prove the theorem by contradiction. Suppose h has no fixed point. Denote

$$d^h_{\min} := \inf_{x \in M} d(x, h(x)).$$

By compactness of M and since h has no fixed point, $d^h_{\min} > 0$. But each ϕ_i must have a fixed point x_i by the Arnold Conjecture, which was proven in [**5**, **14**, **26**]. Hence

$$\overline{d}(h,\phi_i) \geq d(h(x_i),\phi_i(x_i)) = d(h(x_i), x_i) \geq d^h_{\min} > 0$$

for all i. On the other hand, we have

$$\lim_{i\to\infty} \overline{d}(h,\phi_i) = 0,$$

which gives rise to a contradiction. □

Corollary 4.8. *Suppose that (M,ω) carries a symplectic diffeomorphism $\psi \in \mathrm{Symp}_0(M,\omega)$ (or equivalently, $\psi \in \mathrm{Sympeo}_0(M,\omega)$) that has no fixed point. Then $\psi \notin \mathrm{Hameo}(M,\omega)$, and in particular we have*

$$\mathrm{Hameo}(M,\omega) \subsetneq \mathrm{Sympeo}_0(M,\omega).$$

An example of a symplectic manifold (M,ω) satisfying these hypotheses is the torus T^{2n} with the standard symplectic form ω_0. Recall that by identifying $\alpha \in T^{2n}$ with the rotation $x \mapsto x + \alpha$, we can identify T^{2n} with a subgroup of $\mathrm{Symp}_0(T^{2n},\omega_0)$,

$$T^{2n} \hookrightarrow \mathrm{Symp}_0(T^{2n},\omega_0).$$

By Theorem 4.7, we have

$$T^{2n} \cap \mathrm{Hameo}(T^{2n},\omega_0) = \{\mathrm{id}\}.$$

It follows that $\mathrm{Hameo}(T^{2n},\omega_0) \subsetneq \mathrm{Sympeo}_0(T^{2n},\omega_0)$.

5. The two dimensional case

In this section, we will mainly study the case $\dim M = 2$. The first question would be what the relation between the group $\mathrm{Homeo}^{\Omega}(M)$ and its subgroup $\mathrm{Sympeo}(M,\omega)$ is. Similar question can be asked for their identity components, $\mathrm{Homeo}^{\Omega}_0(M)$ and $\mathrm{Sympeo}_0(M,\omega)$. By definition of $\mathrm{Sympeo}(M,\omega)$, this question boils down to approximability of area-preserving homeomorphisms by area-preserving diffeomorphisms in two dimensions. This smoothing result seems to have been known in the dynamical systems community (see [**22**] and [**29**] for a proof). Combined with this smoothing theorem, the following is an immediate translation thereof.

Theorem 5.1. *Let M be a compact orientable surface without boundary and $\omega = \Omega$ be an area form on it. Then we have*

$$\mathrm{Sympeo}(M,\omega) = \mathrm{Homeo}^{\Omega}(M), \quad \mathrm{Sympeo}_0(M,\omega) = \mathrm{Homeo}_0^{\Omega}(M).$$

Next we study the relationship between $\mathrm{Hameo}(M,\omega)$ and $\mathrm{Sympeo}_0(M,\omega)$. We will prove that if $M \neq S^2$ $\mathrm{Hameo}(M,\omega)$ is indeed a proper subgroup of $\mathrm{Sympeo}_0(M,\omega)$. The proof will use the mass flow homomorphism for area-preserving homeomorphisms on a surface, which we recalled in Section 2 in the general context of measure-preserving homeomorphisms. The mass flow homomorphisms can be defined for any isotopy of measure-preserving homeomorphisms preserving a *good* measure, e.g., the Liouville measure on a symplectic manifold (M,ω). The mass flow homomorphism reduces to the dual version of the flux homomorphism for volume-preserving *diffeomorphisms* on a *smooth* manifold [**30**]. Of course in two dimensions, the flux homomorphism coincides with the symplectic flux homomorphism, and so we can compare the mass flow homomorphism and the symplectic flux. One crucial point of considering the mass flow homomorphism instead of the flux homomorphism is that it is defined for an isotopy of area-preserving *homeomorphisms*, not just for diffeomorphisms.

We first recall the definition of the symplectic flux homomorphism. Denote by

$$\mathcal{P}(\mathrm{Symp}_0(M,\omega),\mathrm{id})$$

the space of *smooth* paths $c : [0,1] \to \mathrm{Symp}_0(M,\omega)$ with $c(0) = \mathrm{id}$. This naturally forms a group. For each given $c \in \mathcal{P}(\mathrm{Symp}_0(M,\omega),\mathrm{id})$, the flux of c is defined by

$$\mathcal{P}(\mathrm{Symp}_0(M,\omega),\mathrm{id}) \to H^1(M,\mathbb{R}), \quad \mathrm{Flux}(c) = \int_0^1 \dot{c} \rfloor \omega \, dt. \tag{5.1}$$

This depends only on the homotopy class, relative to the end points, of the path c and therefore projects down to the universal covering space

$$\pi_\omega : \widetilde{\mathrm{Symp}}_0(M,\omega) \to \mathrm{Symp}_0(M,\omega), \quad [c] \mapsto c(1), \tag{5.2}$$

where

$$\widetilde{\mathrm{Symp}}_0(M,\omega) := \{[c] \mid c \in \mathcal{P}(\mathrm{Symp}_0(M),\mathrm{id})\}.$$

Here $[c]$ is the homotopy class of c relative to fixed end points. We recall that $\mathrm{Symp}_0(M,\omega)$ is locally contractible [**33**] and so $\widetilde{\mathrm{Symp}}_0(M,\omega)$ is indeed the universal covering space of $\mathrm{Symp}_0(M,\omega)$. If we put

$$\Gamma_\omega = \mathrm{Flux}\left(\ker\left(\pi_\omega : \widetilde{\mathrm{Symp}}_0(M,\omega) \to \mathrm{Symp}_0(M,\omega)\right)\right),$$

we obtain by passing to the quotient the group homomorphism

$$\text{flux} : \text{Symp}_0(M,\omega) \to \frac{H^1(M,\mathbb{R})}{\Gamma_\omega}. \tag{5.3}$$

The maps (5.1) and (5.3) are also known to be surjective [**1**].

It is also shown in [**4**, Appendix A.5] that $\text{Flux}(c) \in H^1(M,\mathbb{R})$ is the Poincaré dual to the mass flow homomorphism $\widetilde{\theta}(c) \in H_1(M,\mathbb{R})$ recalled in Section 2 (after normalizing ω so that $\int_M \omega = 1$). Since it is also well known [**1**] that

$$\widetilde{\text{Ham}}(M,\omega) = \ker \text{Flux},$$
$$\text{Ham}(M,\omega) = \ker \text{flux},$$

we derive

$$\text{Ham}(M,\omega) \subset \ker\theta \cap \text{Symp}_0(M,\omega). \tag{5.4}$$

Theorem 5.2. *Let* (M,ω) *be a closed orientable surface, where* $\omega = \Omega$ *is a symplectic* (*or area*) *form on* M*. Then we have*

$$\text{Hameo}(M,\omega) \subset \ker\theta \cap \text{Sympeo}_0(M,\omega). \tag{5.5}$$

In particular, if $M \neq S^2$*, we have*

$$\text{Hameo}(M,\omega) \subsetneq \text{Sympeo}_0(M,\omega). \tag{5.6}$$

Proof. Recall (4.1) that $\text{Hameo}(M,\omega) \subset \text{Sympeo}_0(M)$. On the other hand, (5.4) implies $\theta|_{\text{Ham}(M,\omega)} \equiv 0$. From continuity of θ (Theorem 2.2) and the definition of $\text{Hameo}(M,\omega)$, we derive $\theta|_{\text{Hameo}(M,\omega)} \equiv 0$. That proves (5.5).

By the surjectivity of the Flux, the map

$$\theta|_{\text{Sympeo}_0(M,\omega)} : \text{Sympeo}_0(M,\omega) \to \frac{H_1(M,\mathbb{R})}{\Gamma}$$

is surjective. So $\ker\theta|_{\text{Sympeo}_0(M,\omega)} \subsetneq \text{Sympeo}_0(M,\omega)$ when $H_1(M,\mathbb{R}) \neq 0$ (and therefore, $H_1(M,\mathbb{R})/\Gamma \neq 0$ since Γ is discrete), which is the case for $M \neq S^2$. That proves the last statement. □

This theorem verifies that $\text{Hameo}(M,\omega)$ is a *proper* normal subgroup of $\text{Sympeo}_0(M,\omega)$, at least in two dimensions if $M \neq S^2$.

We now propose the following conjecture.

Conjecture 5.3. $\text{Hameo}(M,\omega)$ *is a proper subgroup of* $\ker\theta$ *in general. In particular for* $M = S^2$ *with* $\Omega = \omega$, $\text{Hameo}(S^2,\omega)$ *is a proper normal subgroup of* $\text{Sympeo}_0(S^2,\omega) (= \text{Homeo}_0^\Omega(S^2))$.

The affirmative answer to this conjecture will answer to Question 2.3 negatively and settle the simpleness question of $\text{Homeo}_0^\Omega(S^2)$, which has been open since Fathi's paper [**4**] appeared. In fact, this conjecture is an immediate corollary of the following more concrete conjecture.

Conjecture 5.4. *The answer to Question* 4.3 *on* S^2 *is negative, at least for a suitable choice of* ρ.

The results of this section can be generalized to higher dimensions in many cases. We first recall the flux homomorphism for volume-preserving *diffeomorphisms* on a *smooth* manifold [**30**]. Let Ω be a volume form on M and denote by

$$\mathcal{P}(\mathrm{Diff}_0^\Omega(M), \mathrm{id}),$$

the space of smooth paths $c : [0,1] \to \mathrm{Diff}_0^\Omega(M)$, the group of diffeomorphisms preserving the volume form Ω, with $c(0) = \mathrm{id}$. This also naturally forms a group. For each given $c \in \mathcal{P}(\mathrm{Diff}_0^\Omega(M), \mathrm{id})$, the Volume Flux of c is defined by

$$\mathcal{P}(\mathrm{Diff}_0^\Omega(M), \mathrm{id}) \to H^{2n-1}(M, \mathbb{R}), \quad \tilde{V}(c) = \int_0^1 \dot{c} \rfloor \Omega \, dt.$$

This depends only on the homotopy class relative to the end points of the path c and therefore projects down to the universal covering space

$$\pi_\Omega : \widetilde{\mathrm{Diff}_0^\Omega}(M) \to \mathrm{Diff}_0^\Omega(M), \quad [c] \mapsto c(1),$$

where

$$\widetilde{\mathrm{Diff}_0^\Omega}(M) := \{ [c] \mid c \in \mathcal{P}(\mathrm{Diff}_0^\Omega(M), \mathrm{id})\}.$$

Here $[c]$ again denotes the homotopy class of c relative to fixed end points. It is well-known that $\mathrm{Diff}_0^\Omega(M)$ is locally contractible and so $\widetilde{\mathrm{Diff}_0^\Omega}(M)$ is indeed the universal covering space of $\mathrm{Diff}_0^\Omega(M)$. If we put

$$\Gamma_\Omega = \widetilde{V} \left(\ker \left(\pi_\Omega : \widetilde{\mathrm{Diff}_0^\Omega}(M) \to \mathrm{Diff}_0^\Omega(M) \right) \right),$$

we obtain by passing to the quotient the group homomorphism

$$V : \mathrm{Diff}_0^\Omega(M) \to \frac{H^{2n-1}(M, \mathbb{R})}{\Gamma_\Omega},$$

to which we also refer to as the (*volume*) *flux homomorphism*.

In fact, $\tilde{V}(c) \in H^{2n-1}(M, \mathbb{R})$ is the Poincaré dual to the mass flow homomorphism $\widetilde{\theta}(c) \in H_1(M, \mathbb{R})$ (after normalizing Ω so that $\int_M \Omega = 1$) [**4**].

Now let $\Omega = \frac{1}{n!}\omega^n$ be the Liouville volume form. An easy calculation [**1**] shows that

$$\widetilde{V}(c) = \frac{1}{(n-1)!} \Big(\mathrm{Flux}(c) \Big) \wedge \omega^{n-1}. \tag{5.7}$$

So (5.4) holds in any dimension,

$$\mathrm{Ham}(M, \omega) \subset \ker \theta \cap \mathrm{Symp}_0(M, \omega).$$

By reexamining the proof of Theorem 5.2, we see that (5.5) holds as well, i.e.,

$$\mathrm{Hameo}(M,\omega) \subset \ker\theta \cap \mathrm{Sympeo}_0(M,\omega),$$

for any closed symplectic manifold (M,ω). We also see that

$$\mathrm{Hameo}(M,\omega) \subsetneq \mathrm{Sympeo}_0(M,\omega),$$

if $\theta|_{\mathrm{Sympeo}_0(M,\omega)} \colon \mathrm{Sympeo}_0(M,\omega) \to H_1(M,\mathbb{R})/\Gamma$ is nontrivial. By (5.7) and surjectivity of the Flux, we see that this condition is satisfied if

$$\wedge\omega^{n-1} \colon H^1(M,\mathbb{R}) \to H^{2n-1}(M,\mathbb{R}) \tag{5.8}$$

is nontrivial. The latter condition in particular holds if the map (5.8) is an isomorphism and $H_1(M,\mathbb{R}) \neq 0$ in which case M is said to be of Lefschetz type (for example, Kähler manifolds, or the case $\dim M = 2$ above). Nontriviality of the map (5.8) is equivalent to nontriviality of the pairing

$$H^1(M,\mathbb{R}) \times H^1(M,\mathbb{R}) \to \mathbb{R}, \quad (\alpha,\beta) \mapsto \int_M \alpha\wedge\beta\wedge\omega^{n-1}$$

This holds for the torus T^{2n} and therefore gives another proof of $\mathrm{Hameo}(T^{2n}, \omega_0) \subsetneq \mathrm{Sympeo}_0(T^{2n},\omega_0)$, which was also a consequence of Theorem 4.7. We summarize these results in the following theorem.

Theorem 5.5. *Let (M,ω) be a closed symplectic manifold. Then we have*

$$\mathrm{Ham}(M,\omega) \subset \ker\theta \cap \mathrm{Symp}_0(M,\omega)$$

and

$$\mathrm{Hameo}(M,\omega) \subset \ker\theta \cap \mathrm{Sympeo}_0(M,\omega). \tag{5.9}$$

If in addition the map (5.8) *is nontrivial, then*

$$\mathrm{Hameo}(M,\omega) \subsetneq \mathrm{Sympeo}_0(M,\omega) \subset \mathrm{Homeo}_0^{\Omega}(M).$$

6. The non-compact case and open problems

So far we have assumed that M is closed. In this section, we will indicate the necessary changes to be made for the open case where M is either noncompact or with boundary $\partial M \neq \emptyset$.

There are two possible definitions of compactly supported Hamiltonian diffeomorphisms in the literature. In this paper, we will treat the more standard version, which we call *compactly supported Hamiltonian diffeomorphisms.*

Here is the definition of compactly supported Hamiltonian diffeomorphisms which is mostly used in the literature so far. We denote the set of compactly supported symplectic diffeomorphisms by $\mathrm{Symp}^{\mathrm{c}}(M,\omega) \subset \mathrm{Diff}^{\mathrm{c}}(M,\omega)$.

Definition 6.1. We say that a smooth path $\lambda : [0,1] \to \mathrm{Symp}^{\mathrm{c}}(M,\omega)$ is a compactly supported Hamiltonian path if $\lambda = \phi_H$ for a Hamiltonian function $H : [0,1] \times M \to \mathbb{R}$ such that H is compactly supported in $\mathrm{Int}(M)$, where $\mathrm{supp}(H)$ is defined by

$$\mathrm{supp}(H) = \overline{\bigcup_{t \in [0,1]} \mathrm{supp}(H_t)}.$$

We define

$$\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}^{\mathrm{c}}(M,\omega),\mathrm{id})$$

to be the set of such λ with $\lambda(0) = \mathrm{id}$. A compactly supported symplectic diffeomorphism ϕ is a *compactly supported Hamiltonian diffeomorphism* if $\phi = \mathrm{ev}_1(\lambda)$ for some $\lambda \in \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}^{\mathrm{c}}(M,\omega),\mathrm{id})$. We denote the set of compactly supported Hamiltonian diffeomorphisms by

$$\mathrm{Ham}^{\mathrm{c}}(M,\omega) = \mathrm{ev}_1(\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}^{\mathrm{c}}(M,\omega),\mathrm{id})).$$

We now give descriptions of the Hamiltonian topologies on the path space $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}^{\mathrm{c}}(M,\omega),\mathrm{id})$ and on the group $\mathrm{Ham}^{\mathrm{c}}(M,\omega)$, respectively.

Let $K \subset \mathrm{Int}(M)$ be a compact subset. We denote by $\mathrm{Symp}_K(M,\omega)$ the set of $\psi \in \mathrm{Symp}^{\mathrm{c}}(M,\omega)$ with $\mathrm{supp}\,\psi \subset K$. By definition we have

$$\mathrm{Symp}^{\mathrm{c}}(M,\omega) = \bigcup_{K \subset \mathrm{Int}\, M;\mathrm{compact}} \mathrm{Symp}_K(M,\omega).$$

We denote by

$$\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}_K(M,\omega),\mathrm{id})$$

the set of $\lambda \in \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}^{\mathrm{c}}(M,\omega),\mathrm{id})$ with

$$\mathrm{supp}(\lambda(t)) \subset K \quad \text{for all } t \in [0,1].$$

The Hamiltonian topology on $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}_K(M,\omega),\mathrm{id})$ defined just as in the closed case is equivalent to the metric topology thereon induced by the metric

$$d_{\mathrm{ham},K}(\lambda_0,\lambda_1) = \mathrm{leng}(\lambda_0^{-1}\lambda_1) + \overline{d}(\lambda_0,\lambda_1) \tag{6.1}$$

(see Proposition 3.10), where $\overline{d}$ is the C^0-metric on $\mathcal{P}(\mathrm{Homeo}^{\mathrm{c}}(M),\mathrm{id})$. By definition,

$$\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}^{\mathrm{c}}(M,\omega),\mathrm{id}) = \bigcup_{K \subset \mathrm{Int}\, M;\mathrm{compact}} \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}_K(M,\omega),\mathrm{id}).$$

We also define $\mathrm{Ham}_K(M,\omega)$ to be the image

$$\mathrm{Ham}_K(M,\omega) = \mathrm{ev}_1(\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}_K(M,\omega),\mathrm{id})).$$

Definition 6.2. Suppose M is either noncompact or with boundary $\partial M \neq \emptyset$.

(1) We define the (strong) Hamiltonian topology of $\mathcal{P}^{\text{ham}}(\text{Symp}^{\text{c}}(M,\omega),\text{id})$ by the direct limit topology of the directed system

$$\{\mathcal{P}^{\text{ham}}(\text{Symp}_K(M,\omega),\text{id}) \mid K \subset \text{Int } M, \text{ compact}\}.$$

(2) We define the *Hamiltonian topology* of $\text{Ham}^{\text{c}}(M,\omega)$ by the strongest topology thereon such that the evaluation map

$$\text{ev}_1 \colon \mathcal{P}^{\text{ham}}(\text{Symp}^{\text{c}}(M,\omega),\text{id}) \to \text{Ham}^{\text{c}}(M,\omega).$$

is continuous. We denote the resulting topological space by $\mathcal{H}\text{am}^{\text{c}}(M,\omega)$.

Note that by definition we have

$$\text{Ham}^{\text{c}}(M,\omega) = \bigcup_{K \subset \text{Int } M;\text{compact}} \text{Ham}_K(M,\omega).$$

An easy exercise, using the commutative diagram

$$\begin{array}{lccc} \text{ev}_1 & : \mathcal{P}^{\text{ham}}(\text{Symp}_K(M,\omega),\text{id}) & \longrightarrow & \text{Ham}_K(M,\omega) \\ & \downarrow & & \downarrow \\ \text{ev}_1 & : \mathcal{P}^{\text{ham}}(\text{Symp}^{\text{c}}(M,\omega),\text{id}) & \longrightarrow & \text{Ham}^{\text{c}}(M,\omega), \end{array}$$

shows that the Hamiltonian topology on $\text{Ham}^{\text{c}}(M,\omega)$ is equivalent to the direct limit topology on $\text{Ham}^{\text{c}}(M,\omega)$ induced by the directed system

$$\{\mathcal{H}\text{am}_K(M,\omega) \mid K \subset \text{Int } M, \text{compact}\}.$$

Now the developing map Dev has the form

$$\text{Dev} : \mathcal{P}^{\text{ham}}(\text{Symp}^{\text{c}}(M,\omega),\text{id}) \to C^\infty_{\text{c}}([0,1]\times M,\mathbb{R}).$$

Here $C^\infty_{\text{c}}([0,1]\times M,\mathbb{R})$ is the set of smooth functions such that

$$\overline{\bigcup_{t\in[0,1]} \text{supp}(H_t)} \subset \text{Int}(M)$$

is compact.

We also consider the inclusion map

$$\begin{aligned} \iota_{\text{ham}} : \mathcal{P}^{\text{ham}}(\text{Symp}^{\text{c}}(M,\omega),\text{id}) &\to \mathcal{P}(\text{Symp}^{\text{c}}(M,\omega),\text{id}) \\ &\to \mathcal{P}(\text{Homeo}^{\text{c}}(M),\text{id}). \end{aligned}$$

The unfolding map $(\iota_{\text{ham}}, \text{Dev})$ has the image

$$\mathcal{Q} := \text{Image}(\iota_{\text{ham}},\text{Dev}) \subset \mathcal{P}(\text{Homeo}^{\text{c}}(M),\text{id}) \times L^{(1,\infty)}_c([0,1]\times M,\mathbb{R}).$$

Similarly we define

$$\mathcal{Q}_K := \text{Image}(\iota_{\text{ham},K},\text{Dev}_K) \subset \mathcal{P}(\text{Homeo}_K(M),\text{id}) \times L^{(1,\infty)}_K([0,1]\times M,\mathbb{R}),$$

equipped with the subspace topology induced by the metric topology on the target. Now we equip $\mathcal{Q}$ with the direct limit topology of the system $\{\mathcal{Q}_K\}$. Then it follows that the unfolding map canonically extends to the union

$$\overline{\mathcal{Q}} := \bigcup_{K \subset \text{Int } M;\text{compact}} \overline{\mathcal{Q}}_K,$$

in that we have the following continuous projections

$$\bar{\iota}^{Q}_{\text{ham}} : \overline{\mathcal{Q}} \to \mathcal{P}(\text{Homeo}^{\text{c}}(M), \text{id}), \tag{6.2}$$

$$\overline{\text{Dev}^{Q}} : \overline{\mathcal{Q}} \to L^{(1,\infty)}_{\text{c}}([0,1] \times M, \mathbb{R}), \tag{6.3}$$

with respect to the direct limit topology on $\overline{\mathcal{Q}}$ and the similar topology on the targets. We would like to remark that $\overline{\mathcal{Q}}$ is *not* the closure of Q in the metric topology on $\mathcal{P}(\text{Homeo}^{\text{c}}(M), \text{id}) \times L^{(1,\infty)}_{\text{c}}([0,1] \times M, \mathbb{R})$: the latter product space is not a complete metric space.

By definition we have the extension of the evaluation map

$$\text{ev}_1 : \mathcal{P}^{\text{ham}}(\text{Symp}^{\text{c}}(M,\omega), \text{id}) \to \text{Symp}^{\text{c}}(M,\omega) \to \text{Homeo}^{\text{c}}(M)$$

to

$$\overline{\text{ev}}^{Q}_1 : \overline{\mathcal{Q}} \to \text{Homeo}^{\text{c}}(M), \quad (\lambda, H) \to \lambda(1). \tag{6.4}$$

Definition 6.3. We define the set

$$\mathcal{P}^{\text{ham}}(\text{Sympeo}_K(M,\omega), \text{id}) := \bar{\iota}^{Q}_{\text{ham}}(\overline{\mathcal{Q}}_K) \subset \mathcal{P}(\text{Homeo}_K(M), \text{id}),$$
$$\mathcal{P}^{\text{ham}}(\text{Sympeo}^{\text{c}}(M,\omega), \text{id}) := \bar{\iota}^{Q}_{\text{ham}}(\overline{\mathcal{Q}}) \subset \mathcal{P}(\text{Homeo}^{\text{c}}(M), \text{id}),$$

and call any element of $\mathcal{P}^{\text{ham}}(\text{Sympeo}^{\text{c}}(M,\omega), \text{id})$ a compactly supported *topological Hamiltonian path*. Again we equip the latter with the direct limit topology of the metric topologies on $\mathcal{P}^{\text{ham}}(\text{Sympeo}_K(M,\omega), \text{id})$. We call this the Hamiltonian topology on $\mathcal{P}^{\text{ham}}(\text{Sympeo}^{\text{c}}(M,\omega), \text{id})$.

Then the set of compactly supported *Hamiltonian homeomorphisms* is defined by

$$\begin{aligned} \text{Hameo}^{\text{c}}(M,\omega) = \{h \in \text{Homeo}(M) \mid h = \overline{\text{ev}}_1(\lambda), \\ \lambda \in \mathcal{P}^{\text{ham}}(\text{Sympeo}^{\text{c}}(M,\omega), \text{id})\}. \end{aligned} \tag{6.5}$$

As a topological space, we define it as

Definition 6.4. We define

$$\text{Hameo}_K(M,\omega) = \overline{\text{ev}}^{Q}_1(\overline{\mathcal{Q}}_K),$$

and then

$$\text{Hameo}^{\text{c}}(M,\omega) = \bigcup_{K \subset \text{Int } M;\text{compact}} \text{Hameo}_K(M,\omega).$$

We call the direct limit topology of the metric topologies on $\text{Hameo}_K(M,\omega)$ the *Hamiltonian topology* on $\text{Hameo}^{\text{c}}(M,\omega)$.

With these definitions, the analogs to all the results stated in Sections 2–5 still hold. For example, the following can be proved in the same way as Theorems 4.4 and 4.5.

Theorem 6.5. *The group* $\mathrm{Hameo}^c(M,\omega)$ *is a path-connected normal subgroup of* $\mathrm{Sympeo}_0^c(M,\omega)$.

We would like to point out that this theorem is a sharp contrast to the following interesting observation by Bates [**2**]: *if one takes just the* C^0*-closure of* $\mathrm{Ham}^c(\mathbb{R}^{2n},\omega_0)$ *instead, not with respect to the Hamiltonian topology, the closure becomes the whole* $\mathrm{Sympeo}^c(\mathbb{R}^{2n},\omega_0)$ *even if* $\mathrm{Symp}(\mathbb{R}^{2n},\omega_0)$ *has many connected components*. This is another evidence the Hamiltonian topology is the right topology to take for the study of topological Hamiltonian geometry.

In relation to this definition, we would just like to mention one result by Hofer [**9**] on $\mathbb{R}^{2n}$:

$$\|\phi^{-1}\psi\| \leq C\,\mathrm{diam}(\mathrm{supp}(\phi^{-1}\psi))\|\phi^{-1}\psi\|_{C^0}, \tag{6.6}$$

where C is a constant with the bound $C \leq 128$. This in particular implies that the C^0-topology is stronger than the Hofer topology on $\mathrm{Ham}^c(\mathbb{R}^{2n},\omega_0)$ *if* $\mathrm{supp}(\phi^{-1}\psi)$ *is controlled.*

Finally we list the problems which arise immediately from the various definitions introduced in this paper, and seem to be interesting to investigate. These will be subjects of future study.

Problems

(1) Describe the closed set of length minimizing paths in terms of the geometry and dynamics of the Hamiltonian flows.
(2) Describe the images of $\overline{\mathrm{Tan}^Q}, \overline{\mathrm{Dev}^Q}$ of $\overline{\mathcal{Q}}$ in $L_m^{(1,\infty)}([0,1]\times M,\mathbb{R})$.
(3) Study the structure of the flow of Hamiltonian homeomorphisms in terms of the C^0-Hamiltonian dynamical system or as the high dimensional generalization of area-preserving homeomorphisms with vanishing mass flow or zero mean rotation vector.
(4) Does the identity $[\mathrm{Sympeo}_0, \mathrm{Sympeo}_0] = \mathrm{Hameo}$ hold? Is Hameo simple?
(5) Further investigate the above Hofer's inequality. For example, what would be the optimal constant C in the inequality (6.6)?

7. Appendix 1. Smoothness implies Hamiltonian continuity

We first recall the precise definition of smooth Hamiltonian paths.

Definition A.1. (i) A C^∞-diffeomorphism ϕ of (M,ω) is a *Hamiltonian diffeomorphism* if $\phi = \phi_H^1$ is the time-one map of the Hamilton equation

$$\dot{x} = X_H(t,x),$$

for a C^∞ function $H : \mathbb{R} \times M \to \mathbb{R}$ such that

$$H(t+1, x) = H(t, x)$$

for all $(t, x) \in \mathbb{R} \times M$. We denote by $\mathrm{Ham}(M, \omega)$ the set of Hamiltonian diffeomorphisms with the C^∞-topology induced by the inclusion

$$\mathrm{Ham}(M, \omega) \subset \mathrm{Symp}_0(M, \omega),$$

where $\mathrm{Symp}_0(M, \omega)$ carries the C^∞-topology.

(ii) A (*smooth*) Hamiltonian path $\lambda : [0,1] \to \mathrm{Ham}(M, \omega)$ is a smooth map

$$\Lambda : [0,1] \times M \to M; \quad \Lambda(t, \cdot) := \lambda(t)$$

such that

(1) its derivative $\dot\lambda(t) = \frac{\partial \lambda}{\partial t} \circ (\lambda(t))^{-1}$ is Hamiltonian, i.e., the one form $\dot\lambda(t) \rfloor \omega$ is exact for all $t \in [0,1]$. We call a function $H : \mathbb{R} \times M \to \mathbb{R}$ a *generating Hamiltonian* of λ if it satisfies

$$\lambda(t) = \phi_H^t \circ \lambda(0), \quad \text{or equivalently,} \quad dH_t = \dot\lambda(t) \rfloor \omega.$$

(2) $\lambda(0) := \Lambda(0, \cdot) : M \to M$ is a Hamiltonian diffeomorphism, and therefore, $\lambda(t) = \Lambda(t, \cdot)$ is for all $t \in [0,1]$.

We denote by $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M, \omega))$ the set of Hamiltonian paths $\lambda : [0,1] \to \mathrm{Ham}(M, \omega)$, and by $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M, \omega), \mathrm{id})$ the set of such λ with $\lambda(0) = \mathrm{id}$. We equip $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M, \omega))$ and $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M, \omega), \mathrm{id})$ with the obvious topology induced by the C^∞-topology of the space $C^\infty([0,1] \times M, M)$ of the corresponding maps Λ above. We call this the C^∞-topology of $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M, \omega))$ and $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M, \omega), \mathrm{id})$.

Note that if $\phi = \phi_H^1$ is a Hamiltonian diffeomorphism (in the sense of Definition A.1.(i)), then $t \mapsto \lambda(t) = \phi_H^t$ is a smooth Hamiltonian path (in the sense of Definition A.1(ii)) with $\lambda(0) = \mathrm{id}$ and $\lambda(1) = \phi$. So each $\phi \in \mathrm{Ham}(M, \omega)$ can be connected to the identity by a smooth Hamiltonian path as in A.1.(ii). In particular, $\mathrm{Ham}(M, \omega)$ is the image of the evaluation map ev_1 (1.5). We also note that by Proposition 3.4, each smooth path $\lambda : [0,1] \to \mathrm{Symp}(M, \omega)$ that has its image contained in $\mathrm{Ham}(M, \omega)$ is a smooth Hamiltonian path in the sense of Definition A.1(ii).

In this appendix, we give the proof of the following basic lemma and prove that any smooth path in $\mathrm{Ham}(M, \omega)$ is Hamiltonian continuous. By abuse of notation, we will just denote a smooth Hamiltonian path by

$$\lambda : I \to \mathrm{Ham}(M, \omega),$$

or more generally, a smooth Hamiltonian map from a simplex Δ by

$$\lambda : \Delta \to \mathrm{Ham}(M, \omega).$$

Lemma A.2. *For any Hamiltonian path* $\lambda : I \to \mathrm{Ham}(M,\omega)$ *defined on an interval* $I = [a,b]$ *such that* λ *is flat near* a*, i.e., there exists* $a' > a$ *with*

$$\lambda(s) \equiv \lambda(a) \tag{A.1}$$

for all $a \le s \le a' \le b$*, we can find a smooth map*

$$\Lambda : I \times [0,1] \times M \to M,$$

such that the following hold:

(1) *For each* $s \in I$ *and* $t \in [0,1]$*,* $\Lambda_{(s,t)} \in \mathrm{Ham}(M,\omega)$*, where we denote*

$$\Lambda_{(s,t)}(x) := \Lambda(s,t,x).$$

(2) *For each* $s \in I$*, the path* $\lambda^s : [0,1] \to \mathrm{Ham}(M,\omega)$ *is a Hamiltonian path with* $\lambda^s(0) = \mathrm{id}$ *and* $\lambda^s(1) = \lambda(s)$*, which is flat near* 0*, where we denote*

$$\lambda^s(t) := \Lambda_{(s,t)}.$$

Furthermore, a similar statement holds for a map $\Delta \to \mathrm{Ham}(M,\omega)$ *where* Δ *is a* k*-simplex: in this case* (A.1) *is replaced by the condition that* λ *is flat near the vertex* $0 \in \Delta$*.*

Proof. We may assume $I = [0,1]$. Let $K : I \times M \to \mathbb{R}$ be the (not necessarily normalized) Hamiltonian generating λ such that

$$\lambda(s) = \phi_K^s \circ \lambda(0), \quad s \in [0,1], \tag{A.2}$$

and

$$K(s,\cdot) \equiv 0 \quad \text{for all } 0 \le s \le a'. \tag{A.3}$$

Equation (A.3) is possible because of the assumption (A.1). Next we fix a Hamiltonian $H^0 : [0,1] \times M \to \mathbb{R}$ with $H^0 \mapsto \lambda(0)$. After reparameterization, we may assume that

$$H^0 \equiv 0 \quad \text{near } t = 0,\, 1. \tag{A.4}$$

Now for each $s \in [0,1]$, we define $H^s : [0,1] \times M \to \mathbb{R}$ by the formula

$$H^s(t,x) = \begin{cases} \dfrac{1}{1-s} H^0\left(\dfrac{t}{1-s}, x\right) & \text{for } 0 \le t < 1-s, \\ K(t-(1-s), x) & \text{for } 1-s \le t \le 1. \end{cases} \tag{A.5}$$

Obviously $H : I \times [0,1] \times M \to \mathbb{R}$ is smooth due to the above flatness conditions (A.3) and (A.4) and satisfies

$$\phi^1_{H^s} = \lambda(s).$$

We then define Λ by $\Lambda(s,t) = \phi^t_{H^s}$. It follows from the construction that Λ satisfies all the properties in (1) and (2). The last statement can be proven by a similar argument by considering the retraction of the k-simplex Δ to its vertex 0. □

Remark that if λ is flat also near $t = 1$, then we can assume that λ^s is flat near $t = 1$ for all $s \in I$. The proof goes through the same way.

Corollary A.3. *Any smooth Hamiltonian path $\lambda : [0,1] \to \mathrm{Ham}(M,\omega)$ is Hamiltonian continuous.*

Proof. Let $\lambda = \phi_H : [0,1] \longrightarrow \mathrm{Ham}(M,\omega)$ be a smooth Hamiltonian path (in the sense of Definition A.1(ii)). Here we assume without loss of generality that $\lambda(0) = \mathrm{id}$. We have to show that λ is continuous with respect to the Hamiltonian topology on $\mathrm{Ham}(M,\omega)$, i.e., as a map $\lambda : [0,1] \longrightarrow \mathcal{H}\mathrm{am}(M,\omega)$. Note that λ factors through

$$[0,1] \to \mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id}) \to \mathcal{H}\mathrm{am}(M,\omega),$$
$$s \mapsto \phi_{H^s} \mapsto \phi^1_{H^s} = \phi^s_H = \lambda(s),$$

where the second map is the evaluation map. By definition of the Hamiltonian topology on $\mathrm{Ham}(M,\omega)$, it suffices to prove that the first map is continuous. The topology on $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$ is by Proposition 3.10 equivalent to the metric topology induced by d_{ham}. So we only have to show that the map $s \mapsto \phi_{H^s}$ is continuous with respect to the standard metric on $[0,1]$ and d_{ham} on $\mathcal{P}^{\mathrm{ham}}(\mathrm{Symp}(M,\omega),\mathrm{id})$.

Let H^s be the Hamiltonian and Λ be the smooth map constructed in the proof of Lemma A.2. By definition

$$d_{\mathrm{ham}}\left(\phi_{H^s},\phi_{H^{s'}}\right) = \|H^s - H^{s'}\| + \overline{d}\left(\phi_{H^s},\phi_{H^{s'}}\right). \tag{A.6}$$

If we define the smooth reparameterization functions ζ_1, $\zeta_2 : [0,1] \to [0,1]$, by $\zeta_1(t) = st$, $\zeta_2(t) = s't$, then $\|\zeta_1 - \zeta_2\|_{\mathrm{ham}} = 2|s-s'|$. Hence by Lemma 3.20, the first term in (A.6) is less than $2C|s-s'|$, where C is the constant given in (3.20) in Lemma 3.20. For the second term in (A.6), first note that Λ is Lipschitz continuous since it is smooth and compactly supported. Therefore,

$$d_{C^0}\left(\phi_{H^s},\phi_{H^{s'}}\right) = \max_{(t,x)} d\Big(\Lambda(s,t,x),\Lambda(s',t,x)\Big) < L|s-s'|,$$

where L is a Lipschitz constant for Λ. Since $s \mapsto (\lambda(s))^{-1}$ is also a smooth Hamiltonian path, we can use Lemma A.2 to construct a corresponding map $\Lambda'(s,t) = (\phi^t_{H^s})^{-1}$, and then apply the same argument to obtain

$$d_{C^0}\left((\phi_{H^s})^{-1},(\phi_{H^{s'}})^{-1}\right) < L'|s-s'|,$$

where L' is another Lipschitz constant. This shows that the second term in (A.6) is less than $\max(L,L')|s-s'|$. Altogether, with $c = \max(2C,L,L')$, we have

$$d_{\mathrm{ham}}\left(\phi_{H^s},\phi_{H^{s'}}\right) = \|H^s - H^{s'}\| + \overline{d}(\phi_{H^s},\phi_{H^{s'}}) < c|s-s'|,$$

which completes the proof. ∎

8. Appendix 2. The $L^{(1,\infty)}$-approximation lemma

In this appendix, we give the proof of the $L^{(1,\infty)}$-approximation lemma which is a slight variation of [17, Lemma 5.2].

Lemma A.1 ($L^{(1,\infty)}$-approximation lemma). *Let $H : [0,1] \times M \to \mathbb{R}$ be a smooth Hamiltonian and $\phi = \phi_H^1$ be its time-one map. Then we can reparameterize ϕ_H^t in time so that the Hamiltonian H' generating the reparameterized path satisfies the following properties:*

1. $\phi_{H'}^1 = \phi_H^1$;
2. *$H' \equiv 0$ near $t = 0, 1$, and in particular H' can be extended to be time-periodic on $\mathbb{R} \times M$;*
3. *the norm $\|\overline{H} \# H'\|$ can be made as small as we want;*
4. *for the Hamiltonians H', H'' generating any two such reparameterizations of ϕ_H^t, there is a canonical one–one correspondence between* Per(H') *and* Per(H''), *and* Crit $\mathcal{A}_{H'}$ *and* Crit $\mathcal{A}_{H''}$ *with their actions fixed.*

Furthermore this reparameterization is canonical in the sense that the "smallness" in (3) can be chosen uniformly over H depending only on the C^0-norm and the modulus of continuity of H. In particular, this approximation can be done with respect to the Hamiltonian topology. Moreover, the closeness in the Hamiltonian topology can be made as small as we want independent of H (only the time for which the reparameterized Hamiltonian is flat depends on H).

Proof. We first reparameterize ϕ_H^t in the following way: we choose a smooth function $\zeta : [0,1] \longrightarrow [0,1]$ such that for $\epsilon > 0$

$$\zeta(t) = \begin{cases} 0 & \text{for } \ 0 \leq t \leq \epsilon, \\ 1 & \text{for } \ 1-\epsilon \leq t \leq 1, \end{cases}$$

and

$$\zeta'(t) \geq 0 \quad \text{for all } \ t \in [0,1],$$

and consider the isotopy

$$\psi^t := \phi_H^{\zeta(t)}.$$

It is easy to check that the Hamiltonian generating the isotopy $\{\psi^t\}_{0 \le t \le 1}$ is $H' = \{H'_t\}_{0 \le t \le 1}$ with $H'_t = \zeta'(t) H_{\zeta(t)}$. By definition, it follows that H' satisfies (1) and (2). As always we assume that H is normalized, and then so is H'. In particular, $\int_0^1 \max_x (H'_t - H_t) dt \ge 0$. For (3), we compute

$$\begin{aligned} 0 &\le \int_0^1 \max_x (H'_t - H_t) dt \\ &= \int_0^1 \max_x (\zeta'(t) H_{\zeta(t)} - H_t) dt \\ &\le \int_0^1 \max_x \Big(\zeta'(t)(H_{\zeta(t)} - H_t)\Big) dt + \int_0^1 \max_x \Big((\zeta'(t) - 1) H_t\Big) dt. \end{aligned}$$

For the first term,

$$\begin{aligned} \int_0^1 \max_x \Big(\zeta'(t)(H_{\zeta(t)} - H_t)\Big) dt &= \int_0^1 \zeta'(t) \max_x (H_{\zeta(t)} - H_t) dt \\ &\le \int_0^1 \zeta'(t) \max_{x,t} |H_{\zeta(t)} - H_t| dt \\ &= \max_{x,t} |H_{\zeta(t)}(x) - H_t(x)| \le L \|\zeta - \mathrm{id}\|_{C^0}, \end{aligned}$$

which can be made arbitrarily small by choosing ζ so that $\|\zeta - \mathrm{id}\|_{C^0}$ becomes sufficiently small. Here L is a Lipschitz constant for H in the time variable t (it exists and is finite since H is smooth and supported on the compact set $[0,1] \times M$). We refer to this constant as the modulus of continuity. For the second term,

$$\begin{aligned} \int_0^1 \max_x \Big((\zeta'(t) - 1) H_t\Big) dt &\le \int_0^1 |\zeta'(t) - 1| dt \ \max_{x,t} |H(x,t)| \\ &= \|H\|_{C^0} \int_0^1 |\zeta'(t) - 1| dt. \end{aligned}$$

Again by appropriately choosing ζ (which can be done consistently with the choice above), we can make

$$\int_0^1 |\zeta'(t) - 1| dt$$

as small as we want. Combining these two, we have verified the integral $\int_0^1 \max_x (H'_t - H_t)\, dt$ can be made as small as we want by making the Hamiltonian norm

$$\|\zeta - \mathrm{id}\|_{\mathrm{ham}} = \|\zeta - \mathrm{id}\|_{C^0} + \|\zeta' - 1\|_{L^1}$$

small. This can always be done by choosing ϵ sufficiently small. Similar consideration applies to $\int_0^1 - \min(H' - H)\, dt$ and hence we have finished the proof of (3).

Statement (4) follows from simple comparison of the corresponding actions of periodic orbits. The statements in the last paragraph follow from the construction. For the C^0-closeness, note that similarly to the proof of Corollary A.3, by continuity of the path $t \mapsto \phi_H^t$, the distance $\overline{d}\,(\phi_{H^\zeta}, \phi_H)$ can be made arbitrarily small by choosing ζ so that $\|\zeta - \mathrm{id}\|_{C^0}$ becomes small. This finishes the proof. □

We would like to point out that the above modification does *not* approximate in the L^∞-topology on $[0,1] \times M$ because the derivative of the cut-off function ζ could blow up in the above approximation. In fact, it is easy to see that such an approximation can be done for a given Hamiltonian function H in the L^∞-norm if and only if H_0, $H_1 \equiv$ constant. The proof is essentially the same as above.

Proof of Lemma 3.20. Replace ζ by ζ_1 and id by ζ_2 in the proof of the $L^{(1,\infty)}$-approximation lemma. □

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