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GROUPOIDS, BRANCHED MANIFOLDS AND MULTISECTIONS

Dusa McDuff

Cieliebak *et al.* recently formulated a definition of branched submanifold of Euclidean space in connection with their discussion of multivalued sections and the Euler class. This note proposes an intrinsic definition of a weighted branched manifold $\underline{\boldsymbol{Z}}$ that is obtained from the usual definition of oriented orbifold groupoid by relaxing the properness condition and adding a weighting. We show that if $\underline{\boldsymbol{Z}}$ is compact, finite dimensional and oriented, then it carries a fundamental class $[\boldsymbol{Z}]$. Adapting a construction of Liu and Tian, we also show that the fundamental class $[\boldsymbol{X}]$ of any oriented orbifold $\underline{\boldsymbol{X}}$ may be represented by a map $\underline{\boldsymbol{Z}} \to \underline{\boldsymbol{X}}$, where the branched manifold $\underline{\boldsymbol{Z}}$ is unique up to a natural equivalence relation. This gives further insight into the structure of the virtual moduli cycle in the new polyfold theory recently constructed by Hofer *et al.*

Contents

1. Introduction

Cieliebak *et al.* [**4**] formulate the definition of branched submanifold of $\mathbb{R}^n$ in connection with their discussion of multivalued sections, and use it to represent the Euler class of certain G-bundles (where G is a compact Lie group). In Definition 3.12, we propose an intrinsic definition of a weighted branched manifold $\underline{Z}$ generalizing that in Salamon [**19**]. It is obtained from the usual definition of orbifold groupoid simply by relaxing the properness condition and adding a weighting. Proposition 3.25 states that if $\underline{Z}$ is compact, finite dimensional and oriented, then $\underline{Z}$ carries a fundamental class $[Z]$. Our point of view allows us to deal with a few technical issues that arise when branched submanifolds are not embedded in a finite dimensional ambient space. Our other main result can be stated informally as follows. (For a formal statement, see Propositions 3.16 and 3.25.)

Theorem 1.1. *Any compact oriented orbifold $\underline{Y}$ has a "resolution" $\underline{\phi}: \underline{Z} \to \underline{Y}$ by a branched manifold that is unique up to a natural equivalence relation. Moreover $\underline{\phi}_*([Z])$ is the fundamental class of $\underline{Y}$.*

On the level of groupoids, the resolution is constructed from an orbifold groupoid by refining the objects and also throwing away some of the morphisms; cf. Example 2.10. One can think of resolutions as trading orbifold singularities for branching. One application is to give a simple description of the (Poincaré dual of the) Euler class of a bundle over an orbifold as a homology class represented by a branched manifold; see Proposition 4.19 and Section 4.3. We work in finite dimensions but, as the discussion below indicates, the result also applies in certain infinite dimensional situations.

We give two proofs of Theorem 1.1. The first (in Section 4.1) is an explicit functorial construction that builds $\underline{Z}$ from a set of local uniformizers of $\underline{Y}$. It is a groupoid version of Liu–Tian's [**12**] construction of the virtual moduli cycle. (Also see Lu-Tian [**13**].) The second (in Section 4.2) constructs $\underline{Z}$ as the graph of a multisection of a suitable orbibundle $\underline{E} \to \underline{Y}$. It therefore relates to the construction of the Euler class in [**4**] and to Hofer *et al.*'s new polyfold[1] approach to constructing the virtual moduli cycle of symplectic field theory.

The situation here is the following. The generalized Cauchy–Riemann (or delbar) operator is a global Fredholm section f of a polyfold bundle $\underline{E} \to \underline{Y}$,

[1]For the purposes of the following discussion a polyfold can be understood as an orbifold in the category of Hilbert spaces. For more detail see [**9**, **10**].

and if it were transverse to the zero section one would define the virtual moduli cycle to be its zero set. However, in general, f is not transverse to the zero section. Moreover, because $\underline{Y}$ is an orbifold rather than a manifold, one can achieve transversality only by perturbing f by a multivalued section s. Hence the virtual moduli cycle, which is defined to be the zero set of $f+s$, is a weighted branched submanifold of the infinite dimensional groupoid $\underline{E}$: see [**10**, Ch. 7]. Since s is chosen so that $f+s$ is Fredholm (in the language of [**10**] it is an sc^+-section), this zero set is finite dimensional.

Although one can define the multisections s fairly explicitly, the construction in [**10**] gives little insight into the topological structure of the corresponding zero sets. This may be understood in terms of a resolution $\underline{\phi} : \underline{Z} \to \underline{Y}$. The pullback by $\underline{\phi} : \underline{Z} \to \underline{Y}$ of an orbibundle $\underline{E} \to \underline{Y}$ is a bundle $\underline{\phi}^*(\underline{E}) \to \underline{Z}$ and we show in Lemma 4.16 that its (single valued) sections s push forward to give multivalued sections $\underline{\phi}_*(s)$ of $\underline{E} \to \underline{Y}$ in the sense of [**10**]. Proposition 4.20 shows that if $\underline{E} \to \underline{Y}$ has enough *local* sections to achieve transversality one can construct the resolution to have enough *global* sections to achieve transversality. Hence one can understand the virtual moduli cycle as the zero set of a (single valued) section of $\underline{\phi}^*(\underline{E}) \to \underline{Z}$. In particular, its branching is induced by that of $\underline{Z}$.

This paper is organized as follows. Section 2 sets up the language in which to define orbifolds in terms of groupoids. It is mostly but not entirely review, because we treat the properness requirements in a nonstandard way. In Section 3, we define weighted nonsingular branched groupoids and branched manifolds and establish their main properties. Section 4 gives the two constructions for the resolution. The relation to the work of Cieliebak *et al.* [**4**] is discussed in Section 4.3.

2. Orbifolds and groupoids

Orbifolds (or V-manifolds) were first introduced by Satake [**20**]. The idea of describing them in terms of groupoids and categories is due to Haefliger [**5**, **6**, **7**]. Our presentation and notation is based on the survey by Moerdijk [**15**]. Thus we shall denote the spaces of objects and morphisms of a small topological[2] category $\mathcal{X}$ by the letters X_0 and X_1, respectively. The source and target maps are $s, t : X_0 \to X_1$ and the identity map $x \mapsto \mathrm{id}_x$ is $\mathrm{id} : X_0 \to X_1$. The composition map

$$m\colon X_1 \, {}_s\!\times_t X_1 \;\to\; X_1, \quad (\delta,\gamma) \mapsto \delta\gamma$$

has domain equal to the fiber product $X_1 \, {}_s\!\times_t X_1 = \{(\delta,\gamma) : s(\delta) = t(\gamma)\}$. We denote the space of morphisms from x to y by $\mathrm{Mor}(x,y)$.

[2] i.e., its objects and morphisms form topological spaces and all structure maps are continuous.

2.1. Smooth, stable, étale (sse) groupoids. Throughout this paper we shall work in the smooth category, by which we mean the category of finite dimensional second countable Hausdorff manifolds. If the manifolds have boundary, we assume that all local diffeomorphisms respect the boundary. Similarly, if they are oriented, all local diffeomorphisms respect the orientation.

Definition 2.1. *An* **(oriented) sse groupoid** $\mathcal{X}$ *is a small topological category such that the following conditions hold.*

(Groupoid) *All morphisms are invertible. More formally there is a structure map* $\iota : X_1 \to X_1$ *that takes each* $\gamma \in X_1$ *to its inverse* γ^{-1}.

(Smooth étale) *The spaces* X_0 *of objects and* X_1 *of morphisms are (oriented) manifolds (without boundary), and all structure maps* $(s, t, m, \iota, \mathrm{id})$ *are (oriented) local diffeomorphisms.*

(Stable) *For each* $x \in X_0$*, the set of self-morphisms* $\mathrm{Mor}(x, x) =: G_x$ *is finite.*

Groupoids that also satisfy the properness condition stated below will be called **ep groupoids***, where the initials ep stand for "étale proper".*

(Properness) *The map* $s \times t : X_1 \to X_0 \times X_0$ *that takes a morphism to its source and target is proper.*

The **orbit space** $|X|$ *of* $\mathcal{X}$ *is the quotient of* X_0 *by the equivalence relation in which* $x \sim y$ *iff* $\mathrm{Mor}(x, y) \neq \emptyset$.

Further, $\mathcal{X}$ *is called*

- **nonsingular** *if all stabilizers* $G_x := \mathrm{Mor}(x, x)$ *are trivial.*
- **effective** *if for each* $x \in X_0$ *and* $\gamma \in G_x$ *each neighborhood* $V \subset X_1$ *of* γ *contains a morphism* γ' *such that* $s(\gamma') \neq t(\gamma')$ *(i.e. the action of* γ *is locally effective.)*
- **connected** *if* $|X|$ *is path connected.*

Unless there is specific mention to the contrary, all groupoids $\mathcal{X}$ considered in this paper are sse groupoids, understood in the sense defined above, and all functors are smooth, i.e., they are given by smooth maps on the spaces of objects and morphisms.[3] Many authors call ep groupoids

[3]Moerdijk [**15**] formulates the smooth (or Lie) étale condition in a slightly different but essentially equivalent way. In his context, equivalence is called Morita equivalence. Note also that one can work with the above ideas in categories other than that of finite dimensional Hausdorff manifolds and local diffeomorphisms. For example, as in [**10**], one can work with infinite dimensional M-polyfolds and sc-diffeomorphisms. Haefliger [**5**, **6**, **7**] and Moerdijk–Mrčun [**16**] consider Lie groupoids in which the space X_1 of morphisms is allowed to be a nonHausdorff manifold in order to accommodate examples such as the groupoid of germs of diffeomorphisms. Foliation groupoids also need not be proper. Thus they also develop considerable parts of the theory of étale groupoids without assuming properness, though in the main they are interested in very different manifestations of nonproperness.

orbifold groupoids. Note that stability[4] is a consequence of properness, but we shall often assume the former and not the latter.

Robbin–Salamon [**18**] use the stability condition to show that in any sse groupoid every point $x \in X_0$ has an open neighborhood U_x such that each morphism $\gamma \in G_x$ extends to a diffeomorphism of U_x onto itself.[5] However, in general there could be many other morphisms with source and target in U_x. Their Corollary 2.10 states that $\mathcal{X}$ is proper iff U_x can always be chosen so that this is not so, i.e. so that $(s \times t)^{-1}(U_x \times U_x) \cong U_x \times G_x$. Thus properness is equivalent to the existence of local uniformizers in the sense of Definition 2.11 below.

Another well known consequence of properness is that the orbit space $|X|$ is Hausdorff. The next lemma shows that $|X|$ is Hausdorff iff the equivalence relation on X_0 is closed, i.e. the subset $\{(x, y) \in X_0 \times X_0 : x \sim y\}$ is closed.

Lemma 2.2. *Let $\mathcal{X}$ be an sse groupoid. Then:*

(i) *the projection $\pi : X_0 \to |X|$ is open.*
(ii) *$|X|$ is Hausdorff iff $s \times t$ has closed image.*

Proof. Let $U \subset X_0$ be open. We must show that $|U| := \pi(U)$ is open in the quotient topology, i.e. that $\pi^{-1}(\pi(U)) = t(s^{-1})(U)$ is open in X_0. But $s^{-1}(U)$ is open in X_1 since s is continuous and t is an open map because it is a local diffeomorphism. This proves (i).

Now suppose that $\mathrm{Im}(s \times t)$ is closed and let p, q be any two distinct points in $|X|$. Choose $x \in \pi^{-1}(p)$ and $y \in \pi^{-1}(q)$. Then $(x, y) \notin \mathrm{Im}(s \times t)$ and so there is a neighborhood of (x, y) of the form $U_x \times U_y$ that is disjoint from $\mathrm{Im}(s \times t)$. Then $|U_x|$ and $|U_y|$ are open in $|X|$ by (i). If $|U_x| \cap |U_y| \neq \emptyset$ there is $x' \in \pi^{-1}(|U_x|), y' \in \pi^{-1}(|U_y|)$ such that $x' \sim y'$. Then there is $x'' \in U_x$ such that $x' \sim x''$ and $y'' \in U_y$ such that $y' \sim y''$. Therefore by transitivity $x'' \sim y''$, i.e. $U_x \times U_y$ meets $\mathrm{Im}(s \times t)$, a contradiction. Hence $|U_x|$ and $|U_y|$ are disjoint neighborhoods of p, q, and $|X|$ is Hausdorff. The proof of the converse is similar. Hence (ii) holds. □

Remark 2.3. (i) The properness condition is essential to distinguish orbifolds from branched manifolds. For example, it is easy to define a (nonproper) sse groupoid $\mathcal{B}$ with orbit space $|B|$ equal to the quotient of the disjoint union $B_0 := (0, 2) \sqcup (3, 5)$ by the equivalence relation $x \sim y$ for $y = x + 3, x \in (0, 1)$; cf. Figure 1 and Example 3.3 (i). Note that $|B|$ is nonHausdorff, but can be made Hausdorff by identifying the points 1 and

[4]This terminology, taken from [**18**], is inspired by the finiteness condition satisfied by stable maps.

[5]Note that each $\gamma \in G_x$ extends to a local diffeomorphism ϕ_γ of X_0 in the following way. Let V be a neighborhood of γ in X_1 on which the source and target maps s and t are injective. Then ϕ_γ maps $s(V)$ to $t(V)$ by $s(\delta) \mapsto t(\delta), \delta \in V$. Thus the point of Robbin–Salamon's argument is to show that we may assume that $s(V) = t(V) = U_x$.

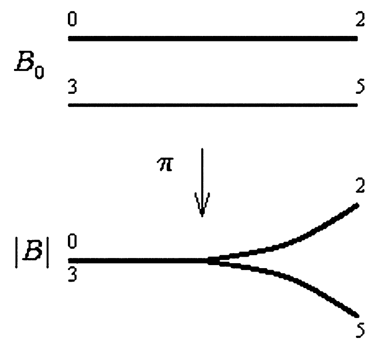

Figure 1. The projection π for a nonproper groupoid.

4; cf. Lemma 3.1. The transverse holonomy groupoid of the Reeb foliation (cf. Haefliger [**6**]) is also of this kind.
(ii) To say that $s \times t$ is a closed map is different from saying that its image is closed. For example, one could define a noneffective groupoid $\mathcal{X}$ with objects $X_0 = S^1 = \mathbb{R}/\mathbb{Z}$ and with $s = t$ but such that X_1 has infinitely many components. Such $\mathcal{X}$ can be stable, for example if $X_1 = X_0 \sqcup (\sqcup_{k\geq 3} V_k)$ where each element in V_k has order 2 and $s(V_k) = t(V_k) = (\frac{1}{3k+2}, \frac{1}{3k})$. Then the image of $s \times t$ is the diagonal in $X_0 \times X_0$ and so is closed. But the map $s \times t$ is not closed. (If $A = \{\gamma_k \in V_k : s(\gamma_k) = \frac{1}{3k+1}, k \geq 3\}$, A is closed but its image under $s \times t$ is not.) Note that, even when $\mathcal{X}$ is proper, the fibers of the projection $\pi : X_0 \to |X|$ need not be finite.
(iii) If an ep groupoid $\mathcal{X}$ is nonsingular, then the orbit space $|X|$ is a manifold.

The groupoid in (ii) above has the awkward property that, despite being connected, some of its points have effective stability groups and some do not. The following lemmas show that this cannot happen in the proper case. Though they are well known, we include them for the sake of completeness.

Lemma 2.4. *Suppose that $\mathcal{X}$ is a connected ep groupoid. Suppose further that for some $x \in X_0$ every element of the stabilizer group G_x acts effectively. Then $\mathcal{X}$ is effective.*

Proof. If every element of G_x acts effectively and $\pi(x) = \pi(y) \in |X|$, then the elements of G_y act effectively. Therefore we may partition $|X|$ into two disjoint sets W_e and W_n, the first being the image of points where G_x acts effectively and the second the image of points where the action of G_x is not effective. The set W_n is always open as it is the image of the open subset

$$V := \{\gamma \in X_1 : s(\gamma') = t(\gamma') \text{ for all } \gamma' \text{ in some neighborhood of } \gamma\}$$

under the composite $\pi \circ s$, which is open by Lemma 2.2(i). If $\mathcal{X}$ is proper, then W_n is also closed. For if not, there is a convergent sequence $p_k \in W_n$

whose limit $p \notin W_n$. Choose $x \in \pi^{-1}(p)$ and a sequence $x_k \in \pi^{-1}(p_k)$ with limit x. For each k there is $\gamma_k \in G_{x_k} \setminus \{\mathrm{id}\}$ that acts as the identity near x_k. Since the sequence (x_k, x_k) converges to (x, x), properness of the map $s \times t$ implies that the γ_k have a convergent subsequence (also called γ_k) with limit γ. Then $\gamma \in G_x$ and hence has finite order. On the other hand, it extends to a local diffeomorphism near x that equals the identity near the x_k. This is possible only if $\gamma = \mathrm{id}$. But this is impossible since $\gamma_k \neq \mathrm{id}$ for all k and the set of identity morphisms is open in X_1. Since $|X|$ is assumed connected and W_e is nonempty by hypothesis, we must have $W_e = |X|$. □

Lemma 2.5. *Suppose that $\mathcal{X}$ is a connected ep groupoid. Then:*

(i) *the isomorphism class of the subgroup K_y of G_y that acts trivially on X_0 is independent of $y \in X_0$;*
(ii) *there is an associated effective groupoid $\mathcal{X}_{\mathrm{eff}}$ with the same objects as $\mathcal{X}$ and where the morphisms from x to y may be identified with the quotient* $\mathrm{Mor}_{\mathcal{X}}(x,y)/K_y$.

Proof. Given $\gamma \in K_y$ denote by V_γ the component of X_1 containing γ. Then, by properness, the image of $s : V_\gamma \to X_0$ is the component U_y of X_0 containing y. Moreover, as in the previous lemma, $s = t$ on V_γ. Since this holds for all $\gamma \in K_y$, the groups $K_z, z \in U_y$, all have the same number of elements. Moreover, they are isomorphic because the operation of composition takes $V_\delta \times V_\gamma$ to $V_{\delta\circ\gamma}$. Statement (i) now follows because $|X|$ is connected, and the groups K_y are isomorphic as y varies in a fiber of $\pi : X_0 \to |X|$.

To prove (ii), define an equivalence relation on X_1 by setting $\delta \sim \delta'$ iff the morphisms $\delta, \delta' : x \to y$ are such that $\delta \circ (\delta')^{-1} \in K_y$. It is easy to check that these equivalence classes form the morphisms of the category $\mathcal{X}_{\mathrm{eff}}$. □

The following definitions are standard.

Definition 2.6. *Let $\mathcal{X}, \mathcal{X}'$ be sse groupoids. A functor $F : \mathcal{X}' \to \mathcal{X}$ is said to be* **smooth** *if the induced maps $X_i' \to X_i, i = 0, 1,$ are smooth. The pair $(\mathcal{X}', F)$ is said to be a* **refinement** *of $\mathcal{X}$ if*

(i) *The induced map $F : X_0' \to X_0$ is a (possibly nonsurjective) local diffeomorphism that induces a homeomorphism $|X'| \to |X|$;*
(ii) *For all $x' \in X_0'$, F induces an isomorphism $G_{x'} \to G_{F(x')}$.*

Two sse groupoids are **equivalent** *if they have a common refinement.*

Remark 2.7. If $(\mathcal{X}', F)$ refines $\mathcal{X}$ then the morphisms in $\mathcal{X}'$ are determined by the map $F : X_0' \to X_0$. Indeed for any pair U, V of components of X_0' the space $\mathrm{Mor}_{\mathcal{X}'}(U, V)$ of morphisms with source in U and target in V is

$$\mathrm{Mor}_{\mathcal{X}'}(U,V) = \{(x,\gamma,y) \in U \times X_1 \times V | s(\gamma) = F(x), t(\gamma) = F(y)\}.$$

For short, we will say that the morphisms in $\mathcal{X}'$ are pulled back from those in $\mathcal{X}$. Moreover $F : X_0' \to X_0$ can be any local diffeomorphism whose image surjects onto $|X|$. In particular, if $\mathcal{U} = \{U_i\}_{i\in A}$ is any collection of open subsets of X_0 that projects to a covering of $|X|$, there is a unique refinement $\mathcal{X}'$ of $\mathcal{X}$ with objects $\sqcup_{i\in A} U_i$. It follows that any category $\mathcal{X}''$ with the same objects as $\mathcal{X}$ but fewer morphisms is significantly different from $\mathcal{X}$; cf. Example 2.10. Later we will see that under certain conditions the corresponding functor $\mathcal{X}'' \to \mathcal{X}$ has the structure of a layered covering; cf. Lemma 3.22.

The proof that the above notion of equivalence is an equivalence relation on sse groupoids is based on the fact that if $F' : \mathcal{X}' \to \mathcal{X}, F'' : \mathcal{X}'' \to \mathcal{X}$ are two refinements of $\mathcal{X}$ their fiber product $\mathcal{X}'' \times_{\mathcal{X}} \mathcal{X}'$ refines both $\mathcal{X}'$ and $\mathcal{X}''$. Here we use the so-called weak fiber product of Moerdijk–Mrčun [**16**] (see also [**15**, Section 2.3]) with objects[6] given by the homotopy pullback

$$\{(x'', \gamma, x') \in X_0'' \times X_1 \times X_0' : F''(x'') = \gamma(F'(x'))\}.$$

Morphisms $(x'', \gamma, x') \to (y'', \delta, y')$ are pairs $(\alpha'', \alpha') \in X_1'' \times X_1'$, where $\alpha' : x' \to y', \alpha'' : x'' \to y''$, such that the following diagram commutes:

$$\begin{array}{ccc} F'(x') & \xrightarrow{F'(\alpha')} & F'(y') \\ \gamma\downarrow & & \delta\downarrow \\ F''(x'') & \xrightarrow{F''(\alpha'')} & F''(y''). \end{array}$$

Thus $(\alpha'', \alpha') : (x'', \gamma, x') \to \big(\alpha''(x''), F''(\alpha'') \circ \gamma \circ F'(\alpha')^{-1}, \alpha'(x')\big)$.

Lemma 2.8. *Let $\mathcal{X}, \mathcal{X}', \mathcal{X}''$ be sse groupoids and $F' : \mathcal{X}' \to \mathcal{X}, F'' : \mathcal{X}'' \to \mathcal{X}$ be smooth functors.*

(i) *If $\mathcal{X}'$ is nonsingular and F'' is injective on each group $G_{x''}$ then $\mathcal{X}'' \times_{\mathcal{X}} \mathcal{X}'$ is nonsingular.*
(ii) *F'' is an equivalence iff the projection $\mathcal{X}'' \times_{\mathcal{X}} \mathcal{X}' \to \mathcal{X}'$ is an equivalence.*

Proof. This is straightforward, and is left to the reader. □

2.2. Orbifolds and atlases.

Definition 2.9. *An* **orbifold structure** *on a paracompact Hausdorff space Y is a pair $(\mathcal{X}, f)$ consisting of an ep groupoid $\mathcal{X}$ together with a homeomorphism $f \colon |X| \to Y$. Two orbifold structures $(\mathcal{X}, f)$ and $(\mathcal{X}', f')$ are*

[6]Note that we cannot take the objects to be the usual (strict) fiber product

$$X_0'' \times_{X_0} X_0' = \{(x'', x') \in X_0'' \times X_0' : F''(x'') = F'(x')\}.$$

This definition is useful only when one of the maps $X_0' \to X_0, X'' \to X_0$ is surjective; for a general equivalence, the above space might be empty. We also cannot use the fiber product

$$X_0'' \times_{|X|} X_0'' = \{(x'', x') \in X_0'' \times X_0' : \pi \circ F''(x'') = \pi \circ F'(x')\}$$

since this is only guaranteed to be a manifold if one of the projections $X_0' \to |X|, X_0'' \to |X|$ is a local submersion.

equivalent if they have a common refinement, i.e., if there is a third structure $(\mathcal{X}'', f'')$ *and refinements* $F : \mathcal{X}'' \to \mathcal{X}, F' : \mathcal{X}'' \to \mathcal{X}'$ *such that* $f'' = f \circ |F| = f' \circ |F'|$.

An **orbifold** $\underline{Y}$ *is a second countable paracompact Hausdorff space* Y *equipped with an equivalence class of orbifold structures. An* **orbifold map** $\underline{\phi} : \underline{X} \to \underline{Y}$ *is an equivalence class of functors* $\Phi : \mathcal{X} \to \mathcal{Y}$, *where* $(\mathcal{X}, f)$ *and* $(\mathcal{Y}, g)$ *are orbifold structures on* $\underline{X}$ *and* $\underline{Y}$, *respectively. The equivalence relation is generated by the obvious notion of refinement of functors. Thus if* $F : \mathcal{X}' \to \mathcal{X}$ *and* $F' : \mathcal{Y}' \to \mathcal{Y}$ *are refinements,* $\Phi' : \mathcal{X}' \to \mathcal{Y}'$ *is said to refine* $\Phi : \mathcal{X} \to \mathcal{Y}$ *if there is a natural transformation between the two functors* $\Phi \circ F, F' \circ \Phi' : \mathcal{X}' \to \mathcal{Y}$.

Each orbifold map $\underline{\phi} : \underline{X} \to \underline{Y}$ induces a well defined continuous map $\phi : X \to Y$ on the spaces X, Y underlying $\underline{X}, \underline{Y}$. Note that it is possible for different equivalence classes of functors to induce the same map $X \to Y$. For more details, see Moerdijk [**15**, Section 2,3]. As pointed out by Lerman [**11**], one really should take a more sophisticated approach to defining orbifolds and orbifold maps in order for them to form some kind of category. Since the focus here is on defining a new class of objects, we shall ignore these subtleties.

Example 2.10. The teardrop orbifold and its resolution. This orbifold has underlying space $Y = S^2$ and one singular point p at the north pole of order k; cf. Figure 2. Cover Y by two open discs D_+, D_- of radius $1 + \varepsilon$ that intersect in the annulus $A = (1 - \varepsilon, 1 + \varepsilon) \times S^1$ and are invariant by rotation about the north/south poles. Denote by $\phi : A \to A$ the k-fold covering map given in polar coordinates by $(r, \theta) \mapsto (2 - r, k\theta)$ and by R_t rotation through the angle $2\pi t$. An orbifold structure on Y is provided by the proper groupoid $\mathcal{X}$ whose space of objects X_0 is the disjoint union

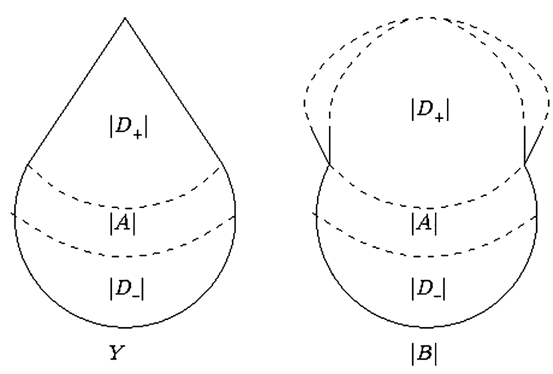

Figure 2. The teardrop orbifold and its resolution $\mathcal{B}$ when $k = 2$.

$D_+ \sqcup D_-$, with morphisms

$$X_1 := \big(X_0 \times \{1\}\big) \ \sqcup\ \big(D_+ \times \{\gamma_1, \dots, \gamma_{k-1}\}\big) \ \sqcup\ \big(A \times \{\phi_+, \phi_-\}\big).$$

Here 1 acts as the identity and $(x, \gamma_j), x \in D_+$, denotes the morphism with target $x \in D_+$ and source $R_{j/k}(x)$ in D_+. Further the pair (x, ϕ_+) is the unique morphism with source $x \in A_+ := A \cap D_+$ and target $\phi(x) \in A_- := A \cap D_-$, By definition $(x, \phi_-) := (x, \phi_+)^{-1}$. Thus $|X| = D_+ \cup D_-/\sim$ where $x \in D_+$ is identified with $R_{j/k}(x)$ and D_+ is attached to D_- over A by a k to 1 map.

By way of contrast, consider the groupoid $\mathcal{B}$ formed from $\mathcal{X}$ by omitting the morphisms $(D_+ \setminus A_+) \times \{\gamma_1, \dots, \gamma_{k-1}\}$. This groupoid is nonsingular, but is no longer proper since the restriction of $(s, t) : B_1 \to B_0 \times B_0$ to the component of morphisms $A_+ \times \{\gamma_1\}$ is not proper. Note that $|B|$ is a branched manifold: k (local) leaves come together along the boundary $\partial|D_-| \subset |B|$. Further, if one weights the leaves over $|D_+|$ by $1/k$ the induced map $|B| \to |X|$ represents the fundamental class of $|X| = Y$. We call $\mathcal{B}$ the resolution of Y.

Often it is convenient to describe an orbifold structure on $\underline{Y}$ in terms of local charts. Here are the relevant definitions.

Definition 2.11. *A* **local uniformizer** (U_i, G_i, π_i) *for an (orientable) orbifold $\underline{Y}$ is a triple consisting of a connected open subset $U_i \subset \mathbb{R}^d$, a finite group G_i that acts by (orientation preserving) diffeomorphisms of U_i and a map $\pi_i : U_i \to Y$ that factors through a homeomorphism from the quotient U_i/G_i onto an open subset $|U_i|$ of Y. Moreover, this uniformizer determines the smooth structure of $\underline{Y}$ over $|U_i|$ in the sense that for one (and hence any) orbifold structure $(\mathcal{X}, f)$ on $\underline{Y}$ the projection $f \circ \pi : X_0 \to Y$ lifts to a local diffeomorphism $(f \circ \pi)^{-1}(|U_i|) \to U_i$.*

A **good atlas** *for $\underline{Y}$ is a collection $\mathcal{A} = \{(U_i, G_i, \pi_i), i \in A\}$ of local uniformizers whose images $\{|U_i|, i \in A\}$ form a locally finite covering of Y.*

It is shown in Moerdijk–Pronk [**17**, Corollary 1.2.5] that every effective orbifold has such an atlas. The argument works equally well in the general case since it is based on choosing an adapted triangulation of $|X|$. In fact, one can also assume that the U_i and their images $|U_i|$ are contractible and that the $|U_i|$ are closed under intersections. Here we assume as always that $|X|$ is finite dimensional. The arguments of [**17**] do not apply in the infinite dimensional case. However Robbin–Salamon [**18**, Lemma 2.10] show that in any category an orbifold has a good atlas. They start from a groupoid structure $(\mathcal{X}, f)$ on $\underline{Y}$ and construct for each $x \in X_0$ a local uniformizer (U, G, π) that embeds in $\mathcal{X}$ in the sense that $U \subset X_0$ and the full subcategory of $\mathcal{X}$ with objects U is isomorphic to $U \times G$.

Let $(U_i, G_i, \pi_i), i \in A$, be a locally finite cover of Y_0 by such uniformizers, and denote by $\mathcal{X}'$ the full subcategory of $\mathcal{X}$ with objects $\sqcup_{i \in A} U_i$. Since this is a refinement of $\mathcal{X}$, it is another orbifold structure on $\underline{Y}$. Observe that the morphisms in $\mathcal{X}'$ with source and target U_i can be identified with $U_i \times G_i$. In this situation, we say that $\mathcal{X}'$ is **constructed from the good atlas** $(U_i, G_i, \pi_i), i \in A$. Many interesting orbifolds, for example, the ep polyfold groupoids considered in [**10**], are constructed in this way.

The following definition will be useful.

Definition 2.12. *Let $\mathcal{X}$ be an ep groupoid. A point $p \in |X|$ is said to be smooth if for one (and hence every) point $x \in \pi^{-1}(p) \subset X_0$ every element $\gamma \in G_x$ acts trivially near x.*

Thus when $\mathcal{X}$ is effective p is smooth iff $G_x = \{\mathrm{id}\}$ for all $x \in \pi^{-1}(p)$. Note that for any local uniformizer (U, G, π) the fixed point set of each $\gamma \in G$ is either the whole of U or is nowhere dense in U. It follows that the smooth points $|X|^{sm}$ form an open and dense subset of $|X|$.

We now show that if $\underline{Y}$ is effective its orbifold structure is uniquely determined by the charts in any good atlas; there is no need for further explication of how these charts fit together. This result is well known: cf. [**16**, Prop. 5.29]. We include a proof to clarify ideas. It is particularly relevant in view of the work of Chen–Ruan [**2**] and Chen [**3**] on orbifold Gromov–Witten invariants, that discusses orbifolds from the point of view of charts. Note that the argument applies to orbifolds in any category, and in particular to polyfold groupoids.

Lemma 2.13. *An effective orbifold $\underline{Y}$ is uniquely determined by the charts $(U_i, G_i, \pi_i), i \in A$, of a good atlas.*

Proof. Here is an outline of the proof. We shall construct an orbifold structure $(\mathcal{X}, f)$ on $\underline{Y}$, such that $\mathcal{X}$ is a groupoid whose objects X_0 are the disjoint union of the sets $U_i, i \in A$, and whose morphisms with source and target in U_i are given by G_i. The construction uses the fact that $\underline{Y}$ has an effective orbifold structure $(\mathcal{Z}, g)$, but the equivalence class of$(\mathcal{X}, f)$ is independent of the choice of $(\mathcal{Z}, g)$. It follows that $(\mathcal{X}, f)$ and $(\mathcal{Z}, g)$ are equivalent, and that the orbifold structure of $\underline{Y}$ is determined by the local charts.

We described above the objects in $\mathcal{X}$ and some of the morphisms. To complete the construction, we must add to X_1 some components C^{ij} of morphisms from U_j to U_i for all $i, j \in A$ such that $|U_i| \cap |U_j| \neq \emptyset$. For smooth points $x \in U_i^{sm}, y \in U_j^{sm}$ the set $\mathrm{Mor}(x, y)$ has at most one element and is nonempty iff $\pi_i(x) = \pi_j(y) \in |X| = Y$. Hence the given data determine the set X_1^{sm} of all morphisms whose source or target is smooth. We shall see that there is a unique way to complete the morphism space X_1.

To begin, fix $i \neq j$ such that $|U_{ij}| := |U_j| \cap |U_i| \neq \emptyset$. Given $y \in U_j$ denote by $V_{ji}^y \subset U_j$ the connected component of $U_j \cap \pi^{-1}(|U_{ij}|)$ that contains y. The key point is that for any point $x \in U_i$ with $\pi_i(x) = \pi_j(y)$ there is a local diffeomorphism ϕ_{xy} from a neighborhood $N_j(y)$ of y in U_j to a neighborhood

$N_i(x)$ of x in U_i such that $\pi_j = \pi_i \circ \phi$. This holds because there is an orbifold structure $(\mathcal{Z}, g)$ on Y. Namely, choose $z \in Z_0$ so that $g(z) = \pi_i(x) = \pi_j(y)$. Shrinking $N_j(y)$ and $N_i(x)$ if necessary, we may suppose that there is a neighborhood N_z of z in Z_0 such that $g(N_z) = \pi_i(N_i(x)) = \pi_j(N_j(y))$. Since (U_i, G_i, π_i) is a local chart for $\pi_i(U_i) \subset Y$, there is a local diffeomorphism $\psi_i : N_z \to U_i$ such that $g = \pi_i \circ \psi_i$. Moreover, since π_i is given by quotienting by the action of G_i there is $\gamma \in G_i$ such that $\psi_{xz}(z) := \gamma \circ \psi_i(z) = x$. Denoting by ψ_{yz} the similar map for $y \in U_j$, we may take $\phi_{xy} := \psi_{xz} \circ (\psi_{yz})^{-1}$.

Since ϕ_{xy} is determined by its restriction to the dense open set of smooth points, it follows that there are precisely $|G_y|$ such local diffeomorphisms, namely the composites $\phi_{xy} \circ \gamma, \gamma \in G_y$. These local diffeomorphisms form the local sections of a sheaf over V_{ji}^y, whose stalk at y consists of the $|G_y|$ elements $\phi_{xy} \circ \gamma$. The existence and uniqueness properties of the ϕ_{xy} imply that each local section is the restriction of a unique, but possibly multivalued, global section σ of this sheaf. We may identify the graph of σ (which is a manifold) with a component C_σ of the space of morphisms from U_j to U_i. Since $|G_y|$ is finite, the source map $s : C_\sigma \to V_{ji}^y$ is a surjective and finite-to-one covering map. The set $\{\sigma_1, \dots, \sigma_\ell\}$ of all such global sections (for different choices of $x \in U_i, y \in U_j$) is invariant under the action of G_j by precomposition and of G_i by postcomposition. We define the space of morphisms in $\mathcal{X}$ from U_j to U_i to consist of the ℓ components $C_{\sigma_k}, 1 \le k \le \ell$.

This defines the morphisms in $\mathcal{X}$ from U_j to U_i. We then complete X_1 by adding the inverses to these elements and all composites. The resulting composition operation is associative because its restriction to the smooth elements is associative. Thus $\mathcal{X}$ is an sse Lie groupoid. Since the atlas is locally finite, the projection $X_0 \to |X|$ is finite to one. Moreover, the induced map $|X| \to Y$ is a homeomorphism. Hence $\mathcal{X}$ is proper because Y is Hausdorff. (See also [**18**, Cor 2.11].) Hence $\mathcal{X}$ is an ep groupoid.

Moerdijk–Mrčun prove that $\mathcal{Z}$ and $\mathcal{X}$ are equivalent by looking at the corresponding groupoids of germs of diffeomorphisms. Alternatively, define $\mathcal{Z}'$ to be the refinement of $\mathcal{Z}$ with objects $\sqcup_{z \in A} N_z$, where $A \subset Z_0$ is large enough that the sets $g(N_z), z \in A$, cover Y. For each $z \in A$ choose one of the corresponding local diffeomorphisms ψ_{xz} and call it f_z. Then define $F : \mathcal{Z}' \to \mathcal{X}$ by setting

$$F|_{N_z} = f_z, \qquad F(\gamma) = f_w \circ \gamma \circ (f_z)^{-1},$$

where $s(\gamma) = z$ and $t(\gamma) = w$. Then F is an equivalence, as required. □

Remark 2.14. This lemma is false when $\underline{Y}$ is not effective. To see this, let K be the cyclic group $\mathbb{Z}/3\mathbb{Z}$ and consider two groupoids $\mathcal{Z}, \mathcal{X}$ both with objects S^1. We assume in both cases that at all objects the stabilizer group is K and that there are no other morphisms. This implies that $s = t$. We define $\mathcal{Z}$ to be topologically trivial, with $Z_1 = S^1 \times K$ (i.e. three copies of S^1) and $s = t$ equal to the projection $S^1 \times K \to S^1$. On the other hand, we define X_1 to be the disjoint union of two copies of S^1. In this case, $s = t : X_1 \to X_0$ is the identity on one circle (the one corresponding to the identity morphisms) and is a double cover on the second. It is easy to check that these groupoids are not equivalent. (One way to see this is to notice that their classifying spaces[7] $B\mathcal{Z}, B\mathcal{X}$ are not homotopy equivalent.) However, they have the same local uniformizers over any proper open subset of S^1.

These groupoids $\mathcal{Z}, \mathcal{X}$ are totally noneffective. According to Henriques–Metzler [**8**], the best way to understand their structure is to think of them as a special kind of gerbe.

2.3. Fundamental cycles and cobordism. We now summarize known facts about homology and cobordism in the context of orbifolds, since we will later generalize them to the branched case. Note first that if a compact space Y admits the structure of an oriented d-dimensional orbifold (without boundary) there is a distinguished d-dimensional cycle $[Y]$ in the singular chain complex of Y that we will call the **fundamental cycle** of Y. One way to see this is to give Y an adapted triangulation as in Moerdijk–Pronk [**17**]. In such a triangulation the open simplices of dimensions d and $d-1$ lie in the smooth[8] part Y^{sm} of Y, and $[Y]$ is represented by the sum of the d-simplices. Another way to define $[Y]$ is to interpret the inclusion $Y^{sm} \to Y$ as a pseudocycle: see Schwarz [**22**] or Zinger [**23**]. These definitions imply that the fundamental cycle of the orbit space $|X|$ of $\mathcal{X}$ is the same as that of its reduction $\mathcal{X}_{\text{eff}}$.[9]

A smooth map of an orbifold $\underline{Y}$ to a manifold M is a map $\phi : Y \to M$ such that for one (and hence every) groupoid structure $\mathcal{X}$ on $\underline{Y}$ the induced map $X_0 \to M$ is smooth. Given two such maps $\phi : Y \to M, \phi' : Y' \to M$, one can perturb ϕ to be transverse to ϕ'. This means that there are adapted triangulations on Y, Y' that meet transversally. In particular, if Y and Y' are oriented and have complementary dimensions, their images in M intersect in smooth points and the intersection number $\phi_*([Y]) \cdot \phi'_*([Y'])$ is defined. Further, if α is a d-form on M that is Poincaré dual to the class $\phi'_*([Y'])$ then

$$\phi_*([Y]) \cdot \phi'_*([Y']) = \int_Y \phi^*(\alpha) := \int_{Y^{sm}} \phi^*(\alpha).$$

Suppose now that $\mathcal{W}$ is an ep groupoid in the category of oriented $(d+1)$-dimensional manifolds with boundary. Then $\mathcal{W}$ has a well defined boundary

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[7]The classifying space $B\mathcal{X}$ is the realization of the nerve of the category, and is denoted $|\mathcal{X}_\bullet|$ in [**15**].

[8]If $\underline{Y}$ is oriented its nonsmooth points have codimension at least 2 since the fixed point submanifold of an orientation preserving linear transformation of $\mathbb{R}^n$ has codimension at least 2.

[9]Note that here we are talking about the ordinary (singular) cohomology of Y, not the stringy cohomology defined by Chen–Ruan [**2**] (cf also Chen–Hu [**1**]).

groupoid $\partial\mathcal{W}$. If in addition $|W|$ is compact and connected, the fundamental cycle of $|W|$ generates $H_{d+1}(|W|,\partial|W|)$. We may therefore make the usual definition of cobordism for orbifolds. We say that two d-dimensional oriented orbifolds Y_1, Y_2 are **cobordant** if there is a $(d+1)$-dimensional groupoid $\mathcal{W}$ whose boundary $\partial\mathcal{W}$ is the union of two disjoint oriented groupoids $-\mathcal{X}_1, \mathcal{X}_2$, that represent $-Y_1, Y_2$ respectively. (Here $-\mathcal{X}$ denotes the groupoid formed from $\mathcal{X}$ by reversing the orientation of X_0.) As usual, if there is a class $\alpha \in H^d(|W|)$ whose restriction to $|\partial W| = -|X_1| \sqcup |X_2|$ is $-\alpha_1 + \alpha_2$, where $\alpha_i \in H^d(Y_i) \equiv H^d(|X_i|) \subset H^d(|\partial W|)$, then $\alpha_1([Y_1]) = \alpha_2([Y_2])$.

3. Weighted nonsingular branched groupoids

3.1. Basic definitions. We saw in Remark 2.3(ii) above that the category of nonproper sse groupoids contains some rather strange objects, not just branched objects such as the teardrop but also groupoids that are not effective in which the trivially acting part K_x of the stabilizer subgroups is not locally constant. In order to get a good class of branched groupoids that carries a fundamental cycle, we need to impose a weighting condition and to ensure that the groupoids are built by assembling well behaved pieces called local branches. Definition 3.2 adapts Salamon [**19**, Def. 5.6] to the present context; it is also very close to the definition in Cieliebak *et al.* [**4**] of a branched submanifold of Euclidean space. For the sake of simplicity, we shall restrict to the nonsingular and oriented case.

We shall use the concept of the **maximal Hausdorff quotient** Y_H of a nonHausdorff topological space Y. This is a pair (f, Y_H) that satisfies the following conditions:

(i) $f : Y \to Y_H$ is a surjection and Y_H has the corresponding quotient topology,
(ii) Y_H is Hausdorff,
(iii) any continuous surjection $f_\alpha : Y \to Y_\alpha$ with Hausdorff image factors through f.

Lemma 3.1. *Every topological space Y has a maximal Hausdorff quotient (f, Y_H).*

Proof. To construct Y_H, denote by A the set of all equivalence classes of pairs (g_α, Y_α) where Y_α is a Hausdorff topological space and $g_\alpha : Y \to Y_\alpha$ is a continuous surjection. (Two such pairs are equivalent if there is a homeomorphism $\phi : Y_\alpha \to Y_\beta$ such that $\phi \circ g_\alpha = g_\beta$.) The product $Y_\pi := \prod_{\alpha\in A} Y_\alpha$ is Hausdorff in the product topology and there is a continuous map $f_\pi : Y \to Y_\pi$. Define $Y_H := \mathrm{Im}\, f_\pi$ with the subspace topology τ_s. Then conditions (ii) and (iii) above hold, and there is a continuous surjection $f : Y \to Y_H$. We must show that (i) holds, i.e., that τ_s coincides with the quotient topology τ_q. Since the identity map $(Y_H, \tau_q) \to (Y_H, \tau_s)$ is continuous, (Y_H, τ_q) is Hausdorff. Therefore by (iii), the map $f : Y \to (Y_H, \tau_q)$ factors through (Y_H, τ_s). Thus the identity map $(Y_H, \tau_s) \to (Y_H, \tau_q)$ is also continuous; in other words, the topologies τ_s and τ_q coincide. □

If $\mathcal{B}$ is a nonproper sse groupoid, then $|B|$ may not be Hausdorff; cf. Lemma 2.2. It will be useful to consider the Hausdorff quotient $|B|_H$.[10] We shall denote by $|\pi|_H$ the projection $|B| \to |B|_H$ and by π_H the projection $B_0 \to |B|_H$. Further $|U|_H := \pi_H(U)$ denotes the image of $U \subset B_0$ in $|B|_H$.

Definition 3.2. *A* **weighted nonsingular branched groupoid** *(or* **wnb groupoid** *for short) is a pair $(\mathcal{B}, \Lambda)$ consisting of an oriented, nonsingular sse Lie groupoid together with a weighting function $\Lambda : |B|_H \to (0, \infty)$ that satisfies the following compatibility conditions. For each $p \in |B|_H$ there is an open neighborhood N of p in $|B|_H$, a collection $U_1, \ldots, U_\ell$ of disjoint open subsets of $\pi_H^{-1}(N) \subset B_0$ (called* **local branches***) and a set of positive weights $m_1, \ldots, m_\ell$ such that:*
(Covering) $|\pi|_H^{-1}(N) = |U_1| \cup \cdots \cup |U_\ell| \subset |B|$*;*
(Local Regularity) *for each $i = 1, \ldots, \ell$ the projection $\pi_H : U_i \to |U_i|_H$ is a homeomorphism onto a relatively closed subset of N;*
(Weighting) *for all $q \in N$, $\Lambda(q)$ is the sum of the weights of the local branches whose image contains q:*

$$\Lambda(q) = \sum_{i: q \in |U_i|_H} m_i.$$

The tuple (N, U_i, m_i) is said to form a **local branching structure** *at p. Sometimes we denote it by (N^p, U_i^p, m_i^p) to emphasize the dependence on p. $\mathcal{B}$ is called* **compact** *if its Hausdorff orbit space $|B|_H$ is compact. The points $p \in |B|_H$ that have more than one inverse image in $|B|$ will be called* **branch points**.

Example 3.3. (i) The groupoid $\mathcal{B}$ of Remark 2.3(i) has a weighting in which $\Lambda = 1$ on the image of $(0, 1]$ in $|B|_H$ and $= 1/2$ on the rest of $|B|_H$. There is one branch point at the image p_1 of 1, and the weighting condition implies that the weights satisfy Kirchoff's law at this branch point. Moreover, we may take the local branches at p_1 to be the two components of B_0 each weighted by $m_1 = m_2 = 1/2$. See Figure 3 for a similar example. Similarly, the groupoid $\mathcal{Z}$ defined in Example 2.10 has a weighting function in which $\Lambda = 1$ on the closure $cl(|D_-|_H)$ of $|D_-|_H$ and $\Lambda = 1/k$ elsewhere on $|Z|_H$. The branch locus is formed by the boundary of $|D_-|_H$.
(ii) If $(\mathcal{B}, \Lambda)$ is compact and zero dimensional, then it is necessarily proper. Hence $|B|_H = |B|$ consists of a finite number of equivalence classes p of the relation $\sim$, each with a label Λ_p. Since the morphisms in $\mathcal{B}$ preserve

[10] Note that $|B| = |B|_H$ if $\mathcal{B}$ is proper.

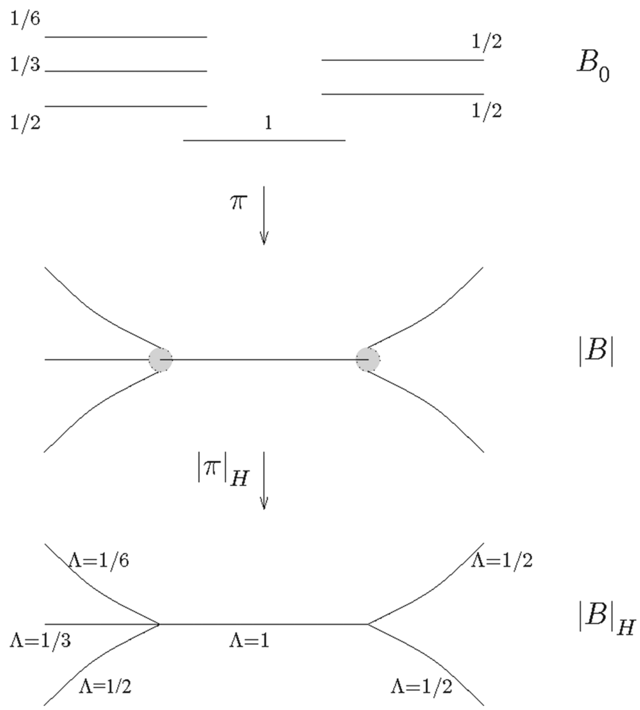

Figure 3. Local branching structures of a weighted branched groupoid.

orientation, all the points $x \in B_0$ in the equivalence class p have the same orientation. Therefore, p has a second label $\mathfrak{o}_p$ consisting of a sign $\mathfrak{o}_p = \pm$ that describes its orientation.[11]

Thus the **number of points** in $\mathcal{B}$ may be defined as $\sum_{p\in|B|_H} \mathfrak{o}_p\Lambda(p)$.

(iii) In both of the above cases Λ is induced by a weighting function λ defined on the components of B_0. However, there are wnb groupoids for which such a function λ is not uniquely defined by Λ: see Remark 3.8 (ii).

The following remark explains some of the technicalities of the above definition.

Remark 3.4. (i) The local branches usually do not comprise the whole of the inverse image of $\pi_H^{-1}(N)$, but it is important that they map onto the full inverse image of N in $|B|$. This condition, together with the fact that each branch injects into $|B|_H$, implies that the projection $|B| \to |B|_H$ is finite-to-one.

(ii) The local regularity condition rules out the trivial example of an arbitrary nonsingular sse groupoid $\mathcal{B}$ with connected object space B_0 (weighted

[11]For further remarks about orientations, see Remark 4.21 and Section 4.3.

by 1) and with $\Lambda \equiv 1$. Note that requiring the U_i to inject into $|B|_H$ is considerably stronger than requiring that they inject into $|B|$. For example, consider the groupoid $\mathcal{B}_\phi$ in which B_0 is the disjoint union of two copies R_1, R_2 of $\mathbb{R}^2$ and the nonidentity morphisms consist of two components each diffeomorphic to the open unit disc D_1 in R_1, one corresponding to a smooth embedding ϕ of D_1 onto a precompact subset of R_2 and the other corresponding to ϕ^{-1}. This groupoid can be weighted by setting $\Lambda(p) = 1$ if $\pi_H^{-1}(p)$ intersects both R_1 and R_2 and $= 1/2$ otherwise. Taking $N_p = |B|_H$ with local branches R_1, R_2 one finds that all the other conditions of Definition 3.2 are satisfied. However, the map $U_i \to |U_i|_H$ is injective iff ϕ extends to a homeomorphism of the closure $cl(D_1)$ onto its image in R_2. In this example, because $\phi(D_1)$ is precompact, it is enough to assume only the injectivity of $U_i \to |U_i|_H$. However, in general, to avoid pathologies, we must assume that this map is a homeomorphism onto its image.

(iii) The other part of regularity is that the branches have closed image in N. This is an essential ingredient of the proof of Lemma 3.10 and of the proof in Proposition 3.6 that Λ is continuous on a dense open subset of $|B|_H$. To see it at work, consider the groupoid $\mathcal{B}$ whose objects are the disjoint union of the plane $\mathbb{R}^2$ and the unit disc (each weighted by 1/2) with two components of nonidentity morphisms given by the inclusion $D \subset \mathbb{R}^2$ and its inverse. Then $|B|_H = \mathbb{R}^2$ and all conditions except for properness are satisfied if we set $\Lambda = 1$ on D and $= 1/2$ elsewhere. Since $\mathcal{B}$ does not accord with our intuitive idea of a branched manifold (for example, if compactified to S^2 it does not carry a fundamental class), it is important to rule it out.

(iv) Because we do not require the local branches to be connected, local branching structures (N, U_i, m_i) behave well under restriction of N: if (N, U_i, m_i) is a local branching structure at p then, for any other neighborhood $N' \subset N$ of p, the sets $U_i \cap \pi_H^{-1}(N')$ are the local branches of a branching structure over N'. Thus each point in $|B|_H$ has a neighborhood basis consisting of sets that support a local branching structure. Note also that, because $|U_i|_H$ is assumed closed in N, U_i is closed in $\pi_H^{-1}(N)$ and so is a union of components of $\pi_H^{-1}(N)$.

(v) We have chosen to impose rather few regularity conditions in order to make the definition of a wnb groupoid as simple and general as possible. However, in order for the quotient space $|B|_H$ to have a reasonable smooth structure (so that, for instance, one can integrate over it) one needs more control over the morphisms; cf. the tameness conditions of Definition 3.20.

We now explain the structure of the Hausdorff quotient $|B|_H$ for wnb groupoids. For general spaces Y it seems very hard to give a constructive definition of its maximal Hausdorff quotient Y_H. However, the covering and local regularity conditions are so strong that the quotient map $B_0 \to |B|_H$ has the following explicit description.

We define $\approx$ to be the equivalence relation on B_0 generated by setting $x \approx y$ if $|x|, |y|$ do not have disjoint open neighborhoods in $|B|$. In particular, if $x \sim y$ (i.e. $\pi(x) = \pi(y) \in |B|$) then $x \approx y$.

Lemma 3.5. *Let $(\mathcal{B}, \Lambda)$ be a wnb groupoid. Then the fibers of $\pi_H : B_0 \to |B|_H$ are the equivalence classes of $\approx$.*

Proof. Since $|B|_H$ is Hausdorff, any two points in $|B|$ that do not have disjoint neighborhoods must have the same image in $|B|_H$. Hence the equivalence classes of $\approx$ are contained in the fibers of π_H. Therefore, there is a continuous surjection $B_\approx \to |B|_H$, where $B_\approx$ denotes the quotient of B_0 by the relation $\approx$, and it suffices to show that $B_\approx$ is Hausdorff. Because of the covering property of the local branches, it suffices to work locally in subsets W of B_0 of the form $W := \pi_H^{-1}(N) = \cup_{i=1}^{\ell} U_i$, where (N, U_i, m_i) is a local branching structure. We show below that for each such W, the quotient $W_\approx$ is Hausdorff. This will complete the proof.

Consider any pair of distinct local branches U_i, U_j and the set $M_{ji} := \mathrm{Mor}(U_i, U_j)$ of morphisms from U_i to U_j. Since U_i, U_j inject into $|B|$ both maps $s : M_{ji} \to U_i, t : M_{ji} \to U_j$ are injective and hence are diffeomorphisms onto their images. Call these $V_{ji} \subset U_i$ and $V_{ij} \subset U_j$. Denote by $cl(V_{ji})$ the closure of V_{ji} in U_i. Since $|U_j|_H$ is relatively closed in $N \supset |U_i|_H \cup |U_j|_H$, the set $\pi_H(V_{ji})$ is contained in $|U_j|_H$. Hence, because π_H is a homeomorphism on each U_i, the diffeomorphism $t \circ s^{-1} : V_{ji} \to V_{ij}$ extends to a homeomorphism $cl(V_{ji}) \to cl(V_{ij})$. Define the ji branch set as

$$Br_{ji} := \partial(V_{ji}) := cl(V_{ji}) \setminus V_{ji} \subset U_i.$$

Then Br_{ji} is closed in U_i. Further there is a homeomorphism $\phi_{ji} : Br_{ji} \to Br_{ij}$.

Now observe that if $x \in Br_{ji}$ then there is a unique $y \in Br_{ij}$ such that $x \approx y$, namely $\phi_{ji}(x)$. Conversely, if $x \in U_i, y \in U_j$ and $x \approx y$ then either $|x| = |y|$ or there are convergent sequences $x_n, y_n \in B_0$ with limits x_∞, y_∞ such that $x_n \sim y_n, x_\infty \sim x, y_\infty \sim y$. The morphism γ from x_∞ to x extends to a neighborhood of x_∞ and so transports (the tail of) the sequence x_n to a sequence $x_n' \in U_i$ that converges to x. Similarly, we may suppose that $y_n \in U_j$. Hence $x \in Br_{ji}, y \in Br_{ij}$. Let us write in this case that $x \approx_{ij} y$.

The equivalence relation $\approx$ therefore has the following description on W. Given $x \in U_i, z \in U_k$, $x \approx z$ iff either $|x| = |z|$ or there is a finite sequence $i_1 := i, i_2, \dots, i_n = k$ of indices (with $n > 1$) and elements $x_j \in Br_{i_{j+1} i_j}$ for $j = 1, \dots, n-1$ such that

$$x_j \approx_{i_j\, i_{j+1}} x_{j+1}, \qquad \text{for all } j.$$

Note that we may assume that all the indices in this chain are different. For, because the maps $\pi_H : U_i \to |B|_H$ are injective and constant on equivalence classes, if $i_j = i_{j'}$ for some $j' > j$ then $x_j = x_{j'}$ so that the intermediate portion of the chain can be omitted. It follows that the number of nonempty chains of this form is bounded by a number depending only on ℓ, the number of local branches. Moreover, for each such chain $I := i_1, \dots, i_n$ the map that takes its initial point to its endpoint is a homeomorphism ϕ_I from a subset $X_I \subset Br_{i_n i_1}$ to a subset $Z_I \subset Br_{i_1 i_n}$, where $\phi_I := \phi_{i_n i_{n-1}} \circ \cdots \circ \phi_{i_2 i_1}$ is the restriction of $\phi_{i_n i_1}$. Note that X_I and Z_I are closed in U_{i_1} and U_{i_n}, respectively. For each $i \neq k$, let C_{ki} be the set of chains from i to k, and consider the set

$$X'_{ki} := \Big(\big(\pi^{-1}(|U_k|) \big) \cap U_i \Big) \;\cup\; \bigcup_{I \in C_{ki}} X_I$$

of all points in U_i that are equivalent to a point in U_k. The above remarks imply that this is closed in U_i.

We now return to the quotient $W_\approx$. To see this is Hausdorff, note first that each equivalence class $\mathbf{x}$ contains at most one element from each U_i. Hence we may write $\mathbf{x} := (x_i)_{i \in J_\mathbf{x}}$, where $x_i \in U_i$ and $J_\mathbf{x} \subset \{1, \dots, \ell\}$. Suppose that $\mathbf{x} \neq \mathbf{y}$. Then $x_i \neq y_i$ for all i. We construct disjoint $\approx$-saturated[12] neighborhoods $V_\mathbf{x}, V_\mathbf{y}$ as follows.

Suppose first that $J_\mathbf{x} \cap J_\mathbf{y} \neq \emptyset$. By renumbering, we may suppose that $1 \in J_\mathbf{x} \cap J_\mathbf{y}$. Choose disjoint open neighborhoods V_{x1}, V_{y1} of x_1, y_1 in U_1 and define V^S_{x1}, V^S_{y1} to be their saturations under $\approx$. Then $V^S_{x1} \cap V^S_{y1} = \emptyset$. To see this note that any point $z \in V^S_{x1} \cap V^S_{y1}$ is equivalent to some point in V_{x1} and some point in V_{y1}. Since z is equivalent to at most one point in U_1 this implies that $V_{x1} \cap V_{y1} \neq \emptyset$, a contradiction.

We now enlarge V^S_{x1}, V^S_{y1} to make them open. To this end, consider the smallest integer i such that either $V^S_{x1} \cap U_i$ or $V^S_{y1} \cap U_i$ is not open. Then $i > 1$ by construction, and for every $j < i$ the disjoint sets $V^S_{x1} \cap X'_{ji}$, $V^S_{y1} \cap X'_{ji}$ are relatively open in the closed set $X'_{ji} \subset U_i$. Therefore we may construct open disjoint neighborhoods $U_{xi} \subset U_i$ of $V^S_{x1} \cap U_i$ and $U_{yi} \subset U_i$ of $V^S_{y1} \cap U_i$ by adding points in $U_i \setminus \big(\cup_{j<i} X'_{ji} \big)$. Now consider the sets $V_{xi} := V^S_{x1} \cup V^S_{xi}$ and $V_{yi} := V^S_{y1} \cup V^S_{yi}$, where A^S denotes the saturation of A. These sets are disjoint as before. Moreover, their intersections with the $U_j, j \leq i$, are open. Hence after a finite number of similar steps, we find suitable disjoint open $V_\mathbf{x} := V_{xn}$ and $V_\mathbf{y} := V_{yn}$.

Next suppose that $J_\mathbf{x}$ and $J_\mathbf{y}$ are disjoint. If $i \in J_\mathbf{x}$ then $x_i \notin X'_{ji}$ for any $j \notin J_\mathbf{x}$. Therefore the set

$$V_\mathbf{x} := \bigcup_{i \in J_\mathbf{x}} \Big(U_i \setminus \big(\cup_{j \notin J_\mathbf{x}} X'_{ji} \big) \Big)$$

is an open neighborhood of $\mathbf{x}$ in $W = \cup_i U_i$. Since $V_\mathbf{x}$ is saturated under $\approx$ by construction, it projects to an open neighborhood of $\mathbf{x}$ in $W_\approx$. Finally

[12] A set is said to be $\approx$-saturated if it is a union of equivalence classes.

note that $V_{\mathbf{x}}$ is disjoint from the similarly defined open set

$$V_{\mathbf{y}} := \bigcup_{i\in J_{\mathbf{y}}} \Big(U_i \setminus \big(\cup_{j\notin J_{\mathbf{y}}} X'_{ji}\big)\Big).$$

This completes the proof. □

Proposition 3.6. (i) *For all $p \in |B|_H$, any open neighborhood in $|B|$ of the fiber over p contains a saturated open neighborhood $|W|$.*

(ii) *$|B|_H$ is second countable and locally compact.*

(iii) *The branch points form a closed and nowhere dense subset of $|B|_H$.*

(iv) *The weighting function Λ is locally constant except possibly at branch points.*

Proof. To prove (i) let (N_p, U_i, m_i) be a local branching structure at p, and $|V| \subset |B|$ be any open neighborhood of the fiber at p. Then $|V_i| := |V| \cap |U_i|$ is open in $|U_i|$. If V_i denotes the corresponding open subset of U_i, the set $|W| := \cap_i |V_i^S|$ satisfies the requirements.

Each open set N that supports a local branching structure is a finite union of locally compact closed sets $|U_i|_H$, and hence is locally compact. Moreover, the $|U_i|_H$ are second countable in the induced topology. Hence so is N. Since we assumed B_0 has a countable dense subset, the same is true for $|B|_H$. Therefore, $|B|_H$ is the union of countable many open sets N_i and so is itself second countable. This proves (ii).

Denote by $|Br|_H$ the set of branch points in $|B|_H$. To prove (iii) it suffices to show that for any N as above the intersection $|Br|_H \cap N$ is closed and nowhere dense in N. It follows from the proof of Lemma 3.5 that

$$|Br|_H \cap |U_i|_H = \pi_H(\cup_{j\neq i} Br_{ij}).$$

We saw earlier that for each i the set of branch points $\cup_{j\neq i} Br_{ij}$ in U_i is relatively closed. It is nowhere dense by construction. Since $|U_i|_H$ is closed in N, and U_i is homeomorphic to $|U_i|_H$, $\pi_H(\cup_{j\neq i} Br_{ij})$ is closed in N for all i. Since there are a finite number of local branches, (iii) holds.

Consider $|Br| := |\pi|_H^{-1}(|Br|_H)$, the set of points in $|B|$ on which $|\pi|_H$ is not injective, and denote by $|N|$ the open set $|\pi|_H^{-1}(N) \subset |B|$. (iv) will follow if we show that for each connected component $|V|$ of $|N| \setminus |Br|$ and each local branch U_i over N either $|U_i| \cap |V| = \emptyset$ or $|V| \subset |U_i|$. But $|U_i|$ is open in $|N|$ by Lemma 2.2. Its intersection with $|N| \setminus |Br|$ is also closed since it is the inverse image of the relatively closed subset $|U_i|_H \setminus |Br|_H$ of $|N| \setminus |Br|$. Hence $|U_i| \setminus |Br|$ is a union of components of $|N| \setminus |Br|$, as required. □

3.2. Layered coverings. There are two useful kinds of functors for wnb groupoids, those that induce homeomorphisms on the orbit space $|B|_H$ and those that simply induce surjections on the orbit space. In the first case, we require Λ to be preserved while in the second we expect the induced map

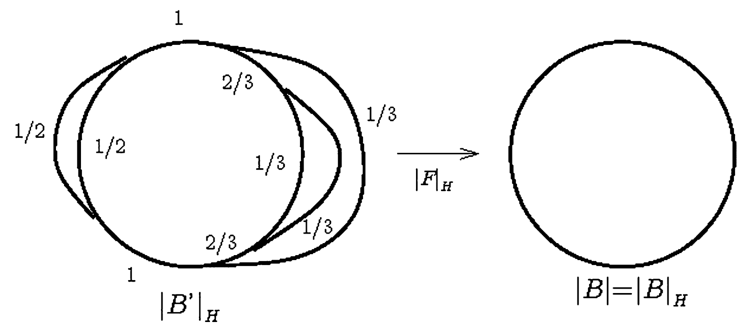

Figure 4. A layered covering of the circle; $|B|$ has $\Lambda \equiv 1$, the values of Λ' are marked.

$|F|_H : |B'|_H \to |B|_H$ to push Λ' forward to Λ. In other words, we expect the identity

$$(|F|_H)_*(\Lambda')(p) := \sum_{q:|F|_H(q)=p} \Lambda'(q) = \Lambda(p) \tag{3.1}$$

to hold at all points $p \in |B|_H$: cf. Figure 4.

Definition 3.7. *Let* $(\mathcal{B}, \Lambda)$ *and* $(\mathcal{B}', \Lambda')$ *be wnb groupoids. A refinement* $F : \mathcal{B}' \to \mathcal{B}$ *is said to be* **weighted** *if* $\Lambda \circ |F|_H = \Lambda'$.

A smooth functor $F : \mathcal{B}' \to \mathcal{B}$ *is said to be a* **layered covering** *if*

(Covering) *F is a local diffeomorphism on objects and induces a surjection* $|F| : |B'| \to |B|$,

(Properness) *the induced map* $|F|_H : |B'|_H \to |B|_H$ *is proper.*

(Weighting) $(|F|_H)_*(\Lambda') = \Lambda$.

Two wnb groupoids are **equivalent** *if they have a common weighted refinement. They are* **commensurate** *if they have a common layered covering. Finally, two compact wnb groupoids* $(\mathcal{B}, \Lambda)$ *and* $(\mathcal{B}', \Lambda')$ *(without boundary) are* **cobordant** *if there is a compact* $(d+1)$*-dimensional wnb groupoid with boundary* $-(\mathcal{B}, \Lambda) \sqcup (\mathcal{B}', \Lambda')$.

Remark 3.8. (i) Any wnb groupoid $(\mathcal{B}, \Lambda)$ is equivalent to a wnb groupoid $(\mathcal{B}', \Lambda')$ in which all local branches needed to describe its branching structure are unions of components of B'_0. To see this, choose a locally finite cover $N_p, i \in A_p$, of $|B|_H$ by sets that support local branching structures and let $U_i^p, i \in A_p$, be the corresponding set of local branches. Then define $\mathcal{B}'$ to be the refinement of $\mathcal{B}$ with objects $\sqcup_{i,p} U_i^p$ as in Remark 2.7. Then $|B'| = |B|$ and $|B'|_H = |B|_H$, and so we may define $\Lambda' := \Lambda$.

(ii) The condition that $|F| : |B'| \to |B|$ is surjective does not follow from the other conditions for a layered covering. Consider for example the wnb groupoid $\mathcal{C}$ with objects two copies of S^1 and morphisms $C_1 = C_0 \cup A^{\pm}$, where $A^+ := S^1 \setminus \{0\}$ identifies one circle with the other except over 0 and $A^- := (A^+)^{-1}$. Then $|C|_H = S^1$ and we give it weight $\Lambda \equiv 1$. The functor $F : \mathcal{C} \to \mathcal{C}$ that identifies both copies of S^1 to the same component of C_0 satisfies all conditions for a layered covering except that the induced map $|C| \to |C|$ is not surjective. This wnb groupoid also illustrates the fact that the local weights m_i on the local branches U_i need not be uniquely determined by Λ, and hence that Λ may not lift to a well defined function on $|C|$.

(iii) If F is a layered covering then although the induced map $|F| : |B'| \to |B|$ is surjective, the induced map on objects need not be surjective. Also, the properness requirement is important. Otherwise, given any wnb groupoid $(\mathcal{B}, \Lambda)$ define $(\mathcal{B}', \Lambda')$ as in (i) above and then consider $(\mathcal{B}'', \Lambda'')$, where $\mathcal{B}''$ has the same objects as $\mathcal{B}'$ but only identity morphisms. Then $|B''|_H = |B''| = \sqcup_{i,p} U_i^p$ and we may define $\Lambda'' := m_{i,p}$ on U_i^p. The inclusion $\mathcal{B}'' \to \mathcal{B}$ satisfies all the conditions for a layered covering except for properness. Since we want commensurate wnb groupoids to have the same fundamental class, we cannot allow this behavior.

Our next aim is to show that layered coverings have the expected functorial properties. In particular, commensurability is an equivalence relation. For this we need some preparatory lemmas.

We shall say that a local branching structure $(N, U_i, m_i)_{i \in I}$ at p is **minimal** if the fiber $(|\pi|_H)^{-1}(p) \subset |B|$ over p is a collection of distinct points $|x_i|, i \in I$, where $x_i \in U_i$. Thus in this case there is a bijective correspondence between the local branches and the points of the fiber.

Lemma 3.9. *Let $(\mathcal{B}, \Lambda)$ be a wnb groupoid. Then every point $p \in |B|_H$ has a minimal local branching structure.*

Proof. Choose any local branching structure (N, U_i, m_i) at p. For each point $w_\alpha \in (|\pi|_H)^{-1}(p) \subset |B|$, let I_α be the set of indices i such that $w_\alpha \in |U_i|$. For each such α, choose one element $i_\alpha \in I_\alpha$, and define

$$m_\alpha := \sum_{i \in I_\alpha} m_i, \qquad |V_\alpha| := \bigcap_{i \in I_\alpha} |U_i|.$$

Each $|V_\alpha|$ is open. Hence $\cup_\alpha |V_\alpha|$ is an open neighborhood of the fiber $(|\pi|_H)^{-1}(p)$ and so by Proposition 3.6 it contains a saturated open neighborhood $|W| := (|\pi|_H)^{-1}(N')$. Define $U'_\alpha := U_{i_\alpha} \cap \pi^{-1}(|W|)$. Then $(N', U'_\alpha, m_\alpha)$ is a minimal local branching structure at p. □

We shall say that two (possibly disconnected) open subsets U_0, U_1 of the space of objects X_0 of an sse groupoid $\mathcal{X}$ are $\mathcal{X}$**-diffeomorphic** if there is a subset $C \subset X_1$ of $s^{-1}(U_0)$ such that the maps $s : C \to U_0$ and $t : C \to U_1$ are both diffeomorphisms. In this situation, we also say that there is a diffeomorphism $\phi : U_0 \to U_1$ in $\mathcal{X}$. If the U_i both inject into $|X|$, there is such ϕ iff $|U_0| = |U_1| \in |X|$.

Lemma 3.10. *Suppose that $F : \mathcal{A} \to \mathcal{B}$ is a layered covering. Let $p \in |B|_H$ and denote the points in $(|F|_H)^{-1}(p)$ by $q_\alpha, 1 \le \alpha \le k$. Then there are minimal local branching structures*

$$(N_p, U_i^p, m_i^p)_{i\in I} \;\; \text{at } p, \qquad (N_\alpha, U_j^\alpha, m_j^\alpha)_{j\in J_\alpha} \;\; \text{at } q_\alpha,$$

that are compatible in the sense that each $F(U_j^\alpha)$ is $\mathcal{B}$-diffeomorphic to some local branch $U_{i_j}^p$. Moreover $U_{i_j}^p$ is unique.

Proof. Let $\{q_1, \dots, q_m\} = (|F|_H)^{-1}(p) \subset |A|_H$. Choose minimal local branching structures (N_p, U_i^p, m_i) at p and $(N_\alpha, U_j^\alpha, m_{j\alpha})$ at q_α for $1 \le \alpha \le m$. Since $|A|_H$ is Hausdorff, we may suppose that the N_α are pairwise disjoint. Moreover, because $|F|_H$ is proper, the union $\cup_\alpha |F|_H(N_\alpha)$ is a neighborhood of p in $|B|_H$. (Otherwise, there would be a sequence of points $q_n \in |A|_H$ lying outside $\cup_\alpha N_\alpha$ but such that $|F|_H(q_n)$ converges to p, which contradicts properness.)

Fix α, and for each $j \in J_\alpha$ denote by x_j^α the point in U_j^α that projects to q_α; this exists by minimality. Since $(|\pi|_H)^{-1}(N_p) = \cup_i |U_i^p|$, for each branch U_j^α there is a morphism $\gamma_\alpha \in B_1$ with source $F(x_j^\alpha)$ and target in some local branch, say $U_{i_j}^{p,\alpha}$, at p. By minimality at p, the index i_j is unique. This morphism extends to a diffeomorphism ϕ_α from an open subset $F(V_j^\alpha)$ of $F(U_j^\alpha)$ onto an open subset $V_{i_j}^p$ of $U_{i_j}^p$. Since $\cup_{j\in J_\alpha} |U_j^\alpha|$ is a neighborhood in $|A|$ of the fiber over q_α, so is the open set $\cup_{j\in J_\alpha} |V_j^\alpha|$. Hence, by Proposition 3.6(i), there is an open neighborhood N'_α of q_α such that $|\pi|_H^{-1}(N'_\alpha) \subset \cup_{j\in J_\alpha} |V_j^\alpha|$. By shrinking the sets V_j^α, we may therefore suppose that $|\pi|_H^{-1}(N'_\alpha) = \cup_j |V_j^\alpha|$, i.e., that $(N'_\alpha, V_j^\alpha, m_{j\alpha})$ is a local branching structure at q_α. We also shrink the $V_{i_j}^{p,\alpha}$ so that they remain $\mathcal{B}$-diffeomorphic to the sets $F(V_j^\alpha)$, $j \in J_\alpha$.

Now observe that because $|F| : |A| \to |B|$ is surjective,

$$(|\pi|_H)^{-1}(p) \;\subset\; |W| := \bigcap_{\alpha,\, j\in J_\alpha} |V_{i_j}^{p,\alpha}| \;\subset\; |B|.$$

The set $|W|$ is open since both F and the map $B_0 \to |B|$ are open. Hence it contains a set of the form $|\pi_H|^{-1}(N'_p)$ for some neighborhood N'_p of p. Shrinking the sets N'_α and V_j^α further as necessary, we may suppose that $|F|_H^{-1}(N'_p) = \cup_\alpha N'_\alpha$. Then the local branching structures $(N'_\alpha, V_j^\alpha, m_{j\alpha})$ and $(N'_p, V_i^p \cap (\pi_H)^{-1} N'_p, m_i)$ satisfy all requirements. □

Proposition 3.11. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be wnb groupoids. Then:*

(i) *If $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ are layered coverings so is the composite $G \circ F$.*

(ii) *If $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{C} \to \mathcal{B}$ are layered coverings, the (weak) fiber product $\mathcal{Z} := \mathcal{A} \times_{\mathcal{B}} \mathcal{C}$ of $\mathcal{A}$ and $\mathcal{C}$ over $\mathcal{B}$ is a wnb groupoid and the induced functors $\mathcal{Z} \to \mathcal{B}, \mathcal{Z} \to \mathcal{A}$ are layered coverings.*

(iii) *Any two compact commensurate wnb groupoids are cobordant through a wnb groupoid.*

Proof. The proof of (i) is straightforward, and is left to the reader. As for (ii), observe first that $\mathcal{Z}$ is a nonsingular sse Lie groupoid with objects Z_0 contained in the product $A_0 \times B_1 \times C_0$.[13] The orbit space $|Z|$ is the (strict) fiber product $|A| \times_{|B|} |C|$. It follows easily that $|Z|_H$ is the fiber product $|A|_H \times_{|B|_H} |C|_H$, since the latter space has the requisite universal property. Hence $|Z|_H = \{(a, c) : |F|_H(a) = |G|_H(c) \in |B|_H\}$ and we set

$$\Lambda_Z(a, c) := \frac{\Lambda_A(a)\,\Lambda_C(c)}{\Lambda_B\big(|F|_H(a)\big)}.$$

Observe that the projections of $|Z|_H$ to $|A|_H$ and $|C|_H$ push Λ_Z forward to Λ_A and Λ_C, respectively. Hence (ii) will follow once we show that $\mathcal{Z}$ has the requisite local branching structures.

Given $z = (a, c) \in |Z|_H$, let $p = |F|_H(a) = |G|_H(c) \in |B|_H$. Applying Lemma 3.10 first to $\mathcal{A} \to \mathcal{B}$ and then to $\mathcal{C} \to \mathcal{B}$ and then restricting as in Remark 3.4(iv), we can find minimal local branching structures $(N_a, U^a_j, m_{ja})_{j\in J}$, $(N_c, U^c_k, m_{kc})_{k\in K}$ and (N_p, U^p_i, m_i) satisfying the following compatibility conditions: for each j, k there are unique i_j, i_k and local diffeomorphisms γ_j, γ_k in $\mathcal{B}$ such that

$$F(U^a_j) = \gamma_j(U^p_{i_j}), \qquad \gamma_k(G(U^c_k)) = U^p_{i_k}.$$

We now show that there is a local branching structure over $N_z := N_a \times_{|B|_H} N_c$. By minimality, for each point $(w_a, w_c) \in |A| \times_{|B|} |C|$, there is a unique pair of local branches (U^a_j, U^c_k) such that $w_a \in |U^a_j|, w_c \in |U^c_k|$. Since $|F|(w_a) = |G|(w_c) \in |B|$, the corresponding indices i_j, i_k coincide. Hence there is an open set

$$U^{ac}_{jk} := \{(x, \gamma, y) : x \in U^a_j, \gamma = \gamma_j \circ \gamma_k|_y, y \in U^c_k\} \subset Z_0.$$

It remains to observe that the collection of all such sets, with weights

$$m^{ac}_{jk} := \frac{m_{ja}\, m_{kc}}{m_{i_j}},$$

forms a local branching structure over N_z. This proves (ii).

To prove (iii), it suffices by (ii) to consider the case when there is a layered covering $F : \mathcal{B}' \to \mathcal{B}$. Define $\mathcal{W}$ by setting $W_0 := B'_0 \times [0, 3/4) \sqcup B_0 \times (1/4, 1]$

[13] Since $\mathcal{B}$ is nonsingular, we may identify Z_0 with the strict fiber product $A_0 \times_{|B|} C_0$. However, later on we will apply this construction to certain nonsingular $\mathcal{B}$, and so it is convenient to retain more general language here.

and

$$W_1 := \Big(B_1' \times [0, 3/4)\Big) \sqcup \Big(B_1 \times (1/4, 1]\Big) \sqcup \Big((B_0')_\pm \times (1/4, 3/4)\Big),$$

where $(x, t)_+ \in (B_0')_+ \times (1/4, 3/4)$ denotes a morphism from $(x, t) \in B_0' \times (1/4, 3/4)$ to $(F(x), t) \in B_0 \times (1/4, 3/4)$ and $(x, t)_-$ is its inverse. The nonsingularity of $\mathcal{B}', \mathcal{B}$ implies that $\mathcal{W}$ is a nonsingular groupoid in the category of oriented manifolds with boundary. In fact, its boundary $\partial\mathcal{W}$ projects to $\partial\mathcal{I}$ under the obvious projection $\mathcal{W} \to \mathcal{I}$, where $\mathcal{I}$ is the category with objects $[0, 1]$ and only identity morphisms. Further $\partial\mathcal{W}$ may obviously be identified with $-\mathcal{B}' \sqcup \mathcal{B}$. Therefore $\mathcal{W}$ is a cobordism from $\mathcal{B}'$ to $\mathcal{B}$.

It remains to check that $\mathcal{W}$ is a wnb groupoid. Since

$$|W|_H \cong \Big(|B'|_H \times [0, 1/4)\Big) \cup \Big(|B|_H \times [1/4, 1]\Big),$$

we define Λ_W to be the pullback of Λ_B' on $|B'|_H \times [0, 1/4)$ and the pullback of Λ_B on $|B|_H \times [1/4, 1]$. It is obvious that $(\mathcal{W}, \Lambda)$ has local branching structures at all points of $|W|_H$ except possibly for $p \in |B|_H \times \{1/4\}$. But here one constructs suitable local branches as in the proof of (ii), using Lemma 3.10 as before. $\square$

3.3. Branched manifolds and resolutions.

Definition 3.12. *A* **branched manifold structure** *on a topological space Z consists of a wnb groupoid $(\mathcal{B}, \Lambda)$ together with a homeomorphism $f : |B|_H \to Z$. Two such structures $(\mathcal{B}, \Lambda, f)$ and $(\mathcal{B}', \Lambda', f')$ are* **equivalent** *if they have a common weighted equivalence, i.e., if there is a third structure $(\mathcal{B}'', \Lambda'', f'')$ and weighted equivalences $F : \mathcal{B}'' \to \mathcal{B}, F' : \mathcal{B}'' \to \mathcal{B}'$ such that $f'' = f \circ |F|_H = f' \circ |F'|_H$.*

A **branched manifold** *$(\underline{Z}, \Lambda_Z)$ is a pair, consisting of topological space Z together with a function $\Lambda_Z : Z \to (0, \infty)$, in which Z is equipped with an equivalence class of branched manifold structures that induce the function Λ_Z.*

Two d-dimensional branched manifolds $(\underline{Z}, \Lambda_Z)$ and $(\underline{Z}', \Lambda_Z')$ are **commensurate** *if they have commensurate branched manifold structures. They are* **cobordant** *if there is a $(d+1)$-dimensional branched manifold $(\underline{W}, \Lambda_W)$ in the category of smooth manifolds with boundary whose (oriented) boundary decomposes into the disjoint union $(-\underline{Z}, \Lambda_Z) \sqcup (\underline{Z}', \Lambda_Z')$.*

Proposition 3.11 implies that if $(\underline{Z}, \Lambda_Z)$ and $(\underline{Z}', \Lambda_Z')$ are commensurate any pair of branched manifold structures on them have a common layered covering. Further any pair of commensurate branched manifolds are cobordant via a branched manifold.

We now consider maps from branched manifolds to orbifolds. These are induced by smooth functors $F : (\mathcal{B}, \Lambda) \to \mathcal{X}$ where $\mathcal{X}$ is an ep groupoid. It is convenient to consider $\mathcal{X}$ as a weighted but *singular* branched groupoid with weighting function that is identically equal to 1. In this case $|X|_H = |X|$ and every point p has a local branch structure with one branch, namely any open set containing a point $x \in \pi^{-1}(p)$. This groupoid satisfies all conditions of Definition 3.2, except that it is singular and the local branches do not inject into $|X|$.

The following definition is motivated by the example of the inclusion $F : \mathcal{Z} \to \mathcal{X}$ of Example 2.10, where $\mathcal{X}$ is the teardrop groupoid and $(\mathcal{Z}, \Lambda_Z)$ is as in Example 3.3(i). We want to consider this as some kind of equivalence, but note that F does not push Λ_Z forward to $\Lambda_X \equiv 1$ at all points. Rather $(|F|_H)_*(\Lambda')(p) = 1$ except at the singular point of $|X|$ where it equals $1/k$. The fact that this is the unique nonsmooth point in the sense of Definition 2.12 motivates the following definition.

Definition 3.13. *Let $(\mathcal{B}, \Lambda)$ be a wnb groupoid and $\mathcal{X}$ an ep groupoid. A functor $F : (\mathcal{B}, \Lambda) \to \mathcal{X}$ is said to be a* **resolution** *of $\mathcal{X}$ if the following conditions hold:*

(Covering) *F is a local diffeomorphism on objects,*

(Properness) *the induced map $|F|_H : |B|_H \to |X|$ is proper.*

(Weighting) *$(|F|_H)_*(\Lambda') = 1$ at all smooth points of $|X|$.*

Similarly, a map $\underline{\phi} : (\underline{Z}, \Lambda_Z) \to (\underline{Y}, \Lambda)$ from a branched manifold to an orbifold is called a **resolution of** *$(\underline{Y}, \Lambda)$ if it is induced by a resolution $F : (\mathcal{B}, \Lambda) \to \mathcal{X}$ where $(\mathcal{B}, \Lambda, f)$ is a branched manifold structure on $(\underline{Z}, \Lambda_Z)$ and $(\mathcal{X}, f')$ is an orbifold structure on $(\underline{Y}, \Lambda)$. Here we require that the diagram*

$$\begin{array}{ccc} |B|_H & \xrightarrow{|F|_H} & |X| \\ f \downarrow & & f' \downarrow \\ Z & \xrightarrow{\phi} & Y \end{array}$$

commute.

Note that for any resolution $F : \mathcal{B} \to \mathcal{X}$ the induced map $|F|_H : |B|_H \to |X| \equiv |X|_H$ is surjective; its image is closed by the properness assumption, and is dense by the weighting property and the fact that the smooth points are dense in $|X|$. Moreover, the following analog of Lemma 3.10 holds.

Lemma 3.14. *For any resolution $F : \mathcal{B} \to \mathcal{X}$ each $x \in X_0$ has a neighborhood U that is evenly covered by F in the sense that for all $q_\alpha \in |F|_H^{-1}(|x|)$ there are local branching structures $(N_\alpha, U_j^\alpha, m_j^\alpha)_{j \in J_\alpha}$ at q_α with the property that $|F|_H(|U_j^\alpha|) = |U|$ for all $\alpha, j \in J_\alpha$. Moreover, each set $F(U_j^\alpha)$ is $\mathcal{X}$-diffeomorphic to U.*

Proof. Since $|F|_H : |B|_H \to |X|$ is proper, it is finite to one, and also open (because $|X|$ is locally compact and normal). In particular, there are a finite number of points q_α. For each q_α choose N'_α so that it supports a local branching structure at q_α. Choose a connected neighborhood U of x so that $|U| \equiv |U|_H \subset \cap_\alpha\big(|F|_H(N'_\alpha)\big)$, and then define $N_\alpha := N'_\alpha \cap (|F|_H)^{-1}(|U|)$. Then $|F|_H(|U^\alpha_j|_H)$ is closed in $|U|$ since $|U^\alpha_j|_H$ is closed in N_α and $|F|_H$ is proper. But $|F|_H(|U^\alpha_j|_H)$ is also open in $|U|$, since it is the image of U^α_j under the composite of the two open maps $F: B_0 \to X_0$ and $\pi: X_0 \to |X|$. Hence $|F|_H(|U^\alpha_j|_H) = |U|$ since $|U|$ is connected. This proves the first statement. Since there are only finitely many pairs (α, j) and $\mathcal{X}$ is sse, one can now restrict U further so that the second statement holds. □

Lemma 3.15. (i) *If $F: \mathcal{A} \to \mathcal{X}$ and $G: \mathcal{C} \to \mathcal{X}$ are resolutions of the ep groupoid $\mathcal{X}$ then the (weak) fiber product $\mathcal{Z} := \mathcal{A} \times_{\mathcal{X}} \mathcal{C}$ of $\mathcal{A}$ and $\mathcal{C}$ over $\mathcal{X}$ may be given the structure of a wnb groupoid, and the induced functors $\mathcal{Z} \to \mathcal{B}, \mathcal{Z} \to \mathcal{A}$ are layered coverings.*

(ii) *If $F: \mathcal{B} \to \mathcal{X}$ is a resolution and $G: \mathcal{X}' \to \mathcal{X}$ is an equivalence then $\mathcal{B}' := \mathcal{B} \times_{\mathcal{X}} \mathcal{X}'$ is a wnb groupoid. Moreover, the induced functor $F': \mathcal{B}' \to \mathcal{X}'$ is a resolution, while $G': \mathcal{B}' \to \mathcal{B}$ is an equivalence.*

Proof. The proof of (i) is very similar to that of Proposition 3.11 (ii), using Lemma 3.14 instead of Lemma 3.10. Its details will be left to the reader.

To see that $\mathcal{B}'$ in (ii) is a wnb groupoid one need only choose the set U of Lemma 3.14 so small that it is $\mathcal{X}$-diffeomorphic to a subset of $G(X'_0)$. Then it can be lifted into $\mathcal{X}'$, and the rest of the proof is clear. □

Note that the diagram

$$\begin{array}{ccc} \mathcal{B}' := \mathcal{B} \times_{\mathcal{X}} \mathcal{X}' & \xrightarrow{F'} & \mathcal{X}' \\ G' \downarrow & & G \downarrow \\ \mathcal{B} & \xrightarrow{F} & \mathcal{X}, \end{array}$$

considered in Lemma 3.15(ii) does *not* in general commute strictly, but only "up to homotopy". In particular, the map $F': B'_0 \to X'_0$ is always surjective, while $F: B_0 \to X_0$ may not be; cf. Remark 4.17(i).

The following result restates the main assertion of Theorem 1.1.

Proposition 3.16. *Every ep groupoid $\mathcal{X}$ has a resolution that is unique up to commensurability and hence up to cobordism.*

The uniqueness statement follows immediately from Lemma 3.15 and Proposition 3.11. We give two proofs of the existence statement in Section 4. The first gives considerable control over the branching structure of the resolution, while the second, which constructs the resolution as a multisection of a bundle $\mathcal{E} \to \mathcal{X}$, is perhaps more direct. However, it gives a resolution as defined above only when $\mathcal{X}$ acts effectively on $\mathcal{E}$.

Remark 3.17. One might argue that the above definition of resolution is not the most appropriate for groupoids that are not effective; cf. the example in Remark 4.5 (ii). We are mostly interested in resolutions because they give easily understood representatives for the fundamental class of an orbifold. But we have defined the fundamental class of $\mathcal{X}$ so that it is the same as that of $\mathcal{X}_{\rm eff}$; see Section 2.3. Therefore we could define a resolution of $\mathcal{X}$ simply to be a resolution $F : (\mathcal{B}, \lambda, \Lambda) \to \mathcal{X}_{\rm eff}$ of $\mathcal{X}_{\rm eff}$. The information about the trivially acting part K_x of the stabilizer groups G_x of $\mathcal{X}$ would then be recorded in the (strict) pullback[14] $(\mathcal{B}', \lambda', \Lambda')$ of the fibration $\mathcal{X} \to \mathcal{X}_{\rm eff}$ by F. Here $\mathcal{B}'$ has the same objects, orbit space and weighting function as $\mathcal{B}$. But the morphisms in $\mathcal{B}'$ from x to y equal the set ${\rm Mor}_{\mathcal{X}}(F(x), F(y))$ if there is a morphism in $\mathcal{B}$ from x to y, and equal the empty set otherwise. If composition is defined by pull back from $\mathcal{X}$, one readily checks that $\mathcal{B}'$ is a groupoid. Since the morphisms in $\mathcal{B}'$ act trivially on the objects, one can easily extend the definition of wnb groupoid to include this case. Hence $(\mathcal{B}', \lambda', \Lambda')$ may also be considered as a kind of resolution of $\mathcal{X}$.

3.4. The fundamental class. We now show that each compact branched manifold $(\underline{Z}, \Lambda_Z)$ of dimension d carries a fundamental class $[Z] \in H_d(Z, \mathbb{R})$ which is compatible with resolutions; that is, if $\underline{\phi} : (\underline{Z}, \Lambda_Z) \to \underline{Y}$ is any resolution, then $\phi_*([Z]) = [Y]$, where $[Y]$ is the fundamental class of the orbifold $\underline{Y}$ discussed in Section 2.3. We shall define $[Z]$ as a singular cycle using triangulation, but also show in Proposition 3.25 that in "nice" (i.e., tame) cases, one can integrate over $[Z]$. In order to construct a suitable integration theory we shall need to consider smooth partitions of unity.

Definition 3.18. *Let $(\underline{Z}, \Lambda_Z)$ be a branched manifold and M a smooth manifold. A map $g : Z \to M$ is smooth iff for one (and hence any) branched manifold structure $(\mathcal{B}, \Lambda, f)$ on $\underline{Z}$ the composite*

$$g_0 : B_0 \stackrel{\pi_H}{\to} |B|_H \stackrel{f}{\to} Z \stackrel{g}{\to} M$$

is smooth. This is equivalent to saying that g is induced by a smooth functor $\mathcal{B} \to \mathcal{M}$, where $\mathcal{M}$ is the category with objects M and only identity morphisms.

A **smooth partition of unity subordinate to the covering** $\mathcal{N} = \{N_k\}_{k \in A}$, *of Z is a family $\beta_k, k \in A$, of smooth functions $Z \to \mathbb{R}$ such that*

(i) ${\rm supp}\, \beta_k \subset N_k$ *for all k,*
(ii) *for each $z \in Z$ only finitely many of the $\beta_k(z)$ are nonzero, and*
(iii) $\sum_k \beta_k(z) = 1$.

The above smoothness condition is quite strong. For example, if $\phi : D_1 \to R_2$ in the example of Remark 3.4(ii) does not extend to a smooth function near ∂D_1, there is no smooth function $g : |B_\phi|_H \to \mathbb{R}^2$ that is injective over the image of R_1. We shall deal with these problems by introducing the notion of tameness.

[14]One could also use the weak pullback $\mathcal{B} \times_{\mathcal{X}_{\rm eff}} \mathcal{X}$, but this is somewhat larger.

Definition 3.19. *Let Ω be a precompact open subset of $\mathbb{R}^d$. It is said to have* **piecewise smooth boundary** *if for every $x \in \partial\Omega$ there is a neighborhood U_x of x and smooth functions $f_1, \dots, f_k : U_x \to \mathbb{R}$ such that*

(i) $U_x \cap \overline{\Omega} = \{y \in U_x : f_i(y) \le 0,$ *for all* $i = 1, \dots, k\}$*;*
(ii) $x \in \partial\Omega \Longleftrightarrow f_i(x) = 0$ *for at least one* i*;*
(iii) *Let $x \in \partial\Omega$ and write $I_x = \{i : f_i(x) = 0\}$. Then the set of vectors $df_i(x), i \in I_x$, is linearly independent.*

Moreover, Ω is said to have **piecewise smooth boundary** *over an open set $N \subset \mathbb{R}^d$ if the above conditions hold for every $x \in N \cap \partial\Omega$.*

Note the following facts about domains Ω with piecewise smooth boundary.

- Any smooth function defined on the closure $\overline{\Omega}$ has a smooth extension to $\mathbb{R}^d$;
- The boundary $\partial\Omega$ has zero measure;
- The intersection $\cap_{j\in J}\Omega_j$ of a generic finite collection $\Omega_j, j \in J$, of such domains also has piecewise smooth boundary. More precisely, suppose that each $\overline{\Omega}_j$ is contained in the interior of the ball B_R of radius R. Then the set of diffeomorphisms $\phi_j, j \in J$ of B_R such that $\cap_{j\in J}\phi_j(\Omega_j)$ has piecewise smooth boundary has second category in the group $\prod_{j\in J} \mathrm{Diff}(B_R)$. Sets $\Omega_j, j \in J$, with this property will be said to be in **general position** or to **intersect transversally**.

Definition 3.20. *A d-dimensional wnb groupoid $(\mathcal{B}, \Lambda)$ is said to be* **tame** *iff if it has local branching structures $(N_{p_k}, U_i^k, m_i^k)_{k\in K}$ such that the following conditions are satisfied.*

(i) $|B|_H = \cup_{k\in K} N_{p_k}$.
(ii) *For each local branch U_i^k there is an injective smooth map $\phi_i^k : U_i^k \to \mathbb{R}^d$ onto the interior of a compact domain $\overline{\Omega}_i^k$ in $\mathbb{R}^d$ with piecewise smooth boundary such that the composite $\pi_H \circ (\phi_i^k)^{-1}$ extends to an injection $\rho_i^k : \overline{\Omega}_i^k \to |B|_H$.*
(iii) *For each pair U_i^k, U_j^ℓ the set $\overline{\Omega}_i^k \cap (\rho_i^k)^{-1}(\rho_j^\ell(\overline{\Omega}_j^\ell))$ has piecewise smooth boundary and the transition map $(\rho_j^\ell)^{-1} \circ \rho_i^k$ extends smoothly to a local diffeomorphism defined on a neighborhood of this set.*

If $\mathcal{B}$ is a groupoid in the category of manifolds with boundary, we replace $\mathbb{R}^d$ in the above by the half space $\mathbb{H}^d = \{x \in \mathbb{R}^d : x_1 \ge 0\}$, and require that all sets meet $\partial\mathbb{H}^d$ transversally.

The main point of this definition is to ensure that the branch locus is piecewise smooth and hence has zero measure.

Lemma 3.21. *Every weighted refinement of a tame wnb groupoid may be further refined to be tame.*

Proof. Suppose that $(\mathcal{B}, \Lambda)$ is tame. Since $|B|_H$ is unchanged under refinement, the branching locus of any refinement $F : \mathcal{B}' \to \mathcal{B}$ is piecewise smooth. If we use the pullback local branching structures on $\mathcal{B}'$, then the taming condition (ii) may not be preserved because it concerns the objects rather than the morphisms of $\mathcal{B}'$. Nevertheless, it is easy to see that there is a further refinement $\mathcal{B}'' \to \mathcal{B}'$ for which it holds. Then (iii) also holds. □

In view of the above lemma, we shall say that the **branched manifold** $(\underline{Z}, \underline{\Lambda}_Z)$ **is tame** if its structure may be represented by a tame wnb groupoid.

In the following lemma, we write $V \sqsubset U$ to mean that the closure $cl(V)$ of V is contained in U.

Lemma 3.22. *Every wnb groupoid $(\mathcal{B}, \Lambda_B)$ has a layered covering that is tame.*

Proof. Choose a locally finite cover of $|B|_H$ by sets $N_k, k \in \mathbb{N}$, as in Definition 3.2. For each k let $U_i^k, i \in A_k$, be the corresponding local branches. Without loss of generality, we may suppose that each local branch U_i^k is identified with a subset of $\mathbb{R}^d$. Since $|B|_H$ is normal we may choose open subsets W_k', N_k', W_k of N_k such that the N_k' cover $|B|_H$ and

$$W_k' \sqsubset N_k' \sqsubset W_k \sqsubset N_k, \qquad \text{for all } k.$$

Since $\pi_H : U_i^k \to |U_i^k|_H$ is a homeomorphism for each i, it follows that

$$U_i^k \cap \pi_H^{-1}(W_k') \sqsubset U_i^k \cap \pi_H^{-1}(N_k') \sqsubset U_i^k \cap \pi_H^{-1}(W_k) \sqsubset U_i^k \sqsubset \mathbb{R}^d.$$

We now construct a sequence of commensurate wnb groupoids $\mathcal{B}^0 \supset \mathcal{B}^1 \supset \mathcal{B}^2 \supset \dots$ and sets $Z_1 \subset Z_2 \subset \cdots \subset |B|_H$ such that, for all $k \ge 1$, $\mathcal{B}^k$ is tame over $Z_k \supset \cup_{j=1}^k W_j'$ and also $\mathcal{B}^{k+1} = \mathcal{B}^k$ over Z_k. Then $\mathcal{B}^\infty := \lim \mathcal{B}^k$ will be the desired tame groupoid. (We weight all these groupoids by pulling back Λ_B. Further to say that $\mathcal{B}^k$ is tame over Z_k means that its full subcategory with objects in $\pi_H^{-1}(Z_k)$ is tame.)

We define $\mathcal{B}^0$ to be the full subcategory of $\mathcal{B}$ with objects

$$B_0^0 := \sqcup_k \Big(\sqcup_{i \in A_k} U_i^k \Big).$$

Because the sets N_p cover $|B|_H$, the inclusion $\mathcal{B}^0 \to \mathcal{B}$ is a weighted equivalence.

The next step is to correct the morphisms over N_1'. Let

$$\mathcal{I}_k := \{ I \subset A_k : |U_I^k|_H := \bigcap_{i \in I} |U_i^k|_H \ne \emptyset \}.$$

Order the sets in $\mathcal{I}_1$ by inclusion. Starting with a maximal $I \in \mathcal{I}_1$ and then working down, choose sets $|V_I^1|_H \subseteq |U_I^1|_H$ satisfying the following conditions for all I, J:

- $I \supset J \Longrightarrow |V_I^1|_H \subseteq |V_J^1|_H$;

- $|V_I^1|_H \cap (N_1 \setminus W_1) = |U_I^1|_H$;
- for one (and hence every) $i \in I$ the subset $\pi_H^{-1}(|V_I^1|_H)$ of $U_i^1 \subset \mathbb{R}^d$ has piecewise smooth boundary over $\pi_H^{-1}(N_1')$.
- $|V_i^1|_H = |U_i^1|_H$ for all i.

For all $k > 1$ and $I \subset A_k$ set $|V_I^k|_H := |U_I^k|_H$. Then define $\mathcal{B}^1$ to be the subcategory of $\mathcal{B}^0$ with the same objects, labeled for convenience as V_i^k instead of U_i^k, and with morphisms determined by the identities

$$\bigcap_{i \in I} |V_i^k|_H = |V_I^k|_H, \quad \text{for all } I \subset A_k, k \geq 1.$$

Then $\mathcal{B}^1$ is a wnb groupoid commensurate to $\mathcal{B}^0$. (Note that the inclusion $|B^1|_H \to |B^0|_H = |B|_H$ is proper because we did not change the morphisms near the boundary ∂N_1.) Now choose Z_1 so that $W_1' \sqsubset Z_1 \sqsubset N_1'$ and so that for each $i \in A_1$ its pullback $\pi_H^{-1}(Z_1) \cap U_i^1$ has piecewise smooth boundary and is transverse to the sets $\pi_H^{-1}(V_I^1) \cap U_i^1$ for all $I \in \mathcal{I}_1$. Then, $\mathcal{B}^1$ is tame over Z_1.

We next repeat this cleaning up process over N_2, making no changes to the morphisms lying over a neighborhood of $Z_1 \cup (N_2 \backslash W_2)$ and taming the morphisms over N_2'. We then choose a suitable set $Z_2 \supset Z_1 \cup W_2'$ to obtain a groupoid $\mathcal{B}^2$ that is tame over Z_2. Continuing this way, we construct the $\mathcal{B}^k$ and hence $\mathcal{B}^\infty$. □

Lemma 3.23. *Let $(\mathcal{B}, \Lambda)$ be a tame wnb groupoid and let $\mathcal{N}$ be any open cover of $|B|_H$. Then $|B|_H$ has a smooth partition of unity subordinate to $\mathcal{N}$.*

Proof. By Remark 3.4(iv), we may suppose that $\mathcal{N}$ consists of sets of the form N_p, where (N_p, U_i^p, m_i) are local branching structures. Pick out a countable subset A such that the sets $N_p, p \in A$, form a locally finite covering of $|B|_H$. Since $|B|_H$ is normal by Proposition 3.6, there are open sets $N_p' \sqsubset N_p'' \sqsubset N_p$ such that $\{N_p'\}_{p \in A}$ is an open cover of $|B|_H$. For each $p \in A$ we shall construct a smooth function $\lambda_p : N_p \to [0,1]$ that equals 1 on N_p' and has support in N_p''. Then $\lambda := \sum_{p \in A} \lambda_p : |B|_H \to \mathbb{R}$ is everywhere positive and smooth. Hence the functions $\beta_p := \lambda_p / \lambda$ form the required partition of unity.

For each p, we construct $\lambda_p : N_p \to \mathbb{R}$ inductively over its subsets $Q_m^p := \cup_{i=1}^m |U_i^p|_H$, where $U_1^p, \dots, U_k^p$ are the local branches at p. To begin, choose a smooth function $f_1^p : U_1^p \to [0,1]$ that equals 1 on $\pi_H^{-1}(cl(N_p'))$ and has support in $\pi_H^{-1}(N_p'')$. This exists because the map $\pi_H : U_1^p \to N_p$ is proper. Since U_1^p injects into N_p, we may define $\lambda_p(q)$ for $q \in Q_1^p$ by

$$\lambda_p(q) := f_1^p(x), \quad \text{where } \pi_H(x) = q.$$

By the tameness hypothesis, the pullback by π_H of λ_p to U_2^p (which is defined over $U_2^p \cap \pi_H^{-1}(|U_1^p|_H))$ may be extended over U_2^p to a smooth function f_2^p

that equals 1 on $\pi_H^{-1}(cl(N_p'))$ and has support in $\pi_H^{-1}(N_p'')$. Now extend λ_p over Q_2^p by setting it equal to the pushdown of f_2^p over $\pi_H(U_2^p)$. Continuing in this way, one extends λ_p to a function on the whole of N_p that equals 1 on $\pi_H^{-1}(cl(N_p'))$ and has support in $\pi_H^{-1}(N_p'')$.

It remains to check that $\lambda_p : |B|_H \to \mathbb{R}$ is smooth. Its pull back to any local branch U_i^p is smooth by construction. Consider any other point $x \in B_0$ such that $\pi_H(x) \in N_p$. Then there is some point $y \in U_i^p$ such that $|x| = |y|$ by the covering property of the local branches. Hence there is a local diffeomorphism ϕ_{yx} of a neighborhood of x to a neighborhood of y and so $\lambda_p \circ \pi_H$ is smooth near x because it is smooth near y. $\square$

Let $(\mathcal{B}, \Lambda, f)$ be a compact tame d-dimensional branched manifold structure on $(\underline{Z}, \Lambda_Z)$. Choose a smooth partition of unity $\{\beta_p\}$ on $|B|_H$ that is subordinate to a covering by sets N_p that support local branching structures (N_p, U_i^p, m_i). If $g : Z \to M$ is any smooth map into a manifold, and μ is a closed d-form on M (where $d = \dim Z$), we define

$$\int_Z g^*\mu := \sum_{p,i} m_i \int_{U_i^p} (\pi_H)^*\beta_p \, (g \circ f \circ \pi_H)^*\mu. \tag{3.2}$$

As we explain in more detail below, the reason why this is well defined and independent of choices is that, because $\mathcal{B}$ is tame, its branching locus is a finite union of piecewise smooth manifolds of dimension $d - 1$ and so has zero measure.[15]

Lemma 3.24. (i) *The number $\int_Z g^*\mu$ defined above is independent of the choice of partition of unity.*

(ii) *If g is bordant to $g' : Z' \to M$ by a bordism through a tame wnb groupoid, then $\int_Z g^*\mu = \int_{Z'} (g')^*\mu$. In particular, it is independent of the choice of $(\mathcal{B}, \Lambda)$.*

Proof. First suppose that the partitions of unity β_p, β_p' are subordinate to the same covering and consider the product groupoid $\mathcal{B} \times \mathcal{I}$ where $\mathcal{I}$ has objects $I_0 := [0,1]$ and only identity morphisms and $\Lambda_I \equiv 1$. There is a partition of unity $\{\beta_p''\}$ on $\mathcal{B} \times \mathcal{I}$ that restricts on the boundary to the two given partitions of unity and is subordinate to the cover $N_p \times |I|$. Hence by Stokes' theorem, it suffices to show that

$$\sum_{p,i} \int_{U_i^p} m_i (\pi_H)^*(\beta_p - \beta_p') \, (\pi_H \circ pr)^*\mu_B \;=\; \sum_{p,i} m_i \int_{U_i^p \times I_0} (\pi_H)^*(d\beta_p'') \, (\pi_H)^*\mu_B \;=\; 0,$$

[15] Achieving an analog of this is also a crucial step in the work of Cieliebak *et al.* cf. [**4**, Lemma 9.10] though they use rather different methods to justify it.

where $\mu_B := (g \circ f)^*\mu$ and $\mathrm{pr} : B_0 \times I_0 \to B_0$ is the projection. Let $V \subset (|B|_H \setminus |Br|_H)$ be a component of the complement of the branching locus, and let $\Lambda(V)$ be the constant value of Λ on V (cf. Proposition 3.6). Because the branching locus in each U_i^p has zero measure it suffices to show that the sum of the integrals over $\big(U_i^p \cap \pi_H^{-1}(V)\big) \times I_0$ vanishes for each V. But $V \cap N_p$ is diffeomorphic to $U_i^p \cap \pi_H^{-1}(V)$ for every i for which the intersection is nonempty. Hence, because $\mathrm{supp}(\beta_p) \subset N_p$, it makes sense to integrate over V and $V \times I_0$, and we find that

$$\sum_{p,i} m_i \int_{(U_i^p \cap \pi_H^{-1}(V)) \times I_0} (\pi_H)^*(d\beta_p'')\ (\pi_H \circ pr)^*\mu_B$$
$$= \sum_p \Lambda(V) \int_{V \times I_0} d\beta_p''\ pr^*\mu_B$$

which vanishes because $\sum_p \beta_p'' = 1$.

The proof of Proposition 3.11 shows that any two covers that support local branching structures have a common refinement that supports a local branching structure. Hence to prove (i) it suffices to show that if $\{\beta_p\}$ is subordinate to $\{N_p\}$ and the cover $\{N_q'\}$ refines $\{N_p\}$ then there is some partition of unity subordinate to $\{N_q'\}$ for which the two integrals are the same. The previous paragraph shows that the first integral may be written as

$$\sum_V \sum_p \int_V \Lambda(V)\beta_p\ \mu_V,$$

and so this statement holds by the standard arguments valid for manifolds.

To prove (ii), note that if $(\mathcal{W}, \Lambda_W)$ is a tame cobordism from $(\mathcal{B}, \Lambda)$ to $(\mathcal{B}', \Lambda')$ then every partition of unity on its boundary extends to a partition of unity on the whole groupoid. Moreover, we can construct the extension to be subordinate to any covering that extends those on the boundary. The rest of the details are straightforward, and are left to the reader. □

Proposition 3.25. *Let $(\underline{Z}, \Lambda_Z)$ be a compact d-dimensional branched manifold with boundary. Then the singular homology group $H_d(Z, \partial Z; \mathbb{R})$ contains an element $[Z]$ called the fundamental class with the following properties:*

(i) *If the weights of all the branches of Z are rational then $[Z] \in H_d(Z; \mathbb{Q})$.*
(ii) *If $\underline{\phi} : (\underline{Z}', \Lambda_{Z'}) \to (\underline{Z}, \Lambda_Z)$ is any layered covering, $\phi_*([Z']) = [Z]$.*
(iii) *The image of $[Z]$ under the boundary map $H_d(Z, \partial Z; \mathbb{R}) \to H_{d-1}(\partial Z; \mathbb{R})$ is $[\partial Z]$.*
(iv) *If $\underline{\phi} : (\underline{Z}, \Lambda_Z) \to \underline{Y}$ is any resolution of the orbifold $\underline{Y}$, then $\phi_*([Z]) = [Y] \in H_d(Y; \mathbb{R})$.*

(v) *Suppose further that $\underline{Z}$ is tame. Then for any smooth map $g : Z \to M$ of Z into a smooth manifold M and any closed d-form μ on M,*

$$[\mu](g_*([Z])) = \int_Z g^*(\mu).$$

Proof. By Lemma 3.22 we may assume that $(\underline{Z}, \Lambda_Z)$ is commensurate with a branched manifold $(\underline{B}, \Lambda_B)$ with structure given by the tame nonsingular wnb groupoid $(\mathcal{B}, \Lambda_B)$. The tameness condition implies that $|B|_H$ may be triangulated in such a way that both the branching locus $|Br|_H$ and the boundary of $|B|_H$ are contained in the $(d-1)$-skeleton. More precisely, we can arrange that any (open) $(d-1)$-simplex that intersects this $|Br|_H \cup \partial(|B|_H)$ is entirely contained in this set, and that no open d-simplex meets it. By first triangulating the boundary we may assume that similar statements (with d replaced by $d-1$) hold for its branching locus. Then, Λ_B is constant on each open d-simplex σ in the triangulation $\mathcal{T}$. To simplify the proof below we will refine $\mathcal{T}$ until each of its $(d-1)$-simplices ρ lies in the support N_p of a local branching structure such that N_p contains all the d-simplices that meet ρ.

Suppose that $\mathcal{B}$ has no boundary. Then we claim that the singular chain defined on $|B|_H$ by

$$c(|B|_H) := \sum_{\sigma \in \mathcal{T}} \Lambda_B(\sigma)\,[\sigma]$$

is a cycle. To see this, consider an open $(d-1)$-simplex ρ. If ρ lies in $|B|_H \setminus |Br|_H$ then it is in the boundary of precisely two oppositely oriented d-simplices with the same weights. Hence it has zero coefficient in $\partial c(|B|_H)$. Suppose now that ρ lies in the branching locus. Choose a local branching structure (N_p, U_i^p, m_i) such that ρ and all the d-simplices that meet it are contained in N_p. Each simplex σ whose boundary contains ρ lies in a component of $|B|_H \setminus |Br|_H$ and so, for each i, $\sigma \cap |U_i^p|_H$ is either empty or is the whole of σ. Moreover,

$$\Lambda_B(\sigma) = \sum_{i : \sigma \cap |U_i^p|_H \neq \emptyset} m_i.$$

Hence $\Lambda_B(\sigma)[\sigma]$ is the pushforward of the chain

$$\sum_{i : \sigma \cap |U_i^p|_H \neq \emptyset} m_i\,[(\pi_H)^{-1}(\sigma) \cap U_i^p] \quad \text{on } B_0.$$

Because the simplices in U_i^p cancel each other out in pairs in the usual way, ρ again makes no contribution to $\partial c(|B|_H)$. We now define $[|B|_H]$ to be the singular homology class represented by $c(|B|_H)$, and $[Z]$ to be its pushforward by $|B|_H \to Z$.

Note that if $\underline{Z}$ and hence $\mathcal{B}$ has boundary, then (iii) holds for the cycles $[|B|_H]$ and $[Z]$ by our choice of triangulation. Any two triangulations of $|B|_H$

can be considered as a triangulation of $|B|_H \times \{0,1\}$ and then extended over $|B|_H \times [0,1]$. Applying (iii) to $[|W|_H]$ where $\mathcal{W} = \mathcal{B} \times \mathcal{I}$, one easily sees that $[Z]$ is independent of the choice of triangulation. A similar argument shows that $[Z]$ is independent of the choice of representing groupoid $(\mathcal{B}, \Lambda_B)$, since any two such groupoids are cobordant by Proposition 3.11(iii). Therefore, the singular homology class $[Z] \in H_d(Z)$ is independent of all choices. It satisfies (i) by definition.

The other statements follow by standard arguments. In particular, (ii) holds by a cobordism argument and (v) holds because we can assume that both the triangulation and the partition of unity are subordinate to the same covering $\{N_p\}$. Hence each d-dimensional simplex in $\mathcal{T}$ lies in some component V of $|B|_H \backslash |Br|_H$ and so we can reduce this to the usual statement for manifolds. □

Remark 3.26. Above we explained the class $g_*([Z])$ for a smooth map $g : Z \to M$ in terms of integration. However, it can also be understood in terms of intersection theory. In fact, because $[Z]$ is represented by the singular cycle $f : c(|B|_H) \to Z$, the number $[\mu](g_*([Z]))$ may be calculated by counting the signed and weighted intersection points of the cycle $g \circ f : c(|B|_H) \to M$ with any singular cycle in M representing the Poincaré dual to $[\mu]$; cf. Example 3.3(ii). Detailed proofs of very similar statements may be found in [**4**].

4. Resolutions

Our first aim in this section is to show that every orbifold $\underline{X}$ has a resolution. We then discuss the relation between resolutions and the (multi)sections of orbibundles. Although we shall assume that $\underline{X}$ is finite dimensional, many arguments apply to orbifolds in any category.

4.1. Construction of the resolution. Let $\underline{Y}$ be a (possibly not effective) orbifold. Choose a good atlas $(U_i, G_i, \pi_i), i \in A$, for $\underline{Y}$ and use it to construct an orbifold structure $\mathcal{X}$ on $\underline{Y}$ with objects $\sqcup U_i$; cf. the discussion after Definition 2.11. For each finite subset $I \subset A$, we denote

$$|U_I| := \bigcap_{i \in I} |U_i| \subset |X|, \qquad G_I := \prod_{i \in I} G_i.$$

We shall not assume that the sets $|U_I|$ are connected, although it would slightly simplify the subsequent argument to do so. We shall identify the countable set A with a subset of $\mathbb{N}$, and shall write I as $\{i_1, \dots, i_k\}$, where $i_1 < i_2 < \cdots < i_k$. The length of I is $|I| := k$.

We now define some sets $\widehat{U}_I$. If $I = \{i\}$ we set $\widehat{U}_i := U_i$. When $|I| > 1$ we define $\widehat{U}_I$ to be the set of composable tuples $(\delta_{k-1}, \dots, \delta_1)$ of morphisms, where

$$s(\delta_j) \in U_{i_j}, \text{ for } 1 \le j \le k-1, \quad \text{and } t(\delta_{k-1}) \in U_{i_k}.$$

Since s and t are local diffeomorphisms, $\widehat{U}_I$ is a manifold.[16] Further, $\widehat{U}_I$ supports an action of the group G_I via:

$$(\delta_{k-1},\dots,\delta_1)\cdot(\gamma_k,\dots,\gamma_1) = (\gamma_k^{-1}\delta_{k-1}\gamma_{k-1},\dots,\gamma_2^{-1}\delta_1\gamma_1).$$

The action of G_I is not in general free. Indeed if $x = s(\delta_1)$ the stabilizer of $(\delta_{k-1},\dots,\delta_1)$ is a subgroup of G_I isomorphic to G_x. (For example, given $\delta \in \widehat{U}_{12}$ and $g_1 \in G_{s(\delta)}$ there is a unique $g_2 \in G_{t(\delta)}$ such that $\delta = g_2^{-1}\delta g_1$.) On the other hand, for each $\ell \in I$ the group

$$G'_{\ell I} := \prod_{i\in I\setminus\ell} G_i \tag{4.1}$$

does act freely on $\widehat{U}_I$. The obvious projection $\pi_I : \widehat{U}_I \to |U_I| \subset |X|$ identifies the quotient $\widehat{U}_I/G_I$ with $|U_I|$. Note that each component of $\widehat{U}_I$ surjects onto a component of $|U_I|$.

The resolution is a groupoid with objects contained in the sets $\widehat{U}_I$ and morphisms given by certain projections $\widehat{\pi}_{JI}$ that we now explain. For each $i_j \in I$ there is a projection $\widehat{\pi}_{i_j I} : \widehat{U}_I \to \widehat{U}_{i_j} = U_{i_j}$ defined as

$$\begin{aligned}\widehat{\pi}_{i_j I}(\delta_{k-1},\dots,\delta_1) &= s(\delta_j), \quad \text{if } j<k\\ &= t(\delta_{k-1}), \quad \text{if } j=k.\end{aligned}$$

If $k = |I| \ge 3$ and $J = I \setminus \{i_\ell\}$ we define a projection $\widehat{\pi}_{JI} : \widehat{U}_I \to \widehat{U}_J$ as follows:

$$\begin{aligned}\widehat{\pi}_{JI}(\delta_{k-1},\dots,\delta_1) &= (\delta_{k-1},\dots,\delta_2) \quad \text{if } \ell=1,\\ &= (\delta_{k-1},\dots,\delta_\ell\delta_{\ell-1},\dots,\delta_1) \quad \text{if } 1<\ell<k,\\ &= (\delta_{k-2},\dots,\delta_1) \quad \text{if } \ell=k.\end{aligned}$$

This map is equivariant with respect to the actions of G_I and G_J, and identifies the image $\widehat{\pi}_{JI}(\widehat{U}_I)$ as the quotient of $\widehat{U}_I$ by a free action of G_{i_ℓ}. If J is an arbitrary subset of I with $|J| > 1$ we define $\widehat{\pi}_{JI}$ as a composite of these basic projections. If $J = \{j\}$ we define $\widehat{\pi}_{JI} := \widehat{\pi}_{jI}$. Clearly $\widehat{\pi}_{jJ} \circ \widehat{\pi}_{JI} = \widehat{\pi}_{jI}$, whenever $\{j\} \subset J \subset I$.

Definition 4.1. *Let $\mathcal{X}$ be an ep groupoid constructed from the good atlas $\mathcal{A}$. A covering $\mathcal{V} = \{|V_I|\}_{I\subseteq A}$ of $|X|$ is called* **$\mathcal{A}$-compatible** *if the following conditions hold:*

---

[16]It is important that $\widehat{U}_I$ consists of morphisms rather than being the corresponding fiber product of the U_i over $|U_I|$, since that is not a manifold in general. For example, suppose that $U_1 = U_2 = \mathbb{R}$ and $G_1 = G_2 = \mathbb{Z}/2\mathbb{Z}$ acting by $x \mapsto -x$ and that each U_i maps to $|X| = [0,\infty)$ by the obvious projection. Then $\widehat{U}_{12}$ can be identified with the two disjoint lines $\{(x,x) : x \in \mathbb{R}\}$ and $\{(x,-x) : x \in \mathbb{R}\}$. On the other hand, the fiber product $U_1 \times_\pi U_2$ consists of two lines that intersect at $(0,0)$. Thus this step reformulates Liu–Tian's concept of desingularization in the language of orbifolds: see [**14**, Section 4.2].

(i) $|V_I| \sqsubset |U_I|$ *for all* I,

(ii) $cl(|V_I|) \cap cl(|V_J|) \neq \emptyset$ *iff* $|V_I| \cap |V_J| \neq \emptyset$ *iff one of* I, J *is contained in the other.*

(iii) *for any* $i \in I$, *any two distinct components of* $\pi_i^{-1}(|V_I|) \subset U_i$ *have disjoint closures in* U_i.

If $\mathcal{V}$ *is* $\mathcal{A}$*-compatible, the* **$\mathcal{V}$-resolution** $\mathcal{X}_\mathcal{V}$ *of* $\mathcal{X}$ *is a nonsingular groupoid defined as follows. Its set of objects is*

$$(X_\mathcal{V})_0 := \sqcup_I \widehat{V}_I,$$

where $\widehat{V}_I = \pi_I^{-1}(|V_I|) \subset \widehat{U}_I$ *and* I *is any subset of* A. *For* $J \subseteq I$ *the space of morphisms with source* $\widehat{V}_I$ *and target* $\widehat{V}_J$ *is given by the restriction of* $\widehat{\pi}_{JI}$ *to* $\pi_I^{-1}(|V_I| \cap |V_J|)$. *When* $I = J$ *these are identity maps. The category is completed by adding the inverses of these morphisms. The projection from the space of objects of* $\mathcal{X}_\mathcal{V}$ *to its Hausdorff orbit space is denoted* $\widehat{\pi}_H : (X_\mathcal{V})_0 \to |X_\mathcal{V}|_H$.

This definition is illustrated in Figures 5 and 6. Observe that we do not need to add composites of the form $\widehat{\pi}_{JI} \circ (\widehat{\pi}_{KI})^{-1}$, for such morphisms would be defined over the intersection $|V_I| \cap |V_J| \cap |V_K|$ which is nonempty only if $K \subset J \subset I$ or $J \subset K \subset I$. In the former case this composite is $(\widehat{\pi}_{KJ})^{-1}$ while in the latter it is $\widehat{\pi}_{JK}$. Hence in either case it is already in the category.

Since $\mathcal{X}_\mathcal{V}$ has at most one morphism between any two objects, it is a nonsingular sse Lie groupoid. It need not be proper. However, we now show that it does have a weighting.

Proposition 4.2. *Let* $\mathcal{X}, \mathcal{A}$ *and* $\mathcal{V}$ *satisfy the conditions in Definition* 4.1, *and form* $\mathcal{X}_\mathcal{V}$ *as above. Then there is a weighting function* $\Lambda_\mathcal{V} : |X_\mathcal{V}|_H \to (0, \infty)$ *such that* $(\mathcal{X}_\mathcal{V}, \Lambda_\mathcal{V})$ *is a wnb groupoid. Moreover,* $(\mathcal{X}_\mathcal{V}, \Lambda_\mathcal{V})$ *is commensurate with* $\mathcal{X}$.

The proof uses the following technical lemma. For each component $|V_I|^\alpha$ of $|V_I| \subset |X|$ we set $\widehat{V}_I^\alpha := \pi_I^{-1}(|V_I|^\alpha) \cap \widehat{U}_I$, where $\pi_I : \widehat{U}_I \to |U_I| \subset |X|$ is the obvious projection.

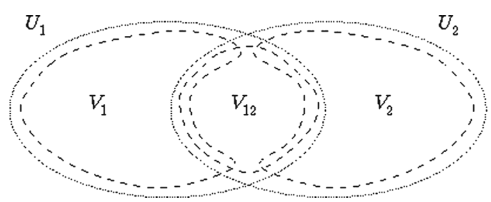

Figure 5. An $\mathcal{A}$-compatible covering $\mathcal{V}$; the sets V_I have dashed boundaries.

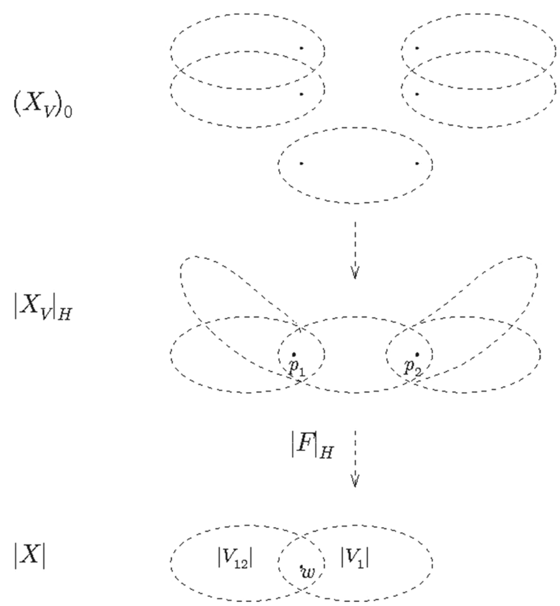

Figure 6. $(X_\nu)_0, |X_\nu|_H$ and $|X|$ with $|G_1| = |G_2| = 2$. The points p_1, p_2 project down to w.

Lemma 4.3. *Assume $\mathcal{X}$ and $\mathcal{V}$ are as in Proposition* 4.2. *Then the following statements hold.*

(i) *For each $\alpha \in \pi_0(|V_I|)$, the group G_I acts transitively on the components of $\widehat{V}_I^\alpha$.*

(ii) *For each $\ell \in I$ the group $G'_{\ell I}$ acts freely on the components of $\widehat{U}_I$.*

(iii) *If $I \subset J$ and $q \in \widehat{V}_I$ projects down to $\pi_I(q) \in |V_I| \cap |V_J|$, there are exactly $|G_J|/|G_I|$ components of $\widehat{V}_J$ whose image by $\widehat{\pi}_{IJ}$ contains q.*

Proof. (i) holds because as we remarked above the projection $\pi_I : \widehat{V}_I^\alpha \to |V_I|^\alpha$ quotients out by the action of G_I.

We shall check (ii) for $\ell = i_1$. (The other cases are similar.) Then the element $(\gamma_k, \dots, \gamma_2, 1)$ of $G'_{i_1 I}$ acts on $\widehat{U}_I$ via the maps

$$(\delta_{k-1}, \dots, \delta_1) \mapsto (\gamma_k^{-1}\delta_{k-1}\gamma_{k-1}, \dots, \gamma_2^{-1}\delta_1).$$

If the source and target of this map lie in the same component of $\widehat{V}_I$ then the morphisms δ_i and $\gamma_{i+1}^{-1}\delta_i\gamma_i$ lie in the same component of X_1 for each i. In particular, δ_1 and $\gamma_2^{-1}\delta_1$ lie in the same component of X_1. Hence composing with δ_1^{-1} we see that $\gamma_2 \in G_2$ is isotopic to an identity map. But because $\mathrm{id} : X_0 \to X_1$ is a section of the local covering map $s : X_1 \to X_0$ the set of identity morphisms form a connected component of X_1 in any groupoid $\mathcal{X}$.

Hence we must have $\gamma_2 = 1$. Repeating this argument we see that $\gamma_i = 1$ for all i. This proves (ii).

Finally (iii) holds because the set of components of $\widehat{V}_J$ whose image by $\widehat{\pi}_{IJ}$ contains q form an orbit of an action by the group $\prod_{i\in J\setminus I} G_i$ which is free by (ii). □

Proof of Proposition 4.2. Without loss of generality we assume that $\mathcal{X}$ is connected and define κ to be the order of the stabilizer of a generic point in X_0. Thus $\kappa = 1$ iff $\mathcal{X}$ is effective.

We first investigate the relation of $\mathcal{X}_{\mathcal{V}}$ to $\mathcal{X}$. To this end, define the functor $F : \mathcal{X}_{\mathcal{V}} \to \mathcal{X}$ on objects as the projection

$$F : \widehat{V}_I \to U_{i_1}, \quad (\delta_{k-1}, \dots, \delta_1) \mapsto s(\delta_1).$$

The morphisms given by $\widehat{\pi}_{JI}$ are taken to identities if $i_1 \in J$. If not, the projection $\widehat{\pi}_{IJ}$ with source $(\delta_{k-1}, \dots, \delta_1)$ is taken to the composite $\delta_{r-1} \circ \dots \circ \delta_1$ where i_r is the smallest element in $I \cap J$.

Since $|X|$ is Hausdorff, the induced map $|X_{\mathcal{V}}| \to |X|$ factors through $|F|_H : |X_{\mathcal{V}}|_H \to |X|$. For each $w \in |X|$, let $I = I_w$ (resp. $J = J_w$) be the subset of minimal length such that $w \in cl(|V_I|)$ (resp. $w \in |V_J|$). (see Figure 7.) Note that $I \subseteq J$ because $\mathcal{V}$ is $\mathcal{A}$-compatible. For each ℓ such that $w \in |U_\ell|$, w has precisely $|G_\ell|/|G_w|$ preimages x in $U_\ell \subset X_0$, where G_w denotes the isomorphism class of the stabilizer groups G_x. If $w \in |V_L|$ and $\ell \in L$, then each such x lies in $U_\ell \cap \widehat{\pi}_{\ell L}(\widehat{V}_L)$. (Here we identify U_ℓ with $\widehat{U}_\ell$.) By Lemma 4.3, each x has $|G'_{\ell L}|$ preimages in $\widehat{V}_L$, each lying in a different component. Thus w has $|G_L|/|G_x|$ preimages in $\widehat{V}_L$: cf. Figure 6. Moreover, the minimality of $J = J_w$ implies that w has $|G_J|/|G_x|$ preimages in $|X_{\mathcal{V}}|$. On the other hand, if $I = I_w \neq J$ two such preimages map to distinct elements in $|X_{\mathcal{V}}|_H$ iff the corresponding elements of $\widehat{V}_J$ map to distinct elements of $\widehat{U}_I$ under the projection $\widehat{\pi}_{IJ}$. Hence w has $|G_I|/|G_x|$ preimages p in $|X_{\mathcal{V}}|_H$.

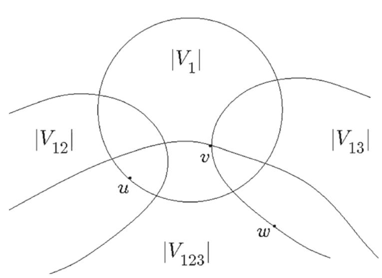

Figure 7. This is a diagram of part of $|X|$. Here $I_u = \{1\}, J_u = \{1,2\}$; $I_v = J_v = \{1\}$; and $I_w = \{1,3\}, J_w = \{1,2,3\}$.

Now consider $p \in |X_{\mathcal{V}}|_H$ and let $|F|_H(p) = w$. Define $\Lambda_{\mathcal{V}} : |X_{\mathcal{V}}|_H \to \mathbb{R}$ by setting $\Lambda_{\mathcal{V}}(p) = \kappa/|G_I|$ where $I = I_w$ is as above. Then, because w has $|G_I|/|G_x|$ preimages p in $|X_{\mathcal{V}}|_H$, $(|F|_H)_*(\Lambda_{\mathcal{V}})(w) = 1$ provided that $\kappa = |G_x|$, i.e., provided that w is smooth. Hence F is a layered equivalence, provided that $(\mathcal{X}_{\mathcal{V}}, \Lambda_{\mathcal{V}})$ is a branched manifold.

Thus it remains to check that the conditions of Definition 3.2 hold for $(\mathcal{X}_{\mathcal{V}}, \Lambda_{\mathcal{V}})$. In the following construction, each local branch at p lies in a component of $\widehat{V}_J$, where $J = J_w$ as defined above. They will all be assigned the same weight $\kappa/|G_J|$. The remarks in the preceding paragraph give rise to the following characterization of the fiber in $|X_{\mathcal{V}}|$ over $p \in |X_{\mathcal{V}}|_H$.

> *Define $I = I_w, J = J_w$ as above, where $w = |F|_H(p)$. Since $p \in cl(|\widehat{V}_I|_H)$ there is a convergent sequence $p_n \to p$ of elements in $|\widehat{V}_I|_H$. Lift this sequence to $\widehat{V}_I$ and choose $q \in cl(\widehat{V}_I) \subset \widehat{U}_I$ to be one of its limit points. Then the points in the fiber over p in $|X_{\mathcal{V}}|$ are in bijective correspondence with the $|G_J|/|G_I|$ distinct elements in $\widehat{V}_J$ that are taken to q by $\widehat{\pi}_{IJ}$.*

In particular, the elements in the fiber over p lie in different components of $|\widehat{V}_J|$. It follows that for any K each component of $\widehat{V}_K$ maps bijectively onto a closed subset of the inverse image $|\widehat{V}_K|_H := (|F|_H)^{-1}(|V_K|)$ of $|V_K|$ in $|X_{\mathcal{V}}|_H$. (Note that each component of $\widehat{V}_K$ injects into $|X_{\mathcal{V}}|$ because $\mathcal{X}_{\mathcal{V}}$ is nonsingular.)

To construct the local branches at p, first suppose that $I_w = J_w$. This hypothesis implies that $p \in |\widehat{V}_I^\alpha|_H$ for some component $|V_I|^\alpha$ of $|V_I| \subset |X|$, but that $p \notin cl(|\widehat{V}_K|_H)$ for any $K \subsetneq I$. Take

$$N_p := |\widehat{V}_I^\alpha|_H \setminus \Big(\bigcup_{K \subsetneq I} cl(|\widehat{V}_K|_H) \Big),$$

and choose a single local branch U_1 equal to the inverse image of N_p in any component of $\widehat{V}_I^\alpha$. (Note that each component of $\widehat{V}_I^\alpha$ surjects onto the connected set $|\widehat{V}_I|_H^\alpha$ since the components of $\widehat{U}_I$ surject onto those of $|U_I|$.) As mentioned above, we set $m_1 := \kappa/|G_I|$. Since $\Lambda_{\mathcal{V}}$ equals $\kappa/|G_I|$ at all points of N_p, the conditions are satisfied in this case.

If $p \in \partial(|\widehat{V}_I|_H^\alpha)$ where $I = I_w$, we choose N to be a connected open neighborhood of p in $|X_{\mathcal{V}}|_H$ that satisfies the following conditions:

(i) $N \subset |\widehat{V}_L|_H$ for all L such that $p \in |\widehat{V}_L|_H$,

(ii) N is disjoint from all sets $cl(|\widehat{V}_K|_H)$ such that $p \notin cl(|\widehat{V}_K|_H)$.
(This is possible because of the local finiteness of $\mathcal{V}$ and the fact that two sets in $\mathcal{V}$ intersect iff their closures intersect.) Now define $L := L_w \supset I$ to be the *maximal* set L such that $p \in |\widehat{V}_L|_H$. Choose $p' \in (N \cap |\widehat{V}_I|_H)$ and a lift $q' \in \widehat{V}_I^\alpha$ of p', i.e., so that $\widehat{\pi}_H(q') = p'$.

Then by Lemma 4.3(iii) there are precisely $|G_L|/|G_I|$ components of $\widehat{V}_L$ whose image by $\widehat{\pi}_{IL}$ contains q'. Choose their intersections with $(\pi_H)^{-1}(N_p)$ to be the local branches at p, where $\pi_H : (X_{\mathcal{V}})_0 \to |X_{\mathcal{V}}|_H$ is the obvious projection. Then the covering and local regularity properties of Definition 3.2 hold by the above discussion of the fibers of the map $|X_{\mathcal{V}}| \to |X_{\mathcal{V}}|_H$. The weighting condition also holds at p since p lies in the image of each local branch.

It remains to check the weighting condition at the other points p'' in N. Define $I'' := I_{w''}$, the minimal index set K such that $p'' \in cl(|\widehat{V}_K|_H)$. Condition (ii) for N implies that $p \in cl(|\widehat{V}_{I''}|_H)$ and hence that $I \subseteq I''$. On the other hand, because $N \subset |\widehat{V}_L|_H$ the minimality of I'' implies that $I'' \subseteq L$. By Lemma 4.3(iii) there are precisely $|G_L|/|G_{I''}|$ components of $\widehat{V}_L$ whose image in $|X_{\mathcal{V}}|_H$ meets $|\widehat{V}_{I''}|_H$ near p''. Since these correspond bijectively to the local branches that intersect $(\pi_H)^{-1}(p'')$ the weighting condition holds for all $p'' \in N$. This completes the proof.

In order to show that every orbifold has a resolution, we need to see that suitable coverings $\mathcal{V}$ do exist. This well-known fact is established in the next lemma: cf. Figure 5.

Lemma 4.4. *Let Y be a normal topological space with an open cover $\{U_i\}$ such that each set U_i meets only finitely many other U_j. Then there are open subsets $U_i^0 \sqsubset U_i$ and $V_I \subset U_I$ with the following properties:*

(i) $Y \subseteq \cup_i U_i^0, \;\; Y \subseteq \cup_I V_I$;

(ii) $V_I \cap U_i^0 = \emptyset$ *if* $i \notin I$;

(iii) *if* $cl(V_I) \cap cl(V_J) \neq \emptyset$ *then one of the sets* I, J *contains the other.*

Proof. First choose an open covering $\{U_i^0\}$ of Y such that $U_i^0 \sqsubset U_i$ for all i. For each i, choose k_i so that $U_i \cap U_J = \emptyset$ for all J such that $|J| > k_i$. Then choose for $n = 1, \ldots, k_i$ open subsets U_i^n, W_i^n of U_i such that

$$U_i^0 \sqsubset W_i^1 \sqsubset U_i^1 \sqsubset W_i^2 \sqsubset \ldots \sqsubset U_i^{k_i} = U_i,$$

and set $U_i^j := U_i^{k_i}$ for $j > k_i$. Then, if $|I| = \ell$, define

$$V_I = W_I^\ell - \bigcup_{J:|J|>\ell} cl(U_J^{\ell+1}).$$

We claim that this covering satisfies all the above conditions. For example, to see that $Y \subseteq \cup_I V_I$, observe that the element $y \in Y$ lies in V_K, where K is a set of maximal length such that $y \in W_K^{|K|}$. To prove (iii), first consider the case when $y \in cl(V_I) \cap cl(V_J)$ where $|I| = |J| = \ell$, but $I \neq J$. Then $y \in cl(W_{I\cup J}^\ell) \subset U_{I\cup J}^{\ell+1}$; but this is impossible because $U_{I\cup J}^{\ell+1}$ is cut out of V_I, which implies that $cl(W_{I\cup J}^\ell)$ does not intersect $cl(V_I)$. The rest of the proof is similar and is left to the reader. $\square$

Proof I of Proposition 3.16. We must show that every orbifold $\underline{Y}$ is commensurate with a tame branched manifold. Let $\mathcal{X}$ be a groupoid structure on $\underline{Y}$ constructed as before from an atlas. By applying Lemma 4.4 to the covering of $|X|$ by the charts $|U_i|$ of this atlas, we may find a covering $\{|V_I|\}$ of $|X| = Y$ that satisfies the conditions in Definition 4.1. We may also choose the $|V_I|$ so that for one (and hence every) $i \in I$ the pullback $\pi^{-1}(|V_I|) \cap U_i$ has piecewise smooth boundary in U_i. Now consider the corresponding wnb groupoid $(\mathcal{X}_{\mathcal{V}}, \Lambda_{\mathcal{V}})$ constructed in Proposition 4.2. This is commensurate to $\mathcal{X}$, and its branch locus is piecewise smooth. Therefore, as in Lemma 3.21 it has a tame refinement. □

Remark 4.5. (i) If $\mathcal{X}$ itself is nonsingular, then the projection $\mathcal{X}_{\mathcal{V}} \to \mathcal{X}$ is an equivalence. Indeed $\mathcal{X}_{\mathcal{V}}$ is just the refinement of $\mathcal{X}$ corresponding to the covering $\mathcal{V}$ of $|X|$.

(ii) Consider the two noneffective groupoids $\mathcal{X}, \mathcal{Z}$ of Remark 2.14. Cover $|X| = |Z| = S^1$ by three open arcs $U_i, i = 1, 2, 3$, whose triple intersection is empty. Then if we choose the same covering $\mathcal{V}$ in both cases, the groupoids $\mathcal{X}_{\mathcal{V}}$ and $\mathcal{Z}_{\mathcal{V}}$ are isomorphic. See Figure 8. Thus if one wants to preserve information about the topological structure of the trivially acting morphisms it might be better to define the resolution of a noneffective groupoid using the approach discussed in Remark 3.17.

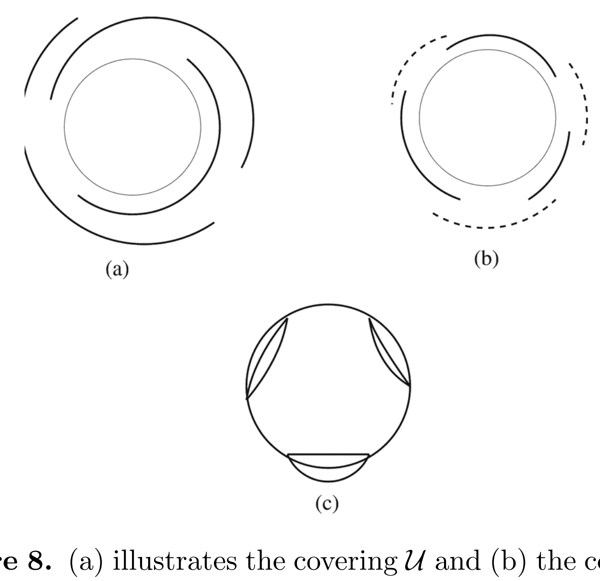

Figure 8. (a) illustrates the covering $\mathcal{U}$ and (b) the covering $\mathcal{V}$, in which the V_i are represented by solid arcs and the V_{ij} by dashed arcs. (c) is the resolution $|X_{\mathcal{V}}|_H = |Z_{\mathcal{V}}|_H$. The arc is tripled over the parts of the circle not covered by the V_i since each set $\widehat{V}_{ij}$ has three components.

Compatibility with SFT operations. In Hofer *et al.* [**10**], it is important that all constructions are compatible with the natural operations of symplectic field theory. These arise from the special structure of the groupoids they consider, which live in a category of manifold-like objects called polyfolds with boundary and corners. Though polyfolds are infinite dimensional, their corners have finite codimension. Moreover, each groupoid $\mathcal{X}$ has a finite formal dimension (or index), and its boundary is build up inductively from pieces of lower index. For example, in the simplest situation where there is a boundary but no corners, the boundary $\partial\mathcal{X}$ of $\mathcal{X}$ decomposes as a disjoint union of products $\mathcal{X}_1 \times \mathcal{X}_2$ of ep groupoids $\mathcal{X}_\alpha$ of lower index (and without boundary). Hence it is important that the operation of constructing a resolution is compatible with taking products and also extends from the boundary of $\mathcal{X}$ to $\mathcal{X}$ itself.

We now discuss these questions in the context of finite dimensional manifolds. First consider products. The construction of the resolution $\mathcal{X}_\mathcal{V} \to \mathcal{X}$ is determined by the choice of good atlas $\mathcal{A}$ (together with an ordering of the set A of charts) and the choice of a subcovering $\mathcal{V}$. If $\mathcal{A}_\alpha = \{(U_{\alpha i}, G_{\alpha i}, \pi_{\alpha i}) : i \in A_\alpha\}$ is an atlas for $\mathcal{X}_\alpha$, where $\alpha = 1, 2$, then there is a product atlas $\mathcal{A}_1 \times \mathcal{A}_2$ for $\mathcal{X}_1 \times \mathcal{X}_2$ with charts $(U_{1i} \times U_{2j}, G_{1i} \times G_{2j}, \pi_{1i} \times \pi_{2j})$ where $(i, j) \in A_1 \times A_2)$. But the product covering by sets $|V_{1I}| \times |V_{2J}|$ is not $\mathcal{A}_1 \times \mathcal{A}_2$-compatible since there will be nonempty intersections of the form $\big(|V_{1I}| \times |V_{2J}|\big) \cap \big(|V_{1I'}| \times |V_{2J'}|\big)$ where $I \subsetneq I'$ and $J' \subsetneq J$.[17] Hence our construction does not commute with taking the product. On the other hand, the following lemma shows that we can take the resolution of a product to be the product of resolutions of the factors.

Lemma 4.6. *For any resolutions $F_\alpha : (\mathcal{B}_\alpha, \Lambda_\alpha) \to \mathcal{X}_\alpha$ for $\alpha = 1, 2$ the product*

$$F_1 \times F_2 : (\mathcal{B}_1 \times \mathcal{B}_2, \Lambda_1, \times \Lambda_2) \to \mathcal{X}_1 \times \mathcal{X}_2$$

is a resolution.

Proof. This is immediate. $\square$

We now show how to extend a resolution from $\partial\mathcal{X}$ to $\mathcal{X}$. Note that $\partial\mathcal{X}$ has collar neighborhood in $\mathcal{X}$ that is diffeomorphic[18] to the product $\partial\mathcal{X} \times \mathcal{I}'$, where $\mathcal{I}'$ denotes the trivial groupoid with objects $(-1, 0]$ as in Proposition 3.11(iii). Further $\mathcal{X}$ is diffeomorphic to the groupoid $\mathcal{X}'$ obtained by extending the collar neighborhood of $\partial\mathcal{X}$ by $\partial\mathcal{X} \times \mathcal{I}$, where $\mathcal{I}$ is the trivial groupoid with objects $[0, 1]$.

[17] The fact that there is no ordering of the new index set $A_1 \times A_2$ that is symmetric in α is less serious, since one could define the resolution without using such an order. For example, the objects of the new resolution would contain $|I|!$ copies of $\widehat{V}_I$, one associated to each possible ordering of the index set I.

[18] Two groupoids $\mathcal{X}, \mathcal{Y}$ are diffeomorphic if there is a functor $F : \mathcal{X} \to \mathcal{Y}$ that is a diffeomorphism on objects and morphisms.

Lemma 4.7. *Let $\mathcal{X}$ be an ep groupoid with boundary. Then any resolution $F : \mathcal{B} \to \partial\mathcal{X}$ extends to a resolution $F' : \mathcal{B}' \to \mathcal{X}$ of $\mathcal{X}$.*

Proof. Construct a resolution $\mathcal{X}_\mathcal{V} \to \mathcal{X}$. As in Proposition 3.11(iii) there is a layered covering $\mathcal{W} \to \partial\mathcal{X} \times \mathcal{I}$ that restricts to $\mathcal{B} \to \partial\mathcal{X}$ over $\partial\mathcal{X} \times \{1\}$ and to $\partial(\mathcal{X}_\mathcal{V}) \to \partial\mathcal{X}$ over $\partial\mathcal{X} \times \{0\}$. Now take $\mathcal{B}'$ to be the union of $\mathcal{X}_\mathcal{V}$ with $\mathcal{W}$ and define F' to be the induced functor $\mathcal{B}' = \mathcal{X}_\mathcal{V} \cup \mathcal{W} \to \mathcal{X} \cup (\partial\mathcal{X} \times \mathcal{I}) = \mathcal{X}' \cong \mathcal{X}$. □

4.2. Orbibundles and multisections. We now show how branched manifolds arise as multivalued sections of (unbranched) orbibundles, a groupoid version of the constructions in Cieliebak *et al.* [**4**] and Hofer *et al.* [**10**].

Recall from Moerdijk [**15**, Section 5] that an **orbibundle** $\underline{E} \to \underline{Y}$ over an orbifold $\underline{Y}$ is given by an equivalence class of functors $\rho : \mathcal{E} \to \mathcal{X}$, where $\mathcal{X}$ is a orbifold structure on Y and $\mathcal{E}$ is a groupoid constructed as follows. The objects E_0 of $\mathcal{E}$ form a vector bundle $\rho_0 : E_0 \to X_0$ and its space E_1 of morphisms arise from a right action of $\mathcal{X}$ on E_0. In other words, E_1 is the fiber product $E \times_{X_0} X_1 := \{(e, \gamma) : \rho(e) = t(\gamma)\}$, the target map is $(e, \gamma) \mapsto e$ and the source map

$$\mu : E_0 \times_{X_0} X_1 \to E_0, \quad (e, \gamma) \mapsto e \cdot \gamma$$

satisfies the obvious identities: namely the diagram

$$\begin{array}{ccc} E_0 \times_{X_0} X_1 & \stackrel{\mu}{\to} & E_0 \\ \rho \downarrow & & \rho \downarrow \\ X_0 \times_{X_0} X_1 & \stackrel{s'}{\to} & X_0 \end{array}$$

commutes (here $s'(t(\gamma), \gamma)) = s(\gamma))$, the identities $\mathrm{id}_x \in X_1$ act by the identity and composition is respected, i.e., $(e, \gamma\delta) = (e, \gamma) \cdot \delta$.

This construction applies when given any sse groupoid $\mathcal{X}$ and any vector bundle $E_0 \to X_0$ that supports a right action of $\mathcal{X}$. We shall call $\rho : \mathcal{E} \to \mathcal{X}$ a vector bundle over $\mathcal{X}$. Clearly, $\mathcal{E}$ is an ep groupoid if $\mathcal{X}$ is. Also, if $(\mathcal{X}, \Lambda)$ is a weighted branched manifold, we can pull Λ back to give a weighting on $\mathcal{E}$. Hence $\mathcal{E}$ is a wnb groupoid if $\mathcal{X}$ is.

Example 4.8. (i) Let $E_0 = TX_0$, the tangent bundle of X_0. Then $\mathcal{X}$ acts on E_0 in the obvious way. The resulting bundle $T\mathcal{X} \to \mathcal{X}$ is called the tangent bundle. Note that $\mathcal{X}$ acts effectively on TX_0 iff $\mathcal{X}$ is effective.

(ii) Every bundle $\mathcal{E} \to \mathcal{X}$ has an effective reduction $\mathcal{E}_{\mathrm{eff}} \to \mathcal{X}'$ where $\mathcal{X}'$ is the quotient of $\mathcal{X}$ by the subgroup of $K \cong K_y$ that acts trivially on E_0. (For notation, see Lemma 2.5.) Thus the functor $\mathcal{X} \to \mathcal{X}_{\mathrm{eff}}$ factors through $\mathcal{X}'$.

A **section** of the bundle $\rho : \mathcal{E} \to \mathcal{X}$ is a smooth functor $\sigma : \mathcal{X} \to \mathcal{E}$ such that $\rho \circ \sigma = id$. In particular, the restriction of each such functor to the space of objects is a (smooth) section σ_0 of the vector bundle $\rho_0 : E_0 \to X_0$.

The conditions imposed by requiring that σ_0 extend to a functor imply that σ_0 must descend to a section of $|E| \to |X|$.

We denote by $\mathcal{S}(\mathcal{E})$ the space of all such sections σ. Since each section σ is determined by the map $\sigma_0 : X_0 \to E_0$, we may identify $\mathcal{S}(\mathcal{E})$ with a subset $\mathcal{S}_0$ of the space $\mathrm{Sect}(E_0)$ of smooth sections of this bundle with the usual Fréchet topology.[19] Let us now suppose that $\mathcal{X}$ is compact. By replacing $\mathcal{X}$ by an equivalent groupoid if necessary, we may suppose that X_0 and X_1 have finitely many components. Hence the subset $\mathcal{S}_0$ is defined by a finite number of smooth compatibility conditions and so is a submanifold of $\mathrm{Sect}(E_0)$.

Definition 4.9. *A section $\sigma : \mathcal{X} \to \mathcal{E}$ is said to be* **transverse to the zero section** *if its image $\sigma(X_0)$ intersects the zero set transversally and if for each intersection point $x \in X_0$ the induced map $d\sigma(x) : T_\sigma\mathcal{S}(\mathcal{X}) \to E_x$ is surjective, where E_x is the fiber of $E_0 \to X_0$ at x and $d\sigma(x)$ is the composite of the derivative of σ with evaluation at x.*

We claim that if the base $\mathcal{B}$ is sufficiently nice there are enough sections.

Lemma 4.10. *Suppose that $\mathcal{E} \to \mathcal{B}$ is a bundle over a tame wnb groupoid. Then a generic element of $\mathcal{S}(\mathcal{E})$ intersects the zero section transversally.*

Proof. We just sketch the proof since the techniques are standard. The idea is the following. Consider an open set $N_p \subset |B|_H$ with local branches $U_i^p, i \in A_p$ as in Definition 3.2. As in Lemma 3.23, the tameness condition allows one to construct enough sections to prove the result for the restriction of $\mathcal{E}$ to $\mathcal{B}_N$, the full subcategory of $\mathcal{B}$ with objects $U_i^p, i \in A_p$. But any section with compact support in $\mathcal{B}_N$ extends uniquely to a section of $\mathcal{E} \to \mathcal{B}$. The result follows. □

If $\mathcal{B}$ is a wnb groupoid, $\mathcal{E}$ is oriented and the section $\sigma : \mathcal{B} \to \mathcal{E}$ is transverse to the zero section, then the zero set $\mathcal{Z}(\sigma)$ of σ inherits a natural structure as a wnb groupoid. Standard arguments shows that its cobordism class is independent of the section chosen. In particular, if the fiber dimension of $\mathcal{E}$ equals $\dim \mathcal{B}$ the Euler number $\chi(\mathcal{E})$ can be calculated as the number of zeros of a generic section of $\mathcal{E} \to \mathcal{B}$, or equivalently as the number of points in the zero-dimensional wnb groupoid $\mathcal{Z}(\sigma)$; cf. Example 3.3(ii). For example, the tangent bundle $T\mathcal{B} \to \mathcal{B}$ of the resolution of the teardrop orbifold described in Example 2.10 has a section with one positively oriented zero in each of D^+, D^-. It follows that $\chi(Y) = 1 + 1/k$.

Definition 4.11. *The* **(homology) Euler class** *of a d-dimensional bundle $\mathcal{E} \to \mathcal{B}$ over an n-dimensional wnb groupoid is the singular homology class*

[19] Since the components of X_0 are noncompact, we take the topology given by uniform C^k-convergence on compact sets for all $k \geq 1$. The induced topology on $\mathcal{S}(\mathcal{E})$ is not very satisfactory, but it is good enough for the present purposes.

$e(\mathcal{E}) \in H_{n-d}(|B|_H;\mathbb{R})$ represented by the image in $|B|_H$ of the fundamental class $[Z(\sigma)] \in H_{n-d}(|Z|_H;\mathbb{R})$ of the zero set $\mathcal{Z}(\sigma)$ of a generic section.

This definition has the expected functorial properties and is consistent with standard definitions; cf. Proposition 4.19 and the discussion in Section 4.3 below. Moreover, if $\mathcal{B}$ has rational weights then $e(\mathcal{E}) \in H_{n-d}(|B|_H;\mathbb{Q})$.

Remark 4.12. The condition on $d\sigma(x)$ in Definition 4.9 ensures the existence of enough local deformations of σ to have a good transversality theory, e.g., one in which the zero sets of two generic sections intersect transversally. As an example, consider the tangent bundle $T\mathcal{X} \to \mathcal{X}$ of the teardrop orbifold. Every section of this bundle vanishes at the singular point p, and so no section is transverse in the sense of the above definition.

One way to deal with the lack of sections over singular $\mathcal{X}$ is to consider multivalued sections. In Cieliebak *et al.* [**4**] and Hofer *et al.* [**10**] these are defined by means of the characteristic function of their graph. The following definition is a mild adaptation of that in [**10**]. Although one could define multisections of bundles over wnb groupoids we shall be content here with the case when the base is unbranched. Further, to be compatible with [**10**] we assume from now on that all weighting functions take rational values.

Definition 4.13. *Let $\rho : \mathcal{E} \to \mathcal{X}$ be a bundle over an ep groupoid and denote by $\mathcal{Q}^{\geq 0}$ the category with objects the nonnegative rational numbers and only identity morphisms. A* **multisection** *of $\mathcal{E} \to \mathcal{X}$ is a smooth functor $\mathcal{L}_S : \mathcal{E} \to \mathcal{Q}^{\geq 0}$ that has the following local description: for each point $x \in X_0$ there is an open neighborhood U and a finite nonempty set of smooth local sections $\sigma_j : U \to E_0$ with positive rational weights m_j such that for all $x \in U$*

$$\mathcal{L}_S((x,e)) = \sum_{j:\sigma_j(x)=e} m_j, \qquad \sum_{e\in E_x} \mathcal{L}_S((x,e)) = 1,$$

where by convention the sum over the empty set is 0. The triple (U, σ_j, m_j) is called a **local section structure for $\mathcal{L}_S$.** *We denote by $\mathcal{S}_m(\mathcal{E})$ the set of all multisections of $\mathcal{E} \to \mathcal{X}$.*

The first condition implies that for each $x \in X_0$ there are a finite number of elements $e \in E_x$ such that $\mathcal{L}_S((x,e)) \neq 0$. The set of such elements $(x,e) \in E_0$ is called the **support** of $\mathcal{L}_S$. The second condition is equivalent to requiring that $\sum m_j = 1$ and implies that $\mathcal{L}_S$ has total weight 1. The sum of two multisections $\mathcal{L}_S, \mathcal{L}_T$ is given by the convolution product:

$$(\mathcal{L}_S + \mathcal{L}_T)(x,e) := \sum_{e'+e''=e} \mathcal{L}_S(x,e')\mathcal{L}_T(x,e'').$$

Similarly, $r\mathcal{L}_S(x,e) := \mathcal{L}_S(x,re)$ for $r \in \mathbb{Q}$. Hence multisections form a $\mathbb{Q}$-vector space. Note also that because $\mathcal{L}_S$ is a functor it takes the same value on all equivalent objects in E_0 and so descends to a function on $|E|_H$. Hence when counting the number of zeros of a multisection, one should count the equivalence classes c lying in the zero section, each weighted by the product $\mathfrak{o}_c \mathcal{L}_S(c)$, where $\mathfrak{o}$ is the orientation (cf. Example 3.3 (ii)).

We say that $\mathcal{L}_S$ is **single valued** if there is just one local section σ_j over each open set U. In this case the support of $\mathcal{L}_S$ is the image of a single valued section $\sigma_0 : X_0 \to E_0$ that extends to a functor $\sigma : \mathcal{X} \to \mathcal{E}$, i.e., a section as we defined it above. However, if $\mathcal{X}_{\text{eff}}$ is singular the values of single valued sections at nonsmooth points are restricted. For example, they must vanish at $x \in X_0$ if every point of E_x is moved by some element of G_x.

We now show that the support of a multisection $\mathcal{L}_S$ is (under a mild hypothesis) the image of a wnb groupoid. We begin with a preliminary lemma that explains the relation between two different local section structures.

Lemma 4.14. *Consider two local section structures $(U, \sigma_j, m_j), (U', \sigma'_j, m'_j)$ for $\mathcal{L}_S$, and let $x \in U \cap U'$. For each pair i, j define $V_{ij} := \{y \in U \cap U' : \sigma_i(y) = \sigma'_j(y)\}$. Then for all i there is j such that $x \in cl\big(\operatorname{Int} V_{ij}\big)$.*

Proof. If not, there is an open neighborhood $V \subset U \cap U'$ of x in B_0 such that $V \cap \operatorname{Int} V_{ij} = \emptyset$ for all j. But $V \subset \cup_j V_{ij}$ and each set $V \cap V_{ij}$ is relatively closed in V. Since there are only finitely many j, at least one intersection $V \cap V_{ij}$ must have nonempty interior. But $\operatorname{Int}(V \cap V_{ij}) = V \cap \operatorname{Int}(V_{ij})$ since V is open, so this contradicts the hypothesis. □

Proposition 4.15. *Let $\rho : \mathcal{E} \to \mathcal{X}$ be an oriented bundle over an ep groupoid $\mathcal{X}$. Then:*

(i) *The support of any multisection $\mathcal{L}_S : \mathcal{E} \to \mathcal{Q}^{\geq 0}$ is the image of an sse groupoid $\mathcal{Y}_S$ by a functor $\Sigma_S : \mathcal{Y}_S \to \mathcal{E}$ such that $\mathcal{L}_S$ pulls back to a (single valued) section σ_S of the bundle $F_S^*(\mathcal{E}) \to \mathcal{Y}_S$, where $F_S := \rho \circ \Sigma_S : \mathcal{Y}_S \to \mathcal{X}$,*

$$\begin{tikzcd}
F_S^*\mathcal{E} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "\rho"'] \\
\mathcal{Y}_S \arrow[u, bend left, "\sigma_S"] \arrow[ur, "\Sigma_S"] \arrow[r, "F_S"] & \mathcal{X}.
\end{tikzcd}$$

(ii) *The groupoid $\mathcal{Y}_S$ is nonsingular iff no open subset of the support of $\mathcal{L}_S$ is contained in the singular set $E_0^{sing} := \{(x, e) \in E_0 : |G_{(x,e)}| > 1\}$.*

(iii) *If $\mathcal{Y}_S$ is nonsingular, it may be given the structure of a wnb groupoid so as to make $F_S : \mathcal{Y}_S \to \mathcal{X}$ a layered covering.*

Proof. To see this, choose a locally finite covering of X_0 by sets $U^\alpha, \alpha \in A$, with the properties of Definition 4.13, and then define $\mathcal{Y}_S$ to be the category whose space of objects is the disjoint union of copies of the sets U^α, with one copy U_i^α of U^α for each section σ_i. These sections give a smooth immersion

$\Sigma_S : Y_0 \to E_0$ that is injective on each component of the domain. We define the morphisms in $\mathcal{Y}_S$ to be the pullback by Σ_S of the morphisms in the *interior* of the full subcategory of $\mathcal{E}$ with objects $\Sigma_S(Y_0)$. (For example, if two local sections agree at an isolated point we ignore the corresponding morphism. We define Z_1 to be the set of all such ignored morphisms.) Then $\mathcal{Y}_S$ is an sse groupoid. The rest of (i) is clear, as is (ii).

Because $|E|$ is Hausdorff, the functor Σ_S induces a continuous map

$$|\Sigma_S|_H : |Y_S|_H \to |\mathrm{supp}(\mathcal{L}_S)| \subset |E|.$$

It is injective over the open dense set $|V|_H := |Y_S|_H \setminus |s(Z_1)|_H$ (where s denotes the source map) because the restriction of Σ_S to the full subcategory of $\mathcal{Y}_S$ with objects $(Y_S)_0 \setminus s(Z_1)$ is a bijection onto a full subcategory of $\mathcal{E}$.

If $\mathcal{Y}_S$ is nonsingular, we define the local branches over the points in

$$N_U := (|F_S|_H)^{-1}(|U|) \subset |Y_S|_H$$

to be the sets U_i with weights m_i, and define

$$\Lambda_Y(p) := |\mathcal{L}_S|\big(|\Sigma_S|_H(p)\big) \ \in \ (0,\infty) \cap \mathbb{Q} \subset |Q^{\geq 0}|.$$

It is now straightforward to check that $(\mathcal{Y}_S, \Lambda_Y)$ is a wnb groupoid. In particular each U_i^α injects into $|Y_S|_H$ because the composite

$$U_i \xrightarrow{\pi_H} |Y_S|_H \xrightarrow{|F_S|_H} |U|,$$

is injective.

To see that $F_S : \mathcal{Y}_S \to \mathcal{X}$ is a resolution, note first that the covering property is immediate and that the weighting property holds because $\sum_{e\in E_x} \mathcal{L}_S((x,e)) = 1$. To see that $|F_S|_H$ is proper, it suffices to show that every sequence $\{p_k\}$ in $|Y_S|_H$ whose image by $|F_S|_H$ converges has a convergent subsequence. Since the covering U_i^α of $(Y_S)_0$ is locally finite, we may pass to a subsequence of $\{p_k\}$ (also called $\{p_k\}$) whose elements all lie in the same set $|U_i^\alpha|_H$. Then, for each k, there is $y_k \in U_i^\alpha$ such that $\pi_H(y_k) = p_k$. Now choose $x_k, x_\infty \in X_0$ so that $|x_k| := \pi(x_k) = |F_S|_H(p_k)$ and $|x_\infty| := \pi(x_\infty) \in |X|$ is the limit of $\{|x_k|\}$. If $|x_\infty| \in |U^\alpha|$, then, because the map $U_i^\alpha \to |U^\alpha| \subset |X|$ is a diffeomorphism, the sequence $\{y_k\}$ converges in U_i^α to the point y_∞ corresponding to x_∞. Hence $\{p_k\}$ has the limit $\pi_H(y_\infty) \in |Y_S|_H$ as required.

Otherwise, choose a local section structure $(U^\beta, \sigma_j', m_j)$ for $\mathcal{L}_S$ near x_∞. Then $x_\infty \in cl(U^\alpha \cap U^\beta)$, and we may suppose that $x_k \in U^\beta$ for all k. By applying Lemma 4.14 to each point $y_k \in U_i^\alpha$ and passing to a further subsequence, we may suppose that there is j such that $y_k \in cl\big(\mathrm{Int}\, V_{ij}\big)$ for all k, where

$$V_{ij} := \{y \in U_i^\alpha \cap U_j^\beta : \sigma_i(y) = \sigma_j'(y)\}.$$

Therefore, for each k there is $z_k \in U_j^\beta$ such that $y_k \approx z_k$. It now follows from Lemma 3.5 that $\pi_H(z_k) = \pi_H(y_k) = p_k$. But now $\{z_k\}$ has a limit in U_j^β. Hence $\{p_k\}$ does too. This completes the proof. □

Conversely, suppose that $F : (\mathcal{B}, \Lambda_B) \to \mathcal{X}$ is a resolution of an ep groupoid, and let $\mathcal{E} \to \mathcal{X}$ be a bundle. Then any (single valued) section $\sigma_S : \mathcal{B} \to F^*(\mathcal{E})$ of the pullback bundle pushes forward to a functor $\Sigma_S : \mathcal{B} \to \mathcal{E}$. Note that its image is not in general the support of a multisection in the sense of Definition 4.13 since it need not contain entire equivalence classes. However, because $\mathcal{E}$ is proper the induced map $|\Sigma_S| : |B| \to |E|$ factors through $|\Sigma_S|_H : |B|_H \to |E|$.

Lemma 4.16. *Let $\mathcal{E} \to \mathcal{X}$ be a bundle over the ep groupoid $\mathcal{X}$ and $F : (\mathcal{B}, \Lambda_B) \to \mathcal{X}$ be a resolution. Each section $\sigma_S : \mathcal{B} \to F^*(\mathcal{E})$ of the pullback bundle gives rise to a multisection $\mathcal{L}_S : \mathcal{E} \to \mathcal{Q}^{\geq 0}$ where*

$$\mathcal{L}_S(x,e) := \sum_{p \in |B|_H : |\Sigma_S|_H(p) = |(x,e)|} \Lambda_B(p),$$

and $\Sigma_S : \mathcal{B} \to \mathcal{E}$ is the composite of σ_S with the push forward $F_* : F^*(\mathcal{E}) \to \mathcal{E}$.

Proof. The definition implies that $\mathcal{L}_S(x,e) = \mathcal{L}_S(x',e')$ whenever $(x,e) \sim (x',e')$ in $\mathcal{E}$. Hence $\mathcal{L}_S$ is a functor. It has the required local structure at $x \in X_0$ by Lemma 3.14. □

Remark 4.17. (i) The above construction of a multisection $\mathcal{L}_S$ of $\mathcal{E} \to \mathcal{X}$ from a section σ_S of $F^*(\mathcal{E}) \to \mathcal{B}$ may be described more formally as follows. Consider the (weak) fiber product $\mathcal{B}' := \mathcal{B} \times_{\mathcal{X}} \mathcal{X}$. This is a wnb groupoid by Lemma 3.15 (ii), and there is a layered equivalence $G' : (\mathcal{B}', \Lambda') \to (\mathcal{B}, \Lambda)$. Consider the diagram

$$\begin{array}{ccc} \mathcal{B}' := \mathcal{B} \times_{\mathcal{X}} \mathcal{X} & \xrightarrow{F'} & \mathcal{X} \\ G' \downarrow & & = \downarrow \\ \mathcal{B} & \xrightarrow{F} & \mathcal{X}. \end{array}$$

The section $\sigma_S : \mathcal{B} \to F^*(\mathcal{E})$ pulls back to a section $\sigma_S' : \mathcal{B}' \to (F')^*(\mathcal{E})$ that gives rise to a functor $\Sigma_S' : \mathcal{B}' \to \mathcal{E}$ whose image is precisely the support of $\mathcal{L}_S$. (The effect of passing to $\mathcal{B}'$ is to saturate the image of Σ_S under $\sim$. Note that the two functors $\Sigma_S' : \mathcal{B}' \to \mathcal{E}$ and $\Sigma_S \circ G' : \mathcal{B}' \to \mathcal{B} \to \mathcal{E}$ do not coincide, because the diagram only commutes up to homotopy.) Thus the two approaches give rise to essentially the same multisections. To distinguish them, we shall call Σ_S a **wp multisection**.

One might think of Σ_S as a stripped down version of $\mathcal{L}_S$, with inessential information removed. For example, in the case of the teardrop with resolution $F : \mathcal{B} \to \mathcal{X}$, the pushforward of a section of $F^*(\mathcal{E}) \to \mathcal{B}$ is single valued

over each component of X_0, while the corresponding $\mathcal{L}_S$ is multivalued over D_+.
(ii) Suppose that $\mathcal{L}_S, \mathcal{L}_T$ give rise as above to the wnb groupoids $(\mathcal{Y}_S, \Lambda_S)$ and $(\mathcal{Y}_T, \Lambda_T)$. Then it is not hard to check that their sum $\mathcal{L}_S + \mathcal{L}_T$ gives rise to the fiber product $\mathcal{Y}_S \times_{\mathcal{X}} \mathcal{Y}_T$. On the other hand, if $\mathcal{Y}_S = \mathcal{Y}_T$ there is a simpler summing operation given by adding the corresponding sections σ_S, σ_T of $F^*(\mathcal{E}) \to \mathcal{Y}_S$.

We say that a multisection $\mathcal{L}_S$ of $\mathcal{E} \to \mathcal{X}$ is **tranverse to the zero section** if it is made from local sections $\sigma_i : U_i \to E_0$ that are transverse to the zero section. It is easy to check that this is equivalent to saying that the corresponding single valued section σ_S of $F^*(\mathcal{E}) \to \mathcal{Y}_S$ is transverse to the zero section. Hence the intersection of σ_S with the zero section has the structure of a wnb groupoid $(\mathcal{Z}_S, \Lambda_S)$ as in the discussion after Lemma 4.10.

Definition 4.18. *Let $\mathcal{E} \to \mathcal{X}$ be a d-dimensional vector bundle $\mathcal{E} \to \mathcal{X}$ over an n-dimensional ep groupoid. If $\mathcal{E}$ is effective, we define its* **(homology) Euler class** *to be the singular homology class $e(\mathcal{E}) \in H_{n-d}(|X|;\mathbb{Q})$ represented by the image under the composite map $\mathcal{Z}_S \to \mathcal{Y}_S \to \mathcal{X}$ of the fundamental class $[Z_S] \in H_{n-d}(|Z_S|_H)$ of the zero set of a generic multisection $\mathcal{L}_S$. In general, we define $e(\mathcal{E}) \in H_{n-d}(|X|;\mathbb{Q})$ to be the Euler class of the corresponding effective bundle $\mathcal{E}_{\rm eff} \to \mathcal{X}'$ (cf. Example* 4.8 *(ii)).*

It is not hard to prove that any two wnb groupoids $(\mathcal{Z}_S, \Lambda_S), (\mathcal{Z}_T, \Lambda_T)$ constructed in this way from multisections $\mathcal{L}_S, \mathcal{L}_T$ are cobordant since the pullbacks of σ_S and σ_T to $\mathcal{Y}_S \times_{\mathcal{X}} \mathcal{Y}_T$ are homotopic. The next result states that Definitions 4.11 and 4.18 are consistent.

Proposition 4.19. *Let $F : (\mathcal{B}, \Lambda_B) \to \mathcal{X}$ be any resolution of the n-dimensional ep groupoid $\mathcal{X}$ and let $\rho : \mathcal{E} \to \mathcal{X}$ be any d-dimensional bundle. Then*

$$(|F|_H)_*\big(e(F^*(\mathcal{E})\big) = e(\mathcal{E}) \ \in \ H_{n-d}(|X|;\mathbb{R}).$$

Moreover, if Λ_B takes rational values, this equality holds in $H_{n-d}(|X|;\mathbb{Q})$.

Proof. This is an immediate consequence of Proposition 3.25; the details of its proof are left to the reader. □

We conclude this section with some constructions. First we explain how to construct "enough" wp multisections using the resolution $\mathcal{X}_{\mathcal{V}} \to \mathcal{X}$. We shall explain this in the context of Fredholm theory and so shall think of the fibers of $E_0 \to X_0$ as infinite dimensional. We start from a Fredholm section $f : \mathcal{X} \to \mathcal{E}$ such that over each chart $U_i \subset X_0$ there is a (possibly finite dimensional) space $\mathcal{S}_i$ of local sections $U_i \to E_0$ that is large enough to achieve transversality over U_i. We then construct a vector space $\mathcal{S}$ of wp multisections with controlled branching that is large enough for global transversality. It is finite dimensional if the $\mathcal{S}_i$ are. This is an adaptation of a result in Liu–Tian [**12**] and was the motivation for their construction of the resolution.[20]

Proposition 4.20. *Let $\mathcal{X}$ be an ep groupoid with a finite good atlas $\mathcal{A} = \{(U_i, G_i, \pi) : i \in A\}$, and choose an $\mathcal{A}$-compatible cover $\mathcal{V} = \{|V_I|\}$ of $|X|$ as in Lemma* 4.4. *Let $F : \mathcal{X}_{\mathcal{V}} \to \mathcal{X}$ be the (tame) resolution constructed in Proposition* 4.2, *and let $\mathcal{E} \to \mathcal{X}$ be an orbibundle. Then:*

(i) *Every section s of the induced vector bundle $\rho : E_i \to U_i$ whose support is contained in $U_i^0 \sqsubset U_i$ extends to a global section σ^s of the pullback bundle $F^*(\mathcal{E}) \to \mathcal{X}_{\mathcal{V}}$.*

(ii) *Let $f : \mathcal{X} \to \mathcal{E}$ be a section. Suppose that for each $i \in A$ there is a space $\mathcal{S}_i$ of sections of the vector bundle $E_0|_{U_i} \to U_i$ such that $f + s : U_i \to E_0$ is transverse to the zero section over U_i for sufficiently small generic $s \in \mathcal{S}_i$. Then there is a corresponding space $\mathcal{S}$ of sections of the pullback bundle $\phi^*(\mathcal{E}) \to \mathcal{X}_{\mathcal{V}}$ such that $F^*(f) + \sigma : \mathcal{X}_{\mathcal{V}} \to F^*(\mathcal{E})$ is transverse to the zero section for sufficiently small generic $\sigma \in \mathcal{S}$.*

Proof. Suppose given a section $s : U_i \to E_i$. If $i \in J$, we define $\sigma^s(x)$ on the object $x = (\delta_{k-1}, \dots, \delta_1) \in \widehat{V}_J$ to be $s(\widehat{\pi}_{iJ}(x))$. Otherwise, $|V_J|$ is disjoint from the support $|U_i^0|$ of s by construction of $\mathcal{V}$, and we set $\sigma^s(x) = 0$. It follows immediately from the definitions that σ^s is compatible with the morphisms in $\mathcal{X}_{\mathcal{V}}$ and so extends to a functor $\mathcal{X}_{\mathcal{V}} \to \phi^*(\mathcal{E})$. This proves (i).

To prove (ii), choose a smooth partition of unity β on $\mathcal{X}_{\mathcal{V}}$, which exists by Lemma 3.23. Then define $\mathcal{S}$ to be the vector space generated by the sections $\beta\sigma, s \in \mathcal{S}_i$. It is easy to check that it has the required properties. □

The previous result used the existence of a resolution to construct multisections. This procedure can be reversed. The following argument applies to any $\mathcal{X}$ that acts effectively on some vector bundle $E_0 \to X_0$.[21]

Proof II of Proposition 3.16 for effective groupoids $\mathcal{X}$. Let $\mathcal{E} \to \mathcal{X}$ be the tangent bundle of $\mathcal{X}$; cf. Example 4.8(i). Then, because $\mathcal{X}$ is effective, the set E_0^{sing} is nowhere dense. (In fact, E_0 may be triangulated in such a way that E_0^{sing} is a union of simplices of codimension ≥ 2.) Each point $x_0 \in X_0$ has a neighborhood U with a smooth compactly supported section $s_U : U \to E_0|_U$ that is nonzero at x_0 and satisfies the condition in Proposition 4.15(ii), i.e., no open subset of $s_U(U)$ is contained in E_0^{sing}. If U is part of a local chart (U, G, π) for $\mathcal{X}$, then, as explained in [**10**], we

[20]The topological aspects of their construction were explored earlier in McDuff [**14**]. However, the current approach using groupoids, when combined with the Fredholm theory of polyfolds, allows for a much cleaner treatment.

[21]Whether one can always find suitable $\mathcal{E}$ is closely related to the presentation problem discussed in [**8**].

may extend s_U to a multisection $\mathcal{L}_S$ as follows. Over $X_0 \setminus U$, $\mathcal{L}_S$ is the characteristic function of the zero section, i.e., $\mathcal{L}_S(x,e) = 0$ for $e \neq 0$ and $\mathcal{L}_S(x,0) = 1$, while if $x \in U$,

$$\mathcal{L}_S(x,e) = \sum_{g \in G\,:\, s_U(x)\cdot g = (x,e)} 1/|G|.$$

Now choose a good atlas $\mathcal{A} = \{(U_i, G_i, \pi_i) : i \in A\}$ for $\mathcal{X}$ and a subordinate partition of unity β_i. Choose for each i and $g_i \in G_i$ a (noncompactly supported) section $s_i : U_i \to E_0$ whose intersection with the nonsmooth set E_0^{sing} has no interior points. Then the sections $r\beta_i s_i$ for any $r \in \mathbb{R}$ have the same property over the support of β_i, at least, since E_0^{sing} consists of lines $(x, \lambda e), \lambda \in \mathbb{R}$. Now construct $\mathcal{L}_{S_i}$ as above from the section $r_i\beta_i s_i$, where $r_i \in \mathbb{R}$ and then define $\mathcal{L}_S := \sum_i \mathcal{L}_{S_i}$. Consider the corresponding groupoid $\mathcal{Y}_S$ as defined in Proposition 4.15. This will be nonsingular for generic choice of the constants r_i and hence a wnb groupoid. We can tame it by Lemma 3.22. □

Remark 4.21. We end with some remarks about the **infinite dimensional case.** There are quite a few places in the above arguments where we used the local compactness of $|X|$. For example we required that layered coverings $F : \mathcal{B}' \to \mathcal{B}$ give rise to proper maps $|F|_H : |B'|_H \to |B|_H$, and used this to obtain the even covering property of Lemmas 3.10 and 3.14. In the infinite dimensional context the notion of layered covering must be formulated in such a way that these lemmas hold.

Both constructions for the resolution of an ep groupoid work in quite general contexts. For example, the construction of $\mathcal{X}_\mathcal{V}$ works for groupoids in any category in which Lemma 4.4 holds. Similarly, the above construction of $\mathcal{Y}_S$ works as long as there are enough local sections of $\mathcal{E} \to \mathcal{X}$. In particular, we need partitions of unity.

Hofer *et al.* [**10**] define a notion of properness that yields a concept of ep polyfold groupoid $\mathcal{X}$ that has all expected properties. In particular, these groupoids admit smooth partitions of unity since they are modelled on M-polyfolds built using Hilbert rather than Banach spaces. The Fredholm theory developed in [**10**] implies that a vector bundle $\mathcal{E} \to \mathcal{X}$ equipped with a Fredholm section f has a good class of Fredholm multisections $\mathcal{L}_S := f + s$ that perturb f and meet the zero section transversally. Since the kernel of the linearization of the operators $f + s$ has a well defined orientation,[22] this intersection can be given the structure of a finite dimensional (oriented) wnb

[22]Note that in this paper we have assumed that all objects (ep groupoids, branched manifolds, bundles) are oriented. We then give the zero set of a multisection the induced orientation. However, when constructing the Euler class it is not necessary to orient the ambient orbibundle $\mathcal{E} \to \mathcal{X}$ provided that one considers a class of multisections whose intersections with the zero set carry a natural orientation; cf. the definition of G-moduli problem in [**4**, Def. 2.1].

groupoid $\mathcal{Z}$. Since any two such multisections are cobordant in the sense that they extend to the pullback bundle over $\mathcal{X} \times \mathcal{I}$, the cobordism class of $\mathcal{Z}$ is independent of choices, as is the fundamental class $[Z]$ defined in Proposition 3.25.

4.3. Branched manifolds and the Euler class. We now discuss the relation of our work on the Euler class to that of Satake and Haefliger and also to the paper of Cieliebak *et al.* [**4**].

First, we sketch a proof that the homological Euler class of Definition 4.18 is Poincaré dual to the usual Euler class for orbibundles. As Haefliger [**6**] points out, one can define cohomology characteristic classes for orbifolds by adapting the usual constructions for manifolds. For example, if $\underline{E} \to \underline{X}$ is an oriented d-dimensional orbibundle, choose the representing functor $\mathcal{E} \to \mathcal{X}$ so that the vector bundle $\rho : E_0 \to X_0$ is trivial over each component of X_0. Each trivialization of this bundle defines a functor $F : \mathcal{X} \to \mathcal{GL}$, where $\mathcal{GL}$ is the topological category with one object and morphisms $GL(d, \mathbb{R})^+$ corresponding to the group $GL(d, \mathbb{R})^+$ of matrices of positive determinant. Then the classifying space $B\mathcal{GL}$ is a model for the classifying space $BGL(d, \mathbb{R})^+$ of oriented d-dimensional bundles, and so carries a universal bundle. The Euler class $\varepsilon(\mathcal{E})$ of $\mathcal{E} \to \mathcal{X}$ is defined to be the pullback by $BF : B\mathcal{X} \to B\mathcal{GL}$ of the Euler class of this universal bundle. Since the projection $B\mathcal{X} \to |X|$ induces an isomorphism on rational homology, we may equally well think of $\varepsilon(\mathcal{E})$ as an element in $H^d(|X|; \mathbb{Q})$. As such, it depends only on $\underline{E} \to \underline{X}$ and so may also be called $\varepsilon(\underline{E})$.

We claim that the homology Euler class $e(\mathcal{E}) \in H_{n-d}(|X|; \mathbb{Q})$ of Definition 4.18 is Poincaré dual to $\varepsilon(\mathcal{E})$. One way to prove this is as follows. In what follows, we assume for simplicity that $\mathcal{X}$ is effective. Note that $\varepsilon(\mathcal{E})$ may be represented on the orbifold $|X|$ in terms of de Rham theory by $i_0^*(\tau)$, where $i : |X| \to |E|$ is the inclusion of the zero section and $\tau \in H^d_c(|E|)$ is a compactly supported smooth form that represents the Thom class.[23] Therefore, it suffices to show that for each smooth $(n-d)$-form β on $|X|$,

$$\int_{|X|} i^*(\tau) \wedge \beta = \int_Z \beta,$$

where the tame branched manifold $(\underline{Z}, \Lambda_Z) \to \underline{X}$ represents $e(\mathcal{E})$ and the right hand integral is defined by equation (3.2). This holds by adapting standard arguments from the smooth case. For example, we may choose a smooth triangulation of $|X|$ so that the singular points lie in the codimension 2 skeleton, and then choose the multisection $\mathcal{L}_S$ so that its zero set $Z_0 \subset X_0$ is transverse to this triangulation. Then $|X|^{sm}$ and $|Z|^{sm} := |Z| \cap |X|^{sm}$ are both pseudocycles in $|X|$. Then the left hand integral above equals the

[23]Thus, for each $x \in X_0$, τ pulls back under the map $E_x \to |E|$ to a generator of the image of $H^d(E_x, E_x \setminus \{0\}; \mathbb{Z})$ in $H^d(E_x, E_x \setminus \{0\}; \mathbb{R})$.

integral of $\tau\wedge\rho^*(\beta)$ over $i(|X|^{sm})$. But the pseudocycle $i(|X|^{sm})$ is cobordant to the weighted pseudocycle given by the image of the multisection $\mathcal{L}_S$. (This is just the image of the fundamental class of $\mathcal{Y}_S$ under $|\Sigma|_S$.) We now perturb the latter cycle straightening it out near the zero set $|Z|^{sm}$ so that its intersection with a neighborhood $U \subset |E|$ of $i(|X|^{sm})$ is precisely $|\rho|^{-1}(|Z|^{sm}) \to |Z|^{sm}$ (with the obvious rational weights.) We may then choose τ so that it vanishes outside U. Then

$$\int_{|X|} i^*(\tau)\wedge\beta = \int_{i(|X|^{sm})} \tau\wedge\rho^*\beta = \int_C \tau\wedge\rho^*\beta = \int_Z \beta.$$

Finally, we compare our approach to that of Cieliebak *et al.* Although they work in the category of Hilbert manifolds, we shall restrict attention here to the finite dimensional case. They consider an orbifold to be a quotient M/G, where G is a compact Lie group acting on a smooth manifold M with finite stabilizers. Hence, for them an orbibundle is a G-equivariant bundle $\rho: E \to M$, where again G acts on E with finite stabilizers. Their Proposition 2.7 shows that there is a homology Euler class $\chi(E)$ for such bundles that lies in the equivariant homology group $H^G_{n-d}(M;\mathbb{Q})$ and has all the standard properties of such a characteristic class, such as naturality, the expected relation to the Thom isomorphism and so on.

We claim that the Euler class described in Definition 4.18 is the same as theirs. To see this, first note the following well known lemma; cf. [**8**].

Lemma 4.22. *Every effective orbifold may be identified with a quotient M/G, where G is a compact Lie group acting on a smooth finite dimensional manifold M with finite stabilizers.*

Sketch of Proof. Given such a quotient one can define a corresponding ep groupoid $\mathcal{X}$ by taking a complete set of local slices for the G action; cf. the discussion at the beginning of section 3 in Moerdijk [**15**]. In the other direction, given an ep groupoid $\mathcal{X}$ one takes M to be the orthonormal frame bundle of $|X|$ with respect to some Riemannian metric on its tangent bundle. The group G is either $\mathrm{O}(d)$ or $\mathrm{SO}(d)$, where $d = \dim X$. □

Similarly, every orbibundle $\mathcal{E} \to \mathcal{X}$ can be identified with a G-equivariant bundle $\rho: E \to M$. The equivariant homology $H^G_*(M;\mathbb{Q})$ is, by definition, the homology of the homotopy quotient $EG \times_G M$ (where $EG \to GB$ is the universal G-bundle, i.e., G acts freely on EG). Because G acts with finite stabilizers, the natural quotient map[24] $EG \times_G M \to M/G$ induces an isomorphism on *rational* homology. Hence, in this case, their Euler class may be considered to lie in $H_*(M/G;\mathbb{Q}) = H_*(|X|;\mathbb{Q})$.

[24]If the ep groupoid $\mathcal{X}$ is an orbifold structure on M/G, then $EG\times_G M$ can be identified with the classifying space $B\mathcal{X}$ (denoted $|\mathcal{X}_\bullet|$ in [**15**]), and the projection $EG\times_G M \to M/G$ can be identified with the natural map $B\mathcal{X} \to |X|$.

In [**4**, Section 10] the Euler class is constructed as the zero set of a multivalued section of the orbibundle $\rho : E \to M$, much as above. However, the definitions in [**4**] are all somewhat different from ours. For example, the authors do not give an abstract definition of a weighted branched manifold but rather think of it as a subobject of some high dimensional manifold M on which the compact Lie group G acts. They also treat orientations a little differently, in that they do not assume the local branches are consistently oriented but rather incorporate the orientation into a signed weighting function on the associated oriented Grassmanian bundle; cf. their Definitions 9.1 and 9.11. As pointed out in their Remark 9.17, their Euler class can be defined in the slightly more restrictive setting in which the orientation is given by a consistent orientation of the branches. Hence below we shall assume this, since that is the approach taken here. A third difference is that their Definition 9.1 describes the analog of a weighted branched groupoid, i.e., they do not restrict to the nonsingular case as we did above.[25]

However, to check that the two Euler classes are the same, it suffices to check that the multivalued sections used in their definition can be described by functors $\mathcal{L}_S$ as in Definition 4.13. Here is the definition of a multivalued section given in [**4**, Def. 10.1] in the oriented case.

Definition 4.23. *Consider an oriented finite dimensional locally trivial bundle $\rho : E \to M$ over an oriented smooth finite dimensional manifold M. Assume that a compact oriented Lie group G acts smoothly, preserving orientation and with finite stabilizers on E and M, and that ρ is G-equivariant. Then a* **multivalued section** *of E is a function*

$$\sigma : E \to \mathbb{Q} \cap [0, \infty)$$

such that

(Equivariance) $\sigma(g^*x, g^*e) = \sigma(x, e)$ *for all* $x \in M, e \in E_x, g \in G$,

(Local structure) *for each $x_0 \in M$ there is an open neighborhood U of x_0 and finitely many smooth sections $s_1, \dots, s_m : U \to E$ with weights $m_i \in \mathbb{Q} \cap (0, \infty)$ such that*

$$\sum m_i = 1, \qquad \sigma(x, e) = \sum_{s_i(x) = e} m_i \ \text{ for all } x \in U,$$

where by convention the sum over the empty set is 0.

To see that every such multisection can be described in terms of a functor $\mathcal{L}_S$ we argue as follows. Choose a locally finite covering of M by sets U^α that have local section structures in the sense of Definition 4.13. By [**4**]

[25] We made this restriction in order to simplify the exposition; certain technical details become harder to describe if one allows the local branches over N_p to have completely arbitrary orbifold structures. This problem is avoided in [**4**] because in this case the local orbifold structures are determined by the global G action.

Proposition 9.8(i), the sets U^α and the sections s_i are locally G-invariant. In other words, we may assume that each U^α is the image of a finite to one map $V^\alpha \times \mathcal{N}_G^\alpha \to U^\alpha$, where $V^\alpha \subset M$ is a local slice for the G action and $\mathcal{N}_G^\alpha$ is a neighborhood of the identity in G. We may use the local slices V^α, with their induced orientations, to build an ep groupoid $\mathcal{X}$ representing M/G with objects $X_0 := \sqcup V^\alpha$ and morphisms induced by the G action. There is a similar groupoid $\mathcal{E}$ representing E/G with objects $\rho^{-1}(V^\alpha) = \cup_{x \in V^\alpha} E_x$. Then, because σ is G-invariant, we may define a functor $\mathcal{L}_S : \mathcal{E} \to \mathcal{Q}^{\geq 0}$ by setting

$$\mathcal{L}_S(x,e) = \sigma(x,e), \quad x \in V^\alpha, e \in E_x.$$

This clearly satisfies the conditions of Definition 4.13. It is now straightforward to check that the definition of Euler class in [**4**] is consistent with the one given here.

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Department of Mathematics
Stony Brook University
Stony Brook
NY 11794-3651
E-mail address: dusa@math.sunysb.edu
http://www.math.sunysb.edu/ dusa

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