Japan J. Indust. Appl. Math., 25 (2008), 83–98 Area ⟨3⟩

A Polynomial-Time Algorithm for a Stable Matching Problem with Linear Valuations and Bounded Side Payments

Rashid Farooq

Centre for Advanced Mathematics and Physics
National University of Sciences and Technology
Campus of College of Electrical and Mechanical Engineering
Peshawar Road Rawalpindi 46000, Pakistan
E-mail: farooq@kurims.kyoto-u.ac.jp

Received June 29, 2006
Revised May 16, 2007

We study an extension of the Gale–Shapley marriage model and the Shapley–Shubik assignment model by considering linear valuations and bounded side payments. Our model includes the Eriksson–Karlander hybrid model as a special case. We propose a polynomial-time algorithm which finds a pairwise-stable outcome.

Key words: stable marriage model, assignment game

1. Introduction

A two-sided matching market consists of two disjoint finite sets of agents. The purpose is to match the agents of opposite sides in pairs. A matching is a set of pairs of opposite sides such that each agent appears at most once and is called stable if there is no pair of agents who are not matched with each other but prefer each other to their partners in the matching.

The marriage model by Gale and Shapley [4] and the assignment model by Shapley and Shubik [10] are well known in the theory of two-sided markets. In [4], Gale and Shapley proposed an algorithm that always finds a (perfect) stable matching for any instance of a stable marriage model problem. In their model, side payments are not allowed, i.e., the agents are rigid. In the assignment model by Shapley and Shubik [10], in contrast to the Gale–Shapley marriage model, side payments are permitted, i.e., the agents are flexible, and they showed that the core is non-empty.

Kaneko [5] unified both the Gale–Shapley marriage model and the Shapley–Shubik assignment model, and proved the non-emptiness of the core, but does not consider lattice property. Roth and Sotomayor [9] proposed a general model that includes the both marriage and assignment model. The existence of stable outcome is not guaranteed in their model. They, however, investigated the lattice property for payoffs in the core. The model of Eriksson and Karlander [1] deals with both rigid and flexible agents, which is a common generalization of the marriage and assignment models. The existence of a stable outcome and the lattice property of the set of stable outcomes are preserved in their model. Following the idea of

Eriksson and Karlander [1], Sotomayor [11] investigated their hybrid model and gave a non-constructive proof of the existence of a pairwise-stable outcome.

Very recently, Fujishige and Tamura [2] proposed a common generalization of the marriage model and the assignment model by utilizing the framework of discrete convex analysis which was developed by Murota [6, 7, 8]. Their model also includes hybrid models of Eriksson and Karlander [1] and Sotomayor [11]. The existence of a pairwise-stable outcome is preserved in their model. They further extended their model in [3] by assuming possibly bounded side payments and proved the existence of a pairwise-stable outcome. In their work, however, structure of the set of pairwise-stable outcome is not discussed.

In the present work, our aim is to formulate a model which includes the Gale–Shapley marriage model, the Shapley–Shubik assignment model and the Eriksson–Karlander model as special cases. We use the notion of bounded side payments and valuations rather than rigidity and flexibility. We list here the main assumptions in our model:

- the set of agents is partitioned into two sets; the set of men and the set of women,
- each agent has at most one partner of opposite side,
- side payments are permitted,
- side payments are bounded by lower and upper bounds,
- valuations of agents for the side payments from opposite side are identified by linear and strictly increasing real valued functions.

We can handle rigid agents by assuming $\mathbf{0}$ as a lower and upper bound for the side payments and the flexible agents can be dealt by considering lower and upper bounds of the side payments sufficiently small and sufficiently large, respectively, and hence the marriage model, the assignment model and the Eriksson–Karlander hybrid model are included in our model. We propose a polynomial-time algorithm in the number of agents to find a pairwise-stable matching.

This paper is organized as follows: In Section 2, we discuss our model and give a comparison between known models and our model. In Section 3, we characterize the pairwise stability. We will use this characterization to develop a polynomial-time algorithm. Section 4 deals with the case when valuations are linear. In this section, first of all, we give several lemmas which will help us to design our algorithm. We then propose our algorithm and finally discuss its correctness and complexity.

2. Model description

Let M and W be two disjoint finite sets of agents and let $E = M \times W$, i.e., the set of all pairs (i, j) of agents $i \in M$ and $j \in W$. A subset $X \subseteq E$ is called a *matching* if every agent appears at most once in X. Given a matching X, $k \in M \cup W$ is called *unmatched in* X if it does not appear in X; otherwise *matched in* X.

Before describing our model, we briefly explain the marriage model by Gale and Shapley [4] and the assignment model by Shapley and Shubik [10]. In the marriage model, M and W represent the sets of men and women, respectively. Each man has preferences on the women and each woman has preferences on the men. Negotiations and side payments are not involved in this model. We represent the preferences of men and women by the numbers $a_{ij} \in \mathbf{R}$ and $b_{ij} \in \mathbf{R}$, respectively, for all $(i, j) \in E$. For $i \in M$ and $j_1, j_2 \in W$, if $a_{ij_1} > a_{ij_2}$ then we say that i *prefers* j_1 to j_2, and i is *indifferent* between j_1 and j_2 if $a_{ij_1} = a_{ij_2}$. Similarly, the preferences of women over men are defined by the vector $\{b_{ij} \mid (i, j) \in E\}$. Here we assume that $a_{ij} > 0$ if j is *acceptable* to i, and $a_{ij} = -\mu$ otherwise, and $b_{ij} > 0$ if i is *acceptable* to j, and $b_{ij} = -\mu$ otherwise, where $\mu > 0$ is a sufficiently large number. A matching X is called *pairwise-stable* if there exist $q \in \mathbf{R}^M$ and $r \in \mathbf{R}^W$ such that

(m1) $q_i = a_{ij}$ and $r_j = b_{ij}$ for all $(i, j) \in X$,
(m2) $q \geq \mathbf{0}$, $r \geq \mathbf{0}$, and $q_i = 0$ (resp. $r_j = 0$) if i (resp. j) is unmatched in X,
(m3) $q_i \geq a_{ij}$ or $r_j \geq b_{ij}$ for all $(i, j) \in E$.

Gale and Shapley [4] presented an algorithm which finds a stable matching in this model.

In the assignment game, M and W represent the sets of sellers and buyers, respectively. The negotiation and side payments between agents of both sides are allowed. Naturally, each agent wants to gain as much profit as possible from his/her partner. Here $a_{ij} \in \mathbf{R}$ and $b_{ij} \in \mathbf{R}$ represent the profits of i and j, respectively, when i and j are matched. The preferences of sellers over buyers and of buyers over sellers can be defined similarly as in the marriage model. A matching X is called *pairwise-stable* if there exist $q \in \mathbf{R}^M$ and $r \in \mathbf{R}^W$ such that

(a1) $q_i + r_j = a_{ij} + b_{ij}$ for all $(i, j) \in X$,
(a2) $q \geq \mathbf{0}$, $r \geq \mathbf{0}$, and $q_i = 0$ (resp. $r_j = 0$) if i (resp. j) is unmatched in X,
(a3) $q_i + r_j \geq a_{ij} + b_{ij}$ for all $(i, j) \in E$.

Shapley and Shubik [10] showed the existence of a stable outcome in this assignment model.

Now we describe our model. For each $(i, j) \in E$ we denote by $\nu_{ij} \colon \mathbf{R} \to \mathbf{R}$ a valuation of agent $i \in M$ for a side payment from $j \in W$ to i, and by $\nu_{ji} \colon \mathbf{R} \to \mathbf{R}$ a valuation of agent $j \in W$ for a side payment from $i \in M$ to j. We assume that ν_{ij} and ν_{ji} are continuous and monotone increasing and that there exist the inverse functions ν_{ij}^{-1} and ν_{ji}^{-1} over $\mathbf{R}$ for all $(i, j) \in E$. We also assume that we are given vectors $l, u \in \mathbf{R}^E$ with $l \leq u$, where l_{ij} and u_{ij} ($(i, j) \in E$) denote the lower and upper bounds of a side payment from $j \in W$ to $i \in M$.

We say that $p = (p_{ij} \colon (i, j) \in E) \in \mathbf{R}^E$ is a *feasible side payment vector* from W to M if $l_{ij} \leq p_{ij} \leq u_{ij}$ for all $(i, j) \in E$. A pair (X, p) of a matching X and a feasible side payment vector p is said to be a *pairwise-stable outcome* if $q \in \mathbf{R}^M$ and $r \in \mathbf{R}^W$ defined by

$$q_i = \begin{cases} \nu_{ij}(p_{ij}) & \text{if } i \text{ is matched with } j \text{ in } X \\ 0 & \text{if } i \text{ is unmatched in } X \end{cases} \qquad (\forall i \in M), \tag{2.1}$$

$$r_j = \begin{cases} \nu_{ji}(-p_{ij}) & \text{if } j \text{ is matched with } i \text{ in } X \\ 0 & \text{if } j \text{ is unmatched in } X \end{cases} \qquad (\forall j \in W) \tag{2.2}$$

satisfy

(S1) $q \geq \mathbf{0}$ and $r \geq \mathbf{0}$,

(S2) $q_i \geq \nu_{ij}(c)$ or $r_j \geq \nu_{ji}(-c)$ for each pair $(i, j) \in E$ and each $c \in [l_{ij}, u_{ij}]$.

Here q and r are regarded as payoff vectors. For a money transfer $c_{ij_1} \in [l_{ij_1}, u_{ij_1}]$ and $c_{ij_2} \in [l_{ij_2}, u_{ij_2}]$ from $j_1 \in W$ to $i \in M$ and $j_2 \in W$ to $i \in M$ respectively, we say that i prefers j_1 to j_2 if $\nu_{ij_1}(c_{ij_1}) > \nu_{ij_2}(c_{ij_2})$ and i is indifferent between j_1 and j_2 if $\nu_{ij_1}(c_{ij_1}) = \nu_{ij_2}(c_{ij_2})$. Similarly, for a money transfer $c_{i_1j} \in [l_{i_1j}, u_{i_1j}]$ and $c_{i_2j} \in [l_{i_2j}, u_{i_2j}]$ from $j \in W$ to $i_1 \in M$ and $j \in W$ to $i_2 \in M$ respectively, we say that j prefers i_1 to i_2 if $\nu_{ji_1}(-c_{i_1j}) > \nu_{ji_2}(-c_{i_2j})$ and j is indifferent between i_1 and i_2 if $\nu_{ji_1}(-c_{i_1j}) = \nu_{ji_2}(-c_{i_2j})$.

In (S1), we assume that payoff vectors are nonnegative. (S2) means the absence of blocking pair where a blocking pair is a pair of agents which are not matched with each other but have incentives to break off their current partnership and to match up with each other. If there exists a pair $(i, j) \in E$ and $c_{ij} \in [l_{ij}, u_{ij}]$ such that $q_i < \nu_{ij}(c_{ij})$ and $r_j < \nu_{ji}(-c_{ij})$ then i and j can improve their payoffs by matching up with each other. (S2) prevent the existence of such a pair.

By defining the linear valuations as

$$\nu_{ij}(p_{ij}) = a_{ij} + p_{ij}, \quad \nu_{ji}(-p_{ij}) = b_{ij} - p_{ij}$$

for all $(i, j) \in E$, where $l_{ij} \leq p_{ij} \leq u_{ij}$ and $a_{ij}, b_{ij} \in \mathbf{R}$, we observe that if $l = u = \mathbf{0}$, we get the marriage model by Gale and Shapley [4]. If $l = (-\mu, \ldots, -\mu)$ and $u = (+\mu, \ldots, +\mu)$ for a sufficiently large $\mu > 0$, then we have the assignment model by Shapley and Shubik [10].

We say that a matching X is *pairwise-stable* if there exists a feasible side payment vector p such that (X, p) is pairwise-stable.

3. Characterization of pairwise-stability

In this section, we characterize pairwise-stability by considering a partition of the set E. We will utilize this characterization to develop our algorithm.

Theorem 3.1. *A matching X is pairwise-stable if and only if there exist a feasible side payment vector p and two subsets E_M and E_W of E such that, defining q and r by* (2.1) *and* (2.2),

(S′1) $q \geq \mathbf{0}$ *and* $r \geq \mathbf{0}$,

(S′2) $q_i \geq \max\{\nu_{ij}(p_{ij}) \mid (i, j) \in E_M\}$ *for all* $i \in M$,

(S′3) $r_j \geq \max\{\nu_{ji}(-p_{ij}) \mid (i, j) \in E_W\}$ *for all* $j \in W$,

(S′4) $E = E_M \cup E_W$,

(S′5) $p_{ij} = l_{ij}$ *for all* $(i, j) \in E \setminus E_M$ *and* $p_{ij} = u_{ij}$ *for all* $(i, j) \in E \setminus E_W$,

where we define the maximum over an empty set to be equal to 0.

Proof. (⇒) Let $(X, \bar{p})$ be a pairwise-stable outcome. We define q and r by (2.1) and (2.2) with $p = \bar{p}$. Let p be a vector defined by

$$p_{ij} = \begin{cases} \nu_{ij}^{-1}(q_i) & \text{if } \nu_{ij}^{-1}(q_i) \in [l_{ij}, u_{ij}] \\ u_{ij} & \text{if } u_{ij} < \nu_{ij}^{-1}(q_i) \\ l_{ij} & \text{if } \nu_{ij}^{-1}(q_i) < l_{ij} \end{cases} \qquad (\forall (i,j) \in E), \tag{3.3}$$

and let E_M and E_W be defined by

$$\begin{aligned} E_M &= E \setminus \{(i,j) \in E \mid \nu_{ij}(p_{ij}) > q_i\}, \\ E_W &= E \setminus \{(i,j) \in E \mid \nu_{ji}(-p_{ij}) > r_j\}. \end{aligned} \tag{3.4}$$

Obviously, p is a feasible side payment vector. We will show that p, E_M and E_W satisfy (S′1)–(S′5). Condition (S′1) holds by (S1). Conditions (S′2) and (S′3) are direct consequences of (3.4). We also have (S′4) by (S2) and (3.4). We next show (S′5). Assume $(i,j) \notin E_M$, that is, $\nu_{ij}(p_{ij}) > q_i$. By (3.3) and the monotonicity of ν_{ij}, $\nu_{ij}(p_{ij}) > q_i$ yields $p_{ij} = l_{ij}$. Assume $(i,j) \notin E_W$, that is, $\nu_{ji}(-p_{ij}) > r_j$. By (S′4) we see $(i,j) \in E_M$, and hence, $\nu_{ij}^{-1}(q_i) \geq l_{ij}$. Suppose $p_{ij} < u_{ij}$ to the contrary. Thus, we have $q_i = \nu_{ij}(p_{ij})$ by (3.3). Since $r_j < \nu_{ji}(-p_{ij})$ holds and ν_{ij} and ν_{ji} are monotone increasing, there exists a sufficiently small positive number ε such that $l_{ij} \leq p_{ij} + \varepsilon \leq u_{ij}$, $q_i < \nu_{ij}(p_{ij} + \varepsilon)$ and $r_j < \nu_{ji}(-(p_{ij} + \varepsilon))$. However, this contradicts (S2). Hence $p_{ij} = u_{ij}$ for all $(i,j) \in E \setminus E_W$.

(⇐) We assume that there exist p, E_M and E_W satisfying (S′1)–(S′5). We will show that (X, p) is pairwise-stable. Obviously, (S1) holds from (S′1). Suppose to the contrary that (S2) does not hold, i.e., there exist $(i,j) \in E$ and $c \in [l_{ij}, u_{ij}]$ such that $q_i < \nu_{ij}(c)$ and $r_j < \nu_{ji}(-c)$. By (S′2), if $q_i < \nu_{ij}(c)$ then either (Case 1) $(i,j) \notin E_M$ or (Case 2) $(i,j) \in E_M$ and $p_{ij} < c$. Similarly by (S′3), if $r_j < \nu_{ji}(-c)$ then either (Case 3) $(i,j) \notin E_W$ or (Case 4) $(i,j) \in E_W$ and $p_{ij} > c$. Trivially (Case 2) and (Case 4) are incompatible. By (S′4), both (Case 1) and (Case 3) do not hold. By (S′5), (Case 1) is incompatible to (Case 4), and (Case 2) incompatible to (Case 3). This is a contradiction. Hence we have (S2). □

4. Linear valuations

In this section, we deal with the case where for all $(i,j) \in E$, valuations ν_{ij} and ν_{ji} are linear, i.e., these are defined by

$$\nu_{ij}(x) = \alpha_{ij}x + \beta_{ij}, \quad \nu_{ji}(x) = \alpha_{ji}x + \beta_{ji} \tag{4.5}$$

where α_{ij} and α_{ji} are given positive reals, and β_{ij} and β_{ji} are given reals. Our main purpose is to show that a pairwise-stable outcome, namely (X, p, E_M, E_W) satisfying (S′1)–(S′5), can be found in polynomial-time in the number of agents.

First, we assume that a given (X, p, E_M, E_W) satisfies (S′1), (S′3), (S′4), (S′5) and the following condition:

(wS′2) $q_i \geq \max\{\nu_{ij}(p_{ij}) \mid (i,j) \in E_M\}$ for each matched man i in X.

We note that if no unmatched man in X has any pair in $E_M \setminus E_0$, then (X, p, E_M, E_W) satisfies (S′1)–(S′5), where

$$E_0 = \{(i,j) \in E \mid \nu_{ij}(p_{ij}) \leq 0\}. \tag{4.6}$$

Initially, one can easily find such a 4-tuple (X, p, E_M, E_W) as follows. Define $p \in \mathbf{R}^E$ by

$$p_{ij} := \begin{cases} u_{ij} & \text{if } \nu_{ji}(-u_{ij}) \geq 0 \\ \max\{l_{ij}, -\nu_{ji}^{-1}(0)\} & \text{if } \nu_{ji}(-u_{ij}) < 0, \end{cases} \tag{4.7}$$

and define E_M, E_W by

$$E_M := \{(i,j) \in E \mid \nu_{ji}(-p_{ij}) \geq 0\}, \tag{4.8}$$

$$E_W := \{(i,j) \in E \mid p_{ij} < u_{ij}\} \cup \{(i,j) \in E \mid l_{ij} = u_{ij},\ \nu_{ji}(-u_{ij}) < 0\}. \tag{4.9}$$

Obviously, p is a feasible side payment vector. By the definition of p, if $\nu_{ji}(-p_{ij}) < 0$ then $p_{ij} = l_{ij}$, that is, $(i,j) \in E_W$ and, if $(i,j) \notin E_W$ then $p_{ij} = u_{ij}$ and $\nu_{ji}(-p_{ij}) \geq 0$, that is, $(i,j) \in E_M$. These imply that E_M and E_W satisfy (S′4) and (S′5).

To define a matching X, we consider $\tilde{q} \in \mathbf{R}^M$ and $\widetilde{E}_M \subseteq E_M$ defined by

$$\tilde{q}_i := \max\{\nu_{ij}(p_{ij}) \mid (i,j) \in E_M \setminus E_0\} \tag{4.10}$$

for all $i \in M$, and

$$\widetilde{E}_M := \{(i,j) \in E_M \setminus E_0 \mid \nu_{ij}(p_{ij}) = \tilde{q}_i\}. \tag{4.11}$$

Recall that the maximum over an empty set is 0 by definition. We also define a set $\widehat{E}_M \subseteq \widetilde{E}_M$ as follows:

$$\widehat{E}_M := \big\{(i,j) \in \widetilde{E}_M \bigm| \nu_{ji}(-p_{ij}) \geq \nu_{ji'}(-p_{i'j})\ \forall (i',j) \in E_W\big\}. \tag{4.12}$$

The fact that $\nu_{ji}(-p_{ij}) \geq 0$ for all $(i,j) \in \widetilde{E}_M$ and $\nu_{ji}(-p_{ij}) \leq 0$ for all $(i,j) \in E_W$ implies that $\widehat{E}_M$ initially coincides with $\widetilde{E}_M$; however, it may be a proper subset of $\widetilde{E}_M$ in further iterations in our algorithm. We take a subset $\widetilde{W}$ of W and initially put $\widetilde{W} = \emptyset$. Let X be a matching in bipartite graph $\big(M, W; \widehat{E}_M\big)$ such that

$$X \text{ matches all members of } \widetilde{W}, \tag{4.13}$$

$$X \text{ maximizes } \sum_{(i,j)\in X} \nu_{ji}(-p_{ij}) \text{ among the matchings having (4.13)}, \tag{4.14}$$

$$X \text{ maximizes } \sum_{(i,j)\in X} (\ln \alpha_{ji} - \ln \alpha_{ij}) \quad \text{among the matchings having (4.14)}. \tag{4.15}$$

In the sequel, we will identify $\widetilde{W}$ as a set of matched women, that is,

$$\widetilde{W} := \{j \in W \mid j \text{ is matched in } X\}. \tag{4.16}$$

Since initially $\widetilde{W} = \emptyset$, any matching satisfies (4.13). If there exists a matching satisfying (4.13) then one can easily find a matching X satisfying (4.13), (4.14), and (4.15) by solving the maximum weight matching problem for a bipartite graph. For a matching X defined as above, define q and r by (2.1) and (2.2). Then q is nonnegative because of $\tilde{q} \geq \mathbf{0}$, and r is nonnegative by (4.8), and hence, (S′1) holds. Moreover, (wS′2) holds because of $\tilde{q} \geq \mathbf{0}$, and (S′3) holds because $\nu_{ji}(-p_{ij}) \leq 0$ for all (i,j) in E_W.

Let (X, p, E_M, E_W) be a 4-tuple satisfying (S′1), (wS′2), (S′3)–(S′5) and, in addition, (4.13)–(4.15). If (S′2) does not hold, then we modify (X, p, E_M, E_W) preserving (S′1), (wS′2), (S′3)–(S′5), and (4.13)–(4.15). Since we initially put p as large as possible, we monotonically decrease p, and hence, preserve $r \geq \mathbf{0}$ and $\nu_{ji}(-p_{ij}) \geq 0$ for all $(i,j) \in E_M$ in our modification. Assume that there exists an unmatched man i_0 in X such that there is a pair $(i_0, j) \in E_M \setminus E_0$. Let D be a directed graph $\big(\{i_0\} \cup \widetilde{E}_M, A\big)$ with arc set A defined by

$$\begin{aligned} A &:= A_0 \cup A_1 \cup A_2, \\ A_0 &:= \big\{(i_0, (i_0, j)) \bigm| (i_0, j) \in \widetilde{E}_M\big\}, \\ A_1 &:= \big\{((i,j),(k,j)) \bigm| (i,j) \in \widetilde{E}_M \setminus X,\ (k,j) \in X,\ \nu_{ji}(-p_{ij}) = \nu_{jk}(-p_{kj})\big\}, \\ A_2 &:= \big\{((i,j),(i,k)) \bigm| (i,j) \in X,\ (i,k) \in \widetilde{E}_M \setminus X\big\}, \end{aligned} \tag{4.17}$$

and assign weights $w(e)$ to each arc e of D as follows:

$$\begin{aligned} e = (i_0, (i_0, j)) \in A_0 &\Rightarrow w(e) = \ln \alpha_{i_0 j}, \\ e = ((i,j),(k,j)) \in A_1 &\Rightarrow w(e) = -\ln \alpha_{ji} + \ln \alpha_{jk}, \\ e = ((i,j),(i,k)) \in A_2 &\Rightarrow w(e) = -\ln \alpha_{ij} + \ln \alpha_{ik}. \end{aligned} \tag{4.18}$$

Let

$$R(i_0) = \big\{(i,j) \in \widetilde{E}_M \bigm| (i,j) \text{ is reachable from } i_0 \text{ in } D\big\}. \tag{4.19}$$

By the definition of D, we have the following lemma.

LEMMA 4.1. *Assume that $(i,j) \in R(i_0) \setminus X$ and $(k,j) \in X$. Then, we have $\nu_{ji}(-p_{ij}) \leq \nu_{jk}(-p_{kj})$.*

Proof. Suppose to the contrary that $(i,j) \in R(i_0) \setminus X$ and $\nu_{ji}(-p_{ij}) > \nu_{jk}(-p_{kj})$ for $(k,j) \in X$. Since (i,j) is reachable from i_0, there exists a sequence S of pairs in $\widetilde{E}_M$:

$$S = (i_0, j_0), (i_1, j_0), (i_1, j_1), \ldots, (i_s, j_s) = (i,j), (i_{s+1}, j_s) = (k,j)$$

such that $((i_h, j_h), (i_{h+1}, j_h)) \in A_1$ for $h = 0, 1, \ldots, s-1$ and $((i_h, j_{h-1}), (i_h, j_h)) \in A_2$ for $h = 1, \ldots, s$. Obviously, the symmetric difference X' of S and X is a matching covering $\widetilde{W}$, and X' is strictly greater than X in the sense of (4.14). This, however, is a contradiction. Thus, the assertion holds. □

We also have the following lemma.

Lemma 4.2. *D has no negative cycle with respect to w.*

Proof. Assume that D has a negative cycle C. By the definition of D, vertices corresponding to $\widetilde{E}_M \setminus X$ and X alternately appear in C. We express C by a sequence of pairs of $\widetilde{E}_M$ as

$$C = (i_1, j_1), (i_2, j_1), (i_2, j_2), \ldots, (i_s, j_s), (i_{s+1}, j_s) = (i_1, j_s), (i_{s+1}, j_{s+1}) = (i_1, j_1),$$

where $(i_h, j_h) \in \widetilde{E}_M \setminus X$ and $(i_{h+1}, j_h) \in X$ for all $h = 1, 2, \ldots, s$. By (4.18), the weight $w(C)$ of C is calculated as

$$\begin{aligned} w(C) &= \sum_{h=1}^{s} (-\ln \alpha_{j_h i_h} + \ln \alpha_{j_h i_{h+1}} - \ln \alpha_{i_{h+1} j_h} + \ln \alpha_{i_{h+1} j_{h+1}}) \\ &= \sum_{(i,j) \in C \cap X} (\ln \alpha_{ji} - \ln \alpha_{ij}) - \sum_{(i,j) \in C \setminus X} (\ln \alpha_{ji} - \ln \alpha_{ij}) < 0. \end{aligned}$$

By the construction of D, the symmetric difference X' of X and C, which is a matching in $(M, W; \widehat{E}_M)$, also satisfies (4.13) and (4.14). The assumption that $w(C) < 0$, however, implies that

$$\sum_{(i,j) \in X} (\ln \alpha_{ji} - \ln \alpha_{ij}) < \sum_{(i,j) \in X'} (\ln \alpha_{ji} - \ln \alpha_{ij}),$$

which contradicts (4.15). Hence, D has no negative cycle. □

By Lemma 4.2, we can consider the shortest distance $d\colon \widetilde{E}_M \to \mathbf{R} \cup \{+\infty\}$ from i_0 to the other vertices in D with respect to w. For convenience, we denote the shortest distance of $(i, j) \in \widetilde{E}_M$ by $d_{(i,j)}$. We now decrease p with a parameter $\varepsilon \geq 0$ as

$$p_{ij}(\varepsilon) := \begin{cases} p_{ij} - \varepsilon \exp(-d_{(i,j)}) & \text{if } (i, j) \in R(i_0) \\ p_{ij} & \text{otherwise.} \end{cases} \tag{4.20}$$

Before discussing how to determine a parameter ε, we give two lemmas.

Lemma 4.3. *For any $(i, j) \in R(i_0)$, we have $(i, k) \in R(i_0)$ for all $(i, k) \in \widetilde{E}_M$ and $\nu_{ij}(p_{ij}(\varepsilon)) = \nu_{ik}(p_{ik}(\varepsilon))$, where $\varepsilon \geq 0$.*

Proof. If $(i,j) \in X$ then $((i,j),(i,k)) \in A_2$, and hence $(i,k) \in R(i_0)$. We assume that $(i,j) \in R(i_0)\backslash X$. If $i = i_0$ then $(i,(i,k)) \in A_0$ and hence $(i,k) \in R(i_0)$. Assume that $i \neq i_0$. By the construction of D, given a vertex of $\widetilde{E}_M \setminus X$ in D, if it has an entering arc, then the arc is unique and leaves some vertex of X. Since $(i,j) \in R(i_0)$, there exists $(i,j') \in X \cap R(i_0)$ such that $((i,j'),(i,j)) \in A_2$, and hence $(i,k) \in R(i_0)$ for all $(i,k) \in \widetilde{E}_M$.

We next show that $\nu_{ij}(p_{ij}(\varepsilon)) = \nu_{ik}(p_{ik}(\varepsilon))$. If $i = i_0$ then the unique path from i_0 to (i_0,j) is $(i_0,(i_0,j))$ and hence

$$\nu_{i_0j'}(p_{i_0j'}(\varepsilon)) = \alpha_{i_0j'}(p_{i_0j'} - \varepsilon\exp(-\ln\alpha_{i_0j'})) + \beta_{i_0j'} = \nu_{i_0j'}(p_{i_0j'}) - \varepsilon \tag{4.21}$$

for all $(i_0,j') \in \widetilde{E}_M$, which means $\nu_{i_0j}(p_{i_0j}(\varepsilon)) = \nu_{i_0k}(p_{i_0k}(\varepsilon))$. Assume that $i \neq i_0$, and, without loss of generality, assume that $(i,j) \in X$. For each $(i,k) \in \widetilde{E}_M \setminus X$, we have $d_{(i,k)} = d_{(i,j)} + (-\ln\alpha_{ij} + \ln\alpha_{ik})$. Hence, we have

$$\begin{aligned}\nu_{ik}(p_{ik}(\varepsilon)) &= \alpha_{ik}(p_{ik} - \varepsilon\exp(-d_{(i,j)} + \ln\alpha_{ij} - \ln\alpha_{ik})) + \beta_{ik} \\ &= \nu_{ik}(p_{ik}) - \varepsilon\alpha_{ij}\exp(-d_{(i,j)}) = \nu_{ij}(p_{ij}(\varepsilon)).\end{aligned}$$

This completes the proof. □

LEMMA 4.4. *Assume that $(i,j) \in R(i_0)$, and that there exists $(k,j) \in X$. For a sufficiently small $\varepsilon \geq 0$, $\nu_{ji}(-p_{ij}(\varepsilon)) \leq \nu_{jk}(-p_{kj}(\varepsilon))$ holds. Moreover, if $((i,j),(k,j)) \in A_1$, then the above inequality holds for all $\varepsilon \geq 0$. The above inequality holds with equality if arc $((i,j),(k,j))$ lies on a shortest path from i_0 to (k,j).*

Proof. In the case where $\nu_{ji}(-p_{ij}) < \nu_{jk}(-p_{kj})$, the assertion obviously holds. In the other case, it follows from Lemma 4.1 that $\nu_{ji}(-p_{ij}) = \nu_{jk}(-p_{kj})$, i.e., $((i,j),(k,j)) \in A_1$ and hence $(k,j) \in X \cap R(i_0)$. Since d is the shortest distance with respect to w, we have $d_{(k,j)} \leq d_{(i,j)} - \ln\alpha_{ji} + \ln\alpha_{jk}$. Hence, we have

$$\begin{aligned}\nu_{jk}(-p_{kj}(\varepsilon)) &= \alpha_{jk}(-p_{kj} + \varepsilon\exp(-d_{(k,j)})) + \beta_{jk} \\ &= \nu_{ji}(-p_{ij}) + \varepsilon\alpha_{jk}\exp(-d_{(k,j)}) \\ &\geq \nu_{ji}(-p_{ij}) + \varepsilon\alpha_{jk}\exp(-d_{(i,j)} + \ln\alpha_{ji} - \ln\alpha_{jk}) = \nu_{ji}(-p_{ij}(\varepsilon)).\end{aligned}$$

Note that if $((i,j),(k,j))$ lies on a shortest path from i_0 to (k,j), then $d_{(k,j)} = d_{(i,j)} - \ln\alpha_{ji} + \ln\alpha_{jk}$ and hence $\nu_{jk}(-p_{kj}(\varepsilon)) = \nu_{ji}(-p_{ij}(\varepsilon))$. This completes the proof. □

Our aim in each iteration of the algorithm is to find a parameter ε as large as possible so that fixed sets X, E_M, $\widetilde{E}_M$, E_0 and E_W preserve conditions (S$'$1), (wS$'$2), (S$'$3)–(S$'$5) and (4.13)–(4.15) for $p(\varepsilon)$ defined by (4.20). By Lemmas 4.3 and 4.4, the following cases give a threshold value of ε.

(1) X can be augmented for $p(\varepsilon)$.
(2) If $\nu_{ij}(p_{ij}(\varepsilon)) = \nu_{ik}(p_{ik}(\varepsilon))$ for some $(i,j) \in R(i_0)$ and $(i,k) \in E_M \setminus R(i_0)$ then (i,k) can be added in $\widetilde{E}_M$ for $p(\varepsilon)$.
(3) If $\nu_{ij}(p_{ij}(\varepsilon)) = 0$ for some $(i,j) \in R(i_0)$ then $p(\varepsilon)$ cannot decrease any more to preserve both $(i,j) \in \widetilde{E}_M$ and (S′1).
(4) If $p_{ij}(\varepsilon) = l_{ij}$ for some $(i,j) \in R(i_0)$, then $p(\varepsilon)$ cannot decrease any more by the feasibility of $p(\varepsilon)$.
(5) If $\nu_{ji}(-p_{ij}(\varepsilon)) = \nu_{jk}(-p_{kj}(\varepsilon))$ for some $(i,j) \in R(i_0) \setminus X$ and $(k,j) \in X \setminus R(i_0)$ then at least one pair of X can be added in $R(i_0)$, which may yield an augmentation of X.

To determine parameter ε, we explicitly write the above here:

Case 1. Let $\varepsilon_1 \geq 0$ be the minimum for which $\nu_{ij}(p_{ij}(\varepsilon_1)) = \nu_{ik}(p_{ik}(\varepsilon_1))$ for some $(i,j) \in R(i_0)$ and $(i,k) \in E_M$ with $k \in W \setminus \widetilde{W}$. If such ε_1 does not exist, then put $\varepsilon_1 = +\infty$.

Case 2. Let $\varepsilon_2 \geq 0$ be the minimum for which $\nu_{ij}(p_{ij}(\varepsilon_2)) = \nu_{ik}(p_{ik}(\varepsilon_2))$ for some $(i,j) \in R(i_0)$ and $(i,k) \in E_M \setminus R(i_0)$ with $k \in \widetilde{W}$. If such ε_2 does not exist, then put $\varepsilon_2 = +\infty$.

Case 3. Let $\varepsilon_3 \geq 0$ be the minimum for which $\nu_{ij}(p_{ij}(\varepsilon_3)) = 0$ for some $(i,j) \in R(i_0)$.

Case 4. Let $\varepsilon_4 \geq 0$ be the minimum for which $p_{ij}(\varepsilon_4) = l_{ij}$ holds for some $(i,j) \in R(i_0)$.

Case 5. Let $\varepsilon_5 \geq 0$ be the minimum for which $\nu_{ji}(-p_{ij}(\varepsilon_5)) = \nu_{jk}(-p_{kj}(\varepsilon_5))$ holds for some $(i,j) \in R(i_0) \setminus X$ and $(k,j) \in X \setminus R(i_0)$. If such ε_5 does not exist, then put $\varepsilon_5 = +\infty$.

Since $l \in \mathbf{R}^E$, we have $\varepsilon_4 \in \mathbf{R}$. Thus, ε determined by

$$\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5\} \tag{4.22}$$

is well-defined. It follows from $\varepsilon \leq \varepsilon_4$ that $p(\varepsilon)$ is a feasible side payment vector. From the definitions of ε_1 and ε_2, and by the Lemma 4.3, we observe that for any $(i,j), (i,k) \in \widetilde{E}_M$, we get

$$\nu_{ij}(p_{ij}(\varepsilon)) = \nu_{ik}(p_{ik}(\varepsilon)) \geq \nu_{ij'}(p_{ij'}(\varepsilon)) \quad (\forall (i,j') \in E_M). \tag{4.23}$$

Note that (4.23) is a key to preserve (wS′2) during each iteration of our algorithm. Put

$$\tilde{p} = p(\varepsilon). \tag{4.24}$$

Now, our algorithm is described as below.

Algorithm. `Stable_Outcome`

Step 0: Let $\widetilde{W} = \emptyset$. Initially define (X, p, E_M, E_W), E_0 and $\widetilde{E}_M$ by (4.6), (4.7), (4.8), (4.9), (4.11), and (4.13)–(4.15).

Step 1: If no unmatched man in X has any pair in $E_M \setminus E_0$ then stop.

Step 2: Let i_0 be an unmatched man in X such that there is a pair $(i_0, j) \in E_M \setminus E_0$. Construct a directed graph $D = \big(\{i_0\} \cup \widetilde{E}_M, A\big)$ and weight w by (4.11), (4.17) and (4.18). Calculate the shortest distances d of all vertices in D from i_0, and ε and $\tilde{p}$ by (4.22) and (4.24).

Step 3: Let $R(i_0)$ and $\widetilde{W}$ be the sets defined by (4.19) and (4.16) for the current matching X. Suppose that (i, j) and k denote a pair and an agent defined in the above five cases.

(a) If $\varepsilon = \varepsilon_1$ then set $\widetilde{W} := \widetilde{W} \cup \{k\}$ and go to Step 4; else go to (b),

(b) If $\varepsilon = \varepsilon_2$ then go to Step 4; else go to (c),

(c) If $\varepsilon = \varepsilon_3$ then set $E_0 := E_0 \cup \big\{(i, j') \in \widetilde{E}_M \bigm| j' \in W\big\}$ and go to Step 4; else go to (d),

(d) If $\varepsilon = \varepsilon_4$ then set $E_M := E_M \setminus \{(i, j)\}$ and $E_W := E_W \cup \{(i, j)\}$ and go to Step 4; else go to (e),

(e) If $\varepsilon = \varepsilon_5$ then go to Step 4.

Step 4: Replace p by $\tilde{p}$, and calculate $\widetilde{E}_M$ and $\widehat{E}_M$ by (4.11) and (4.12) for the updated p. Augment E_W by adding all $(i, j) \in E$ that satisfy (4.9). Find a matching X in $\big(M, W; \widehat{E}_M\big)$ satisfying (4.13)–(4.15). Go to Step 1.

We here mention that $\nu_{ij}(p_{ij}) > 0$, for all $(i, j) \in \widetilde{E}_M$, before the execution of Step 3 in the first iteration. Since we update E_0 at Step 3 (c), if the `Stable_Outcome` goes from Step 3 (a) to Step 4 or from Step 3 (b) to Step 4 or from Step 3 (c) to Step 4 in some iteration, then it is possible that there exists $(i, j) \in \widetilde{E}_M$ such that $\nu_{ij}(p_{ij}) = 0$ in the next iteration. However, such a pair does not violate (S$'$1). It is also worthy of note that E_M is updated when `Stable_Outcome` goes from Step 3 (d) to Step 4.

Before showing the correctness, we give here few characteristics of `Stable_Outcome`. We will use the notation [Step AA $\to$ Step 4] which means `Stable_Outcome` goes from Step AA to Step 4.

Lemma 4.5. *In each iteration of* `Stable_Outcome`*, the following hold:*

(i) *If* [Step 3 (a) $\to$ Step 4] *or* [Step 3 (b) $\to$ Step 4] *or* [Step 3 (e) $\to$ Step 4] *is executed, then $\widetilde{E}_M$ remains the same or enlarges at* Step 4*. In particular, if $\varepsilon_1 = 0$ or* [Step 3 (e) $\to$ Step 4] *is executed, then $\widetilde{E}_M$ remains the same at* Step 4*,*

(ii) *If* [Step 3 (c) $\to$ Step 4] *or* [Step 3 (d) $\to$ Step 4] *is executed, then $\widetilde{E}_M$ reduces at* Step 4*,*

(iii) *$\widetilde{W}$, E_0 and E_W enlarge or remain the same, and a pair eliminated from $\widetilde{E}_M$ in the previous iterations never appears again in $\widetilde{E}_M$,*

(iv) *If a pair* (i,k) *is added in* $\widetilde{E}_M$ *at* Step 4, *then* $p_{ik}(\varepsilon)$ *is equal to the initial* p_{ik} *defined in* (4.7). *In particular, if* $\nu_{ki}(-p_{ik}(\varepsilon)) > 0$ *then* $(i,k) \notin E_W$ *at* Step 4.

Proof. Let $\widetilde{W}$, E_0, E_M, E_W, $\widetilde{E}_M$ and p be the values before the algorithm goes from Step 3 to Step 4.

(i) We consider the case when [Step 3 (a) $\to$ Step 4] is executed. In this case, $\varepsilon = \varepsilon_1$. The case when $\varepsilon = 0$ is easy to see. Suppose that $\varepsilon > 0$. Then there exist $(i,j) \in R(i_0)$ and $(i,k) \in E_M$ $\big(k \in W \setminus \widetilde{W}\big)$ such that

$$\nu_{ij}(p_{ij}(\varepsilon)) = \nu_{ik}(p_{ik}(\varepsilon)) \geq \nu_{ij'}(p_{ij'}(\varepsilon)) \quad (\forall (i,j') \in E_M) \tag{4.25}$$

where the last inequality is by Lemma 4.3 and by the definitions of ε, ε_1 and ε_2. Note that $(i,k) \notin R(i_0)$ by the minimality of ε_1 and the fact that $\varepsilon_1 > 0$. (4.25) says that $\widetilde{E}_M$ is enlarged at Step 4. If [Step 3 (b) $\to$ Step 4] is executed, then $\varepsilon = \varepsilon_2$. In this case, $\varepsilon > 0$ since $(i,k) \notin R(i_0)$. Analogously, we can prove that $\widetilde{E}_M$ enlarges. To prove the result when [Step 3 (e) $\to$ Step 4] is executed, it is enough to note that $\varepsilon_5 < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$.

(ii) Next, let [Step 3 (c) $\to$ Step 4] be executed. Then obviously $\varepsilon = \varepsilon_3 < \min\{\varepsilon_1, \varepsilon_2\}$ and hence for any $(i,j) \in \widetilde{E}_M$, we have

$$\nu_{ij}(p_{ij}(\varepsilon)) > \nu_{ik}(p_{ik}(\varepsilon)) \quad \big(\forall (i,k) \in E_M \setminus \big(E_0 \cup \widetilde{E}_M\big)\big) \tag{4.26}$$

by the definitions of ε_1 and ε_2. Also by Lemma 4.3, we have

$$\nu_{ij}(p_{ij}(\varepsilon)) = \nu_{ij'}(p_{ij'}(\varepsilon)) \quad \big(\forall (i,j') \in \widetilde{E}_M\big). \tag{4.27}$$

From (4.26) and (4.27), we see that no new element is added in $\widetilde{E}_M$ and since $\varepsilon = \varepsilon_3$, the set E_0 enlarges at Step 3 and consequently $\widetilde{E}_M$ reduces at Step 4. Similarly, if [Step 3 (d) $\to$ Step 4] is executed then $\varepsilon = \varepsilon_4$ and E_M reduces at Step 3 and hence $\widetilde{E}_M$ also reduces at Step 4.

(iii) By the modification of $\widetilde{W}$ at Step 3 (a) and by (4.13), we see that $\widetilde{W}$ remains the same or enlarges. E_0 enlarges when [Step 3 (c) $\to$ Step 4] is executed else it remains the same. In each iteration, p remains the same or decreases, and E_W is modified by (4.9) or when [Step 3 (d) $\to$ Step 4] is executed. In either case, E_W enlarges or remains the same.

Note that $\widetilde{E}_M$ is modified by (4.11) and if E_0 enlarges then at least one pair is deleted from $\widetilde{E}_M$. Since p remains the same or decreases, the deleted pair never appears again. Similarly, if E_M reduces then one pair is deleted from $\widetilde{E}_M$ and never appears again.

(iv) We observe from (i) that if $(i,k) \in E_M$ is a new pair added in $\widetilde{E}_M$ at Step 4 then $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\} > 0$ which yields $(i,k) \notin R(i_0)$. This, together with (4.20) implies $p_{ik}(\varepsilon) = p_{ik}$. By (iii), a pair eliminated from $\widetilde{E}_M$ in the previous iterations never appears again in $\widetilde{E}_M$, p_{ik} must be the initial value defined in (4.7).

Initial definition of p in (4.7) and the fact that $(i,k) \in E_M$ imply that $p_{ik} = u_{ik}$ or $\nu_{ki}(-p_{ik}) = 0$. Now, if $\nu_{ki}(-p_{ik}) > 0$ then $p_{ik} = u_{ik}$. Thus the equation (4.9) implies that $(i,k) \notin E_W$. This completes the proof. □

Now we show the correctness of `Stable_Outcome`.

LEMMA 4.6. *In each iteration of* `Stable_Outcome`*,* (X, p, E_M, E_W) *at* Step 1 *satisfies* (S′1)*,* (wS′2)*,* (S′3)–(S′5) *and* (4.13)–(4.15)*, and furthermore, there exists a matching in* $\big(M, W; \widehat{E}_M\big)$ *satisfying* (4.13)–(4.15) *at* Step 4*.*

Proof. We will use the terminology $(old)*$ and $(new)*$ below, which means the sets before and after update, respectively, in some iteration.

By the initial selection of (X, p, E_M, E_W) at Step 0, obviously (S′1), (wS′2) and (S′3)–(S′5) and (4.13)–(4.15) hold prior to the execution of Step 1 in the first iteration. Thus, the first assertion holds in the first iteration. Suppose that (X, p, E_M, E_W) satisfies (S′1), (wS′2) and (S′3)–(S′5) and (4.13)–(4.15) before the start of the t-th iteration, $t \geq 1$. We first show the second assertion. It is enough to show that there exists a matching $\widetilde{X}$ in $\big(M, W; (new)\widehat{E}_M\big)$ satisfying (4.13). Observe that, since $\varepsilon \leq \varepsilon_5$ and by Lemma 4.4, for any $(i', j') \in (old)X$, we get

$$\nu_{j'i'}(-p_{i'j'}(\varepsilon)) \geq \nu_{j'k'}(-p_{k'j'}(\varepsilon)) \quad (\forall (k', j') \in R(i_0)). \tag{4.28}$$

We first consider the case when [Step 3 (b) → Step 4] or [Step 3 (e) → Step 4] is executed. In this case, Lemma 4.5 (i) implies that $(old)X \subseteq (new)\widetilde{E}_M$. We prove that $(old)X \subseteq (new)\widehat{E}_M$. Suppose there exists $(i', j') \in (old)X$ but $(i', j') \notin (new)\widehat{E}_M$. This means there exists $(k', j') \in (new)E_W$ such that

$$\nu_{j'i'}(-p_{i'j'}) < \nu_{j'i'}(-p_{i'j'}(\varepsilon)) < \nu_{j'k'}(-p_{k'j'}(\varepsilon)). \tag{4.29}$$

(4.28) and (4.29) imply $(k', j') \notin R(i_0)$, that is, $\nu_{j'k'}(-p_{k'j'}(\varepsilon)) = \nu_{j'k'}(-p_{k'j'})$ by (4.20). This means $(k', j') \in (old)E_W$. (4.29) implies $\nu_{j'i'}(-p_{i'j'}) < \nu_{j'k'}(-p_{k'j'})$ which contradicts (S′3). Therefore $(old)X \subseteq (new)\widehat{E}_M$, that is, $(old)X$ satisfies (4.13). Analogously, $(old)X$ satisfies (4.13) when [Step 3 (d) → Step 4] is executed and $(i,j) \notin (old)X$. We next consider the case when [Step 3 (d) → Step 4] is executed and $(i,j) \in (old)X$. In this case, there exists a shortest path S from i_0 to (i,j) in D, which is denoted by

$$S = (i_0, j_0), (i_1, j_0), (i_1, j_1), \ldots, (i_s, j_s), (i_{s+1}, j_s) = (i,j) \tag{4.30}$$

where $((i_h, j_h), (i_{h+1}, j_h)) \in A_1$ for $h = 0, \ldots, s$ and $((i_h, j_{h-1}), (i_h, j_h)) \in A_2$ for $h = 1, \ldots, s$. In the same way as above, Lemmas 4.4, 4.5 and (S′3) guarantee that all pairs in $S \cup (old)X$ other than (i,j) are contained in $(new)\widehat{E}_M$. Let $\widetilde{X}$ be the symmetric difference of S and $(old)X$. Obviously, $(i,j) \notin \widetilde{X}$ and $\widetilde{X}$ is a matching in $\big(M, W; (new)\widehat{E}_M\big)$ satisfying (4.13). Analogously we can deal the case when [Step 3 (c) → Step 4] is executed. We finally consider the case when [Step 3 (a) → Step 4] is executed. In this case $\varepsilon = \varepsilon_1$, which together Lemma 4.5 (i) implies that $(i,k) \in (new)\widetilde{E}_M$. We first show that $(i,k) \in (new)\widehat{E}_M$. Suppose that $(i,k) \notin (new)\widehat{E}_M$. Then, there exists $(i',k) \in (new)E_W$ such that

$$0 \leq \nu_{ki}(-p_{ik}) \leq \nu_{ki}(-p_{ik}(\varepsilon)) < \nu_{ki'}(-p_{i'k}(\varepsilon)). \tag{4.31}$$

It is easy to see that $(i',k) \notin R(i_0)$, that is, $\nu_{ki'}(-p_{i'k}(\varepsilon)) = \nu_{ki'}(-p_{i'k})$ by (4.20). This means $(i',k) \in (old)E_W$. (4.31) yields $\nu_{ki'}(-p_{i'k}) > 0$ which contradicts (S′3) since $k \in W\backslash(old)\widetilde{W}$. Hence, $(i,k) \in (new)\widehat{E}_M$. It is obvious to see that there exists a matching $\widetilde{X}$ satisfying (4.13) when $i = i_0$. We suppose that $i \neq i_0$. Then, without loss of generality, assume that $(i,j) \in (old)X$. Let S be a shortest path from i_0 to (i,j) denoted as (4.30). In the same way as above, we see that $S \cup (old)X \subseteq (new)\widehat{E}_M$ and the symmetric difference $\widetilde{X}$ of $S \cup \{(i,k)\}$ and $(old)X$ is a matching covering the $(new)\widetilde{W}$.

Next we prove that (X, p, E_M, E_W) defined at Step 4 satisfies (S′1), (wS′2) and (S′3)–(S′5). We have $q \geq \mathbf{0}$ since $\varepsilon \leq \varepsilon_3$, and $r \geq \mathbf{0}$ since (4.14) holds for X and $\nu_{ji}(-p_{ij}) \geq 0$ for all $(i,j) \in E_M$. Hence (S′1) does hold. Definition of $\tilde{q}$ guarantees (wS′2). The definition of ε_1 along with (4.20) implies that, for any $(i,j) \in E_M$, we have $p_{ij}(\varepsilon) = p_{ij}$ if $j \in W \setminus (old)\widetilde{W}$ or $(i,j) \notin R(i_0)$. This, together with the fact that $X \subseteq \widehat{E}_M$ and (S′3) holds at Step 1 implies that (X, p, E_M, E_W) satisfies (S′3). At Step 3, E_M and E_W remain the same or one element is removed from E_M and added into E_W, and at Step 4 E_W may be enlarged. Thus (S′4) holds true. We note that if any $(i,j) \in E$ is removed from E_M and added into E_W, then $p_{ij} = l_{ij}$. Hence, the first part of (S′5) holds. Also $(i,j) \in E_W$ if and only if either $p_{ij} < u_{ij}$ or ($l_{ij} = u_{ij}$ and $\nu_{ji}(-u_{ij}) < 0$). Thus, we have $p_{ij} = u_{ij}$ for all $(i,j) \in E \setminus E_W$, the second part of (S′5). This completes the proof. □

Theorem 4.7. *If* `Stable_Outcome` *terminates, then it outputs a pairwise-stable outcome* (X, p, E_M, E_W) *satisfying* (S′1)–(S′5).

Proof. Lemma 4.6 guarantees that (X, p, E_M, E_W) at Step 1 satisfies (S′1), (wS′2) and (S′3)–(S′5). If `Stable_Outcome` terminates, then there is no unmatched man in X having any pair in $E_M \setminus E_0$. This says that (X, p, E_M, E_W) also satisfies (S′2). □

Until now, we have spent our efforts to show the correctness of `Stable_Outcome`. We now finally show that `Stable_Outcome` terminates in polynomial-time in the number n of agents.

Lemma 4.8. `Stable_Outcome` *terminates in* $O(n^3)$ *iterations.*

Proof. During `Stable_Outcome`, $\widetilde{W}$, E_0 and E_W enlarge or remain the same, and a pair eliminated from $\widetilde{E}_M$ will never appear again in $\widetilde{E}_M$ by Lemma 4.5.

If $\varepsilon = \varepsilon_1$ then $\widetilde{W}$ is enlarged. Thus, [Step 3 (a) → Step 4] is executed at most $|W|$ times.

If $\varepsilon = \varepsilon_2$, then at least one pair is added to $\widetilde{E}_M$, and hence, [Step 3 (b) → Step 4] is executed at most $|E|$ times.

Since E_0 is enlarged if $\varepsilon = \varepsilon_3$, [Step 3 (c) $\to$ Step 4] is executed at most $|M|$ times.

If $\varepsilon = \varepsilon_4$, then one pair is deleted from E_M and added in E_W and is not selected for X in the subsequent iterations. Hence, [Step 3 (d) $\to$ Step 4] is executed at most $|E|$ times.

Summing up the above discussion, the cases other than $\varepsilon = \varepsilon_5$ occurs at most $O(n^2)$ times.

We finally consider the case where $\varepsilon = \varepsilon_5$ but the other cases do not occur. In this case, [Step 3 (e) $\to$ Step 4] is executed and a pair in X is added into $R(i_0)$. Such case successively occurs at most $\left|\widetilde{W}\right|$ times, because $R(i_0)$ does not reduce if $\varepsilon = \varepsilon_5 < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$. This means that `Stable_Outcome` terminates in $O(n^3)$ iterations. $\square$

In each iteration of `Stable_Outcome`, we solve the maximum weight matching problem in bipartite graph $\left(M, W; \widehat{E}_M\right)$ and the single source shortest path problem in D. We note that one can execute `Stable_Outcome` without calculating logarithms nor exponentials in practice, because maximizing $\sum_{(i,j)\in X}(\ln \alpha_{ji} - \ln \alpha_{ij})$ is equivalent to maximizing $\prod_{(i,j)\in X} \frac{\alpha_{ji}}{\alpha_{ij}}$, and because $\exp(-d_{(i,j)})$ in (4.20) can be expressed by products and divisions of α_{ij}'s. It is well known that the maximum weight matching problem in a bipartite graph can be solved in $O(n^3)$. Since the arcs in graph D have general weights (positive or negative) and D does not have negative cycles with respect to w, we utilize the Moore–Bellman–Ford Algorithm which finds the shortest distances in $O\left(\left|\widetilde{E}_M\right| \cdot |A|\right)$. Since each pair $(i,j) \in \widetilde{E}_M \setminus X$ has at most one entering arc and one leaving arc in D, $|A|$ is bounded by $O(n^2)$ from above. Thus, the shortest distances can be calculated in $O(n^4)$. By Lemma 4.8, we can easily derive the following Theorem.

THEOREM 4.9. *The complexity of* `Stable_Outcome` *is* $O(n^7)$ *where* n *denotes the number of agents.*

Acknowledgements. The author is grateful to Satoru Fujishige and Akihisa Tamura for their skilful guidance which paved the way for this research task. Thanks are also due to the anonymous referees for their useful comments that improved the presentation of this paper.

References

[1] K. Eriksson and J. Karlander, Stable matching in a common generalization of the marriage and assignment models. Discrete Math., **217** (2000), 135–156.

[2] S. Fujishige and A. Tamura, A general two-sided matching market with discrete concave utility functions. Discrete Appl. Math., **154** (2006), 950–970.

[3] S. Fujishige and A. Tamura, A two-sided discrete-concave market with possibly bounded side payments: An approach by discrete concave analysis. Math. Oper. Res., **32** (2007), 136–155.

[4] D. Gale and L.S. Shapley, College admissions and the stability of marriage. Amer. Math. Monthly, **69** (1962), 9–15.

[5] M. Kaneko, The central assignment game and the assignment markets. J. Math. Econom., **10** (1982), 205–232.

[6] K. Murota, Convexity and Steinitz's exchange property. Adv. Math., **124** (1996), 272–311.

[7] K. Murota, Discrete convex analysis. Math. Programming, **83** (1998), 313–371.

[8] K. Murota, Discrete Convex Analysis. Society for Industrial and Applied Mathematics, Philadelphia, 2003.

[9] A.E. Roth and M.A.O. Sotomayor, Stable outcomes in discrete and continuous models of two-sided matching: A unified treatment. Rev. Econom., **16** (1996), 1–24.

[10] L.S. Shapley and M. Shubik, The assignment game I: The core. Internat. J. Game Theory, **1** (1972), 111–130.

[11] M. Sotomayor, Existence of stable outcomes and the lattice property for a unified matching market. Math. Social Sci., **39** (2000), 119–132.