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BOOK REVIEW

Géométrie différentielle et Mécanique, by Pierre Aimé, Ellipses, Paris, 2005, viii + 544pp. ISBN 9-782729-822613.

It is well known that differential geometry plays an important role in modern theoretical physics. In general relativity we are confronted with manifolds equipped with a metric and we have to manipulate the Riemann tensor and forms derived from it (e.g., the Ricci curvature). In classical mechanics in its symplectic incarnation we have to deal with the symplectic form. And in gauge theories we have to deal with principal fibre bundles with connections and the derived objects (such as holonomy and curvature). The newcomer to one of these fields is often confronted with the (sad) reality that modern differential geometry seems to be an unsurmountable amount of formalism. I would argue that it is like a new language that one has to learn to speak. Once one knows the language, it becomes natural and the initial obstacles seem trivial.

Given this situation, it is not surprising that several authors have written books with titles like "differential geometry for physicists", in order to explain to the (accomplished or beginning) physicist the intricacies of differential geometry. The level of these books varies from very elementary to tough. It is often the case that these books are written with a particular kind of application in physics in mind, in particular classical mechanics or relativity. The subjects treated thus differ, even though the basics remain the same.

The title of the book under review is "Differential Geometry and Mechanics", but it should not be interpreted as restrictive to standard classical mechanics. It also includes the formalism of fluid mechanics and mechanics of deformable bodies. The book is meant to be self contained, even though it is part of a five volume series on "geometry and applications". Moreover, the reader is not supposed to have any prior knowledge of mechanics as he/she should have only a minimal knowledge of real analysis and some knowledge of curves and surfaces in $\mathbb{R}^3$. Each of the five chapters starts or finishes with one or two sections on mechanics to illustrate the notions described in it. The treatment of the subjects is rather encyclopedic in the sense that all kind of circumstantial details are given. This means that this book is not meant to get a quick overview of the subject, but rather to get a thorough understanding of the underlying mathematics involved.

The first chapter treats the notion of a manifold, first in the sense of a submanifold of some Euclidean space $\mathbb{R}^n$ but later as an abstract notion. It includes the notions of manifolds with boundary, submanifolds, differential maps, immersions and submersions. It also treats Lie groups and actions of Lie groups on manifolds, as well as fibred manifolds and quotient manifolds (by a Lie group action). These notions are used to describe mechanical models as well as (geometric) constraints. Roughly speaking such a model is given by a standard configuration of the object to be described in terms of a submanifold of $\mathbb{R}^3$ (such as a wheel or a piece of deformable clay) and the "group" of immersions of this standard configuration into $\mathbb{R}^3$. The idea being that the positions this object can take are given by these immersions. For instance, a particle is described as a single point for the standard configuration and the group is the translation group $\mathbb{R}^3$. A (non-symmetric) solid is described by its form (a particular position in $\mathbb{R}^3$) as the standard configuration and the group is the full group of isometries of $\mathbb{R}^3$ including the rotations. If we go to a deformable body instead of a solid, the group gets replaced (enlarged) by the full semi-direct product of $\mathbb{R}^3$ and $\mathrm{GL}(3, \mathbb{R})$. In this way the dynamics of the model should be described in terms of curves in the "group" of configurations.

The second chapter deals with tangent structures in a large sense because, besides the tangent and cotangent bundle, it includes general fibre bundles and vector bundles, distributions and foliations. It also deals with flows of vector fields and Frobenius' theorem (and the generalization of Sussmann and Stefan), as well as the notion of derivations (Lie derivative of tensor fields, exterior derivative of differential forms). The notion of fundamental vector field associated to the action of a Lie group is interpreted as the generalization of the flow of a vector field, which can be seen as the fundamental vector field of the action of $\mathbb{R}$ on a manifold (provided this vector field is complete).

Chapter three is completely devoted to the notion of an (Ehresmann) connection in a fibre bundle and the special cases of a principal connection on a principal fibre bundle and a linear connection on a vector bundle. The curvature and torsion of a connection are not neglected, though the notion of holonomy is not mentioned. In the applications we encounter the idea of kinematic constraints in a mechanical system with the standard example of rolling without sliding (e.g. a coin rolling on a flat surface without sliding).

Chapter four naturally splits into two parts. In the fist part the idea of integration of a volume form (a differential form of top rank) over a manifold (with or without boundary) is introduced and the important Stokes' theorem is proven. The notion of integration of a volume form is cast in a measure theoretic context by interpreting it as inducing a (Radon-)measure on the manifold in such a way that integration with respect to this measure corresponds exactly to integration of volume forms. As a consequence the standard machinery of (Lebesgue) integration over a measure space applies. It should be noted that the idea of integration of a differential k-form over a k-dimensional submanifold (k-chain) is not mentioned, but this is hardly an omission as it is a direct consequence of the previous notion applied to the submanifold in question.

The second part of chapter four deals with metric structures on a manifold. Here one finds the notion of a (pseudo-)metric and its associated distance function, complete metrics, isometries, the exponential map (providing the geodesics), the Levi-Civita connection and the canonical volume form associated to a metric, as well as some properties of these notions (e.g. theorems of Myers-Steenrod and Rinow-Hopf). Combining these notions with integration of volume forms, we get the Hodge duality, the divergence and rotation of a vector field, the Laplace operator and the special cases of Stokes theorem known as the Green-Riemann-Ostrogradski theorems. In the applications we find the notions of work and acceleration applied to the various models (solid bodies, deformable bodies etc.).

In the fifth and last chapter we find a description of symplectic and Poisson geometry as well as their use in mechanics. This includes the definition of the Schouten bracket of multi vectors as well as the definition of Lagrange subspaces and abstract (symplectic and Poisson) reduction. Symmetry groups and conserved quantities (momentum maps) are not neglected and the well-known Marsden-Weinstein reduction theorem is given. Here again we find the rolling disc as an example.

The book concludes with two Appendices (50 pages in total) on algebra and topology which contains the (standard) material on multi-linear algebra, exterior algebra and duality (for the algebra part) and a crash course on topology including locally compact spaces, paracompact spaces, Baire spaces and the quotient topology. A third, four page Appendix on categories just gives the definitions of a category and a functor and provides some examples.

This book is a strange mix of a highly abstract presentation of differential geometry and very down-to-earth examples, which are cast in the form of the abstract theory. As said, it is encyclopedic, but in a rather dense form. It gives the definitions and the basic theorems (and then some more, I even saw one or two I did not know before), but leaves out a lot of examples that could have illustrated these definitions. Hence I do not think this book is suited for self study. It could be used as a supplementary text to a course, but then the lecturer should be inclined to an abstract presentation besides the more mundane level of applications, else the connection between the course and the book will be lost.

A final word on the lay-out of this book. The editor/publisher has done a very poor job at editing the text. On nearly every page one finds typos (sometimes even sentences that are missing whole words) and the lay-out of the pages is horrible (even though it obviously is done in TEX). This makes that reading the text is no pleasure. Readability would have been improved enormously by a different presentation in terms of indentations and white spaces.

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