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THE ISOMORPHISM BETWEEN MOTIVIC COHOMOLOGY AND K-GROUPS FOR EQUI-CHARACTERISTIC REGULAR LOCAL RINGS

YUKI KATO

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Abstract

One of the well-known problems in algebraic K-theory is the comparison of higher Chow groups and K-groups. In this paper, using the motivic complex defined by Voevodsky–Suslin–Friedlander, we prove the comparison theorem for equi-characteristic regular local rings.

1. Introduction

Voevodsky–Suslin–Friedlander [8] defined the motivic cohomology $\mathrm{CH}^r_{\mathrm{Zar}}(X,n)$ by using equi-dimensional cycle groups $\mathcal{Z}_{\mathrm{equi}}(X\times\Delta^\bullet\times\mathbb{A}^r/X\times\Delta^\bullet,0)$ for smooth noetherian schemes X over a field and showed the contravariant functoriality for morphisms of schemes. Friedlander–Suslin [2] proved that $\mathrm{CH}^r_{\mathrm{Zar}}(X,n)=\mathrm{CH}^r(X,n)$ for smooth quasi-projective schemes X over a field, where $\mathrm{CH}^r(X,n)$ is the higher Chow group of X defined by Bloch [1]. For smooth quasi-projective schemes X over a field, Bloch [1] proved that $\bigoplus_{r\geqslant 0}\mathrm{CH}^r(X,n)$ coincides with the n-th algebraic K-group $K_n(X)$ after tensoring with $\mathbb{Q}$. We use the subscript $-_{\mathbb{Q}}$ to mean $-\otimes_{\mathbb{Z}}\mathbb{Q}$.

In this paper, we consider the motivic cohomology groups $\mathrm{CH}^r_{\mathrm{Zar}}(X,n)$ for regular schemes by using an equi-dimensional cycle group [8] and prove that there is an isomorphism between the K-group $K_n(X)$ and the motivic cohomology group $\mathrm{CH}^r_{\mathrm{Zar}}(X,n)$ for the spectrum of an arbitrary regular local ring containing a field after tensoring with $\mathbb{Q}$.

Theorem 1.1. *Let R be a regular local ring containing a prime field. Then the cycle class map*

$$\mathrm{cl}^{(r)}\colon K_n(R)^{(r)}_{\mathbb{Q}}\to\mathrm{CH}^r_{\mathrm{Zar}}(R,n)_{\mathbb{Q}}$$

is an isomorphism for any $n,r\geqslant 0$, where $\mathrm{cl}^{(r)}$ is the cycle-class map constructed in Section 3.1 and $K_n(R)^{(r)}_{\mathbb{Q}}$ is the eigenspace of the Adams operation $\Psi^k\colon K_n(R)_{\mathbb{Q}}\to K_n(R)_{\mathbb{Q}}$ with the eigenvalue k^r for $k=2,3,\dots$.

This theorem is proved by using Popescu's result [6, Corollary 2.7] that says that any equi-characteristic regular local ring R is a directed inductive limit of smooth

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sub-algebras R_α over a field F. Since we may assume that F is perfect $R = \varinjlim R_\alpha$ and $K_n(R) = \varinjlim K_n(R_\alpha)$, we can reduce Theorem 1.1 to the case of a smooth F-algebra R. Then we have to prove that the functor $\mathrm{CH}^r_{\mathrm{Zar}}(-,n)_{\mathbb{Q}}$ commutes with directed inductive limits of algebras, and this is proved by Proposition 2.2.

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2. Motivic cohomology of equi-dimensional cycles

In this section, we always assume that all schemes are regular noetherian and separated. A morphism $p\colon X \to S$ of schemes of finite type is said to be *equi-dimensional of dimension* r, if $\dim p^{-1}(p(x)) = r$ for any $x \in X$ and any irreducible component of X dominates an irreducible component of S. In particular, any equi-dimensional morphism of dimension zero is a quasi-finite morphism and dominates an irreducible component.

Let $\mathcal{Z}_{\mathrm{equi}}(X/S, r)$ be the free abelian group generated by closed integral subschemes of X which are equi-dimensional of dimension r over S. We call $\mathcal{Z}_{\mathrm{equi}}(X/S, r)$ the *equi-dimensional cycle group of dimension* r.

Let X be an S-scheme of finite type. According to [8, Chapter 2, Theorem 3.3.1, Lemma 3.3.6 and Corollary 3.4.3], for any morphism of regular noetherian schemes $f\colon T \to S$, we have a homomorphism $f^*\colon \mathcal{Z}_{\mathrm{equi}}(X/S, r) \to \mathcal{Z}_{\mathrm{equi}}(X \times_S T/T, r)$ and $\mathcal{Z}_{\mathrm{equi}}(X \times_S -/-, r)$ is a contravariant functor for morphisms of regular noetherian schemes. Furthermore, the functor $\mathcal{Z}_{\mathrm{equi}}(X \times_S -/-, r)$ is an étale-sheaf [2, p. 816] on S, hence this is a Zariski-sheaf on S. We define the motivic cohomology $\mathrm{CH}^r_{\mathrm{Zar}}(X, n)$ for finite dimensional regular noetherian schemes X:

Definition 2.1. Let X be a regular noetherian scheme of finite dimension. Write $\Delta^n = \operatorname{Spec} \mathbb{Z}[t_0, \ldots, t_n]/(t_0 + \cdots + t_n - 1)$. Then $X \times \Delta^\bullet$ is a regular, noetherian cosimplicial scheme in the usual sense, and $\mathcal{Z}_{\mathrm{equi}}(- \times \Delta^\bullet \times \mathbb{A}^r/ - \times \Delta^\bullet, 0)$ is a simplicial sheaf on X. We define the motivic cohomology to be the Zariski-hypercohomology:

$$\mathrm{CH}^r_{\mathrm{Zar}}(X,\, n) = \mathbb{H}^{-n}_{\mathrm{Zar}}(X, \mathcal{Z}_{\mathrm{equi}}(- \times \Delta^\bullet \times \mathbb{A}^r/ - \times \Delta^\bullet, 0)).$$

Let $(T_\alpha, f_{\alpha\beta})$ be a directed inverse system of smooth schemes over a regular noetherian scheme S with a directed ordered index set I, where each transition map $f_{\alpha\beta}\colon T_\beta \to T_\alpha$ is affine and dominant ($\beta \geqslant \alpha$). Assume that $T = \varprojlim T_\alpha$ is regular and noetherian. Then we have the following:

Proposition 2.2. *Let X be a scheme of finite type over T and assume that there exists a scheme X_0 of finite type over S such that $X = X_0 \times_S T$. Then the canonical morphism of Zariski sheaves on T*

$$\varinjlim_\alpha (f_\alpha^* \mathcal{Z}_{\mathrm{equi}}(X_\alpha \times_{T_\alpha} -/-, 0)_{\mathbb{Q}}) \to \mathcal{Z}_{\mathrm{equi}}(X \times_T -/-, 0)_{\mathbb{Q}}$$

is an isomorphism, where $f_\alpha\colon X \to X_\alpha = X_0 \times_S T_\alpha$ denotes the canonical morphism

induced by $T \to T_\alpha$ *and* $f_\alpha^* \mathcal{Z}_{\mathrm{equi}}(X_\alpha \times_{T_\alpha} -/-, 0)_{\mathbb{Q}}$ *is the inverse image of the Zariski sheaf* $\mathcal{Z}_{\mathrm{equi}}(X_\alpha \times_{T_\alpha} -/-, 0)_{\mathbb{Q}}$ *on* T_α.

Proof. Let $\mathcal{T}_\alpha$ be the category of Zariski-open subschemes of T_α. Note that the family of inverse images

$$\{f_\alpha^{-1}(U_\alpha) \mid U_\alpha \in \mathcal{T}_\alpha, f_{\alpha\beta}^{-1}(U_\alpha) = U_\beta \text{ for } \beta \geqslant \alpha, \alpha \in I\}$$

is an open basis of $X \times_S T$. We prove that the canonical morphism

$$\varinjlim_{\beta \geqslant \alpha} \mathcal{Z}_{\mathrm{equi}}(X_\beta \times_S U_\beta/U_\beta, 0)_{\mathbb{Q}} \to \mathcal{Z}_{\mathrm{equi}}(X \times_S U/U, 0)_{\mathbb{Q}}$$

is bijective. The injectivity is obvious. We prove its surjectivity. Let $[W] \in \mathcal{Z}_{\mathrm{equi}}(X \times_T U/U, 0)_{\mathbb{Q}}$ be the cycle of an integral scheme $W \subset X \times_T U$. Since $W \to U$ is quasi-finite, there exists an index γ and a closed integral subscheme $W_\gamma \subset X_\gamma \times_{T_\gamma} U_\gamma$ such that $W = W_\gamma \times_{U_\gamma} U$ and each $W \times_{T_\gamma} T_{\gamma'} \to U_\gamma \times_{T_\gamma} T_{\gamma'}$ is quasi-finite for $\gamma' \geqslant \gamma$ by [4, Theorem 8.10.5]. Since $W \to U$ and $U \to U_\gamma$ are dominant, $W_\gamma \to U_\gamma$ is dominant. Hence the cycle $[W_\gamma]$ is in $\mathcal{Z}_{\mathrm{equi}}(X_\gamma/U_\gamma, 0)$. By [8, Chapter 2, Lemma 3.3.6], $f_\gamma[W_\gamma]$ is a formal linear combination of irreducible components of $W = W_\gamma \times_{U_\gamma} U$, and by [8, Chapter 2, Lemma 3.5.9] there exists a positive integer m such that $f_\gamma^*[W_\gamma] = m[W]$. Thus $[W] = f_\gamma^*(m^{-1}[W_\gamma])$. □

3. The proof of main theorem

3.1. The cycle class maps

In this section, we assume that all schemes are noetherian and separated. Let $\mathcal{CP}(X)$ be the category of bounded complexes of big vector bundles on X. Let $\mathcal{F}$ be a family of closed subschemes of X and $\mathcal{CP}^{\mathcal{F}}(X)$ the full subcategory of $\mathcal{CP}(X)$ consisting of complexes acyclic outside of $\bigcup_{W \in \mathcal{F}} W$. We make $\mathcal{CP}^{\mathcal{F}}(X)$ into a Waldhausen category by cofibrations and weak equivalences to be degree-wise split monomorphisms and quasi-isomorphisms, respectively. (See [7] and [9].)

Assume further that $f\colon Y \to X$ is a morphism of schemes and $\mathcal{F}'$ is a family of closed subschemes of Y. The functor f^* takes $\mathcal{CP}^{\mathcal{F}}(X)$ to $\mathcal{CP}^{\mathcal{F}'}(Y)$ provided that $f^{-1}(W) \subset \bigcup_{W' \in \mathcal{F}'} W'$ for all $W \in \mathcal{F}$. Furthermore, for a composition $Z \xrightarrow{g} Y \xrightarrow{f} X$ of morphisms of X-schemes, one has $(g \circ f)^* = f^* \circ g^*$ if f^*, g^* and $(g \circ f)^*$ are defined.

Let S be a regular noetherian scheme. For any regular noetherian schemes X, $S_\bullet \mathcal{CP}^{\mathcal{Q},S}(X)$ denotes the Waldhausen's S-construction (cf. [9]) of $\mathcal{CP}^{\mathcal{Q},S}(X) := \mathcal{CP}^{\mathcal{Q}_X(X \times_{\mathbb{Z}} S)}(X \times S)$ with the family of supports $\mathcal{Q}_X(X \times_{\mathbb{Z}} S)$ consisting of all closed subschemes quasi-finite over X. Further, $K_n^{\mathcal{Q},\mathcal{S}}(X)$ denotes the n-th K-group of $\mathcal{CP}^{\mathcal{Q},S}(X)$.

For any abelian group A, $B_\bullet(A)$ denotes the classifying space of A. For any small category $\mathcal{C}$, $N_\bullet(\mathcal{C})$ denotes the nerve of $\mathcal{C}$. If $S = \mathbb{A}^r$, we define a map $\mathrm{cl}_0^r\colon B_\bullet(K_0^{\mathcal{Q},\mathbb{A}^r}(X)) \to B_\bullet(\mathcal{Z}_{\mathrm{equi}}(X \times \mathbb{A}^r/X, 0))$ of simplicial sets by the formula

$$\mathrm{cl}_0^r(\mathcal{F}) = \sum_{W \subset X \times_k \mathbb{A}^r} \mathrm{length}(\mathcal{F}_W)[W],$$

where the sum is over all closed integral subschemes W of $X \times \mathbb{A}^r$ which are quasi-finite and dominant over a component of X. We consider the composition

$$\mathrm{cl}^r \colon N_\bullet \mathbf{w} S_\bullet \mathcal{CP}^{\mathcal{Q},\mathbb{A}^r}(X) \to B_\bullet(K_0^{\mathcal{Q},\mathbb{A}^r}(X)) \xrightarrow{B_\bullet(\mathrm{cl}_0^r)} B_\bullet(\mathcal{Z}_{\mathrm{equi}}(X \times \mathbb{A}^r/X, 0)),$$

where $\mathbf{w}S.\mathcal{CP}^{\mathcal{Q},\mathbb{A}^r}(X)$ is the subcategory of weak-equivalences in $S.\mathcal{CP}^{\mathcal{Q},\mathbb{A}^r}(X)$, and $\mathbf{w}S.\mathcal{CP}^{\mathcal{Q},\mathbb{A}^r}(X) \to (K_0^{\mathcal{Q},\mathbb{A}^r}(X))^n$ is the canonical map of bisimplicial sets. (See [7, Section 1].)

For any morphism $f \colon Y \to X$ of regular noetherian schemes, $f^* \colon K_0^{\mathcal{Q},\mathbb{A}^r_{\mathbb{Z}}}(X) \to K_0^{\mathcal{Q},\mathbb{A}^r_{\mathbb{Z}}}(Y)$ coincides with the map $\mathcal{F} \mapsto \sum_{i \geqslant 0}(-1)^i \mathbb{L}_i f^*(\mathcal{F})$, where each $\mathbb{L}_i f^*$ is the i-th left derived functor of the inverse image f^*. Using [8, Theorem 3.3.1 and Lemma 3.5.9], we have that the map $B_\bullet(\mathrm{cl}_0^r)$ is functorial for any morphism of regular noetherian schemes by the direct calculation. Hence cl^r is functorial for any regular noetherian schemes. In particular, cl^r commutes with all coface maps and codegeneracy maps of the regular noetherian cosimplicial scheme $X \times \Delta^\bullet$. Thus we obtain the map

$$\begin{aligned}\mathrm{cl}^r \colon N_\bullet \mathbf{w} S_\bullet \mathcal{CP}^{\mathcal{Q},\mathbb{A}^r}(X \times \Delta^\bullet) &\to B_\bullet(K_0^{\mathcal{Q},\mathbb{A}^r}(X \times \Delta^\bullet)) \\ &\to B_\bullet(\mathcal{Z}_{\mathrm{equi}}(X \times \Delta^\bullet \times \mathbb{A}^r/X \times \Delta^\bullet, 0))\end{aligned}$$

called the *cycle-class map*. Here $B_\bullet(A_\bullet)$ is the classifying space of a simplicial abelian group $A_\bullet$, and $B_\bullet(A_\bullet)$ is a bisimplicial set.

3.2. Friedlander–Suslin's spectral sequence

In this section, we consider the case where X is smooth over a field F. Let $K_n^{\mathcal{Q},\mathbb{A}^r}(X \times \Delta^\bullet)$ denote the $n+1$-th homotopy group of the diagonal of a 3-fold simplicial set $N_\bullet \mathbf{w} S_\bullet \mathcal{CP}^{\mathcal{Q},\mathbb{A}^r}(X \times \Delta^\bullet)$. In the case that X is a smooth scheme over a field, Friedlander–Suslin [2] proved that there exists a strongly convergent spectral sequence:

$$E_2^{pq} = \mathrm{CH}^{-q}_{\mathrm{Zar}}(X, -p-q) \Longrightarrow K_{-p-q}(X)$$

by an exact couple $(D_2^{p,q}, E_2^{p,q}, i, j, k)$ defined by the following:

$$D_2^{p,q} = K_{-p-q}^{\mathcal{Q},\mathbb{A}^{-q+1}}(X \times \Delta^\bullet), \quad E_2^{p,q} = \mathrm{CH}^{-q}_{\mathrm{Zar}}(X, -p-q),$$

where j is the cycle-class map. (See [2, Section 13].) We have that Friedlander–Suslin's spectral sequence admits Adams operations:

Proposition 3.1 (cf. [3, Theorem 7])**.** *Let X be a smooth scheme over a field F. Then the spectral sequence*

$$E_2^{pq} = \mathrm{CH}^{-q}_{\mathrm{Zar}}(X, -p-q) \Longrightarrow K_{-p-q}(X)$$

admits Adams operations Ψ^k with the following properties:

1. *The Ψ^k are natural in Sm_F.*
2. *The $\Psi^k \colon K_*^{\mathcal{Q},\mathbb{A}^q}(X \times \Delta^\bullet) \to K_*^{\mathcal{Q},\mathbb{A}^q}(X \times \Delta^\bullet)$ are compatible with the Adams operations Ψ^k on $K_*(X) = K_*^{\mathcal{Q},\mathbb{A}^0}(X)$.*
3. *On the E_2-term $\mathrm{CH}^{-q}_{\mathrm{Zar}}(X, -p-q)$, Ψ^k acts by multiplication by k^{-q}.*

Proof. The proof is similar to [3, Theorem 7]. □

Corollary 3.2. *Let X be a smooth scheme over a field F. The cycle-class map $\mathrm{cl}^r\colon K_n^{\mathcal{Q},\mathbb{A}^r}(X\times\Delta^\bullet)\to \mathrm{CH}^r_{\mathrm{Zar}}(X,n)_{\mathbb{Q}}$ induces an isomorphism*

$$\mathrm{cl}^{(r)}\colon K_n(X)^{(r)}_{\mathbb{Q}}\to \mathrm{CH}^r_{\mathrm{Zar}}(X,\,n)_{\mathbb{Q}}$$

for any $n, r\geqslant 0$.

3.3. The proof of Theorem 1.1

By Popescu's result [6, Corollary 2.7], there exist a prime field F and a directed inductive system $(R_\alpha,\psi_{\beta\alpha})$ of smooth F-algebras of R such that its inductive limit is R. Since each $\psi^\sharp_{\beta\alpha}\colon \operatorname{Spec} R_\beta\to\operatorname{Spec} R_\alpha$ is affine, we have that $\varinjlim \mathrm{CH}^r_{\mathrm{Zar}}(R_\alpha,n)_{\mathbb{Q}}=\mathrm{CH}^r_{\mathrm{Zar}}(R,n)_{\mathbb{Q}}$ follows from [5, Theorem 5.7] and Proposition 2.2. By the functoriality of cycle-class maps and Corollary 3.2, we obtain

$$K_n(R)^{(r)}_{\mathbb{Q}}=\varinjlim K_n(R_\alpha)^{(r)}_{\mathbb{Q}}=\varinjlim \mathrm{CH}^r_{\mathrm{Zar}}(R_\alpha,n)_{\mathbb{Q}}=\mathrm{CH}^r_{\mathrm{Zar}}(R,n)_{\mathbb{Q}}.$$

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Yuki Kato `ykato.math@gmail.com`

Mathematical Institute, Tohoku University, Aoba, Sendai 980-8578 Japan