

POTENTIALS OF HOMOTOPY CYCLIC A_∞ -ALGEBRAS

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Abstract

For a cyclic A -infinity algebra, a potential recording the structure constants can be defined. We define an analogous potential for a homotopy cyclic A -infinity algebra and prove its properties. On the other hand, we find another different potential for a homotopy cyclic A -infinity algebra, which is related to the algebraic analogue of generalized holonomy map of Abbaspour, Tradler and Zeinalian.

1. Introduction

We first recall the definition of cyclic inner products due to Kontsevich [Ko], which may be understood as constant invariant symplectic structures in non-commutative geometry.

Definition 1.1. An A_∞ -algebra $(A, \{m_*\})$ is said to have a *cyclic inner product* if there exists a skew symmetric non-degenerate, bilinear map

$$\langle, \rangle: A \otimes A \rightarrow \mathbf{k}$$

such that for all integers $k \geq 1$,

$$\langle m_k(x_1, \dots, x_k), x_{k+1} \rangle = (-1)^{K(\vec{x})} \langle m_k(x_2, \dots, x_{k+1}), x_1 \rangle. \quad (1)$$

Here, $(-1)^{K(\vec{x})}$ denotes the sign given by Koszul sign convention. Namely,

$$(-1)^{K(\vec{x})} = (-1)^{|x_1|'(|x_2|'+\dots+|x_{k+1}|')}, \quad (2)$$

where $|x|'$ is the shifted degree of x .

This notion for the A_∞ -algebras and A_∞ -categories is crucial in homological mirror symmetry, for example, as in the work of Kontsevich-Soibelman [KS] or of Costello [Cos]. In particular, Costello has proved in [Cos] that the category of open topological conformal field theory is homotopy equivalent to the category of Calabi-Yau categories, where the Calabi-Yau category is a categorical generalization of a cyclic A_∞ -algebra.

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The first application of this gadget is to define a potential for a cyclic A_∞ -algebra, which in physics, is called an action of a string field theory: Let (A, m_\star^A) be a cyclic A_∞ -algebra. Let e_i be generators of A as a vector space, which is assumed to be finite dimensional. Define $\mathbf{x} = \sum_i e_i x_i$ where x_i are formal parameters with $\deg(x_i) = -\deg(e_i)$.

Definition 1.2. Define

$$\Phi^A(\mathbf{x}) = \sum_{k=1}^{\infty} \frac{1}{k+1} \langle m_k^A(\mathbf{x}, \mathbf{x}, \dots, \mathbf{x}), \mathbf{x} \rangle. \quad (3)$$

This may be considered as a systematic way of gathering structure constants of a cyclic A_∞ -algebra. In the case of toric manifolds, this potential, when restricted to the Maurer-Cartan elements, becomes the Landau-Ginzburg superpotential of the mirror B-model (see [CO, FOOO1]).

The notion of cyclicity is not a homotopy invariant notion. For example, an A_∞ -algebra which is homotopy equivalent to a cyclic A_∞ -algebra may not be cyclic. Instead, it has a strong homotopy inner product, which was defined by the first author in [C]: for this, a cyclic inner product on A may be understood as a special kind of A_∞ -bimodule map $A \rightarrow A^*$, where A^* is the linear dual of A , which is an A -bimodule (see, for example, Lemma 3.1 [C]). An A_∞ -bimodule quasi-isomorphism $A \rightarrow A^*$ is called as an infinity inner product by Tradler (see [T, TZ] for example).

Definition 1.3 ([C, Definition 3.6]). Let A be an A_∞ -algebra. We call an A_∞ -bimodule map $\phi: A \rightarrow A^*$ a *strong homotopy inner product in the sense of [C]* if there exists a cyclic A_∞ -algebra B with $\psi: B \rightarrow B^*$ and an A_∞ -quasi-isomorphism $f: A \rightarrow B$ such that the following diagram of A_∞ -bimodules over A commutes:

$$\begin{array}{ccc} A & \xrightarrow{\quad g = \tilde{f} \quad} & B \\ \phi \downarrow & & \psi \downarrow \text{cyc} \\ A^* & \xleftarrow{\quad g^* \quad} & B^* \end{array} \quad (4)$$

Here, by $g: A \rightarrow B$, we denote the induced A_∞ -bimodule map $\tilde{f} = g$ where B is considered as an A_∞ -bimodule over A .

In this paper, we give a definition of the potential for strong homotopy inner products and prove its properties. It turns out that the definition of the potential in Definition 3.1 is very similar to that of (3):

$$\begin{aligned} \Phi^A(\mathbf{x}) &= \sum_{N=1}^{\infty} \Phi_N^A(\mathbf{x}) \\ &:= \sum_{N=1}^{\infty} \sum_{p+q+k=N} \frac{1}{N+1} \langle \mathbf{x}, \mathbf{x}, \dots, \mathbf{x}, \underline{m_k^A(\mathbf{x}, \mathbf{x}, \dots, \mathbf{x})}, \mathbf{x}, \dots, \mathbf{x} \mid \mathbf{x} \rangle_{p,q}, \end{aligned}$$

but the proof that they are indeed related is non-trivial and involves quite combinatorial arguments. Beyond the fact that it is quite natural to work with homotopy notions when dealing with homotopy algebras, sometimes it is necessary to work directly with homotopy notions. For example, in the work of Kontsevich and

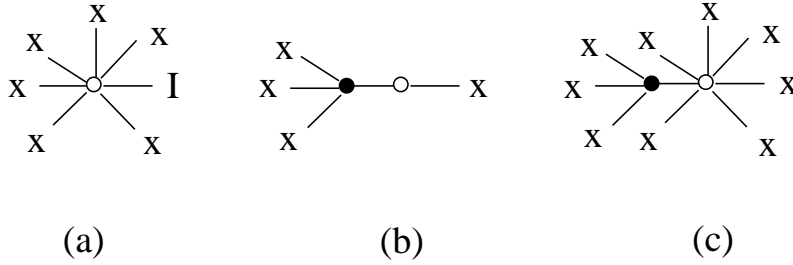


Figure 1: (a) Potential Ψ . (b) Cyclic Potential Φ . (c) Homotopy cyclic potential Φ .

Soibelman [KS], they find a relation between cyclic cohomology of an A_∞ -algebra A and cyclic symmetry. Given a cyclic cohomology class, one first obtains a homotopy inner product on A and then a cyclic inner product in the minimal model. We refer readers to [CL] for the explicit formulas of this correspondence in terms of negative cyclic cohomology $HC_-^\bullet(A)$ and strong homotopy inner products.

Now, let us assume that the A_∞ -algebra is unital (see Definition 4.1), and assume that the A_∞ -bimodule maps are also unital. Then, from the strong homotopy inner products $\{\langle \cdot, \cdot \rangle_{p,q}\}$, we can define another potential as follows: (Here, $\langle \cdot, \cdot \rangle_{p,q}$ is obtained from the (p, q) -component of the bimodule map ϕ ; see (6).)

Definition 1.4. Define

$$\Psi^A(\mathbf{x}) = \sum_{p,q \geq 0} \frac{1}{p+q+1} \langle \underbrace{\mathbf{x}, \dots, \mathbf{x}}_p, \underbrace{\mathbf{x}, \dots, \mathbf{x}}_q \mid I \rangle_{p,q}.$$

We prove that this potential is, in fact, *invariant under the gauge equivalence* for Maurer-Cartan elements. We also find its relation to the work of Abbaspour, Tradler and Zeinalian [ATZ], where this map corresponds to the algebraic analogue of the generalized holonomy map from the negative cyclic cohomology to the function ring of Maurer-Cartan elements. The following Figure 1 explains the differences of the expressions used in these potentials (without the coefficients). In the figure, the circle represents the strong homotopy inner product (following that of Tradler [T]) whose horizontal arrows are for the inputs from modules. The filled circle represents the A_∞ -operation m .

This paper may be considered as a continuation of the paper [C] to which we refer readers for the notations and further introductions, especially about the signs. Throughout the paper we assume that $H^\bullet(A)$ is finite dimensional.

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2. Strong homotopy inner products

We begin by proposing a modified definition of strong homotopy inner products and discuss their equivalences and pullbacks.

We first make an observation that there exist certain subtleties in the direction of arrows in the diagram (4) in the definition of strong homotopy inner products. One could try to make the definition with the arrow $A \leftarrow B$ instead of $A \rightarrow B$, but the resulting diagram would become weaker as there may exist elements of A which are not covered by the image of the map $A \leftarrow B$ in general. The subtlety actually disappears if we have non-degeneracy in the chain level. The correct definition (which corresponds to exactly the non-commutative invariant symplectic two form) is rather in between these two definitions: to make the correct definition, we first recall the following characterization theorem of strong homotopy inner products in the sense of [C].

Theorem 2.1 ([C, Theorem 5.1]). *An A_∞ -algebra A has a strong homotopy inner product in the sense of [C] if and only if there exist an A_∞ -bimodule map $\phi: A \rightarrow A^*$, satisfying the following three conditions:*

1. (Skew symmetry). For any $a_i, v, b_j, w \in A$,

$$\phi_{k,l}(\vec{a}, v, \vec{b})(w) = -(-1)^K \phi_{l,k}(\vec{b}, w, \vec{a})(v),$$

$$\text{with } |K| = (\sum_{i=1}^k |a_i|' + |v|') (\sum_{j=1}^l |b_j|' + |w|')$$

2. (Closedness). For any choice of a family (a_1, \dots, a_{l+1}) and any choice of indices $1 \leq i < j < k \leq l+1$, we have

$$\begin{aligned} (-1)^{K_i} \phi(\dots, \underline{a_i}, \dots)(a_j) + (-1)^{K_j} \phi(\dots, \underline{a_j}, \dots)(a_k) + (-1)^{K_k} \\ \times \phi(\dots, \underline{a_k}, \dots)(a_i) = 0, \end{aligned}$$

where the arguments inside ϕ are uniquely given by the cyclic order of the family (a_1, \dots, a_{l+1}) , and the signs K_* are given by the Koszul convention

$$K_* = (|a_1|' + \dots + |a_*|')(|a_{*+1}|' + \dots + |a_k|').$$

3. (Homological non-degeneracy). For any non-zero $[a] \in H^\bullet(A)$ with $a \in A$, there exists a $[b] \in H^\bullet(A)$ with $b \in A$, such that $\phi_{0,0}(a)(b) \neq 0$.

For non-degeneracy on the chain level, ϕ itself gives the strong homotopy inner product in the sense of [C], otherwise the inner product obtained $\phi': A \rightarrow A^*$ is only equivalent to ϕ .

The second condition is called a *closed condition* since it is equivalent to the closed condition of the related non-commutative symplectic 2-form, and this plays a crucial role in proving the properties of the potential defined in this paper.

We also remark that in the proof of Theorem 2.1, ϕ satisfying the three conditions does not always become exactly a strong homotopy inner product in the sense of [C] as itself, but is only equivalent to a strong homotopy inner product in the sense of [C] (the equivalence is defined below).

Hence, we propose to define the strong homotopy inner products by Theorem 2.1 because such a definition is equivalent to that of the non-commutative symplectic form as explained in [C].

Definition 2.2. Let A be an A_∞ -algebra. We call an A_∞ -bimodule map $\phi: A \rightarrow A^*$ a *strong homotopy inner product* if it is skew-symmetric, closed and homologically non-degenerate as in Theorem 2.1. And A is called *homotopy cyclic A_∞ -algebra*, if there exists a strong homotopy inner product of A .

Then, the main result of [C] can be phrased as the following theorem:

Theorem 2.3. *Let $\phi: A \rightarrow A^*$ be an A_∞ -bimodule map.*

1. *If ϕ is a strong homotopy inner product in the sense of Definition 2.2, then there exists an A_∞ -algebra B with a cyclic inner product $\psi: B \rightarrow B^*$ and an A_∞ -quasi-isomorphism $\iota: B \rightarrow A$ satisfying the following commutative diagram of A_∞ -bimodule homomorphisms:*

$$\begin{array}{ccc}
 A & \xleftarrow{\tilde{\iota}} & B \\
 \phi \downarrow & & \psi \downarrow \text{cyc} \\
 A^* & \xrightarrow{\tilde{\iota}^*} & B^*
 \end{array} \tag{5}$$

2. *If there exists a cyclic A_∞ -algebra B with $\psi: B \rightarrow B^*$ and an A_∞ -quasi-isomorphism $f: A \rightarrow B$ such that the following diagram of A_∞ -bimodules over A commutes:*

$$\begin{array}{ccc}
 A & \xrightarrow{g=\tilde{f}} & B \\
 \phi \downarrow & & \psi \downarrow \text{cyc} \\
 A^* & \xleftarrow{g^*} & B^*
 \end{array}$$

then ϕ is a strong homotopy inner product in the sense of Definition 2.2.

If $\phi_{0,0}$ is non-degenerate on the chain level, then the definition of ‘strong homotopy in the sense of [C]’ and the new definition of strong homotopy inner product (in the sense of Definition 2.2) are equivalent.

Remark 2.4. Hence the new definition of the strong homotopy inner product is a little stronger than the diagram using $A \leftarrow B$, a little weaker than the diagram using $A \rightarrow B$ and equivalent to the non-commutative symplectic two form.

Proof. If $\phi_{0,0}$ is non-degenerate on the chain level, then one can find B with an A_∞ -isomorphism $f: A \rightarrow B$ from the proof of Theorem 2.1 making the commuting diagram (4). Hence one can find the exact inverse of f to make the commuting diagram (5).

Also, statement 2. can be checked without much difficulty from the commuting diagram, so we only consider statement 1. We explain that the proof of Theorem 2.1, given in [C], is enough to prove the existence of the diagram (5): We recall from [C] that the first step of the construction of the cyclic A_∞ -algebra B , when A is only homologically non-degenerate, is to consider the minimal model $\iota: H^\bullet(A) \rightarrow A$ and

consider the pullback $\iota^*\phi$,

$$\begin{array}{ccccc} A & \xleftarrow{\tilde{\iota}} & H^\bullet(A) & \xrightarrow{\tilde{f}} & H^\bullet(A) \\ \phi \downarrow & & \iota^*\phi \downarrow & & \text{cyc} \downarrow \\ A^* & \xrightarrow{\tilde{\iota}^*} & (H^\bullet(A))^* & \xleftarrow{\tilde{f}^*} & (H^\bullet(A))^* \end{array}$$

Then $\iota^*\phi$ is non-degenerate and skew symmetric and closed, and one proves the theorem for $\iota^*\phi$ to find $f: H^\bullet(A) \rightarrow H^\bullet(A)$ with the above commutative diagram. As the quasi-isomorphism f on $H^\bullet(A)$ is, in fact, an isomorphism, hence there exists the explicit inverse f^{-1} , and we obtain the diagram (5). \square

We can also prove the following corollary:

Corollary 2.5. *Let $\phi: A \rightarrow A^*$ be a strong homotopy inner product. Suppose we have an A_∞ -quasi-isomorphism $f: A \rightarrow H^\bullet(A)$ with the commuting diagram*

$$\begin{array}{ccc} A & \xrightarrow{g=\tilde{f}} & H^\bullet(A) \\ \phi \downarrow & & \psi \downarrow \text{cyc} \\ A^* & \xleftarrow{\tilde{g}^*} & H^\bullet(A)^* \end{array}$$

Then, there exists an A_∞ -quasi-isomorphism $h: H^\bullet(A) \rightarrow A$ with the commuting diagram (with the same ψ as the above)

$$\begin{array}{ccc} A & \xleftarrow{\tilde{h}} & H^\bullet(A) \\ \phi \downarrow & & \psi \downarrow \text{cyc} \\ A^* & \xrightarrow{\tilde{h}^*} & (H^\bullet(A))^* \end{array}$$

Proof. By the decomposition theorem of A_∞ -algebras, the map f has a right inverse A_∞ -quasi-homomorphism, say $h: H^\bullet(A) \rightarrow A$ such that $f \circ h = id$. To see this, consider an A_∞ -isomorphism η ,

$$\eta: A \rightarrow A^{dc} := A^H \oplus A^{lc}$$

to the direct sum of the minimal A_∞ -algebra A^H and the linear contractible A^{lc} .

Let $\pi: A^{dc} \rightarrow A^H$ be the projection and $i: A^H \rightarrow A^{dc}$ be the inclusion where both are A_∞ -quasi-isomorphisms with $\pi \circ i = id$. As f is an A_∞ -quasi-isomorphism, $f \circ \eta^{-1} \circ i: A^H \rightarrow H^\bullet(A)$ is an A_∞ -isomorphism and hence has an A_∞ -inverse say ξ . Then, we define the right A_∞ inverse $h = \eta^{-1} \circ i \circ \xi$. The property $f \circ h = id$ can be checked immediately. The second diagram then follows from the first commuting diagram. \square

Now, we define equivalences between strong homotopy inner products.

Definition 2.6. Let $\phi: A \rightarrow A^*$ and $\psi: B \rightarrow B^*$ be strong homotopy inner products. They are called *equivalent* if there exists a cyclic symmetric A_∞ -algebra H with a

commutative diagram

$$\begin{array}{ccccc} A & \xleftarrow{qis} & H & \xrightarrow{qis} & B \\ \phi \downarrow & & \downarrow cyc & & \downarrow \psi \\ A^* & \longrightarrow & H^* & \longleftarrow & B^*. \end{array}$$

One can actually choose H to be a minimal (or canonical) model.

Given a strong homotopy inner product $\phi: B \rightarrow B^*$ and an A_∞ -quasi-isomorphism $f: A \rightarrow B$, we may define a pullback $f^*\phi: A \rightarrow A^*$

$$\begin{array}{ccc} A & \xrightarrow{\quad} & B \\ f^*\phi \downarrow & \tilde{f} & \downarrow \phi \\ A^* & \xleftarrow{\quad} & B^* \end{array}$$

as a composition: $f^*\phi = \tilde{f}^* \circ \hat{\phi} \circ \tilde{f}$ where $\hat{\phi}$ and \tilde{f} denote the extensions to higher tensor powers; see [C, Section 3].

Proposition 2.7. *$f^*\phi$ defines a strong homotopy inner product on A which is equivalent to ϕ .*

Proof. Since $\phi: B \rightarrow B^*$ is skew-symmetric and closed, so is $f^*\phi$ by Lemma 5.6 of [C]. It is not hard to check that $f^*\phi$ is also homologically non-degenerate as f is a quasi-isomorphism. Hence, by Definition 2.2, $F^8\phi$ is a strong homotopy inner product. Hence there exist an A_∞ -algebra C which is cyclic symmetric ($\psi: C \rightarrow C^*$) and an A_∞ -quasi-homomorphism $h: C \rightarrow A$ with the following commutative diagrams:

$$\begin{array}{ccccc} C & \xrightarrow{\quad} & A & \xrightarrow{\quad} & B \\ \psi \downarrow & \tilde{h} & \downarrow f^*\phi & \tilde{f} & \downarrow \phi \\ C^* & \xleftarrow{\quad} & A^* & \xleftarrow{\quad} & (B)^* \end{array}$$

From the diagram, it is easy to see that ϕ and $f^*\phi$ are equivalent in the sense of Definition 2.6. \square

3. Potentials

In this section we define a potential of a homotopy cyclic A_∞ -algebra and prove its properties. Let (A, m_*^A) be given a strong homotopy inner product $\phi: A \rightarrow A^*$. Recall that an A_∞ -bimodule map ϕ is given by a family of maps

$$\phi_{p,q}: \underline{A}^{\otimes p} \otimes \underline{A} \otimes A^{\otimes q} \rightarrow A^*,$$

where the underlined A is to emphasize that it is an A -bimodule for the readers' convenience. Let

$$\langle x_1, \dots, x_p, \underline{v}, y_1, \dots, y_q \mid w \rangle_{p,q} := \phi(x_1, \dots, x_p, \underline{v}, y_1, \dots, y_q)(w). \quad (6)$$

As in the cyclic case, let e_i be generators of A as a vector space, which is assumed to be finite dimensional. (One may use the pullback defined in the previous section

using the inclusion $\iota: H^\bullet(A) \rightarrow A$ in the case that $H^\bullet(A)$ is finite dimensional.) Define $\mathbf{x} = \sum_i e_i x_i$, where x_i are formal parameters with $\deg(x_i) = -\deg(e_i)$. Now we give a definition of a potential for strong homotopy inner products.

Definition 3.1. The *potential* of an A_∞ -algebra (A, m_*^A) with a strong homotopy inner product $\phi: A \rightarrow A^*$ is defined as

$$\begin{aligned} \Phi^A(\mathbf{x}) &= \sum_{N=1}^{\infty} \Phi_N^A(\mathbf{x}) \\ &:= \sum_{N=1}^{\infty} \sum_{p+q+k=N} \frac{1}{N+1} \langle \mathbf{x}, \mathbf{x}, \dots, \mathbf{x}, \underline{m_k^A(\mathbf{x}, \mathbf{x}, \dots, \mathbf{x})}, \mathbf{x}, \dots, \mathbf{x} \mid \mathbf{x} \rangle_{p,q}. \end{aligned}$$

The definition itself is somewhat similar to that of cyclic case (3). But in (3), the fraction $1/k$ was to cancel out repetitive contributions to the potential due to cyclic symmetry (1), whereas in the strong homotopy case, such cyclic symmetry of the rotation of arguments do not exist. Namely, in general,

$$\begin{aligned} &\langle e_1, \dots, \underline{m_i(e_j, \dots, e_{j+i-1})}, \dots, e_k \mid e_{k+1} \rangle \\ &\quad \neq \pm \langle e_2, \dots, \underline{m_i(e_{j+1}, \dots, e_{j+i})}, \dots, e_{k+1} \mid e_1 \rangle. \\ &\langle e_1, \dots, \underline{m_i(e_j, \dots, e_{j+i-1})}, \dots, e_k \mid e_{k+1} \rangle \\ &\quad \neq \pm \langle e_2, \dots, \underline{m_i(e_{j+1}, \dots, e_{j+i})}, \dots, e_{k+1} \mid e_1 \rangle. \end{aligned}$$

We later show that the combination of the A_∞ -bimodule equation, skew-symmetry and the closed condition will compensate the absence of the strict cyclic symmetry.

We explain how the potential behaves under pullbacks, and this will show the relation between the potentials of equivalent strong homotopy inner products. For an A_∞ -quasi-isomorphism $h: B \rightarrow A$, the pullback of a potential is defined as follows: We assume B is finite dimensional as a vector space and denote by $\{f_i\}$ its basis and introduce corresponding formal variables y_i as before. Suppose

$$h_k(f_{j_1}, \dots, f_{j_k}) = h_{j_1, \dots, j_k}^i e_i, \quad h_{j_1, \dots, j_k}^i \in \mathbf{k}.$$

Then, we set

$$x_i \mapsto h_{j_{11}}^i y_{j_{11}} + h_{j_{21}, j_{22}}^i y_{j_{21}} y_{j_{22}} + \dots + h_{j_{11}, \dots, j_{1k}}^i y_{j_{11}} \cdots y_{j_{1k}} + \dots.$$

Then, one defines the pullback $h^* \Phi^A$ by using the above change of coordinate formula. Namely, $h^* \Phi^A$ is given by the replacement of \mathbf{x} by $\sum_{k \geq 1} h_k(\mathbf{y}^{\otimes k})$ in the formula of Φ^A .

Theorem 3.2. Let $\phi: A \rightarrow A^*$ be a strong homotopy inner product. Let B be a cyclic A_∞ -algebra with a quasi-isomorphism $h: B \rightarrow A$ providing the commutative diagram (5). Then, we have

$$\Phi^B = h^* \Phi^A.$$

Proof. The overall scheme of the proof, which is first to differentiate and then to compare, follows that of [C] (idea due to Kajiuira [Kaj] in the unfiltered case). The

main difficulty, and the essential part of the proof, is the first step where we take (formal) partial derivatives on each side. The following lemma shows that after partial differentiation, the fraction on each summand disappears.

Lemma 3.3.

$$\begin{aligned} \frac{\partial}{\partial x_i} \Phi_N^A(\mathbf{x}) &= \frac{\partial}{\partial x_i} \sum_{p+q+k=N} \frac{1}{N+1} \langle \mathbf{x}, \mathbf{x}, \dots, \mathbf{x}, \underline{m_k^A(\mathbf{x}, \mathbf{x}, \dots, \mathbf{x})}, \mathbf{x}, \dots, \mathbf{x} \mid \mathbf{x} \rangle_{p,q} \\ &= \sum_{p+q+k=N} \langle \mathbf{x}, \mathbf{x}, \dots, \mathbf{x}, \underline{m_k^A(\mathbf{x}, \mathbf{x}, \dots, \mathbf{x})}, \mathbf{x}, \dots, \mathbf{x} \mid e_i \rangle_{p,q}. \end{aligned}$$

We assume the lemma for a moment and show the proof of the theorem using the lemma. Let $\{f_i\}$ be a basis of $H^\bullet(A)$, and let $\{y_i\}$ be the corresponding formal variables for $\{f_i\}$, namely $\mathbf{y} := \sum_i y_i f_i$.

We let $h^{sum}(\mathbf{y}) := \sum_{k \geq 1} h_k(\mathbf{y}^{\otimes k})$. Then

$$\frac{\partial}{\partial y_i} \Phi^{H^\bullet(A)} = \sum_{k \geq 1} \left\langle m_k^{H^\bullet(A)}(\mathbf{y}, \dots, \mathbf{y}), f_i \right\rangle$$

by cyclic symmetry, and

$$\begin{aligned} &\frac{\partial}{\partial y_i} h^* \Phi^A \\ &= \frac{\partial}{\partial y_i} \sum_{\substack{k \geq 1 \\ p+q+k=N}} \frac{1}{N+1} \langle h^{sum}(\mathbf{y})^{\otimes p}, \underline{m_k^A(h^{sum}(\mathbf{y}), \dots, h^{sum}(\mathbf{y}))}, h^{sum}(\mathbf{y})^{\otimes q} \mid h^{sum}(\mathbf{y}) \rangle \\ &= \sum_{\substack{N \geq 1 \\ p+q+k=N}} \langle h^{sum}(\mathbf{y})^{\otimes p}, \underline{m_k^A(h^{sum}(\mathbf{y}), \dots, h^{sum}(\mathbf{y}))}, h^{sum}(\mathbf{y})^{\otimes q} \mid \frac{\partial}{\partial y_i} h^{sum}(\mathbf{y}) \rangle \end{aligned}$$

by the above lemma. From the diagram (5), we have $\psi = \tilde{h}^* \circ \hat{\phi} \circ \hat{h}$, where all maps are $H^\bullet(A)$ -bimodule homomorphisms, consider the following:

$$\begin{aligned} &\sum_{\substack{p,q \geq 0 \\ k \geq 1}} \psi(\mathbf{y}^{\otimes p}, \underline{m_k^{H^\bullet(A)}(\vec{\mathbf{y}})}, \mathbf{y}^{\otimes q})(f_i) \\ &= \sum_{\substack{p,q \geq 0 \\ k \geq 1}} (\tilde{h}^* \circ \hat{\phi} \circ \hat{h})(\mathbf{y}^{\otimes p}, \underline{m_k^{H^\bullet(A)}(\vec{\mathbf{y}})}, \mathbf{y}^{\otimes q})(f_i) \\ &= \sum_{\substack{p,q \geq 0 \\ k \geq 1}} \sum_{\substack{p_1+p_2+p_3=p \\ q_1+q_2+q_3=q}} \tilde{h}^*(\mathbf{y}^{\otimes p_3}, \phi(\hat{h}(\mathbf{y}^{\otimes p_2}), h_{p_1+q_1+1}(\mathbf{y}^{\otimes p_1}, \underline{m_k^{H^\bullet(A)}(\vec{\mathbf{y}})}, \mathbf{y}^{\otimes q_1}), \hat{h}(\mathbf{y}^{\otimes q_2})), \mathbf{y}^{\otimes q_3})(f_i) \\ &= \sum_{\substack{p,q \geq 0 \\ k \geq 1}} \sum_{\substack{p_1+p_2+p_3=p \\ q_1+q_2+q_3=q}} \phi(\hat{h}(\mathbf{y}^{\otimes p_2}), h_{p_1+q_1+1}(\mathbf{y}^{\otimes p_1}, \underline{m_k^{H^\bullet(A)}(\vec{\mathbf{y}})}, \mathbf{y}^{\otimes q_1}), \hat{h}(\mathbf{y}^{\otimes q_2})) (h_{p_3+q_3+1}(\mathbf{y}^{\otimes q_3}, \underline{f_i}, \mathbf{y}^{\otimes p_3})) \\ &= \sum_{\substack{p,q \geq 0 \\ k \geq 1}} \sum_{\substack{p_1+p_2+p_3=p \\ q_1+q_2+q_3=q}} \langle \hat{h}(\mathbf{y}^{\otimes p_2}), h_{p_1+q_1+1}(\mathbf{y}^{\otimes p_1}, \underline{m_k^{H^\bullet(A)}(\vec{\mathbf{y}})}, \mathbf{y}^{\otimes q_1}), \hat{h}(\mathbf{y}^{\otimes q_2}) \mid h_{p_3+q_3+1}(\mathbf{y}^{\otimes q_3}, \underline{f_i}, \mathbf{y}^{\otimes p_3}) \rangle \end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{N \geq 1 \\ p+q+k=N}} \langle h^{sum}(\mathbf{y})^{\otimes p}, \widehat{m}_k^A(h^{sum}(\mathbf{y}), \dots, h^{sum}(\mathbf{y})), h^{sum}(\mathbf{y})^{\otimes q} \mid \frac{\partial}{\partial y_i} h^{sum}(\mathbf{y}) \rangle \\
&= \frac{\partial}{\partial y_i} h^* \Phi^A.
\end{aligned} \tag{7}$$

Here, we denote by $m_k(\vec{\mathbf{y}})$ the expression $m_k(\mathbf{y}, \dots, \mathbf{y})$ for simplicity. The last identity holds because the sum is over all $p_1 + p_2 + p_3 = p$ and $q_1 + q_2 + q_3 = q$ where p and q run over all non-negative integers, and there is the A_∞ -bimodule relation $\widehat{m}^A \circ \widehat{h} = \widehat{h} \circ \widehat{m}^{H^\bullet(A)}$. We also use the fact that

$$\frac{\partial}{\partial y_i} h_k(\mathbf{y}^{\otimes k}) = \sum_{p_3+q_3+1=k} h_{p_3+q_3+1}(\mathbf{y}^{\otimes p_3}, f_i, \mathbf{y}^{\otimes q_3}).$$

The summands of (7) are all zero except for $(p, q) = (0, 0)$ because ψ is a cyclic symmetric inner product. Hence,

$$\frac{\partial}{\partial y_i} h^* \Phi^A = \sum_{k \geq 1} \psi(m_k^{H^\bullet(A)}(\vec{\mathbf{y}}))(f_i) = \sum_{k \geq 1} \langle m_k^{H^\bullet(A)}(\mathbf{y}, \dots, \mathbf{y}), f_i \rangle = \frac{\partial}{\partial y_i} \Phi^{H^\bullet(A)}.$$

This proves the theorem. \square

Proof. We prove Lemma 3.3. Before we proceed, we give some remarks on the signs. The sign convention used in this paper and in [C] is the Koszul convention after the degree one shift. For simplicity, we omit the Koszul sign factor, and the expression will appear with $+$ if it agrees with the Koszul sign rule and $-$ if it is the negative of the Koszul sign. We illustrate this for two examples, from which the general convention can be easily understood. The first example is the A_∞ -equation with two inputs. We write

$$m_1 m_2(x_1, x_2) + m_2(m_1(x_1), x_2) + m_2(x_1, m_1(x_2)) = 0, \tag{8}$$

whereas the actual equation is

$$m_1 m_2(x_1, x_2) + m_2(m_1(x_1), x_2) + (-1)^{|x_1|'} m_2(x_1, m_1(x_2)) = 0.$$

Equation (8) will also be written as

$$m_1 m_2(x_1, x_2) = -m_2(m_1(x_1), x_2) - m_2(x_1, m_1(x_2)).$$

The second example is the equation for $\langle m_2(\underline{x}_1, x_2) \mid x_3 \rangle$. Note that ϕ being an A_∞ -bimodule map $\phi: A \rightarrow A^*$ with the induced A_∞ -bimodule structure on A^* (see expression (3.3) [C] for the precise definition) implies the following actual equation:

$$\begin{aligned}
&\langle m_2(\underline{x}_1, x_2) \mid x_3 \rangle + \langle m_1(\underline{x}_1), x_2 \mid x_3 \rangle \\
&\quad + (-1)^{|x_1|'} \langle \underline{x}_1, m_1(x_2) \mid x_3 \rangle + (-1)^{|x_1|'+|x_2|'} \langle \underline{x}_1, x_2 \mid m_1(x_3) \rangle \\
&\quad + (-1)^{|x_1|'} \langle \underline{x}_1 \mid m_2(x_2, x_3) \rangle = 0.
\end{aligned}$$

In this paper, the above equation will be written simply as

$$\begin{aligned}
&\langle m_2(\underline{x}_1, x_2) \mid x_3 \rangle + \langle m_1(\underline{x}_1), x_2 \mid x_3 \rangle + \langle \underline{x}_1, m_1(x_2) \mid x_3 \rangle \\
&\quad + \langle \underline{x}_1, x_2 \mid m_1(x_3) \rangle + \langle \underline{x}_1 \mid m_2(x_2, x_3) \rangle = 0.
\end{aligned}$$

Now, we begin the proof of the lemma. From now on, we replace m_k^A by m_k if there is no ambiguity. By taking a derivative, the expression becomes

$$\frac{\partial}{\partial x_i} \sum_{p+q+k=N} \langle \mathbf{x}, \mathbf{x}, \dots, \mathbf{x}, \underline{m_k(\mathbf{x}, \mathbf{x}, \dots, \mathbf{x})}, \mathbf{x}, \dots, \mathbf{x} \mid \mathbf{x} \rangle_{p,q} \quad (9)$$

$$= \sum_{\substack{p+q+k=N \\ r+s=k-1}} \langle \mathbf{x}, \dots, \mathbf{x}, \underline{m_k(\overbrace{\mathbf{x}, \dots, \mathbf{x}}^r, e_i, \overbrace{\mathbf{x}, \dots, \mathbf{x}}^s)}, \mathbf{x}, \dots, \mathbf{x} \mid \mathbf{x} \rangle_{p,q} \quad (10)$$

$$+ \sum_{\substack{p+q+k=N \\ r+s=p-1}} \langle \underbrace{\mathbf{x}, \dots, \mathbf{x}}_r, e_i, \underbrace{\mathbf{x}, \dots, \mathbf{x}}_s, \underline{m_k(\mathbf{x}, \dots, \mathbf{x})}, \mathbf{x}, \dots, \mathbf{x} \mid \mathbf{x} \rangle_{p,q} \quad (11)$$

$$+ \sum_{\substack{p+q+k=N \\ r+s=q-1}} \langle \mathbf{x}, \dots, \mathbf{x}, \underline{m_k(\mathbf{x}, \dots, \mathbf{x})}, \underbrace{\mathbf{x}, \dots, \mathbf{x}}_r, e_i, \underbrace{\mathbf{x}, \dots, \mathbf{x}}_s \mid \mathbf{x} \rangle_{p,q} \quad (12)$$

$$+ \sum_{p+q+k=N} \langle \mathbf{x}, \dots, \mathbf{x}, \underline{m_k(\mathbf{x}, \dots, \mathbf{x})}, \mathbf{x}, \dots, \mathbf{x} \mid e_i \rangle_{p,q}. \quad (13)$$

Now, the lemma can be proved by the following lemma. \square

Lemma 3.4. *The sum of the terms in (10), (11) and (12) equals to N times of the expression (13).*

Proof. To prove the lemma, we recall the A_∞ -bimodule equation. The equation for A_∞ -bimodule homomorphism $A \rightarrow A^*$ is

$$\phi \circ \widehat{b}_A = b_{A^*} \circ \widehat{\phi} \quad (14)$$

with $b_A = m^A$, when A is considered to be an A_∞ -bimodule, and b_{A^*} is defined by canonical construction of the dual of the A_∞ -bimodule A . Here $\widehat{\phi}$ is the coalgebra homomorphism induced from ϕ . (We refer readers to [C, T] or [GJ] for details.) Let us restrict equation (14) to the case $(\mathbf{x}, \dots, \mathbf{x}, e_i, \mathbf{x}, \dots, \mathbf{x}) \in A^{\otimes n} \otimes \underline{A} \otimes A^{\otimes m}$ where $n + m + 1 = N$. Then it becomes

$$\sum_{\substack{p+j_1=n \\ j_2+q=m}} \langle \mathbf{x}, \dots, \mathbf{x}, \underline{m_{j_1+j_2+1}(\overbrace{\mathbf{x}, \dots, \mathbf{x}}^{j_1}, \underline{e_i}, \overbrace{\mathbf{x}, \dots, \mathbf{x}}^{j_2})}, \mathbf{x}, \dots, \mathbf{x} \mid \mathbf{x} \rangle_{p,q} \quad (15)$$

$$+ \sum_{\substack{k_1+k_2+j=n \\ p=k_1+k_2+1}} \langle \underbrace{\mathbf{x}, \dots, \mathbf{x}}_{k_1}, m_j(\mathbf{x}, \dots, \mathbf{x}), \underbrace{\mathbf{x}, \dots, \mathbf{x}}_{k_2}, e_i, \mathbf{x}, \dots, \mathbf{x} \mid \mathbf{x} \rangle_{p,m}^{dum} \quad (16)$$

$$+ \sum_{\substack{l_1+l_2+h=m \\ q=l_1+l_2+1}} \langle \mathbf{x}, \dots, \mathbf{x}, e_i, \underbrace{\mathbf{x}, \dots, \mathbf{x}}_{l_1}, m_h(\mathbf{x}, \dots, \mathbf{x}), \underbrace{\mathbf{x}, \dots, \mathbf{x}}_{l_2} \mid \mathbf{x} \rangle_{n,q}^{dum} \quad (17)$$

$$= \sum_{\substack{p+k_1=m \\ k_2+q=n}} \langle \mathbf{x}, \dots, \mathbf{x}, \underline{m_{k_1+k_2+1}(\overbrace{\mathbf{x}, \dots, \mathbf{x}}^{k_1}, \underline{\mathbf{x}}, \overbrace{\mathbf{x}, \dots, \mathbf{x}}^{k_2})}, \mathbf{x}, \dots, \mathbf{x} \mid e_i \rangle_{p,q}. \quad (18)$$

It is important to note that the expression in summand (18) is obtained in $k := k_1 + k_2 + 1$ different ways according to the position of the (underlined) bimodule

element \underline{x} . Namely, different choices of a bimodule element still give rise to equivalent expressions. We also observe that (15) = (10) after summing over $n + m + 1 = N$.

We apply skew-symmetry to (16) and (17). Namely we have

$$-(16) = \sum_{p+j+k_1+k_2+1=N} \langle \mathbf{x}^{\otimes p}, \underline{\mathbf{x}}, \mathbf{x}^{\otimes k_1}, m_j(\vec{\mathbf{x}}), \mathbf{x}^{\otimes k_2} \mid e_i \rangle, \quad (19)$$

$$-(17) = \sum_{p+j+k_1+k_2+1=N} \langle \mathbf{x}^{\otimes p}, m_j(\vec{\mathbf{x}}), \mathbf{x}^{\otimes k_1}, \underline{\mathbf{x}}, \mathbf{x}^{\otimes k_2} \mid e_i \rangle. \quad (20)$$

Here we set $m_j(\vec{\mathbf{x}}) := m_j(\mathbf{x}, \dots, \mathbf{x})$.

In summary, we have the following:

$$(10) = k \cdot (13) + (19) + (20),$$

hence

$$(9) = k \cdot (13) + (19) + (20) + (11) + (12) + (13).$$

Now it remains to show that

$$(19) + (20) + (11) + (12) = (N - k) \cdot (13),$$

which proves the theorem.

Let us list the remaining terms first:

$$(11) \quad \sum_{p+k+j_1+j_2+1=N} \langle \mathbf{x}^{\otimes p}, e_i, \mathbf{x}^{\otimes j_1}, \underline{m_k(\vec{\mathbf{x}})}, \mathbf{x}^{\otimes j_2} \mid \mathbf{x} \rangle,$$

$$(12) \quad \sum_{p+k+j_1+j_2+1=N} \langle \mathbf{x}^{\otimes p}, \underline{m_k(\vec{\mathbf{x}})}, \mathbf{x}^{\otimes j_1}, e_i, \mathbf{x}^{\otimes j_2} \mid \mathbf{x} \rangle,$$

$$(19) \quad \sum_{p+k+j_1+j_2+1=N} \langle \mathbf{x}^{\otimes p}, \underline{\mathbf{x}}, \mathbf{x}^{\otimes j_1}, m_k(\vec{\mathbf{x}}), \mathbf{x}^{\otimes j_2} \mid e_i \rangle,$$

$$(20) \quad \sum_{p+k+j_1+j_2+1=N} \langle \mathbf{x}^{\otimes p}, m_k(\vec{\mathbf{x}}), \mathbf{x}^{\otimes j_1}, \underline{\mathbf{x}}, \mathbf{x}^{\otimes j_2} \mid e_i \rangle.$$

Now we use the closed condition with these terms:

1. By applying the closed condition from Theorem 2.1 to (12) and (19), we obtain (here (a_i, a_j, a_k) corresponds to $(e_i, m_k(\vec{\mathbf{x}}), \mathbf{x})$)

$$\begin{aligned} & \langle \underbrace{\mathbf{x}, \dots, \mathbf{x}, \underline{\mathbf{x}}, \mathbf{x}, \dots, \mathbf{x}}_s, m_k(\vec{\mathbf{x}}), \mathbf{x}^{\otimes r} \mid e_i \rangle + \langle \mathbf{x}, \dots, \mathbf{x}, \underline{m_k(\vec{\mathbf{x}})}, \mathbf{x}, \dots, \mathbf{x}, e_i, \mathbf{x}, \dots, \mathbf{x} \mid \mathbf{x} \rangle \\ & \quad + \langle \mathbf{x}^{\otimes r}, e_i, \mathbf{x}^{\otimes s} \mid m_k(\vec{\mathbf{x}}) \rangle = 0. \end{aligned}$$

In fact, we obtain s different such equations depending on the position of $\underline{\mathbf{x}}$ in the first line. Hence, the sum of expressions (12) and (19) produces s times that of (13) as the last term equals the minus of (13):

$$\langle \mathbf{x}^{\otimes r}, e_i, \mathbf{x}^{\otimes s} \mid m_k(\vec{\mathbf{x}}) \rangle = -\langle \mathbf{x}^{\otimes s}, \underline{m_k(\vec{\mathbf{x}})}, \mathbf{x}^{\otimes r} \mid e_i \rangle.$$

2. Similarly, by applying the closed condition to (11) and (20),

$$\begin{aligned} & \langle \mathbf{x}^{\otimes s}, m_k(\vec{\mathbf{x}}), \underbrace{\mathbf{x}, \dots, \mathbf{x}, \underline{\mathbf{x}}, \mathbf{x}, \dots, \mathbf{x}}_r \mid e_i \rangle \\ & + \langle \mathbf{x}, \dots, \mathbf{x}, e_i, \mathbf{x}, \dots, \mathbf{x}, m_k(\vec{\mathbf{x}}), \mathbf{x}, \dots, \mathbf{x} \mid \mathbf{x} \rangle \\ & + \langle \mathbf{x}^{\otimes r}, \underline{e_i}, \mathbf{x}^{\otimes s} \mid m_k(\vec{\mathbf{x}}) \rangle = 0, \end{aligned}$$

we obtain r different such equations depending on the position of $\underline{\mathbf{x}}$ in the first line.

Hence we obtain $r + s = N - k$ times the expression of (13), which proves Lemma 3.4. \square

4. Potential Ψ and the generalized holonomy map

In this section, we consider another potential Ψ defined in Definition 1.4 for a unital homotopy cyclic A_∞ -algebra. We discuss its gauge invariance and its relationship with the algebraic analogue of generalized holonomy map in [ATZ].

Let us first recall the definition of a unit for an A_∞ -algebra.

Definition 4.1. An element $I \in C^0 = C^{-1}[1]$ is called a unit if

$$\begin{cases} m_{k+1}(x_1, \dots, I, \dots, x_k) = 0 & \text{for } k \geq 2 \text{ or } k = 0 \\ m_2(I, x) = (-1)^{\deg x} m_2(x, I) = x. \end{cases}$$

We assume that the strong homotopy inner product $\phi: A \rightarrow A^*$ is a unital A_∞ -bimodule map, or $\phi_{k,l}(\vec{a}, v, \vec{b})(w)$ vanishes if one of a_i 's or b_i 's is a constant multiple of I .

We also recall the Maurer-Cartan elements and its gauge equivalences.

Definition 4.2. Let A be an A_∞ -algebra. An element $b \in A^1$ satisfying $m(e^b) = \sum_k m_k(b, \dots, b) = 0$ is called a Maurer-Cartan element, and we denote by $MC(A)$ the set of all Maurer-Cartan elements. Let $\mathcal{MC} := MC / \sim$ be the moduli space of Maurer-Cartan elements, whose gauge equivalence is defined as follows (Definition 2.3 of [Fu]): b is *gauge equivalent* to \tilde{b} if there are one-parameter families $b(t) \in A^1[t], c(t) \in A^0[t]$ such that

- $b(0) = b, b(1) = \tilde{b}$, and
- $\frac{d}{dt}b(t) = \sum_{k \geq 1} m_k(b(t), \dots, b(t), c(t), b(t), \dots, b(t))$.

We remark that $b(t)$ is also a Maurer-Cartan element for any t (Lemma 4.3.7 of [FOOO]). Now, we prove the gauge invariance of the potential Ψ for Maurer-Cartan elements.

Proposition 4.3. *The potential $\Psi(x) = \sum_{p,q \geq 0} \frac{1}{p+q+1} \langle x^{\otimes p} \otimes \underline{x} \otimes x^{\otimes q} \mid I \rangle$, when restricted to the Maurer-Cartan elements MC is invariant under gauge equivalences. That is, if $x(t)$ is a one-parameter family in the Maurer-Cartan solution space, then*

$$\frac{d}{dt} \Psi(x(t)) = 0.$$

Proof. We prove this proposition with the help of following lemmas:

Lemma 4.4. $\Psi(x)$ equals the following expression: $\Psi(x) = \sum_{k \geq 0} \langle \underline{x} \otimes x^{\otimes k} \mid I \rangle$.

Proof. By the closedness condition of ϕ , for any p and q we have

$$\begin{aligned} \langle x^{\otimes p} \otimes \underline{x} \otimes x^{\otimes q} \mid I \rangle + \langle x^{\otimes p+q} \otimes \underline{I} \mid x \rangle \\ + \langle x^{\otimes q} \otimes I \otimes \underline{x} \otimes x^{\otimes p-1} \mid x \rangle = 0. \end{aligned}$$

By definition of unital A_∞ -bimodule homomorphisms, we have

$$\langle x^{\otimes q} \otimes I \otimes \underline{x} \otimes x^{\otimes p-1} \mid x \rangle = 0,$$

and the above equation gives

$$\langle x^{\otimes p} \otimes \underline{x} \otimes x^{\otimes q} \mid I \rangle = -\langle x^{\otimes p+q} \otimes \underline{I} \mid x \rangle = \langle \underline{x} \otimes x^{\otimes p+q} \mid I \rangle,$$

where the last equality follows from the skew-symmetry of ϕ . This proves the lemma. \square

Lemma 4.5.

$$\sum_{\sigma \in \mathbb{Z}/n\mathbb{Z}} \langle \underline{a_{\sigma(1)}}, a_{\sigma(2)}, \dots, a_{\sigma(n-1)} \mid a_{\sigma(n)} \rangle = 0.$$

Proof. Fix a_1, \dots, a_n and denote $[i, j] := \langle \dots, \underline{a_i}, \dots \mid a_j \rangle$. Then what we need to prove is

$$[1, n] + [2, 1] + \dots + [n, n-1] = 0.$$

The closedness condition of strong homotopy inner products gives

$$[i, j] + [j, k] = [i, k].$$

Hence, it follows that

$$[1, n] + [n, n-1] + \dots + [2, 1] = [1, n] + [n, 1] = 0. \quad \square$$

Now we prove the above proposition. First, assume

$$\frac{d}{dt} x(t) = \sum_{i+j=k \geq 0} m_{k+1} (x(t)^{\otimes i} \otimes c(t) \otimes x(t)^{\otimes j}).$$

We denote x by $x(t)$ and c by $c(t)$ for they cause no problem in this proof.

Applying Lemma 4.4, the fraction disappears and we get

$$\frac{d}{dt} \Psi(x) = \sum_{l \geq 0} \left\langle \sum_{i+j=k \geq 0} m_{k+1} (x^{\otimes i} \otimes c \otimes x^{\otimes j}) \otimes x^{\otimes l} \mid I \right\rangle \quad (21)$$

$$+ \sum_{l, m \geq 0} \left\langle \underline{x} \otimes x^{\otimes l} \otimes \sum_{i+j=k \geq 0} m_{k+1} (x^{\otimes i} \otimes c \otimes x^{\otimes j}) \otimes x^{\otimes m} \mid I \right\rangle. \quad (22)$$

To prove that it is zero, we use the A_∞ -bimodule equation. Namely, we compute

$$(\phi \circ \widehat{m} - m^* \circ \widehat{\phi}) \left(\sum_{l \geq 0} \underline{c} \otimes x^{\otimes l} + \sum_{l, m \geq 0} \underline{x} \otimes x^{\otimes l} \otimes c \otimes x^{\otimes m} \right) (I),$$

which is *a priori* zero:

$$(\phi \circ \widehat{m})\left(\sum_{i \geq 0} \underline{c} \otimes x^{\otimes i}\right)(I) = \sum_{l \geq 0} \left\langle \sum_{k \geq 0} m_{k+1}(c \otimes x^{\otimes k}) \otimes x^{\otimes l} \mid I \right\rangle \quad (23)$$

$$+ \sum_{l, m \geq 0} \left\langle \underline{c} \otimes x^{\otimes l} \otimes \left(\sum_{k \geq 1} m_k(x^{\otimes k})\right) \otimes x^{\otimes m} \mid I \right\rangle, \quad (24)$$

and (24) is zero by the Maurer-Cartan equation:

$$\begin{aligned} & (\phi \circ \widehat{m})\left(\sum_{i, j \geq 0} \underline{x} \otimes x^{\otimes i} \otimes c \otimes x^{\otimes j}\right)(I) \\ &= \sum_{l, m \geq 0} \left\langle \sum_{k \geq 1} m_k(x^{\otimes k}) \otimes x^{\otimes l} \otimes c \otimes x^{\otimes m} \mid I \right\rangle \end{aligned} \quad (25)$$

$$+ \sum_{l \geq 0} \left\langle \sum_{i \geq 1, j \geq 0} m_k(x^{\otimes i} \otimes c \otimes x^{\otimes j}) \otimes x^{\otimes l} \mid I \right\rangle \quad (26)$$

$$+ \sum_{l, m \geq 0} \left\langle \underline{x} \otimes x^{\otimes l} \otimes \sum_{i+j=k \geq 0} m_{k+1}(x^{\otimes i} \otimes c \otimes x^{\otimes j}) \otimes x^{\otimes m} \mid I \right\rangle \quad (27)$$

$$+ \sum_{l, m, n \geq 0} \left\langle \underline{x} \otimes x^{\otimes l} \otimes c \otimes x^{\otimes m} \otimes \sum_{k \geq 1} m_k(x^{\otimes k}) \otimes x^{\otimes n} \mid I \right\rangle \quad (28)$$

$$+ \sum_{l, m, n \geq 0} \left\langle \underline{x} \otimes x^{\otimes l} \otimes \sum_{k \geq 1} m_k(x^{\otimes k}) \otimes x^{\otimes m} \otimes c \otimes x^{\otimes n} \mid I \right\rangle. \quad (29)$$

Remark again that (25), (28) and (29) vanish by the Maurer-Cartan equation. Observe also that

- (23) + (26) = (21),
- (27) = (22).

It remains to show that

$$(m^* \circ \widehat{\phi})\left(\sum_{l \geq 0} \underline{c} \otimes x^{\otimes l} + \sum_{l, m \geq 0} \underline{x} \otimes x^{\otimes l} \otimes c \otimes x^{\otimes m}\right)(I) = 0.$$

Since I is the unit, we may easily verify that

$$(m^* \circ \widehat{\phi})\left(\sum_{l \geq 0} \underline{c} \otimes x^{\otimes l}\right)(I) = \sum_{l \geq 0} \langle \underline{c} \otimes x^{\otimes l} \mid x \rangle, \quad (30)$$

$$(m^* \circ \widehat{\phi})\left(\sum_{l \geq 0} \underline{x} \otimes x^{\otimes l} \otimes c\right)(I) = \sum_{l \geq 0} \langle \underline{x} \otimes x^{\otimes l} \mid c \rangle, \quad (31)$$

$$(m^* \circ \widehat{\phi})\left(\sum_{l \geq 0, m \geq 1} \underline{x} \otimes x^{\otimes l} \otimes c \otimes x^{\otimes m}\right)(I) = \sum_{l, m \geq 0} \langle \underline{x} \otimes x^{\otimes l} \otimes c \otimes x^{\otimes m} \mid x \rangle.$$

In (30) and (31), for $l = 0$, we have

$$\langle c \mid x \rangle + \langle x \mid c \rangle = 0$$

by skew-symmetry. For remaining parts, we collect terms appropriately and use closedness condition to show that they all vanish. More precisely, for $k \geq 1$, we claim

that

$$\langle \underline{c} \otimes x^{\otimes k} \mid x \rangle + \langle \underline{x} \otimes x^{\otimes k} \mid c \rangle + \sum_{l+m=k-1} \langle \underline{x} \otimes x^{\otimes l} \otimes c \otimes x^{\otimes m} \mid x \rangle = 0.$$

But this follows from the previous Lemma 4.5 by setting

$$a_1 = c, a_2 = \cdots = a_{k+2} = x. \quad \square$$

Lemma 4.6. *Let $\phi: B \rightarrow B^*$ be a strong homotopy inner product, and let $f: A \rightarrow B$ an A_∞ -quasi-isomorphism, with the pullback strong homotopy inner product $f^*\phi: A \rightarrow A^*$. Given a Maurer-Cartan element $x \in A$, denote by $f_*(x) = \sum_k f_k(x, \dots, x)$ the corresponding Maurer-Cartan element of B . Then, we have*

$$\Psi^A(x) = \Psi^B(f_*(x)).$$

Proof. This can be checked from the Lemma 4.4 as in the case of the potential Φ . We leave the details to the readers. \square

Now, we discuss the potential Ψ and the algebraic generalized holonomy map. We refer readers to [ATZ] or [CL] for the relevant definitions of this construction.

First, recall from Proposition 6.1 of [CL] that given a negative cyclic cohomology class α of an A_∞ -algebra A , one obtains a bimodule map $\widetilde{\alpha}_0: A \rightarrow A^*$. This provides a strong homotopy inner product, if α is, in addition, homologically non-degenerate. Definition 1.4 thus provides the potential Ψ^α using α . Combined with the above proposition, we prove

Theorem 4.7. *The potential Ψ provides a map $\Psi: HC_-^\bullet(A) \rightarrow \mathcal{O}(\mathcal{MC})$ defined by $\alpha \mapsto \Psi^\alpha|_{MC}$. Furthermore, this agrees with the algebraic analogue of generalized holonomy map of Abbaspour, Tradler and Zeinalian [ATZ].*

Proof. We only need to prove the relation with that of [ATZ], and we recall the construction of a map $\rho: HC_-^\bullet(A) \rightarrow \mathcal{O}(\mathcal{MC})$. Here we always work with reduced versions of negative cyclic or Hochschild (co)homologies.

Given a Maurer-Cartan element a of a unital A_∞ -algebra A , consider the expression (Definition 8 of [ATZ])

$$P(a) := \sum_{i \geq 0} I \otimes a^{\otimes i} = (I \otimes I) + (I \otimes a) + (I \otimes a \otimes a) + \cdots.$$

One can check that $P(a)$ is a Hochschild homology cycle from the unital property of I and the Maurer-Cartan equation. Note that the Connes-Tsygan operator B of $P(a)$ vanishes on the reduced complex, due to the unit I . Hence, $P(a)$ can be considered as a negative cyclic homology cycle.

Hence, given a negative cyclic cohomology cycle $\alpha \in HC_-^\bullet(A)$, one can use the pairing $\langle, \rangle: HC_-^\bullet(A) \otimes HC_-^\bullet(A) \rightarrow \mathbf{k}$ to define the map ρ as

$$\rho([\alpha])([a]) := \langle \alpha, \sum_{i \geq 0} I \otimes a^{\otimes i} \rangle.$$

Now, we compare the above expression with that of Lemma 4.4. We recall the following proposition from [CL]:

Proposition 4.8 ([**CL**, Proposition 6.1]). *Let $\alpha \in C_{red}^\bullet(A, A^*)$ be a negative cyclic cocycle. We define*

$$\widetilde{\alpha}_0(\vec{a}, \underline{v}, \vec{b})(w) := \alpha_0(\vec{a}, v, \vec{b})(w) - \alpha_0(\vec{b}, w, \vec{a})(v).$$

Then $\widetilde{\alpha}_0$ is an A_∞ -bimodule map from A to A^ , satisfying the skew-symmetry and closedness condition.*

Negative cyclic cocycles lie in the 2nd and 3rd quadrant of the (b^*, B^*) -bicomplex (see (2.12) of [**CL**] including the 0-th column (y -axis). By α_0 , we mean the 0-th column of α in the (b^*, B^*) -bicomplex. It is easy to see that Hochschild cocycles ($\text{Ker } b^*$) at the 0-th column become negative cyclic cocycles. For a general negative cyclic cocycle α , $b^*\alpha_0$ may not vanish, but equals $B^*\alpha_1$, and it is shown in [**CL**] that $\widetilde{B^*\alpha_1} = 0$.

Also, from the unital property, we have

$$\widetilde{\alpha}_0(\underline{a}, a, \dots, a)(I) = \alpha_0(a, \dots, a)(I) - \alpha_0(a, \dots, a, I)(a) = \alpha_0(a, \dots, a)(I).$$

Hence,

$$\langle \alpha, I \otimes a^{\otimes i} \rangle = \langle \alpha_0, I \otimes a^{\otimes i} \rangle = \alpha_0(a, \dots, a)(I) = \widetilde{\alpha}_0(\underline{a}, a, \dots, a)(I) = \langle \underline{a}, a, \dots, a \mid I \rangle,$$

where the second equality follows from the identification

$$\text{Hom}(A \otimes (A[1]/k \cdot 1)^{\otimes n}, k) \cong \text{Hom}((A[1]/k \cdot 1)^{\otimes n}, A^*).$$

Hence, each term of the function ρ of [**ATZ**] equals the potential Ψ in the paper given in the Lemma 4.4. This proves the theorem. \square

Remark 4.9. From [**CL**], one can observe that the homological non-degeneracy condition is well-defined for negative cyclic cohomology classes (independent of coboundary), and it is shown there that the negative cyclic cohomology class (with homological non-degeneracy) determines an equivalence class of strong homotopy inner products. The value of potential at Maurer-Cartan elements are well-defined up to equivalence classes of strong homotopy inner product from the Lemma 4.6. Thus the map $\Psi: HC_-(A) \rightarrow \mathcal{O}(\mathcal{MC})$, when restricted to the subset with homological non-degeneracy conditions, factors through the equivalence classes of strong homotopy inner products.

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